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Crossconvergence: A new bounded topological presentation

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Abstract

We present crossconvergence, which commonly generalize bornology, preuniform convergence, orderconvergence, b-uniform filtration and grill-determined prenearness. The corresponding construct **CCONV** *forms a strong topological universe, respectively quasitopos, in which quotients are stable under arbitrary products. We also discuss its enlargement to semicrossconvergence and precrossconvergence, respectively.*

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1. Introduction

The new considered bounded, respectively bornological approach to *convergence* provides a high level framework that enables the analysis of convergence-like concepts from the perspective of boundedness rather that the traditional convergence of filters to points or the convergence of uniform filters, which determines uniformity in *special* cases. This shift in viewpoint has proven especially useful in the study of non-normable structures and generalizations of topological and uniform spaces. Over the years, several *classical* frameworks have been introduced in this context, including metric spaces, proximity spaces, uniform spaces, merotopic spaces, supertopological spaces and set-convergence spaces, see 2. Classical

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Concepts. Each of this models offers different perspective on the concept of point-convergence or uniform convergence in mathematical structures.

This striking similarities among them are leading to their embeddings into certain ones, called crossconvergence, semicrossconvergence or precrossconvergence, respectively. Here, it is important to note that **CCONV**, the category of crossconvergence spaces and corresponding maps is forming a strong topological universe [27] or quasitopos [25], respectively, in which quotients are stable under arbitrary products and containing some of the former mentioned important constructs as *nicely* embedded subcategories. Since **CCONV** is *cartesian closed*, thus in that it possesses a natural and well-behaved complete way of forming spaces of functions (which Standard Topology can not boast).

Moreover, **CCONV** is *extensional*, as it possesses a way of extending any space by a single point, which consistency. (Similarly, such consistency can fail to exist). Additionally, **CCONV** corrects particular troubles that occur in any of the mathematical frameworks. For example, it can correctly manage the "uniform components" of a space, which are problematic in uniform spaces (**UNIF**). Furthermore, subspaces of topological spaces behave better when they are formed in **CCONV**. But in **TOP** products of quotients need not be quotients. But in **CCONV** this statement holds (see 6. Convenient properties of former considered constructs). Uniform concepts, such as uniform continuity, uniform convergence, Cauchy sequences (Cfilter, see 7) and completeness are not available in **TOP**. Furthermore, the localization of the concept of precompactness (=total boundedness) leads to a cartesian closed category in which quotients are hereditary, compare with 6.

Other interesting reports [2], [3] and [7] consider closure operators in Categorical Topology and Categorical Algebra. So it is possible to give the characterization of both closed and strongly closed subobjects of an object in the category of semiuniform convergence spaces to show that they form appropriate closure operators which enjoy the basic properties like idempotency, (weak) hereditariness, and productivity in the category of semiuniform convergence spaces. In a category equipped with notion of subobject and closure one may pursue topological concepts in a context no longer confined to "**TOP**-like categories". Par example, universal compactification of topological positively convex sets and moduls can be described as well as Banach limits with respect to convex analysis [24], [29].

This framework now presents a contribution to the area of "Convenient Topology" and builds a *base* that is *both* useful and general, and possesses solid, reliable properties for use in *higher* structures. In addition, this paper serves also as a *revised* edition of the older published papers of b-convergence by the first author [14], [15].

In comparing this revised edition with some other published papers of b-convergence by the first outhor we point out that the new one establishes carefully a hierarchy of setting convergences by starting in *general* with **precrossconvergence** (in short pcc), as the first item, motivated by the fact that set-convergence in the sense of Wyler [32] can be considered as *special* case and thus, the corresponding category **SETCONV** can be nicely embedded into **PCCONV**, the *topological* construct whose objects are the *precrossconvergence spaces* and whose morphisms are the *bounded equiform maps* (in short beq-maps), see section 5. **SCCONV**, the full and isomorphism-closed subcategory of **PCCONV** contains in particular **b-TOP**, the category of *b-topological spaces* and *bounded continuous maps* (in short bc-maps) as nicely embedded subcategory and extends the results in [14], [15]. **SCCONV** is bireflective in **PCCONV**, see section 5. **CCONV**, the full and isomorphism closed subcategory of **SCCONV** is now focus of our considered view, bicoreflective embedded into **SCCONV** and containing **PUCONV**, the category of *preuniform convergence spaces* and *uniformly continuous* maps [27], **b-UFIL**, the category of *b-uniform filter spaces* and *bounded uniformly continuous* maps (in short buc-maps)[17], **ORDC**, the category of *order convergence spaces* and *bounded continuous* maps (in short bic-maps) [21], **G-PNEAR**, the category of *grill-determined prenearness* spaces and corresponding maps, see section 4. **BOUND**, the category of *bounded* spaces and *bounded* maps [32], respectively **BORN**, the full and isomorphism-closed subcategory of **BOUND**, whose objects are the *bornological* spaces [10], see section 4. And last but not least we consider **ECCONV**, the full and isomorphism-closed subcategory of **CCONV**, whose objects are the *epicrossconvergence* spaces, in which **ULIM**, the full and isomorphism-closed subcategory of **PUCONV** whose objects are the *uniform limit* spaces [27] is nicely embedded.

In section 2 we remined the reader at certain *classical* concepts with notation, convention and corresponding references. The here listed members of them are serving now in form of isomorphic components in its corresponding convergences like pcc, scc, cc or ecc, respectively.

In section 3 we consider categorical notions and statements, transferred into the concept of topological constructs (here especially for crossconvergences). In section 4 and 5 the basics of crossconvergences are introduced and important "embedding-theorems" are presented. The purpose of section 6 is to show that the topological construct **CCONV** fulfil *all* the properties for being a *strong topological universe*, meaning that it is quotent-stable, cartesian closed and extensional. And at the end, in section 7, we inform the reader about the possibility to enlarge uniform limit spaces to epicrossconvergence and studying generalized completeness by using so called Cauchy-filter which are convergent. In this context we renounce on the properties for being fixed and saturated, see section 4.

To simplify certain expressions, we provide now a *notion table (NT)* as follows:

For a pair $(\mathcal{B}^X, \mu)$, where $\mathcal{B}^X$ is boundedness and $\mu : \mathcal{B}^X \longrightarrow \underline{P}(FIL(X \times X))$ an operator (map) from $\mathcal{B}^X$ into $\underline{P}(FIL(X \times X))$, $(\mathcal{B}^X, \mu)$ is called

(i) *saturated* iff $\mathcal{B}^X$ is saturated;

(ii) *fixed* iff $B_1, B_2 \in \mathcal{B}^X \backslash \{\emptyset\}$ implying $\mu(B_1) = \mu(B)$;

(iii) *set-based* iff $B \in \mathcal{B}^X$ implies $\dot{B} \times \dot{B} \in \mu(B)$, where $\dot{B} := \{A \subset X : A \supset B\}$;

(iv) *fil-determined* iff $x \in X$ and $\mathcal{U} \in \mu(\{x\})$ implying the existence of an filter $\mathcal{F} \in \tau_\mu(\{x\})$, $\dot{x} \times \mathcal{F} \in \mu(\{x\})$ and $\dot{x} \times \mathcal{F} \subset \mathcal{U}$, where $\tau_\mu(\{x\}) := \{\mathcal{F} \in FIL(X) : \exists\, \mathcal{V} \in \mu(\{x\}), \mathcal{V} \subset \dot{x} \times \mathcal{F}\}$;

(v) *point-bounded* iff $x \in X$ and $\mathcal{U} \in \mu(\{x\})$ implying $\{x\} \times \{x\} \in \mathcal{U}$;

(vi) *grill-based* iff $B \in \mathcal{B}^X$ and $\mathcal{U} \in \mu(B)$ implying the existence of an $\mathcal{G} \in GRL(X) \cap \gamma_\mu$, $\mathrm{sec}\mathcal{G} \times \mathfrak{sec}\mathcal{G} \subset \mathcal{U}$ and $\mathrm{sec}\mathcal{G} \times \mathfrak{sec}\mathcal{G} \in \mu(B)$, where $\gamma_\mu := \{\mathcal{A} \subset \underline{P}X : \exists \mathcal{G} \in GRL(X)\ \exists\ B \in \mathcal{B}^X, \mathcal{A} \subset \mathcal{G}$ and $\mathrm{sec}\mathcal{G} \times \mathfrak{sec}\mathcal{G} \in \mu(B)\}$; here $GRL(X) := \{\mathcal{G} \subset \underline{P}X : \mathcal{G}$ is grill $\}$ with $\mathcal{G}$ is called *grill* (X-grill) iff

  (g1) $\emptyset \notin \mathcal{G}$;

  (g2) $B_1 \cup B_2 \in \mathcal{G}$ iff $B_1 \in \mathcal{G}$ or $\mathcal{B}_2 \in \mathcal{G}$;

  And for $\mathcal{A} \subset \underline{P}X$ and $\mathcal{R} \subset \underline{P}(X \times X)$ we set, $\mathfrak{sec}\mathcal{A} := \{B \subset X : \forall A \in \mathcal{A}\ A \cap B \neq \emptyset\}$, and $\mathfrak{sec}_X\mathcal{R} := \{U \subset X \times X : \forall R \in \mathcal{R}\ R \cap U \neq \emptyset\}$;

(vii) *limited* iff $B \in \mathcal{B}^X \backslash \{\emptyset\}$ and $\mathcal{U} \in \mu(B)$ implying the existence of $\mathcal{F} \in FIL(X)$ s.t. $\dot{B} \times \mathcal{F} \subset \mathcal{U}$ and $\mathcal{F}\ q_\mu\ B$, where for $B \in \mathcal{B}^X \backslash \{\emptyset\}$, $\mathcal{F}\ q_\mu\ B$ iff there exists $\mathcal{V} \in \mu(B)$, $\mathcal{V} \subset \dot{B} \times \mathcal{F}$ and $\mathcal{F}\ q_\mu \emptyset$ iff $\mathcal{F} = \underline{P}X$.

2. Classical Concepts (Notation, Convention and References)

Definition 2.1. A boundedness on a set X is a non-empty subset $\mathcal{B}^X \subset \underline{P}X$, where $\underline{P}X$ denotes the powerset of X, which possesses the following properties:

(b1) $B_1 \subset B \in \mathcal{B}^X$ implies $B_1 \in \mathcal{B}^X$;

(b2) $x \in X$ implies $\{x\} \in \mathcal{B}^X$. Then the pair $(X, \mathcal{B}^X)$ is called a *bounded space* [5], [32].

Sometimes $\mathcal{B}^X$ is called $\underline{B}$-set. [32]

Definition 2.2. A bounded space $(X, \mathcal{B}^X)$ and $\mathcal{B}^X$, respectively are called

(i) *bornological* provided that $\mathcal{B}^X$ satisfies

  (b) $B_1, B_2 \in \mathcal{B}^X$ implying $B_1 \cup B_2 \in \mathcal{B}^X$.

(ii) *saturated*, provided that $X \in \mathcal{B}^X$ is holding and thus $\mathcal{B}^X = \underline{P}X$.

Remark 2.3. Note, that $\mathcal{D}^X := \{\emptyset\} \cup \{\{x\} : x \in X\}$ defines a boundedness on a set X and $\mathcal{E}^X := \{B \subset X : B$ is finite $\}$ is bornology on X. For bounded spaces $(X, \mathcal{B}^X)$, $(Y, \mathcal{B}^Y)$, a map $f : X \longrightarrow Y$ is called *bounded*, provided it satisfies

(bf) $B \in \mathcal{B}^X$ implies $f[B] \in \mathcal{B}^Y$.

BOUND denotes the category, whose objects are the bounded spaces and whose morphisms are the bounded maps [32]. **BORN** denotes the full subcategory of **BOUND**, whose objects are bornological [10].

Definition 2.4. For a set X, $X \times X$ denotes the crossproduct with itself and $\dot{x} := \{A \subset X : x \in A\}$. By $\dot{x} \times \dot{x}$ we denote the filter generated by $\{(x, x)\}$. And for filter $\mathcal{U}, \mathcal{V} \in FIL(X \times X)$; where $FIL(X \times X)$ denotes the set of all filters on $X \times X$, we set $\mathcal{U} \times \mathcal{V} := \{R \subset X \times X : \exists\, U \in \mathcal{U}\, \exists V \in \mathcal{V}, U \times V \subset R\}$, representing the *crossproduct* of $\mathcal{U}, \mathcal{V}$. Notice, that $\underline{P}(X \times X)$ is allowed for being a filter on $X \times X$.

Definition 2.5. A *preuniform convergence* on a set X is a subset $J_X \subset FIL(X \times X)$, where $FIL(X \times X)$ denotes the set of all filters on $(X \times X)$, such that the following are satisfied:

(PUC1) $\mathcal{U}_1 \in J_X$, whenever $\mathcal{U} \in J_X$ and $\mathcal{U} \subset \mathcal{U}_1$;

(PUC2) The filter $\dot{x} \times \dot{x}$ belongs to J_X for each $x \in X$.

Remark 2.6. If $(X, \mathcal{U})$ is a uniform space and $[\mathcal{U}] := \{\mathcal{V} \in FIL(X \times X) : \mathcal{V} \supset \mathcal{U}\}$, then $(X, [\mathcal{U}])$ is a preuniform convergence space. For preuniform convergence spaces (X, J_X), (Y, J_Y), a map $f : X \longrightarrow Y$ is called *uniformly continuous* provided that $(f \times f)(\mathcal{U}) \in J_Y$ for each $\mathcal{U} \in J_X$, where $(f \times f)(\mathcal{U}) := \{V \subset Y \times Y : \exists\, U \in \mathcal{U}$ s.t. $(f \times f)[U] \subset V\}$ with $(f \times f)[U] := \{(f \times f)(x_1, x_2) : (x_1, x_2) \in U\} = \{f(x_1), f(x_2) : (x_1, x_2) \in U\}$. **PUCONV** denotes the category whose objects are the preuniform convergence spaces and whose morphisms are the uniformly continuous maps. [27]

Definition 2.7. For a set $X \neq \emptyset$, a pair $(\mathcal{B}^X, \mu)$ consisting of a boundedness $\mathcal{B}^X$ and a non-empty set $\mu \subset FIL(X \times X)$ is called a *b-uniform filter structure* (*b-uniform filtration*) on X, and the triple $(X, \mathcal{B}^X, \mu)$ a *b-uniform filter space* provided that the following axioms are satisfied:

(buf1) $\mathcal{U} \in \mu$ and $\mathcal{U} \subset \mathcal{U}_1 \in FIL(X \times X)$ implying $\mathcal{U}_1 \in \mu$;

(buf2) $B \in \mathcal{B}^X \setminus \{\emptyset\}$ implies $\dot{B} \times \dot{B} \in \mu$, where $\dot{B} := \{A \subset X : A \supset B\}$.

Remark 2.8. Given a pair of b-uniform filter spaces $(X, \mathcal{B}^X, \mu_X)$, $(Y, \mathcal{B}^Y, \mu_Y)$,a map $f : X \longrightarrow Y$ is called *b-uniformly continuous*, in short *buc*, if f satisfies the following conditions:

(buc1) $B \in \mathcal{B}^X$ implies $f[B] \in \mathcal{B}^Y$;

(buc2) $\mathcal{U} \in \mu_X$ implies $(f \times f)(\mathcal{U}) \in \mu_Y$.

b-UFIL denotes the topological construct of b-uniform filter spaces and b-uniformly continuous maps.[17]

Definition 2.9. An *orderconvergence* is a pair $(\mathcal{B}^X, \tau)$, where $\mathcal{B}^X$ is $\underline{\text{B}}$-set in the sense of Wyler [32], hence a boundedness, and $\tau : \mathcal{B}^X \longrightarrow \underline{P}(FIL(X)) := \{\mathcal{F} \subset \underline{P}X : \mathcal{F}$ is filter $\}$ is function from $\mathcal{B}^X$ into $\underline{P}(FIL(X))$ such that the following properties are satisfied:

(OC1) $\tau(\emptyset) := \{\underline{P}X\}$, here the nullfilter $\underline{P}X$ is *allowed* to be an element of $FIL(X)$.

(OC2) $B \in \mathcal{B}^X$, $\mathcal{F} \in \tau(B)$ and $F \subset \mathcal{F}_1 \in FIL(X)$ implying $\mathcal{F}_1 \in \tau(B)$;
(OC3) $x \in X$ implies $\dot{x} \in \tau(\{x\})$;
(OC4) $B \in \mathcal{B}^X \backslash \{\emptyset\}$ implies $\tau(B) = \cup\{\tau(\{x\}) : x \in B\}$.

Then the triple $(X, \mathcal{B}^X, \tau)$, where $(\mathcal{B}^X, \tau)$ represents an orderconvergence, is called *orderconvergence space*.

Remark 2.10. For orderconvergence spaces $(X, \mathcal{B}^X, \tau_X), (Y, \mathcal{B}^Y, \tau_Y)$, a map $f : X \longrightarrow Y$ is said to be *b-continuous* (in short bc-map), provided that f is bounded and continuous by

(C) $B \in \mathcal{B}^X \backslash \{\emptyset\}$ and $\mathcal{F} \in \tau_X(B)$ implying $f(\mathcal{F}) \in \tau_Y(f[B])$.

ORDC denotes the category whose objects are the orderconvergence spaces and whose morphisms are the b-continuous maps. In this context notice that point-convergence spaces can be considered as specific orderconvergence spaces. [21]

Definition 2.11. A *prenearness* on a set X is a subset $\xi \subset \underline{P}(\underline{P}(\underline{P}X))$ satisfying the following conditions:
(PN1) $\mathcal{A}_1 \ll \mathcal{A} \in \xi$ implies $\mathcal{A}_1 \in \xi$, where $\mathcal{A}_1 \ll \mathcal{A} \Leftrightarrow \forall A_1 \in \mathcal{A}_1 \; \exists A \in \mathcal{A}, A_1 \supset A$;
(PN2) $\{\emptyset\} \notin \xi$ and $\xi \neq \emptyset$;
(PN3) $\mathcal{A} \subset \underline{P}X$ with $\cap\mathcal{A} \neq \emptyset$ implying $\mathcal{A} \in \xi$.

For a prenearness ξ on X, the pair (X, ξ) is called a *prenearness space* [9].
A prenearness ξ is said to be *grill-determined* provided it satisfies

(g) $\mathcal{A} \in \xi$ implies the existence of a grill $\mathcal{G} \subset \underline{P}X$ such that $\mathcal{A} \subset \mathcal{G}$,

where $\mathcal{G} \subset \underline{P}X$ is called a *grill*, iff
(gri1) $\emptyset \notin \mathcal{G}$;
(gri2) $G_1, G_2 \in \mathcal{G} \Leftrightarrow G_1 \in \mathcal{G}$ or $G_2 \in \mathcal{G}$.

Remark 2.12. For prenearness spaces $(X, \xi), (Y, \eta)$, a map $f : X \longrightarrow Y$ is called *near map*, in short *n-map*, provided $f\mathcal{A} \in \eta$, whenever $\mathcal{A} \in \xi$, where $f\mathcal{A} := \{f[A] : A \in \mathcal{A}\}$. By **PNEAR** we denote the category of prenearness spaces and n-maps and by **G-PNEAR** its full subcategory of grill-determined prenearness spaces. In that context we point out that the important construct **CHY** of Cauchy spaces and corresponding maps has a corresponding counterpart in **G-PNEAR** [4].

Definition 2.13. A *supertopology* on a set X is a pair $(\mathcal{B}^X, \theta)$, where $\mathcal{B}^X$ is $\underline{B}$-set and $\theta : \mathcal{B}^X \longrightarrow FIL(X)$ function from $\mathcal{B}^X$ into $FIL(X)$ s.t. the following properties are satisfied:
(STOP1) $\theta(\emptyset) = \underline{P}X$;
(STOP2) $B \in \mathcal{B}^X$ and $V \in \theta(B)$ implying $V \supset B$;
(STOP3) $B \in \mathcal{B}^X$ and $U \in \theta(B)$ implying the existence of $V \in \theta(B)$ that always $U \in \theta(D)$ for any $D \in \mathcal{B}^X$ with $D \subset V$.

Remark 2.14. For supertopological spaces $(X, \mathcal{B}^X, \theta)$, $(Y, \mathcal{B}^Y, \phi)$, a map $f : X \longrightarrow Y$ is called *b-continuous* (in short *bc-map*) provided that for any $B \in \mathcal{B}^X \backslash \{\emptyset\}$ and $V \in \phi(f[B])$, $f^{-1}[V] \in \theta(B)$. By **STOP** we denoting the corresponding construct. In this context notice that in the *special* case $\mathcal{B}^X = \mathcal{D}^X$, θ represents the *Hausdorff-surrounding system* for each $x \in X$. [5]

3. Categorical Background

By a construct we mean a category C whose objects are structured sets, i.e. pairs (X, ξ), where X is a set and ξ a C-structure on X, whose morphisms $f : (X, \xi) \longrightarrow (Y, \eta)$ are suitable maps between X and Y and whose composition law is the *usual* composition of maps.

Definition 3.1. A construct is called *topological* iff it satisfies the following conditions:

(tc1) Existence of *initial* structures:
For any set X, any family $((X_i, \xi_i))_{i \in I}$ of C-objects indexed by a class I and any family $(f_i : X \longrightarrow X_i)_{i \in I}$ of maps indexed by I there exists *an unique* C-structure ξ on X which is *initial* with respect to $(X, f_i, (X_i, \xi_i), I)$, i.e. such that for any ξ-object (Y, η), a map $g : (Y, \eta) \longrightarrow (X, \xi)$ is a C-morphism iff for every $i \in I$ the composite map $f_i \circ g : (Y, \eta) \longrightarrow (X_i, \xi_i)$ is a C-morphism;

(tc2) For any set X, the class $\{(Y, \eta) \in |C| : X = Y\}$ of all C-objects with underlying set X is a set;

(tc3) For any set X with cardiality at most one, there exists *exactly* one C-object with underlying set X (i.e. there exists exactly one C-structure on X).

Definition 3.2. A category C is called *cartesian closed* provided that the following conditions are satisfied:

(cc1) For each pair (A, B) of C-objects there exists a product $A \times B$ in C;

(cc2) For each C -object A holds: For each C object B, there exists some C-object B^A (called *power object*) and some C-morphism $e_{A,B} : A \times B^A \longrightarrow B$ (called *evaluation morphism*) such that for each C-object C and each C-morphism $f : A \times C \longrightarrow B$, there exists an *unique* C- morphism $\overline{f} : C \longrightarrow B^A$ such that the diagram commutes.

$$\begin{tikzcd}
A \times B^A \arrow[rr, "e_{A,B}"] & & B \\
& A \times C \arrow[ul, "1_A \times \overline{f}"] \arrow[ur, "f"'] &
\end{tikzcd}$$

Remark 3.3. For a topological construct C the following statements are equivalent:

(i) C is cartesian closed;

(ii) For any pair $(A, B) \in |C| \times |C|$, the set $[A, B]_e$ can be *endowed* with the structure of a C-object denoted by $\mathcal{B}^A$ such that the following are satisfied:

(α) The evaluation map $e_A, B : A \times B^A \longrightarrow B$ defined by $e_{A,B}(a, g) = g(a)$ for each $(a, g) \in A \times B^A$ is a C-morphism;

(β) For each C-object C and each C morphism $f : A \times C \longrightarrow B$ the map $\overline{f} : C \longrightarrow B^A$ defined by $\overline{f}(c)(a) := f(a, c)$ for each $c \in C$ and each $a \in A$ is a C-morphism. $[A, B]_C$ denotes the set of all C-morphism between A and B.

Note 3.4. *Let $\mathcal{A}$ be a subcategory of a category C and $F_e : \mathcal{A} \longrightarrow C$ be the inclusion functor. Then we will use the theorem that $\mathcal{A}$ is reflective in C iff one of the two equivalent conditions is satisfied:*

(i) *F_e has a left adjoint R;*

(ii) *Each object $X \in |C|$ has a universal map with respect to F_e, i.e. for each $X \in |C|$ there exists an $\mathcal{A}$-object $X_{\mathcal{A}}$ and a C-morphism $r_X : X \longrightarrow X_{\mathcal{A}}$ such that for each $\mathcal{A}$-object Y and each C-morphism $f : X \longrightarrow Y$ there exists an unique $\mathcal{A}$-morphism $\overline{f} : X_{\mathcal{A}} \longrightarrow Y$ such that the diagram*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "r_X"'] & & Y \\ & X_{\mathcal{A}} \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$

commutes. (The functor R is called a reflector).

And it is called bireflective in C iff for each $X \in |C|$ the C-morphism $r_X : X \longrightarrow X_{\mathcal{A}}$ is a bimorphism. Then r_X is said to be the bireflection of X with respect to $\mathcal{A}$.

Note 3.5. *Analogously, we will use the theorem that $\mathcal{A}$ is coreflective in C iff one of the two equivalent conditions is satisfied:*

(i) *F_e has a right adjoint R_c;*

(ii) *Each object $X \in |C|$ has a couniversal map with respect to F_e, i.e. for each $X \in |C|$ there exists an $\mathcal{A}$-object $X_{\mathcal{A}}$ and a C-morphism $c_X : X_{\mathcal{A}} \longrightarrow X$ such that for each $\mathcal{A}$-object Y and each C-morphism $f : Y \longrightarrow X$ there exists an unique $\mathcal{A}$-morphism $\overline{f} : Y \longrightarrow X_{\mathcal{A}}$ such that the diagram commutes. (The functor R_c is called a coreflector).*

$$\begin{tikzcd} X_{\mathcal{A}} \arrow[rr, "c_X"] & & X \\ & Y \arrow[ul, "\overline{f}"] \arrow[ur, "f"'] & \end{tikzcd}$$

And it is called bicoreflective in C iff for each $X \in |C|$ the C-morphism $r_X : X_{\mathcal{A}} \longrightarrow X$ is a bimorphism. Then c_X is said to be the bicoreflection of X with respect to $\mathcal{A}$.

4. Topological constructs

Definition 4.1. A *crossconvergence* (in short *cc*) is a pair $(\mathcal{B}^X, \mu)$, where $\mathcal{B}^X$ is a $\underline{\mathrm{B}}$-set in the sense of Wyler, and thus it is a non-empty subset of $\underline{P}X$, the power set of a set X, stable under subsets and containing all singletons, and $\mu : \mathcal{B}^X \longrightarrow \underline{P}(FIL(X \times X))$ is a function from $\mathcal{B}^X$ into $\underline{P}(FIL(X \times X))$, where $FIL(X \times X)$ denotes the set of all filters on $X \times X$, including the nullfilter, such that the following properties are satisfied:

(CC1) $\mu(\emptyset) := \{\underline{P}(X \times X)\}$;
(CC2) $B \in \mathcal{B}^X$, $\mathcal{U} \in \mu(B)$ and $\mathcal{U} \subset \mathcal{U}_1 \in FIL(X \times X)$ implying $\mathcal{U}_1 \in \mu(B)$;
(CC3) $x \in X$ implies $\overset{\bullet}{x} \times \overset{\bullet}{x} \in \mu(\{x\})$;
(CC4) $\emptyset \neq B_1 \subset B \in \mathcal{B}^X$ implying $\mu(B_1) \subset \mu(B)$;
(CC5) $B \in \mathcal{B}^X \backslash \{\emptyset\}$ implies $\mu(B) \subset \cup\{\mu(\{x\}) : x \in B\}$.

Then we call the triple $(X, \mathcal{B}^X, \mu)$, where $(\mathcal{B}^X, \mu)$ represents a cc, *crossconvergence space* (in short cc-space).
For cc-spaces $(X, \mathcal{B}^X, \mu_X)$, $(Y, \mathcal{B}^Y, \mu_Y)$ a map $f : X \longrightarrow Y$ is said to be *b-equiform* (in short *beq-map*)provided that f is *bounded*, i.e. $B \in \mathcal{B}^X$ implies $f[B] \in \mathcal{B}^Y$ and *equiform*, i.e. $B \in \mathcal{B}^X \backslash \emptyset$ and $\mathcal{U} \in \mu_X(B)$ implying $(f \times f)(\mathcal{U}) \in \mu_Y(f[B])$. **CCONV** denotes the construct of cc-spaces and beq-maps.

Remark 4.2. Notice, that the underlying space $(X, \mathcal{B}^X)$ of an cc-space is bounded. Furthermore, for a boundedness $\mathcal{B}^X$, the triple $(X, \mathcal{B}^X, \mu_b^X)$ constitutes an cc-space, where $\mu_b^X(\emptyset) := \{\underline{P}(X \times X)\}$ and for $B \in \mathcal{B}^X \backslash \{\emptyset\}$ we put: $\mu_b^X(B) := \{\mathcal{U} \in FIL(X \times X) : \exists\, x \in B,\ \mathfrak{sec}_X\, \mathcal{U} \subset \overset{\bullet}{x} \times \overset{\bullet}{x}\}$.
In this context we note, that $(\mathcal{B}^X, \mu_b^X)$ constitutes an cc, which in addition is *point-bounded*, see (NT).

Definition 4.3. By **PB-CCONV** we denote the full subcategory of **CCONV**, whose objects are the point-bounded cc-spaces.

Theorem 4.4. *The constructs **PB-CCONV** and **BOUND** are isomorphic.*

Proof. At the first, compare with 3. Categorical Background. We construct a functor F : **BOUND** $\longrightarrow$ **PB-CCONV** by setting for any bounded space $(X, \mathcal{B}^X)$, $F((X, \mathcal{B}^X)) := (X, \mathcal{B}^X, \mu_b^X)$, as defined in 4.2. And for any bounded map $f : (X, \mathcal{B}^X) \longrightarrow (Y, \mathcal{B}^Y)$ between pairs of bounded spaces we put $F(f) := f$. Then for any bounded space $(Z, \mathcal{B}^z)$ and any bounded map $g : (Y, \mathcal{B}^Y) \rightarrow (Z, \mathcal{B}^Z)$ we have $F(g \circ f) = g \circ f = F(g) \circ F(g)$ since $F(f) : (X, \mathcal{B}^X, \mu_b^X) \rightarrow (Y, \mathcal{B}^Y, \mu_b^Y)$ is beq-map and the composition of beq-maps is beq-map again. For $B \in \mathcal{B}^X \backslash \{\emptyset\}$ let $\mathcal{U} \in \mu_b^X(B)$, hence there exists $x \in B$, $\mathfrak{sec}_X \mathcal{U} \subset \dot{x} \times \dot{x}$. $f(x) \in f[B]$ follows and $\mathfrak{sec}_Y (f \times f)(\mathcal{U}) \subset \dot{f(y)} \times \dot{f(y)}$ is true, because of $\{x\} \times \{x\} \in \mathcal{U}$ implies $\{f(y)\} \times \{f(y)\} \in (f \times f)(\mathcal{U})$, and thus $(f \times f)(\mathcal{U}) \in \mu_b^Y(f[B])$. Conversely, for any point-bounded crossconvergence space $(X, \mathcal{B}^X, \eta)$ we set $G((X, \mathcal{B}^X, \eta)) := (X, \mathcal{B}^X)$ and for any beq-map $f : (X, \mathcal{B}^X, \eta) \rightarrow (Y, \mathcal{B}^Y, \mu)$, $G(f) := f$ as the "forgetfull functor". It remains to prove that $F \circ G = 1_{\textbf{PB-CCONV}}$ and $G \circ F = 1_{\textbf{BOUND}}$, where $1_{\textbf{PB-CONV}}$ denotes the identity functor on **PB-CCONV**, and $1_{\textbf{BOUND}}$ the identity functor on **BOUND**. For a bounded space $(X, \mathcal{B}^X)$ we have $(G \circ F)((X, \mathcal{B}^X)) = G(F((X, \mathcal{B}^X))) = G((X, \mathcal{B}^X, \mu_X^b)) = (X, \mathcal{B}^X) = 1_{\textbf{BOUND}}((X, \mathcal{B}^X))$. And conversely for an point-bounded crossconvergence space $(X, \mathcal{B}^X, \eta)$ we have $(F \circ G)((X, \mathcal{B}^X, \eta)) = F(G((X, \mathcal{B}^X, \eta))) = F((X, \mathcal{B}^X)) = (X, \mathcal{B}^X, \mu_b^X)$, so it remains to verify $\eta = \mu_b^X$. For $B \in \mathcal{B}^X \backslash \{\emptyset\}$ let $\mathcal{U} \in \mu_b^X(B)$, hence $\mathfrak{sec}_X \mathcal{U} \subset \dot{x} \times \dot{x}$ follows for some $x \in B$. But $\dot{x} \times \dot{x} \in \eta(\{x\}) \subset \eta(B)$ implies $\mathcal{U} \in \eta(B)$. Conversely, $\mathcal{U} \in \eta(B)$ implies $\mathcal{U} \in \eta(\{x\})$ for some $x \in B$. By the hypothesis we get $\{x\} \times \{x\} \in \mathcal{U}$ which implying $\mathcal{U} \in \mu_b^X(B)$. □

Remark 4.5. Now, we still adding the fact, that the fundamental construct **BORN** of bornological spaces and bounded maps has an counterpart in **CCONV**, too [10].

Example 4.6. For a preuniform convergence space (X, J) let $\mathcal{B}^X$ be a $\underline{B}$-set. Then we consider the pair $(\mathcal{B}^X, \mu_J)$, where $\mu_J : \mathcal{B}^X \longrightarrow \underline{P}(FIL(X \times X))$ is defined by setting $\mu_J(\emptyset) := \{\underline{P}(X \times X))$ and for $B \in \mathcal{B}^X \backslash \{\emptyset\}$, $\mu_J(B) := J$. Hence $(\mathcal{B}^X, \mu_J)$ constitutes an cc, which in addition is fixed. If $\mathcal{B}^X$ is saturated, then $(\mathcal{B}^X, \mu_J)$ constitutes a fixed and saturated cc, see (NT).

Theorem 4.7. *The full subcategory **FSAT-CCONV**, of **CCONV**whose objects are the fixed, saturated cc-spaces is isomorphic to the construct **PUCONV**.*

Proof. We construct a functor F : **PUCONV** $\longrightarrow$ **FSAT-CCONV** by setting for any preuniform convergence space (X, J_X), $F((X, J)) := (X, \underline{P}X, \mu_{J_X})$, compare with 4.6. And for any uniformly continuous map f between pairs of preuniform convergence spaces (X, J_X), (Y, J_Y) we put $F(f) := f$. Then for any preuniform convergence space (Z, J_Z) and any uniformly continuous map $g : (Y, J_Y) \rightarrow (Z, J_Z)$, we have $F(g \circ f) = g \circ f = F(g) \circ F(f)$, since $F(f) : (X, \underline{P}X, \mu_{J_X}) \rightarrow (Y, \underline{P}Y, \mu_{J_Y})$ is beq-map. Also note that for maps f, g, the equation $(g \circ f) \times (g \circ f) = (g \times g) \circ (f \times f)$ is valid. For $B \in \underline{P}X \backslash \{\emptyset\}$ let $\mathcal{U} \in \mu_{J_X}(B)$, our goal is $(f \times f)(\mathcal{U}) \in \mu_{J_Y}(f[B])$. By the hypothesis, $\mathcal{U} \in J_X$ implies $(f \times f)(\mathcal{U}) \in J_Y = \mu_{J_Y}(f[B])$, which shows the claim. Conversely, we define a functor G : **FSAT-CCONV** $\longrightarrow$ **PUCONV** by setting for any fixed and saturated cc-space $(X, \mathcal{B}^X, \eta)$, $G((X, \mathcal{B}^X, \eta)) := (X, \eta(X))$. $(X, \eta(X)) \in$ **PUCONV** (note that $X \in \mathcal{B}^X$ is holding). And for any beq-map $f : (X, \mathcal{B}^X, \eta) \longrightarrow (Y, \mathcal{B}^Y, \xi)$ between pairs of fixed and saturated cc-spaces we put $G(f) := f$. Then for any fixed and saturated cc-space $(Z, \mathcal{B}^Z, \mu)$ and any beq-map $g : (Y, \mathcal{B}^Y, \xi) \to (Z, \mathcal{B}^Z, \mu)$, we have $G(g \circ f) = g \circ f = G(g) \circ G(f)$, since $G(f) : (X, \eta(X)) \to (Y, \xi(Y))$ is uniformly continuous, and the composition of uniformly continuous maps is uniformly continuous again. So let $\mathcal{U} \in \eta(X)$, then $(f \times f)(\mathcal{U}) \in \xi(f[B])$, since f is beq-map. $Y \in \mathcal{B}^Y$ is valid by the saturation, and $\xi(f[B]) = \xi(Y)$ follows, since $(\mathcal{B}^Y, \xi)$ is fixed. Consequently, $(f \times f)(\mathcal{U}) \in \xi(Y)$ is true. It remains to show that the following equations hold, i.e.

(i) $G \circ F = 1_{\mathbf{PUCONV}}$ and

(ii) $F \circ G = 1_{\mathbf{FSAT\text{-}CCONV}}$.

to (i): For a preuniform convergence space (X, J) we consider $(G \circ F)((X, J)) = G(F((X, J))) = G((X, \underline{P}X, \mu_J)) = (X, \mu_J(X)) = (X, J) = 1_{\mathbf{PUCONV}}(X, J)$. On the other hand for a fixed and saturated cc-space $(X, \mathcal{B}^X, \eta)$ we consider $(F \circ G)((X, \mathcal{B}^X, \eta)) = F(G(X, \mathcal{B}^X, \eta))) = F(X, \eta(X)) = (X, \underline{P}X, \mu_{\eta(X)}) = (X, \underline{P}X, \eta)$. Notice, that for any $B \in \mathcal{B}^X = \underline{P}X$, $\eta(B) = \eta(X) = \mu_{\eta(X)}(B)$. □

Theorem 4.8. *By denoting* ***SAT-CCONV*** *the full and isomorphism-closed subcategory of* ***CCONV*** *whose objects are saturated, then we claim that* ***FSAT-CCONV*** *is bireflective in* ***SAT-CCONV***.

Proof. At the first compare with 3. Categorical Background. For a saturated cc-space $(X, \mathcal{B}^X, \mu)$ of **SAT-CCONV** we consider the triple $(X, \mathcal{B}^X, \mu_{fX})$, where $\mu_{fX} : B^X \to \underline{P}(FIL(X \times X))$ is defined by setting $\mu_{f_X}(\emptyset) := \{\underline{P}(X \times X)\}$ and $\mu_{fX}(B) := \mu(X)$ for any $B \in \mathcal{B}^X \backslash \{\emptyset\}$. Then $1_X : (X, \mathcal{B}^X, \mu) \to (X, \mathcal{B}^X, \mu_{fX})$ is the bireflection of $(X, \mathcal{B}^X, \mu)$ with respect to **FSAT-CCONV**. Notice also, that $\underline{P}X$ equals $\mathcal{B}^X$.
$(X, \underline{P}X, \mu_{fX})$ is a fixed and saturated cc-space such that $1_X : (X, \mathcal{B}^X, \mu) \to (X, \underline{P}X, \mu_{fX})$ is beq-map. Now, let $(Y, \mathcal{B}^Y, \eta)$ be a fixed, saturated cc-space and $f : (X, \mathcal{B}^X, \mu) \to (Y, \mathcal{B}^Y, \eta)$ be beq-map, we have to show that $f : (X, \underline{P}X, \mu_{fX}) \to (Y, \mathcal{P}Y, \eta)$ is equiform. Compare with the following diagram:

$$\begin{tikzcd}
& (X, \mathcal{B}^X, \mu) \arrow[dl, "f"'] \arrow[dr, "1_X"] & \\
(Y, \mathcal{B}^Y, \eta) & & (X, \mathcal{B}^X, \mu_{fX}) \arrow[ll, "f"']
\end{tikzcd}$$

Let for $B \in \underline{P}X \backslash \{\emptyset\} \mathcal{U} \in \mu_{fX}(B)$, hence $\mathcal{U} \in \mu(X)$ implies $(f \times f)(\mathcal{U}) \in \eta(f[X])$ by the hypothesis. Consequently, $(f \times f)(\mathcal{U}) \in \eta(Y)$ implies $(f \times f)(\mathcal{U}) \in \eta(f[B])$, since $(Y, \mathcal{B}^Y, \eta)$ is fixed and saturated. Thus the claim results. □

Theorem 4.9. *The construct* ***SAT-CCONV*** *is a bireflective in* ***CCONV****.*

Proof. At the first compare with 3. Categorical Background. For an object $(X, \mathcal{B}^X, \mu)$ in **CCONV** we are setting $\mu_{sat}(\emptyset) := \{\underline{P}(X \times X)\}$, $\mu_{sat}(B) = \mu(B)$ for any $B \in \mathcal{B}^X \backslash \{\emptyset\}$, and for each $B \in \underline{P}X \backslash \mathcal{B}^X$, $\mu_{sat}(B) = \{\mathcal{U} \in FIL(X \times X) : \exists x \in B, \mathcal{U} \in \mu(\{x\})\}$. Then $1_X : (X, \mathcal{B}^X, \mu) \to (X, \underline{P}X, \mu_{sat})$ is the bireflection of $(X, \mathcal{B}^X, \mu)$ with respect to **SAT-CCONV**. $(X, \underline{P}X, \mu_{sat})$ is a saturated cc-space such that $1_X : (X, \mathcal{B}^X, \mu) \to (X, \underline{P}X, \mu_{sat})$ is beq-map. Now, let $(Y, \mathcal{B}^Y, \eta)$ be a saturated cc-space and $f : (X, \mathcal{B}^X, \mu) \to (Y, \mathcal{B}^Y, \eta)$ be beq-map, we have to show that $f : (X, \underline{P}X, \mu_{sat}) \to (Y, \mathcal{B}^Y, \eta)$ is equiform. Compare with the following diagram:

$$\begin{tikzcd}
 & (X, \mathcal{B}^X, \mu) \arrow[dl, "f"'] \arrow[dr, "1_X"] & \\
(Y, \mathcal{B}^Y, \eta) & & (X, \underline{P}X, \mu_{sat}) \arrow[ll, "f"']
\end{tikzcd}$$

For $B \in \mathcal{B}^X \backslash \{\emptyset\}$ and $\mathcal{U} \in \mu_{sat}(B)$ we have $\mathcal{U} \in \mu(B)$ implying that $(f \times f)(\mathcal{U}) \in \mu(f[B])$ is holding by the hypothesis. In the case that $B \in \underline{P}X \backslash \mathcal{B}^X$ and $\mathcal{U} \in \mu_{sat}(B)$ are valid, we can find an element $x \in B$ such that $\mathcal{U} \in \mu(\{x\}) = \mu_{sat}(\{x\})$. And by the hypothesis $(f \times f)(\mathcal{U}) \in \eta(\{f(x)\})$ follows, which implies that $(f \times f)(\mathcal{U}) \in \eta(f[B])$ is true. □

Corollary 4.10. ***FSAT-CCONV*** *is a full and isomorphism closed subconstruct, which is bireflective in* ***CCONV*** *and thus,* ***PUCONV*** *can be essentially considerd as a bireflective full embedded subcategory of* ***CCONV****.*

Example 4.11. For a b-uniform filter spaces $(X, \mathcal{B}^X, \pi)$ we consider the pair $(\mathcal{B}^X, \mu_\pi)$, where $\mu_\pi : \mathcal{B}^X \longrightarrow \underline{P}(FIL(X \times X))$ is defined by setting $\mu_\pi(\emptyset) := \{\underline{P}(X \times X)\}$ and for $B \in \mathcal{B}^X \backslash \{\emptyset\}$, $\mu_\pi(B) := \pi$, then $(\mathcal{B}^X, \mu_\pi)$ constitutes an cc, which in addition is *set-based*, compare with (NT).

Remark 4.12. In that context we point out, that any *discrete* crossconvergence $(\mathcal{D}^X, \mu)$ is set-based, with $\mathcal{D}^X := \{\emptyset \cup \{\{x\} : x \in X\}$.

Theorem 4.13. *The full subcategory* ***FSET-CCONV*** *of* ***CCONV****, whose objects are the fixed, set-based cc-spaces is isomorphic to the construct* ***b-UFIL****.*

Proof. We compare also with 2., respectively 3. and construct a functor F : **b-UFIL** $\longrightarrow$ **FSET-CCONV** by setting for any b-uniform filter space $(X, \mathcal{B}^X, \pi)$, $F((X, \mathcal{B}^X, \pi)) := (X, \mathcal{B}^X, \mu_\pi)$, compare with 4.12. By the definition $(\mathcal{B}^X, \mu_\pi)$ is fixed, too. And for any b-uniformly continuous map f between pairs of b-uniform filter spaces $(X, \mathcal{B}^X, \pi)$, $(Y, \mathcal{B}^Y, \psi)$ we put $F(f) := f$. Then for any b-uniform filter space $(Z, \mathcal{B}^Z, \emptyset)$ and any buc-map $g : (Y, \mathcal{B}^Y, \psi) \to (Z, \mathcal{B}^Z, \emptyset)$ we have $F(g \circ f) = g \circ f = F(g) \circ F(f)$, since $F(f) : (X, \mathcal{B}^X, \mu_\pi) \to (Y, \mathcal{B}^Y, \mu_\psi)$ is beq-map. For $B \in \mathcal{B}^X \setminus \{\emptyset\}$ let $\mathcal{U} \in \mu_\pi(B)$, our goal is $(f \times f)(\mathcal{U}) \in \mu_\psi(f[B])$. By the hypothesis, $\mathcal{U} \in \pi$ implies $(f \times f)(\mathcal{U}) \in \psi = \mu_\psi(f[B])$, which shows the claim. Conversly, we define a functor G : **FSET-CCONV** $\to$ **b-UFIL** by setting for any fixed and set based cc-space $(X, \mathcal{B}^X, \eta)$, $G((X, \mathcal{B}^X, \eta)) := (X, \mathcal{B}^X, \sigma_\eta)$, where $\sigma_\eta := \{\mathcal{U} : FIL(X \times X) : \exists B \in \mathcal{B}^X \setminus \{\emptyset\}, \mathcal{U} \in \eta(B) \cup \{\underline{P}(X \times X)\}$. $(\mathcal{B}^X, \sigma_\eta)$ constitutes a b-uniform filtration on X. And for any beq-map $f : (X, \mathcal{B}^X, \eta) \to (Y, \mathcal{B}^Y, \xi)$ between fixed and set-based cc-spaces we put $G(f) := f$. Then for any fixed and set-based cc-space $(Z, \mathcal{B}^Z, \mu)$ and any beq-map $g : (Y, \mathcal{B}^Y, \xi) \to (Z, \mathcal{B}^Z, \mu)$ we have $G(g \circ f) = g \circ f = G(g) \circ G(f)$, since $G(f) : (X, \mathcal{B}^X, \sigma_\eta) \to (Y, \mathcal{B}^Y, \sigma_\xi)$ is uniformly continuous. So let $\mathcal{U} \in \sigma_\eta$ and without restriction $\mathcal{U} \in \eta(B)$ for some $B \in \mathcal{B}^X \setminus \{\emptyset\}$. Then $(f \times f)(\mathcal{U}) \in \xi(f[B])$, since f is beq-map. Consequently, $(f \times f)(\mathcal{U}) \in \sigma_\xi$ follows, since f is especially bounded. It remains to show that the following equations hold, i.e.

(i) $G \circ F = 1_{\textbf{b-UFIL}}$ and
(ii) $F \circ G = 1_{\textbf{FSET-CCONV}}$

to (i): For a b-uniform filter space $(X, \mathcal{B}^X, \pi)$ we consider $(G \circ F)((X, \mathcal{B}^X, \pi)) = G(F(X, \mathcal{B}^X, \pi)) = G((X, \mathcal{B}^X, \mu_\pi)) = (X, \mathcal{B}^X, \sigma_{\mu_\pi}) = (X, \mathcal{B}^X, \pi) = 1_{\textbf{b-UFIL}}(X, \mathcal{B}^X, \pi)$, because of $\mathcal{U} \in \sigma_{\mu_\pi}$ implies $\mathcal{U} = \underline{P}(X \times X)$ or $\mathcal{U} \in \mu_\pi(B)$ for some $B \in \mathcal{B}^X \setminus \{\emptyset\}$. But in both cases $\mathcal{U} \in \pi$ is true. Conversely, $\mathcal{U} \in \pi$ implies $\mathcal{U} \in \mu_\pi(\{x\})$ for some $x \in X$, and thus $\mathcal{U} \in \sigma_{\mu_\pi}$.
to (ii): For a fixed and set-based cc-space $(X, \mathcal{B}^X, \eta)$ we consider $(F \circ G)((X, \mathcal{B}^X, \eta)) = F(G(X, \mathcal{B}^X, \eta)) = F(X, \mathcal{B}^X, \sigma_\eta) = (X, \mathcal{B}^X, \mu_{\sigma_\eta}) = (X, \mathcal{B}^X, \eta) = 1_{\textbf{FSET-CCONV}}(X, \mathcal{B}^X, \eta)$ because of $\mathcal{U} \in \mu_{\sigma_\eta}(B)$ for $B \in \mathcal{B}^X \setminus \{\emptyset\}$ implying $\mathcal{U} \in \sigma_\eta$. Without restriction let $\mathcal{U} \in \eta(B_1)$ for some $B_1 \in \mathcal{B}^X \setminus \{\emptyset\}$. Consequently, $\mathcal{U} \in \eta(B)$ is true by the hypothesis. Conversely, let $\mathcal{U} \in \eta(B)$ for some $B \in \mathcal{B}^X \setminus \{\emptyset\}$ hence $\mathcal{U} \in \sigma_\eta = \mu_{\sigma_\eta}(B)$ follows which concludes the proof. □

Example 4.14. For an orderconvergence space $(X, \mathcal{B}^X, \tau)$ we consider the pair $(\mathcal{B}^X, \mu_\tau)$, where $\mu_\tau : \mathcal{B}^X \longrightarrow \underline{P}(FIL(X \times X))$ is defined by setting $\mu_\tau(\emptyset) := \{\underline{P}(X \times X)\}$ and for $B \in \mathcal{B}^X \setminus \{\emptyset\}$, $\mu_\tau(B) := \{\mathcal{U} \in FIL(X \times X) : \exists x \in B\ \exists \mathcal{F} \in \tau(\{x\}), \dot{x} \times \mathcal{F} \subset \mathcal{U}\}$. Hence $(\mathcal{B}^X, \mu_\tau)$ constitutes a crossconvergence on X, which in addition is *fil-determined*, compare also with (NT).

Theorem 4.15. *The full subcategory **FIL-CCONV** of **CCONV**, whose objects are fil-determined is isomorphic to the construct **ORDC**.*

Proof. At the first compare with 2., 3. and (NT). We construct a functor F : **ORDC** $\longrightarrow$ **FIL-CCONV** by setting for any orderconvergence space $(X, \mathcal{B}^X, \tau)$, $F((X, \mathcal{B}^X, \tau)) := (X, \mathcal{B}^X, \mu_\tau)$, compare with 4.15.

And for any bc-map $f : (X, \mathcal{B}^X, \tau_X) \longrightarrow (Y, \mathcal{B}^Y, \tau_Y)$ between pairs of orderconvergence spaces we put $F(f) := f$. Then for any orderconvergence space $(Z, \mathcal{B}^Z, \tau_Z)$ and any bc-map $g : (Y, \mathcal{B}^Y, \tau_Y) \to (Z, \mathcal{B}^Z, \tau_Z)$ we have $F(g \circ f) = g \circ f = F(g) \circ F(f)$, since $F(f) : (X, \mathcal{B}^X, \mu_{\tau_X}) \to (Y, \mathcal{B}^Y, \mu_{\tau_X})$ is beq-map. For $B \in \mathcal{B}^X \backslash \{\emptyset\}$ let $\mathcal{U} \in \mu_{\tau_X}(B)$. There exists $x \in B$ and $\mathcal{F} \in \tau_X(\{x\})$, $\dot{x} \times \mathcal{F} \subset \mathcal{U}$. By the hypothesis $f(\mathcal{F}) \in \tau_Y(\{f(x)\}))$ and $f(x) \in f[B]$ with $\dot{f(x)} \times f(\mathcal{F}) = (f \times f)(\dot{x} \times \mathcal{F}) \subset (f \times f)(\mathcal{U})$ implying $(f \times f)(\mathcal{U}) \in \mu_{\tau_Y}(f[B])$, which shows the claim. Conversely, we define a functor G : **FIL-CCONV** $\to$ **ORDC** by setting for any fil-determined crossconvergence space $(X, \mathcal{B}^X, \eta)$, $G(((X, \mathcal{B}^X, \eta)) = (X, \mathcal{B}^X, \psi_\eta)$ where $(\mathcal{B}^X, \psi_\eta)$ is defined by setting $\psi_\eta(\emptyset) := \{\underline{P}X\}$, and for $B \in \mathcal{B}^X \backslash \{\emptyset\}$ we put: $\psi_\eta(B) := \{\mathcal{F} \in FIL(X) : \exists\, x \in B\ \exists\, \mathcal{U} \in \eta(\{x\}),\ \mathcal{U} \subset \dot{x} \times \mathcal{F}\}$. Consequently, $(X, \mathcal{B}^X, \psi_\eta)$ constitutes an orderconvergence space, compare with (NT). And for any beq-map $f : X, \mathcal{B}^X, \eta) \to (Y, \mathcal{B}^Y, \xi)$ between pairs of fil-determined crossconvergence spaces we put $G(f) := f$. Then for any fil-determined cc-space $(Z, \mathcal{B}^Z, \mu)$ and any beq-map $g : (Y, \mathcal{B}^Y, \xi) \to (Z, \mathcal{B}^Z, \mu)$ we have $G(g \circ f) = g \circ f = G(f) \circ G(f)$, since $G(f) : (X, \mathcal{B}^X, \psi_\eta) \to (Y, \mathcal{B}^Y, \psi_\xi)$ is bc-map, and the composition of bc-maps is bounded continuous again. So let for $B \in \mathcal{B}^X \backslash \{\emptyset\}$, $\mathcal{U} \in \psi_\eta(B)$. Hence there exists an element $x \in B$ and $\mathcal{U} \in \eta(\{x\})$ such that $\mathcal{U} \subset \dot{x} \times \mathcal{F}$. By the hypothesis $(f \times f)(\mathcal{U}) \in \xi(\{f(x)\})$. But then, $f(x) \in f[B]$ and $f \times f)(\mathcal{U}) \subset (f \times f)(\dot{x} \times \mathcal{F}) = \dot{f(x)} \times f(\mathcal{F})$ implying $f(\mathcal{F}) \in \psi_\xi(f[B])$. Now, it remains to show that the following equations hold, i.e.

(i) $G \circ F = 1_{\mathbf{ORDC}}$;
(ii) $F \circ G = 1_{\mathbf{FIL\text{-}CCONV}}$.

to (i): For an orderconvergence space $(X, \mathcal{B}^X, \tau)$ we consider $(G \circ F)((X, \mathcal{B}^X, \tau)) = G(F((X, \mathcal{B}^X, \tau))) = G((X, \mathcal{B}^X, \mu_\tau)) = (X, \mathcal{B}^X, \psi_{\mu_\tau}) = (X, \mathcal{B}^X, \tau) = 1_{\mathbf{ORDC}}((X, \mathcal{B}^X, \tau))$, because of : $\mathcal{F} \in \psi_{\mu_\tau}(B)$ for $B \in \mathcal{B}^X \backslash \{\emptyset\}$ implies the existence of $x \in B$ and $\mathcal{U} \in \mu_\tau(\{x\})$ such that $\mathcal{U} \subset \dot{x} \times \mathcal{F}$. Choose $\mathcal{X} \in \tau(\{x\})$ with $\dot{x} \times \mathcal{X} \subset \mathcal{U}$. Consequently, $\dot{x} \times \mathcal{X} \subset \dot{x} \times \mathcal{F}$ implies $\mathcal{F} \in \tau(B)$. Conversely, let $\mathcal{F} \in \tau(B)$, hence $\mathcal{F} \in \tau(\{x\})$ for some $x \in B$ implies $\dot{x} \times \mathcal{F} \in \mu_\tau(\{x\})$, and thus $\mathcal{F} \in \psi_{\mu_\tau}(B)$, which closing this end.
to (ii): For a fil-determined cc-space $(X, \mathcal{B}^X, \eta)$ we consider $(F \circ G)(((X, \mathcal{B}^X, \eta))) = F(G((X, \mathcal{B}^X, \eta))) = F((X, \mathcal{B}^X, \psi_\eta)) = (X, \mathcal{B}^X, \mu_{\psi_\eta}) = (X, \mathcal{B}^X, \eta) = 1_{\mathbf{FIL\text{-}CCONV}}(X, \mathcal{B}^X, \eta)$, because of: For $\mathcal{U} \in \mu_{\psi_\eta}(B)$, $B \in \mathcal{B}^X \backslash \{\emptyset\}$ we can find $x \in B$ and $\mathcal{F} \in \psi_\eta(\{x\})$ such that $\dot{x} \times \mathcal{F} \subset \mathcal{U}$. Choose $\mathcal{V} \in \eta(\{x\})$ with $\mathcal{V} \subset \dot{x} \times \mathcal{F}$. But then $\mathcal{U} \in \eta(B)$ follows. Conversely $\mathcal{U} \in \eta(B)$ implies the existence of an element $x \in B$ with $\mathcal{U} \in \eta(\{x\})$. By the hypothesis we can find a filter $\mathcal{F} \in \psi_\eta(\{x\})$ with $\dot{x} \times \mathcal{F} \in \eta(\{x\})$ and $\dot{x} \times \mathcal{F} \subset \mathcal{U}$. Consequently $\mathcal{U} \in \mu_{\psi_\eta}(B)$ is true. □

Remark 4.16. As an *fundamental* application we state, that the well-known point-convergence spaces, such as Kent convergence spaces, limit spaces, pretopological as well as topological convergence spaces [23] have a corresponding counterpart in **CCONV**, too.

Theorem 4.17. *The construct **FIL-CCONV** is a full and isomorphism-closed subcategory, which is bicoreflective in **CCONV**.*

Proof. At the first compare with 3. and (NT). For an cc-space $(X, \mathcal{B}^X, \eta)$ we consider the triple $(X, \mathcal{B}^X, \eta_{fil})$, where $\eta_{fil}(\emptyset) := \{\underline{P}(X \times X)\}$ and for $B \in \mathcal{B}^X \backslash \{\emptyset\}$, $\eta_{fil}(B) := \{\mathcal{U} \in FIL(X \times X) : \exists\, x \in B\, \exists\, \mathcal{F} \in \tau_\eta(\{x\}), \dot{x} \times \mathcal{F} \subset \mathcal{U}\}$. Then $1_X : (X, \mathcal{B}^X, \eta_{fil}) \longrightarrow (X, \mathcal{B}^X, \eta)$ is the bicoreflection of $(X, \mathcal{B}^X, \eta)$ with respect to **FIL-CCONV**.

$(X, \mathcal{B}^X, \eta_{fil})$ is an fil-determined cc-space such that $1_X : (X, \mathcal{B}^X, \eta_{fil}) \longrightarrow (X, \mathcal{B}^X, \eta)$ is beq-map. Now, let $(Y, \mathcal{B}^Y, \mu)$ be an fil-determined cc-space and $f : (Y, \mathcal{B}^Y, \mu) \longrightarrow (X, \mathcal{B}^X, \eta)$ be beq-map, we have to show that $f : (Y, \mathcal{B}^Y, \mu) \longrightarrow (X, \mathcal{B}^X, \eta_{fil})$ is equiform, in square see:

$$\begin{tikzcd}
& (X, \mathcal{B}^X, \eta) & \\
(X, \mathcal{B}^X, \eta_{fil}) \arrow[ur, "1_X"] & & (Y, \mathcal{B}^Y, \mu) \arrow[ul, "f"'] \arrow[ll, "f"']
\end{tikzcd}$$

So let for $D \in \mathcal{B}^Y \backslash \{\emptyset\}$, $\mathcal{V} \in \mu(D)$, hence $\mathcal{V} \in \mu(\{y\})$ for some $y \in D$. By the hypothesis we can find a filter $\mathcal{F} \in \psi_\mu(\{y\})$ with $\dot{y} \times \mathcal{F} \in \mu(\{y\})$ and $\dot{y} \times \mathcal{F} \subset \mathcal{V}$. We have $f(y) \in f[D]$ and $f(\mathcal{F}) \in \psi_\eta(\{f(y)\})$ with $\dot{f(y)} \times f(\mathcal{F}) \subset (f \times f)(\mathcal{V})$, and thus the claim results. Notice especially, that any beq-map between cc-spaces is bc-map between the *underlying* order convergence spaces. □

Example 4.18. For a grill-determined prenearness space (X, ξ) let B^X be $\underline{B}$-set. Then we consider the pair $(\mathcal{B}^X, \mu_\xi)$, where $\mu_\xi(\emptyset) := \{\underline{P}(X \times X)\}$ and for $B \in \mathcal{B}^X \backslash \{\emptyset\}$ we put, $\mu_\xi(B) := \{\mathcal{U} \in FIL(X \times X) : \exists \mathcal{G} \in GRL(X) \cap \xi, \mathfrak{sec}\mathcal{G} \times \mathfrak{sec}\mathcal{G} \subset \mathcal{U}\}$, where for $\mathcal{G} \subset \underline{P}X$, $\mathfrak{sec}\mathcal{G} := \{A \subset X : \forall\, F \in \mathcal{G}\; A \cap F\} \neq \emptyset\}$. Then $(\mathcal{B}^X, \mu_\xi)$ constitutes an crossconvergence which in addition is *grill-based*, compare with 4. and (NT).

In obtaining an *useful* correspondence between **PNEAR** and **CCONV** we will now modify the definition of maps between the objects of their prevailing constructs.

Definition 4.19. By **G-PNEAR**$^\triangledown$ we denote the construct, whose objects are the grill-determined prenearness spaces and whose morphisms are the *sected* n-maps. Here a function $f : (X, \xi) \longrightarrow (Y, \eta)$ between prenearness spaces (X, ξ), (Y, η) is said to be *sected n-map* (in short *sn-map*) provided that for $A \in \xi$, $\mathfrak{sec}(f\,\mathfrak{sec}\mathcal{A}) \in \eta$.

Note 4.20. *Especially, we point out that any sn-map between prenearness spaces always is a n-map. In this context we further note that the pair $(\mathcal{B}^X, \mu_\xi)$ in 4.19 is fixed by the definition, and we denote by*

GFSAT-CCONV *the full subcategory of* ***CCONV****, whose objects are the grill-based, fixed and saturated cc-spaces.*

Theorem 4.21. *The category* ***G-PNEAR***$^{\triangledown}$ *is isomorphic to the construct* ***GFSAT-CCONV****.*

Proof. We construct a functor F : **G-PNEAR**$^{\triangledown}$ $\longrightarrow$ **GFSAT-CCONV** by setting for any grill-determined prenearness space (X,ξ), $F((X,\xi)) := (X, \underline{P}X, \mu_{\xi})$, compare with 4.19. And for any sn-map $f : (X,\xi) \to (Y,\eta)$ between pairs of grill-determined prenearness spaces we put $F(f) := f$. Then for any grill-determined prenearness space (Z,γ) and any sn-map $g : (Y,\eta) \to (Z,\gamma)$ we have $F(g \circ f) = g \circ g = F(g) \circ F(f)$, since $F(f) := (X, \underline{P}X, \mu_{\xi}) \to (Y, \underline{P}Y, \mu_{\eta})$ is beq-map. For $B \in \mathcal{B}^X \backslash \{\emptyset\}$ let $\mathcal{U} \in \mu_{\xi}(B)$. Hence we can find $\mathcal{G} \in GRL(X) \cap \xi$ such that $\mathfrak{sec}\mathcal{G} \times \mathfrak{sec}\mathcal{G} \subset \mathcal{U}$. By the hypothesis $\mathfrak{sec}(f\mathfrak{sec}\mathcal{G}) \in \eta$ follows. Choose $\mathcal{H} \in GR(Y) \cap \eta$ with $\mathfrak{sec}(f\mathrm{sec}\mathcal{G}) \subset \mathcal{H}$, since η is grill-determined. But $\mathfrak{sec}\mathcal{G} \subset f\mathfrak{sec}\mathcal{G}$ implies $\mathfrak{sec}\mathcal{H} \times \mathfrak{sec}\mathcal{H} \subset f(\mathfrak{sec}\mathcal{G}) \times f(\mathfrak{sec}\mathcal{G}) = (f \times f)(\mathfrak{sec}\mathcal{G} \times \mathfrak{sec}\mathcal{G}) \subset (f \times f)(\mathcal{U})$, thus $(f \times f)(\mathcal{U}) \in \mu_{\eta}(f[B])$. Conversely, we define a functor G : **GFSAT-CCONV** $\to$ **G-PNEAR** by setting for any grill-based, fixed and saturated cc-space $(X, \mathcal{B}^X, \mu)$, $G(X, \mathcal{B}^X, \mu) := (X, \gamma_{\mu})$, where $\gamma_{\mu} := \{\mathcal{A} \subset \underline{P}X : \exists \mathcal{G} \in GRL(X)\ \exists\ B \in \mathcal{B}^X, \mathcal{A} \subset \mathcal{G}$ and $\mathfrak{sec}\mathcal{G} \times \mathfrak{sec}\mathcal{G} \in \mu(B)\}$. And for any beq-map $f : (X, \mathcal{B}^X, \mu) \to (Y, \mathcal{B}^Y, \eta)$ between pairs of grill-based, fixed and saturated cc-spaces we put $G(f) := f$. Then for any grill-based, fixed and saturated cc-space $(Z, \mathcal{B}^Y, \psi)$ and any beq-map $g : (Y, \mathcal{B}^Y, \eta) \to (Z, \mathcal{B}^Z, \psi)$ we have $G(g \circ f) = g \circ f = G(g) \circ G(f)$, since $G(f) := (X, \gamma_{\mu}) \to (Y, \gamma\eta)$ is sn-map. $\mathcal{A} \in \gamma\mu$ implies the existence of $\mathcal{G} \in GRL(X)$ and $B \in \mathcal{B}^X$ with $\mathcal{A} \subset \mathcal{G}$ and $\mathfrak{sec}\mathcal{G} \times \mathfrak{sec}\mathcal{G} \in \mu(B)$. Consequently $(f \times f)(\mathfrak{sec}\mathcal{G} \times \mathfrak{sec}\mathcal{G}) \in \eta(f[B])$, $f[B] \in \mathcal{B}^Y$ and $f\mathcal{A} \subset f(\mathcal{G}) \in GRL(Y)$ are valid. But $f(\mathfrak{sec}\mathcal{G}) \subset \mathfrak{sec}f(\mathcal{G}) \in \eta(f[B])$implies $f\mathcal{A} \in \gamma_{\eta}$. (Note also that the composition of sn-maps is a sected n-map again). Now, it remains to show that the following equations hold, i.e.

(i) $G \circ F = 1_{\textbf{P-NEAR}^{\triangledown}}$;
(ii) $F \circ G = 1_{\textbf{GFSAT-CCONV}}$.

to (i): For a grill-determined prenearness space (X,ξ) we have $(G \circ F)((X,\xi)) = G(F((X,\xi)) = G((X, \underline{P}X, \mu_{\xi}))$ $= (X, \gamma_{\mu_{\xi}}) = (X,\xi) = 1_{\textbf{G-PNEAR}}(X,\xi)$ by proving the equation $\xi = \gamma_{\mu_{\xi}}$.
$\mathcal{A} \in \gamma_{\mu_{\xi}}$ implies the existence of $\mathcal{G} \in GRL(X)$ and $B \in \mathcal{B}^X$, $\mathcal{A} \subset \mathcal{G}$ and $\mathfrak{sec}\mathcal{G} \times \mathfrak{sec}\mathcal{G} \in \mu_{\xi}(B)$. Then we can find a grill $\mathcal{H} \in GRL(X) \cap \xi$ with $\mathfrak{sec}\mathcal{H} \times \mathfrak{sec}\mathcal{H} \subset \mathfrak{sec}\mathcal{G} \times \mathfrak{sec}\mathcal{G}$. But $\mathcal{G} \subset \mathcal{H}$ implies $\mathcal{A} \in \xi$. Conversely, let $\mathcal{A} \in \xi$. Since ξ is grill- determined we can choose a grill $\mathcal{G} \in GRL(X) \cap \xi$ such that $\mathcal{A} \subset \mathcal{G}$. Since $X \in \mathcal{B}^X$ and $\mathfrak{sec}\mathcal{G} \times \mathfrak{sec}\mathcal{G} \in \mu_{\xi}(X)$. we are getting $\mathcal{A} \in \gamma_{\mu_{\xi}}$. For a grill-based, fixed and saturated cc-space $(X, \mathcal{B}^X, \xi)$ we have $(F \circ G)((X, \mathcal{B}^X, \eta)) = F(G((X, \mathcal{B}^X, \eta)) = F((X, \gamma_{\eta})) = (X, \mathcal{B}^X, \mu_{\gamma_{\eta}}) = (X, \mathcal{B}^X, \eta) = 1_{\textbf{GFSAT-CCONV}}(X, \mathcal{B}^X, \eta)$ by proving the equation $\mu_{\gamma_{\eta}} = \eta$.
For $B \in \mathcal{B}^X \backslash \{\emptyset\}$ and $\mathcal{U} \in \mu_{\gamma_{\eta}}(B)$ we can find a $\mathcal{G} \in GRL(X) \cap \gamma_{\eta}$ with $\mathfrak{sec}\mathcal{G} \times \mathfrak{sec}\mathcal{G} \subset \mathcal{U}$. Moreover we can choose $\mathcal{H} \in GRL(X)$ and a bounded set $D \in \mathcal{B}^X$, $\mathcal{G} \subset \mathcal{H}$ and $\mathfrak{sec}\mathcal{G} \times \mathfrak{sec}\mathcal{G} \in \eta(D)$. But $\mathfrak{sec}\mathcal{H} \subset \mathfrak{sec}\mathcal{G})$

implies $\mathfrak{sec}\mathcal{G} \times \mathfrak{sec}\mathcal{G} \in \eta(D)$, and the claim results. Conversely, for $B \in \mathcal{B}^X \setminus \{\emptyset\}$ let $\mathcal{U} \in \eta(B)$. Then we can find $\mathcal{G} \in GRL(X) \cap \gamma_\eta$ with $\mathfrak{sec}\mathcal{G} \times \mathfrak{sec}\mathcal{G} \subset \mathcal{U}$ and $\mathfrak{sec}\mathcal{G} \times \mathfrak{sec}\mathcal{G} \in \eta(B)$ by the hypothesis, and the claim results. □

Note 4.22. *On the other hand, in generalizing beq-maps between cc-spaces* $(X, \mathcal{B}^X, \mu_X)$, $(Y, \mathcal{B}^Y, \mu_Y)$, *we call a bounded map* $f : \mathcal{B}^X \longrightarrow \mathcal{B}^Y$ *grillequiform (in short geq-map), provided that for* $\mathcal{G} \in GRL(X)$, $B \in \mathcal{B}^X \setminus \{\emptyset\}$ *and* $\mathfrak{sec}\mathcal{G} \times \mathfrak{sec}\mathcal{G} \in \mu_X(B)$, $\mathfrak{sec} f(\mathcal{G}) \times \mathfrak{sec} f(\mathcal{G}) \in \mu_Y(f[B])$.
In this context we note, that any beq-map between cc-spaces is grillequiform, since for any $\mathcal{G} \in GRL(X)$, $f(\mathfrak{sec}\mathcal{G})$ *is subset of* $\mathfrak{sec} f(\mathcal{G})$. *In addition we mention that for pairs of grill-determined prenearness spacess* (X, ξ_X), (Y, ξ_Y) *and a map* $f : X \longrightarrow Y$, $f : (X, \xi_X) \longrightarrow (Y, \xi_Y)$ *is near map if and only if* $f : (X, \underline{P}X, \mu_{\xi_X}) \longrightarrow (Y, \underline{P}Y, \mu_{\xi_Y})$ *is grillequiform.*

Definition 4.23. **CCONV**$^\triangle$ denotes the category, whose objects are the cc-spaces and whose morphisms are the geq-maps.

Remark 4.24. **CCONV** is obviously a subcategory of **CCONV**$^\triangle$, and the following theorem is valid:

Theorem 4.25. *The category* ***GFSAT-CCONV***$^\triangle$ *is isomorphic to* ***G-PNEAR***.

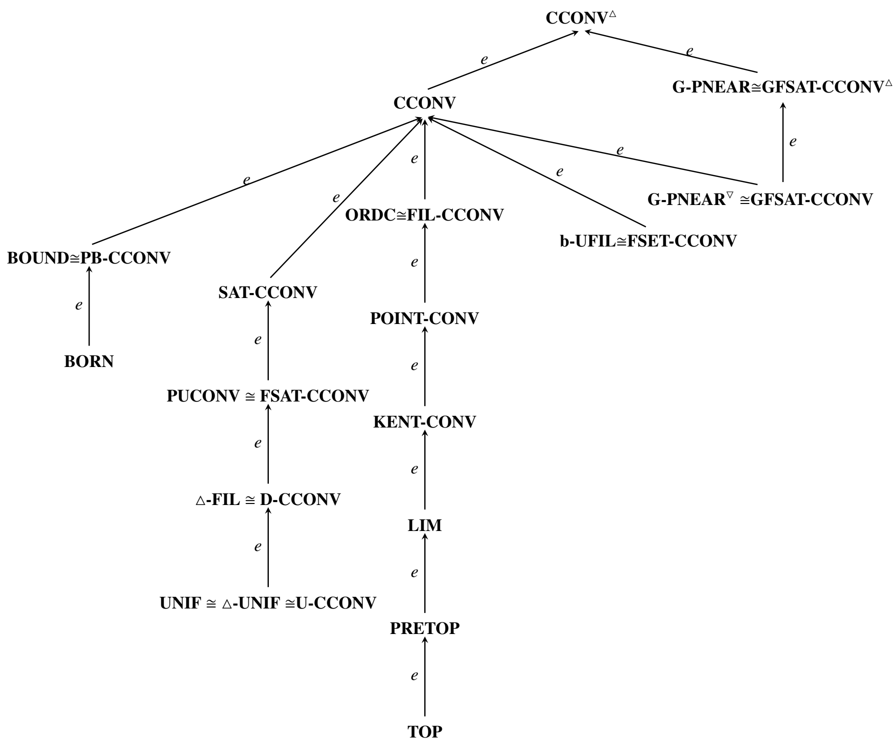
Proof. By applying the former results. □

Example 4.26. At the end of this section we are looking at two *extraordinary* crossconvergences on a set X presented by $(\mathcal{D}^X, \mu_d)$, where $\mathcal{D}^X := \{\emptyset \cup \{\{x\} : x \in X\}$ and μ_d is defined by setting $\mu_d(\emptyset) := \{\underline{P}(X \times X)\}$, and for $x \in X$, $\mu_d(\{x\}) := \{\mathcal{U} \in FIL(X \times X) : \{x\} \times \{x\} \in \mathcal{U}\}$. And the second one by $(\underline{P}X, \mu_P)$, where μ_P is defined by setting $\mu_P(\emptyset) := \{\underline{P}(X \times X)\}$ and for $B \in \underline{P}X \setminus \{\emptyset\}$, $\mu_P(B) := FIL(X \times X)$.

5. Semicrossconvergence and Precrossconvergence

If we omit the axiom (cc5) in the definition 4.1., then we get the notion of a so-called *semicross-convergence* (in short *scc*), and the corresponding space is called *semicrossconvergence space*, (in short *scc-space*). By considering *b-topologies* respectively *b-topological spaces* $(X, \mathcal{B}^X, t)$, [19], which are a *natural* generalization of Kuratowski closure spaces and playing an *important* rule if one is considering *enlarged* topological extensions, [19], we look at the following assignments, see 5.1. Here, a b-topology is a pair $(\mathcal{B}^X, t)$, where $\mathcal{B}^X$ is bornology and $t : \mathcal{B}^X \longrightarrow \underline{P}X$ a map, satisfying the following conditions:

(bt1) $t(\emptyset) = \emptyset$;
(bt2) $B \in \mathcal{B}^X$ implies $t(B) \in \mathcal{B}^X$;
(bt3) $B_1 \subset B \in \mathcal{B}^X$ implying $t(B_1) \subset t(B)$;



Legend

$\xrightarrow{e}$ agree with embedding

$\cong$ agree with isomorphism

FIGURE 1. Diagram of some important categories

(bt4) $B \in \mathcal{B}^X$ implies $B \subset t(B)$;

(bt5) $B \in \mathcal{B}^X$ implies $t(t(B)) \subset t(B)$;

(bt6) $B_1, B_2 \in \mathcal{B}^X$ implying $t(B_1 \cup B_2) \subset t(B_1) \cup t(B_2)$.

In that context we note, that for pairs $(X, \mathcal{B}^X, t_X)$, $(Y, \mathcal{B}^Y, t_Y)$ of b-topological spaces, a bounded map $f : (X, \mathcal{B}^X, t_X) \longrightarrow (Y, \mathcal{B}^Y, t_Y)$ is said to be *continuous* provided that $B \in \mathcal{B}^X \backslash \{\emptyset\}$ implies $f[t_X(B)] \subset t_Y f[B])$, and by **b-TOP** we denoting the corresponding category.

Example 5.1. For any b-topological space $(X, \mathcal{B}^X, t)$ we put:
$\mu_t(\emptyset) := \{\underline{P}(X \times X)\}$, and for $B \in \mathcal{B}^X \backslash \{\emptyset\}$, $\mu_t(B) := \{\mathcal{U} \in FIL(X \times X) : \exists\, \mathcal{F} \in UFIL(X), \mathcal{F} \times \mathcal{F} \subset \mathcal{U}$ and $t(B) \in \mathcal{F}\}$, where $UFIL(X) := \{\mathcal{F} \in \underline{P}^2 X : \mathcal{F}$ is ultrafilter $\}$. $(\mathcal{B}^X, \mu_t)$ constitutes an semicrossconvergence.

Remark 5.2. For an semicrossconvergence $(\mathcal{B}^X, \eta)$ we set: $cl_\eta(\emptyset) := \emptyset$ and for $B \in \mathcal{B}^X \backslash \{\emptyset\}$, $cl_\eta(B) := \{x \in X : \exists \mathcal{F} \in UFIL(X), \mathcal{F} \times \mathcal{F} \in \eta(B) \text{ and } x \in \cap \mathcal{F}\}$. Then $cl_\eta : \mathcal{B}^X \longrightarrow \underline{P}X$ satisfies (bt1), (bt3), (bt4), (bt5)and (bt6), respectively.

Definition 5.3. An semicrossconvergence $(\mathcal{B}^X, \eta)$ is said to be *bornotopic* by satisfying the following conditions:

(bt0) $\mathcal{B}^X$ is bornology;
(bt1) $B \in \mathcal{B}^X$ implies $cl_\eta(B) \in \mathcal{B}^X$, (covered);
(bt2) $B \in \mathcal{B}^X \backslash \{\emptyset\}$ and $\mathcal{U} \in \eta(B)$ implying the existence of $\mathcal{F} \in UFIL(X), \mathcal{F} \times \mathcal{F} \subset \mathcal{U}$ and $cl_\eta(B) \in \mathcal{F}$, (crossfiltered);
(bt3) $B \in \mathcal{B}^X, \mathcal{F} \in UFIL(X)$ and $cl_\eta(B) \in \mathcal{F}$ implying $\mathcal{F} \times \mathcal{F} \in \eta(B)$, (dense);
(bt4) $B \in \mathcal{B}^X$ implies $\eta(cl_\eta(B)) = \eta(B)$, (closed);
(bt5) $B_1, B_2 \in \mathcal{B}^X \backslash \{\emptyset\}$ implying $\eta(B_1 \cup B_2) = \eta(B_1) \cup \eta(B_2)$,(additive).

Remark 5.4. Now by stating $(\mathcal{B}^X, \eta_t)$ is bornotopic, then any bornotopic semicrossconvergence $(\mathcal{B}^X, \eta)$ leads to the following equations,i.e. (i) $cl_{\mu_t} = t$ and (ii) $\mu_{cl_\eta} = \eta$. Thus, we obtain a bijection between the corresponding constructs. And furthermore, for pairs $(X, \mathcal{B}^X, t_X)$, $(Y, \mathcal{B}^Y, t_Y)$ of b-topological spaces, a bounded map $f : (X, \mathcal{B}^X) \longrightarrow (Y, \mathcal{B}^Y)$ is continuous from $(X, \mathcal{B}^X, t_X) \longrightarrow (Y, \mathcal{B}^Y, t_Y)$ iff $f : (X, \mathcal{B}^X, \mu_{t_X}) \longrightarrow (Y, \mathcal{B}^Y, \mu_{t_Y})$ is equiform.
BT-SCCONV denotes the full subcategory of **SCCONV**, the category of scc-spaces and beq-maps, whose objects are the bornotopic semicrossconvergence spaces, then we pose the following theorem:

Theorem 5.5. *The categories **b-TOP** and **BT-SCCONV** are isomorphic.*

Proof. We define a functor F : **b-TOP** $\rightarrow$ **BT-SCCONV** by setting for any b-topological space $(X, \mathcal{B}^X, t)$, $F(X, \mathcal{B}^X, t) := (X, \mathcal{B}^X, \mu_t)$ and conversely a functor G : **BT-SCCONV** $\rightarrow$ **b-TOP** by setting for any bornotopic scc-space $(X, \mathcal{B}^X, \eta) := (X, \mathcal{B}^X, cl_\eta)$. By 5.4., $G \circ F = 1_{\textbf{b-TOP}}$ and $F \circ G = 1_{\textbf{BT-SCCONV}}$.For the prevailing morphisms f between the corresponding spaces, we set $F(f) := f$ and $G(f) := f$. Then the following equations are valid: $F(g \circ f) = F(g) \circ F(f)$ and $G(g \circ f) = G(g) \circ G(f)$ compare with 5.4. □

Note 5.6. *Here, we remind again, that saturated bornotopic semicrosssconvergences and saturated b-topologies are essentially the same species, and thus, they can both be considered as Kuratowski closure operators and vice versa.*

Note 5.7. *If we further omit in definition 4.1. the axiom (cc4), we get the notion of a so-called precross-convergence (in short pcc), and the corresponding space is called precrossconvergence space (in short pcc-space). By considering set-convergence spaces* $(X, \mathcal{B}^X, q)$ *in the sense of Wyler,* [32], *which present*

a general theory of convergences, we look at the following assignment:
For any set-convergence space $(X, \mathcal{B}^X, q)$, where $\mathcal{B}^X$ is $\underline{B}$-set and $q \subset FIL(X) \times \mathcal{B}^X$, a relation between filters and bounded sets on X, such that the following is valid:

(sc1) *$B \in \mathcal{B}^X$ implies $(\overset{\bullet}{B}, B) \in q \Leftrightarrow \overset{\bullet}{B}\, qB$;*
(sc2) *$\mathcal{F} \in FIL(X)$ and $\mathcal{F} q \emptyset$ implying $\mathcal{F} = \underline{P}X$;*
(sc3) *$B \in \mathcal{B}^X$, $\mathcal{F} qB$ and $\mathcal{F} \subset \mathcal{F}_1 \in FIL(X)$ implying $\mathcal{F}_1 qB$,*

We put: $\mu_q(\emptyset) := \{\underline{P}(X \times X)\}$ and for $B \in \mathcal{B}^X \backslash \{\emptyset\}$, $\mu_q(B) := \{\mathcal{U} \in FIL(X \times X) : \exists\, \mathcal{F} \in FIL(X), \mathcal{F} qB$ and $\overset{\bullet}{B} \times \mathcal{F} \subset \mathcal{U}\}$. Then, $(X, \mathcal{B}^X, \mu_q)$ constitutes an pcc-space, which in addition is set-based and limited, (NT).

Remark 5.8. Note, that for a limited set-based pcc $(\mathcal{B}^X, \eta)$, $(X, \mathcal{B}^X, p_\eta)$ is forming the so-called *underlying* set-convergence. Furthermore, we are getting the equations $p_{\mu_q} = q$ and $\mu_{p_\eta} = \eta$. And for pairs $(X, \mathcal{B}^X, q_X)$, $(Y, \mathcal{B}^Y, q_Y)$ of set-convergence spaces and a bounded map $f : (X, \mathcal{B}^X) \longrightarrow (Y, \mathcal{B}^Y)$, $f : (X, \mathcal{B}^X, q_X) \longrightarrow (Y, \mathcal{B}^Y, q_Y)$ is continuous, which means $f(\mathcal{F}) q_Y f[B]$ whenever $\mathcal{F} q_X B$, iff $f : (X, \mathcal{B}^X, \mu_{q_X}) \longrightarrow (Y, \mathcal{B}^Y, \mu_{q_Y})$ is equiform. **LSET-PCCONV** denotes the full subcategory of **PCCONV**, the category of pcc-spaces and beq-maps, whose objects are limited and set-based, then we claim the following theorem:

Theorem 5.9. *The category **SETCONV** of set-convergence spaces and continuous maps is isomorphic to **LSET-PCCONV**.*

Proof. We define a functor F : **SETCONV** $\rightarrow$ **LSET-PCCONV** by setting for any set-convergence space $(X, \mathcal{B}^X, q)$, $F(X, \mathcal{B}^X, q) := (X, \mathcal{B}^X, \mu_q)$ and for any bounded continuous map between pairs of set-convergence spaces $F(f) := f$. Conversely, we set for any limited and set-based cc-space $(X, \mathcal{B}^X, \eta)$, $G((X, \mathcal{B}^X, \eta)) := (X, \mathcal{B}^X, p_\eta)$. Hence $G \circ F = 1_{\textbf{SETCONV}}$, $F \circ G = 1_{\textbf{LSET-PCCONV}}$ and $F(g \circ f) = F(g) \circ F(f)$, $G(g \circ f) = G(g) \circ G(f)$ are true by applying 5.8. □

Remark 5.10. In that context we point out, that any *discrete* pcc $(\mathcal{D}^X, \eta)$ is *especially* set-based, where $\mathcal{D}^X := \{\emptyset\} \cup \{\{x\} : x \in X\}$.

Note 5.11. *Now, let **SET-PCCONV** be denoting the full subcategory of **PCCONV**, whose objects are the set-based precrossconvergence spaces, then we note the following corollary:*

Corollary 5.12. ***LSET-PCCONV** is bicoreflective in **SET-PCCONV**.*

Proof. For an set-based pcc-space $(X, \mathcal{B}^X, \eta)$ we consider the triple $(X, \mathcal{B}^X, \eta_{lim})$, where $\eta_{lim}(\emptyset) := \{\underline{P}(X \times X)\}$, and for $B \in \mathcal{B}^X \backslash \{\emptyset\}$, $\eta_{lim}(B) := \{\mathcal{U} \in FIL(X) : \exists\, \mathcal{F} \in FIL(X), \overset{\bullet}{B} \times \mathcal{F} \subset \mathcal{U}$ and $\mathcal{F} p_\eta B\}$. Then

$(\mathcal{B}^X, \eta_{lim})$ is limited and set-based such that $1_X : (X, \mathcal{B}^X, \eta_{lim}) \longrightarrow (X, \mathcal{B}^X, \eta)$ is the bicorefletion of $(X, \mathcal{B}^X, \eta)$ with respect to **SET-PCCONV**.
Now, let $(Y, \mathcal{B}^Y, \mu)$ be a limited and set-based pcc-space and $f : (Y, \mathcal{B}^Y, \mu) \longrightarrow (X, \mathcal{B}^X, \eta)$ be beq-map, we have to show that $f : (Y, \mathcal{B}^Y, \mu) \longrightarrow (X, \mathcal{B}^X, \eta_{lim})$ is equiform. So let for $D \in \mathcal{B}^Y \backslash \{\emptyset\}$, $\mathcal{V} \in \mu(D)$, hence we can find a filter $\mathcal{F} \in FIL(X)$ such that $\overset{\bullet}{B} \times \mathcal{F} \subset \mathcal{U}$ and $\mathcal{F} p_\mu B$. Hence $f(\mathcal{F}) p_\eta f[B]$ with $\overset{\bullet}{f[B]} \times f(\mathcal{F}) \subset (f \times f)(\mathcal{U})$. Notice, that any beq-map between pcc-spaces is continuous between the underlying set-convergence spaces, and the claim results, compare with the diagram:

$$\begin{tikzcd}
& (X, \mathcal{B}^X, \eta) & \\
(Y, \mathcal{B}^Y, \mu) \arrow[ur, "f"] \arrow[rr, "f"] & & (X, \mathcal{B}^X, \eta_{lim}) \arrow[ul, "1_X"']
\end{tikzcd}$$

□

Remark 5.13. Thus, set-convergence spaces are now considered as *special* cases of set-based precross-convergence spaces, and furthermore, *pseudostop spaces* have also an *counterpart* in **SET-PCCONV**, which offers now a new platform for studying, par example pseudotopological spaces from a *more general* point of view, [32].
At last, we mention that supertopological spaces and its corresponding maps, [5] are *integrated* as well.

Note 5.14. ***SCCONV** is a full and and isomorphism-closed subcategory, which is bireflective in **PC-CONV**.*

Proof. For an pcc-space $(X, \mathcal{B}^X, \mu)$ we set: $\mu_{sc}(\emptyset) := \{\underline{P}(X \times X)\}$ and for $B \in \mathcal{B}^X \backslash \{\emptyset\}$, $\mu_{sc}(B) := \{\mathcal{U} \in FIL(X \times X) : \exists\, B_1 \in \mathcal{B}^X \backslash \{\emptyset\}, B_1 \subset B$ and $\mathcal{U} \in \mu(B_1)\}$. Then $(X, \mathcal{B}^X, \mu_{sc})$ is scc-space, and $1_X : (X, \mathcal{B}^X, \mu) \longrightarrow (X, \mathcal{B}^X, \mu_{sc})$ the bireflection of $(X, \mathcal{B}^X, \mu)$ with respect to **SCCONV**.
Now let $(Y, \mathcal{B}^Y, \eta)$ be scc-space and $f : (X, \mathcal{B}^X, \mu) \longrightarrow (Y, \mathcal{B}^Y, \eta)$ be beq-map, we have to show that $f : (X, \mathcal{B}^X, \mu_{sc}) \longrightarrow (Y, \mathcal{B}^Y, \eta)$ is equiform. So let for $B \in \mathcal{B}^X \backslash \{\emptyset\}$, $\mathcal{U} \in \mu_{sc}(B)$, hence we can find a bounded set $B_1 \in \mathcal{B}^X \backslash \{\emptyset\}$, $B_1 \subset B$ and $\mathcal{U} \in \mu(B_1)$. By the hypothesis, $(f \times f)(\mathcal{U}) \in \eta(f[B_1])$, implies $(f \times f)(\mathcal{U}) \in \eta(f[B])$, and the claim results, compare with the diagram: □

Note 5.15. ***CCONV** is a full and isomorphism-closed subcategory, which is bicoreflective in **SCCONV**.*

Proof. For any scc-space $(X, \mathcal{B}^X, \mu)$ we set: $\mu_{cc}(\emptyset) := \{\underline{P}(X \times X)\}$ and for $B \in \mathcal{B}^X \backslash \{\emptyset\}$, $\mu_{cc}(B) := \{\mathcal{U} \in FIL(X \times X) : \exists x \in B, \mathcal{U} \in \mu(\{x\})\}$. Then $(X, \mathcal{B}^X, \mu_{cc})$ is cc-space, and $1_X : (X, \mathcal{B}^X, \mu_{cc}) \longrightarrow$

$$\begin{tikzcd}[column sep=small]
 & (X,\mathcal{B}^X,\mu) \arrow[dl, "f"'] \arrow[dr, "1_X"] & \\
(Y,\mathcal{B}^Y,\eta) & & (X,\mathcal{B}^X,\mu_{sc}) \arrow[ll, "f"']
\end{tikzcd}$$

$(X,\mathcal{B}^X,\mu)$ the bicoreflection of $(X,\mathcal{B}^X,\mu)$ with respect to **CCONV**. Now let $(Y,\mathcal{B}^Y,\eta)$ be scc-space and $f:(Y,\mathcal{B}^Y,\eta)\longrightarrow(X,\mathcal{B}^X,\mu)$ be beq-map, we have to show that $f:(Y,\mathcal{B}^Y,\eta)\longrightarrow(X,\mathcal{B}^X,\mu_{cc})$ is equiform. So let for $D\in\mathcal{B}^Y\backslash\{\emptyset\}$, $\mathcal{U}\in\eta(D)$. By the hypothesis we can find $y\in D$ with $\mathcal{U}\in\eta(\{y\})$. Hence $(f\times f)(\mathcal{U})\in\mu(\{y\})$ follows, since f is beq-map. But $f(y)\in f[D]$ implies the claim, compare with the diagram. □

$$\begin{tikzcd}[column sep=small]
 & (X,\mathcal{B}^X,\mu) & \\
(Y,\mathcal{B}^Y,\eta) \arrow[ur, "f"] \arrow[rr, "f"] & & (X,\mathcal{B}^X,\mu_{cc}) \arrow[ul, "1_X"']
\end{tikzcd}$$

Example 5.16. In addition, for some examples let $\mathcal{S}\subset\underline{P}X$ and $\mathcal{B}_\mathcal{S}$ be the bornology generated by $\mathcal{S}$, then $(X,\mathcal{B}^\mathcal{S},\mu_b^X)$ forms a point-bounded crossconvergence on X. And for a symmetric topological space (X,t), where $t$denotes the corresponding closure operator, we consider (X,ξ_t), where $\xi_t:=\{\mathcal{A}\subset\underline{P}X:\cap\{t(A):A\in\mathcal{A}\}\neq\emptyset\}$ and then looking at $(\underline{P}X,\mu_{\xi_t})$, see 4.19. Consequently, $\mathcal{A}\in\xi_t$ implies the existence of $x\in X$ such that $x\in\cap\{t(A):A\in\mathcal{A}\}$ and $\mathcal{G}:=\{D\subset X:x\in t(D)\}\in GRL(X)$ with $\mathcal{A}\subset\mathcal{G}\in\xi_t$. Thus $(\underline{P}X,\mu_{\xi_t})$ is object of **GFSAT-CCONV**.
Let a supertopological space be given in the sense of Doitchinov [5], see also 2.13. Then a corresponding precrossconvergence $(\mathcal{B}^X,\mu_\theta)$ will be presented at next, compare also with (NT) and remark 5.8.
Here at first we consider a so-called *neighborhood space* $(X,\mathcal{B}^X,\theta)$, where $\theta:\mathcal{B}^X\longrightarrow FIL(X)$ satisfies the following conditions:

(n1) $\theta(\emptyset)=\underline{P}X$;
(n2) $B\in\mathcal{B}^X$ and $V\in\theta(B)$ implying $V\supset B$;
(n3) $B_1\subset B\in\mathcal{B}^X$ implying $\theta(B)\subset\theta(B_1)$.

Then we consider $(\mathcal{B}^X,\mu_\theta)$, where $\mu_\theta:\mathcal{B}^X\longrightarrow\underline{P}FIL(X\times X)$ is defined for $B\in\mathcal{B}^X$ by setting: $\mu_\theta(B):=\{\mathcal{U}\in FIL(X\times X):\exists\,\mathcal{F}\in t_\theta(B),\overset{\bullet}{B}\times\mathcal{F}\subset\mathcal{U}\}$, where $t_\theta(B):=\{\mathcal{F}\in FIL(X):\mathcal{F}\supset\theta(B)\}$. On the other hand let $(X,\mathcal{B}^X,\eta)$ be pcc-space, then for $B\in\mathcal{B}^X$ we look at $\Theta_\eta(B):=\cap\{\mathcal{F}\in FIL(X):\mathcal{F}\in\tau_\eta(B)\}$,

where $\tau_\eta(B) := \{\mathcal{F} \in FIL(X) : \exists\, \mathcal{V} \in \eta(B)\ \mathcal{V} \overset{\bullet}{\subset B} \times \mathcal{F}\}$. Notice, that $(\mathcal{B}^X, \mu_\theta)$ is especially limited and set-based, but furthermore it satisfies the condition for being *surrounded*.
Here, a corresponding pcc-space $(X, \mathcal{B}^X, \eta)$ is said to be *surrounded* iff for $B \in \mathcal{B}^X$, $\cap\{\mathcal{U} \in FIL(X \times X) : \mathcal{U} \in \mu(B)\} \in \mu(B)$. Thus, neighborhood spaces and corresponding surrounded, limited and set set-based pcc-spaces are *essentially* the same, up to isomorphism. Then, in consequence such a space is said to be *supertopic*, provided that its underlying neighborhood space is a supertopological one. Hence **STOP** and **ST-PRECCONV**, the full subcategory of **PRECCONV**, whose objects are the supertopic pcc-spaces , are isomorphic.

6. Convenient properties of the former considered constructs

At the first, will show that **PCCONV** satisfies the conditions for being a *topological* construct. (see 3. Categorical Background).

Theorem 6.1. *PCCONV is a topological construct.*

Proof. For a one point set $X := \{x\}$ we put: $\mathcal{B}_X := \{\emptyset, \{x\}\}$, $\mu_X(\emptyset) := \{\underline{P}(X \times X)\}$ and $\mu_X(\{x\}) := \{\{(x \times x)\}\}$. Then, $(\mathcal{B}_X, \mu_X)$ is the *only one* precrossconvergence on X. Moreover, for any set X the class of all precrossconvergences on X forms a set. And at last, for a set X and a class I let $(\mathcal{B}^{X_i}, \mu_i)_{i \in I}$ be a family of pcc-spaces and $(f_i : X \longrightarrow X_i)_{i \in I}$ a family of functions. We set $\mathcal{B}^X_{in} := \{B \subset X : \forall\, i \in I$, $f_i[B] \in \mathcal{B}^{X_i}\}$, $\mu_{in}(\emptyset) := \{\underline{P}(X \times X)\}$ and for any $B \in \mathcal{B}^X_{in} \backslash \{\emptyset\}$, $\mu_{in}(B) := \{\mathcal{U} \in FIL(X \times X) : \forall\, i \in I$, $(f_i \times f_i)(\mathcal{U}) \in \mu_i(f_i[B])\}$. Then $(\mathcal{B}^X_{in}, \mu_{in})$ is the coarsest precrossconvergence on X such that for any $i \in I$, $f_i : (\mathcal{B}^X_{in}, \mu_{in}) \longrightarrow (X_i, \mathcal{B}^{X_i}, \mu_i)$ is beq-map. Thus $(\mathcal{B}^X_{in}, \mu_{in})$ constitutes the *initial* precrossconvergence on X with respect to the given data. □

Corollary 6.2. *The constructs* ***SCCONV*** *and* ***CCONV*** *are both topological categories.*

Proof. This is valid by using only categorical arguments with respect to the results in 5.14. and 5.15., respectively. □

Note 6.3. *In CCONV the initial crossconvergence $(\mathcal{B}^X_{in}, \mu_{in})$ with respect to the given data is defined by setting: $\mathcal{B}^X_{in} := \{B \subset X : \forall\, i \in I, f_i[B] \in \mathcal{B}^{X_i}\}$, $\mu_{in}(\emptyset) := \{\underline{P}(X \times X)\}$ and for any $B \in \mathcal{B}^X_{in} \backslash \{\emptyset\}$, $\mu_{in}(B) := \{\mathcal{U} \in FIL(X \times X) : \forall\, i \in I\ \exists\, x_i \in B_i,\ (f_i \times f_i)(\mathcal{U}) \in \mu_i(\{f_i(x_i)\})\}$. Furthermore, in* ***CCONV*** *the final crossconvergence $(\mathcal{B}^X_{fin}, \mu_{fin})$ with respect to the given data is defined by setting: $\mathcal{B}^X_{fin} := \{B \subset X : \exists\, i \in I\ \exists B_i \in \mathcal{B}^{X_i}, B \subset f_i[B_i]\} \cup \mathcal{D}^X$, $\mu_{fin}(\emptyset) := \{\underline{P}(X \times X)\}$ and for $B \in \mathcal{B}^X_{fin} \backslash \{\emptyset\}$, $\mu_{fin}(B) := \{\mathcal{U} \in FIL(X \times X) : \exists\, i \in I\ \exists B_i \in \mathcal{B}^{X_i} \backslash \{\emptyset\}\ \exists\, \mathcal{U}_i \in \mu_i(B_i),\ f_i[B_i] \subset B$ and $(f_i \times f_i)(\mathcal{U}_i) \subset \mathcal{U}\} \cup \{\mathcal{U} \in FIL(X \times X) : \exists\, x \in B, \{x\} \times \{x\} \in \mathcal{U}\}$.*

At the next, we consider the latter property, but more precisely described in some application. So let $f : (X, \mathcal{B}^X, \mu) \longrightarrow (Y, \mathcal{B}^Y, \eta)$ be a quotient map in **CCONV**, i.e. $f : X \longrightarrow Y$ is surjective and $(\mathcal{B}^Y, \eta)$ the final crossconvergence on Y with respect to $(f, \mathcal{B}^X, \mu)$, compare with 6.3., then we state:

Note 6.4. *In **CCONV** products of quotient maps are quotient maps.*

Proof. Let $(f_i : (X_i, \mathcal{B}^{X_i}, \mu_i) \longrightarrow (Y_i, \mathcal{B}^{Y_i}, \eta_i) i \in I$ be a non-empty family of quotient maps in **CCONV** and let

$$\begin{tikzcd}
(X, \mathcal{B}^X, \mu) \arrow[r, "f^{\#}"] \arrow[d, "p_i"'] & (Y, \mathcal{B}^Y, \eta) \arrow[d, "q_i"] \\
(X_i, \mathcal{B}^{X_i}, \mu_i) \arrow[r, "f_i"] & (Y_i, \mathcal{B}^{Y_i}, \eta_i)
\end{tikzcd}$$

be the corresponding product diagram in **CCONV**, where $(X, \mathcal{B}^X, \mu) := \Pi_{i \in I}(X_i, \mathcal{B}^{X_i}, \mu_i)$, $(Y, \mathcal{B}^Y, \eta) := \Pi_{i \in I}(Y_i, \mathcal{B}^{Y_i}, \eta_i)$, p_i, q_i are denoting the corresponding projections for any $i \in I$ and $f^{\#}$ the existing *product* map. Since all f_i are surjective, $f^{\#}$ is surjective. By 5.3., $\mathcal{B}^{Y_i} = \{D_i \subset Y_i : \exists B_i \in \mathcal{B}^{X_i}, D_i \subset f_i[B_i]\}$ for all $i \in I$, $\eta_i(\emptyset) := \{\underline{P}(Y_i \times Y_i)\}$ and for $D_i \in \mathcal{B}^{Y_i} \backslash \{\emptyset\}$, $\eta_i(D_i) = \{\mathcal{V}_i \in FIL(Y_i \times Y_i) : \exists B_i \in \mathcal{B}^{X_i} \backslash \{\emptyset\}, \exists \mathcal{U}_i \in \mu_i(B_i), (f_i \times f_i)(\mathcal{U}_i) \subset \mathcal{V}_i$ and $\mathcal{D}_i \supset f_i[B_i]\}$ for all $i \in I$, because f_i is quotient map for all $i \in I$. Now, we show that $\mathcal{B}^Y$ equals with $\mathcal{B}^Y_{f^{\#}} := \{D \subset Y : \exists B \in \mathcal{B}^X, D \subset f^{\#}[B]\}$ and η equals with $\eta_{f^{\#}}$ where $\eta_{f^{\#}}(\emptyset) := \{\underline{P}(Y \times Y)\}$ and for $D \in \mathcal{B}^Y_{f^{\#}} \backslash \{\emptyset\}$, $\eta_{f^{\#}}(D) = \{\mathcal{W} \in FIL(Y \times Y) : \exists H \in \mathcal{B}^X \backslash \{\emptyset\}, \exists \mathcal{H} \in \mu(H), (f^{\#} \times f^{\#})(\mathcal{H}) \subset \mathcal{W}$ and $D \supset f^{\#}[B]\}$ Thus, the product map $f^{\#}$ is *quotient* map in **CCONV**.
At the first let $D \in \mathcal{B}^Y$, hence $q_i[D] \in \mathcal{B}^{Y_i}$ for each $i \in I$, since $\mathcal{B}^Y$ is the product boundedness for each $i \in I$. But then we can find $B_i \in \mathcal{B}^{X_i}$ with $q_i[D] \subset f_i[B_i]$ for each $i \in I$ by the hypothesis.Thus $B := \prod_{i \in I} B_i \in \mathcal{B}^X$, since $\mathcal{B}^X$ is the product boundedness on X. Consequently, $f^{\#}[B] \supset D$, since $y \in D$ implies $y_i(y) = y_i \in q_i[D] \subset f_i[B_i]$. Hence $q_i(y) = f_i(x_i)$ for $x_i \in B_i$, for every $i \in I$, and then $x = (x_i)_{i \in I} \in B$ with $f^{\#}(x) = y$ follows by using the commutativity of the product diagram. At the second let $D \in \mathcal{B}^Y_{f^{\#}}$. Then we can find $B \in \mathcal{B}^X$ with $f^{\#}[B] \supset D$. Consequently, $q_i[D] \subset q_i[f^{\#}[B]] \in \mathcal{B}^{Y_i}$ for each $i \in I$, and $q_i[D] \in \mathcal{B}^{Y_i}$ follows, showing that $D \in \mathcal{B}^Y$ is valid. Now, we prove that η equals with $\eta_{f^{\#}}$, where $\eta_{f^{\#}}(\emptyset) = \{\underline{P}(Y \times Y)\}$ and $\eta_{f^{\#}}(D) := \{\mathcal{W} \in FIL(Y \times Y) : \exists H \in \mathcal{B}^X \backslash \{\emptyset\} \exists \mathcal{H} \in \mu(H), (f^{\#} \times f^{\#})(\mathcal{H}) \subset \mathcal{W}$ and $f^{\#}[H] \subset D\}$ for any $D \in \mathcal{B}^Y \backslash \emptyset\}$. So let for $D \in \mathcal{B}^Y \backslash \{\emptyset\}$ at first $\mathcal{W} \in \eta_{f^{\#}}(D)$. Then we can find $H \in \mathcal{B}^X \backslash \{\emptyset\}$ and $\mathcal{H} \in \mu(H)$ with $(f^{\#} \times f^{\#})(\mathcal{H}) \subset \mathcal{W}$ and $f^{\#}[H] \subset D$. Consequently, $(q_i \times q_i)((f^{\#} \times f^{\#})(\mathcal{H})) = (f_i \times f_i)((p_i \times p_i)(\mathcal{H})) \in \eta_i(f_i[p_i[H]])$ for each $i \in I$. Otherwise, we have $f_i[p_i[H]] = q_i[f^{\#}[H]]$ for any $i \in I$, hence $\eta_i(f_i[p_i[H]]) = \eta_i(q_i[f^{\#}[H]]) \subset \eta_i(q_i[D])$ follows with $(q_i \times q_i)((f^{\#} \times f^{\#}))(\mathcal{H}) \subset (q_i \times q_i)(\mathcal{W})$,

which implying $(q_i \times q_i)(\mathcal{W}) \in \eta_i(q_i[D])$ for each $i \in I$. Thus $\mathcal{W} \in \eta(D)$, see 5.4.
Conversely, let for $D \in \mathcal{B}^Y \backslash \{\emptyset\}$, $\mathcal{W} \in \eta(D)$, then $(q_i \times q_i)(\mathcal{W}) \in \eta_i(q_i[D])$ for each $i \in I$. Thus, for each $i \in I$ there is some $B_i \in \mathcal{B}^{X_i} \backslash \{\emptyset\}$ and some $\mathcal{H}_i \in \mu_i(B_i)$, with $(f_i \times f_i)(\mathcal{H}_i) \subset (q_i \times q_i)(\mathcal{W})$ and $f_i[B_i] \subset q_i[D]$. If $j : \prod_{i \in I}(X_i \times X_i) \longrightarrow \prod_{i \in I} X_i \times \prod_{i \in I} X_i$ denotes the canonical isomorphism. (i.e. $(j((x_i, z_i)) := ((x_i), (z_i))$ and $\prod_{i \in I} \mathcal{H}_i$ the product filter on $\prod_{i \in I}(X_i \times X_i)$, then $j(\prod_{i \in I} \mathcal{H}_i)$ is a filter on $\prod_{i \in I} X_i \times \prod_{i \in I} X_i$ with $(p_i \times p_i)(j(\prod_{i \in I} \mathcal{H}_i)) = \mathcal{H}_i$ for each $i \in I$. Thus $j(\prod_{i \in I} \mathcal{H}_i) \in \mu(\prod_{i \in I} B_i)$. If $t : \prod_{i \in I}(Y_i \times Y_i) \longrightarrow \prod_{i \in I} Y_i \times \prod_{i \in I} Y_i$ denotes the canonical isomorphism, then $t^{-1}(f^{\#} \times f^{\#})(j(\prod_{i \in I} \mathcal{H}_i)) \subset \prod_{i \in I}(f_i \times f_i)(\mathcal{H}_i) \subset \prod_{i \in I}(q_i \times q_i)(\mathcal{W}) \subset t^{-1}(\mathcal{W})$. Thus, $(f^{\#} \times f^{\#})(j(\prod_{i \in I} \mathcal{H}_i))$ and $f^{\#}[\prod_{i \in I} B_i] \subset D$ shows the claim. □

Remark 6.5. As pointed out by Preuss, [27], the theory of connection and disconnection profits from the *better* behavior of quotients in topological constructs which are *strong* in the sense that quotients are stable under arbitrary products. In **TOP** we have, that *finite* products of quotient maps must not be quotient maps again. Furthermore, *extensionality*, i.e. the existence of one-point extensions, which will be studied at the next, implies that quotients are hereditary. And the theory of connection and disconnection profits from this fact, i.e. the statement, that the quotient space of a uniform space X, obtained by the decomposition of X into its uniform components is uniformly disconnected, is true whenever quotients are formed in **CCONV**, where they are hereditary, but it is false, when quotients are formed in **UNIF**.

Theorem 6.6. *The strong topological construct **CCONV** is extensional.*

Proof. Let $(X, \mathcal{B}^X, \mu)$ be a crossconvergence space. We put $X^* := X \cup \{\infty\}$ with $\infty \notin X$. Let us denote by $i : X \longrightarrow X^*$ the inclusion map. Then we define a crossconvergence $(\mathcal{B}^{X^*}, \mu^*)$ on X^* as follows:
$\mathcal{B}^{X^*} := \mathcal{B}^X \cup \{\{\infty\}\}$, $\mu^*(\emptyset) := \{\underline{P}(X^* \times X^*)\}$, for $B \in \mathcal{B}^{X^*}$ with $\infty \in B$: $\mu^*(B) := FIL(X^* \times X^*)$ and for $B \in \mathcal{B}^{X^*} \backslash \{\emptyset, \{\infty\}\}$: $\mu^*(B) := \{\mathcal{U}^* \in FIL(X^* \times X^*) : \{(\infty, \infty)\}^* \in \mathcal{U}^*$ or $\{(\infty, \infty)\}^* \notin \mathcal{U}^*$ with $(i \times i)^{-1}(\mathcal{U}^*) \in \mu(B)\}$. Note, that for $R \subset X^* \times X^*$, $R^* := R \cup (X^* \times \{\infty\}) \cup (\{\infty\} \times X^*)$, and for $\mathcal{U} \in \mu(B)$, $B \in \mathcal{B}^X \backslash \{\emptyset\}$, the filter $\mathcal{U}^* := \{R^* : R \in \mathcal{U}\} \in \mu^*(B)$, respectively that $\mu^* : \mathcal{B}^{X^*} \longrightarrow FIL(X^* \times X^*)$ is defined for all $B \in \mathcal{B}^{X^*}$, since especially $B \in \mathcal{B}^X \backslash \{\emptyset\}$ iff $B \in \mathcal{B}^{X^*} \backslash \{\emptyset, \{\infty\}\}$. $\mathcal{B}^{X^*}$ is boundedness on X^* Then $(X^*, \mathcal{B}^{X^*}, \mu^*)$ is the one-point extension of $(X, \mathcal{B}^X, \mu)$, which especially means that $(\mathcal{B}^X, \mu)$ is *initial* with respect to $(X, i, (X^*, \mathcal{B}^{X^*}, \mu^*)$ or in other words $(X, \mathcal{B}^X, \mu)$ is *subspace* of $(X^*, \mathcal{B}^{X^*}, \mu^*)$, see also 4.2.
to (cc1): evident by the definition;
to (cc2): Let $B \in \mathcal{B}^{X^*}$ with $\infty \in B$, $\mathcal{U} \in \mu^*(B)$ and $\mathcal{U} \subset \mathcal{U}_1 \in FIL(X^* \times X^*)$, then evidently $\mathcal{U} \in \mu^*(B)$ by the definition;
Now, let $B \in \mathcal{B}^{X^*} \backslash \{\emptyset, \{\infty\}\}$ and for $\mathcal{U} \in \mu^*(B)\{(\infty \times \infty)\}^* \in \mathcal{U}^*$, then for $\mathcal{U}_1 \in FIL(X^* \times X^*)$ with $\mathcal{U} \subset \mathcal{U}_1$ the claim immediately follows. At last let $\{(\infty, \infty)\}^* \notin \mathcal{U}_1$ for $\mathcal{U} \subset \mathcal{U}_1 \in FIL(X^* \times X^*)$ with

$\mathcal{U} \in \mu^*(B)$, $\{(\infty,\infty)\}^* \notin \mathcal{U}$ implies that the trace of $\mathcal{U}$ exists with $(i \times i)^{-1}(\mathcal{U}) \in \mu(B)$. By the hypothesis the trace of $\mathcal{U}_1$ also exists with $(i \times i)^{-1}(\mathcal{U}) \subset (i \times i)^{-1}(\mathcal{U}_1)$. Thus $(i \times i)^{-1}(\mathcal{U})_1 \in \mu(B)$, which implies the claim.
to (cc3): Let $x \in X^*$, then in the case of $x \in X$, we have $\dot{x} \times \dot{x} \in \mu^*(\{x\})$, since $\dot{x} \times \dot{x} \in \mu(\{x\})$ is valid. Otherwise $X = \infty$ implies $\dot{\infty} \times \dot{\infty} \in FIL(X^* \times X^*) = \mu^*(\{\infty\})$ by the definition.
to (cc4): Now let for $\emptyset \neq B_1 \subset B \in \mathcal{B}^{X^*}$, $\infty \in B_1$, then $\mu^*(B_1) = \mu^*(B)$ follows. In the case $\infty \in B$ and $\infty \notin B_1$ we have $B_1 \in \mathcal{B}^X \backslash \{\emptyset\}$, and $\mathcal{U}^* \in \mu^*(B_1)$ implies immediately $\mathcal{U}^* \in \mu^*(B)$.
to (cc5): At last let $B \in \mathcal{B}^{X^*} \backslash \{\emptyset\}$. For $\infty \in B$ we have $\mu^*(\{\infty\}) = FIL(X^* \times X^*) = \mu^*(B)$. On the other hand $\infty \notin B$ implies $B \in \mathcal{B}^{X^*} \backslash \{\emptyset, \{\infty\}\}$. Then $B \in \mathcal{B}^X \backslash \{\emptyset\}$ implies $\mu(B) \subset \mu(\{x\})$ for some $x \in B$. $\mathcal{U}^* \in \mu^*(B)$ and $\{(\infty,\infty)\}^* \in \mathcal{U}^*$, implying $\mathcal{U}^* \in \mu^*(\{x\})$, since $x \neq \infty$. $\{\infty,\infty)\}^* \notin \mathcal{U}^*$ implies the existence of the trace of $\mathcal{U}^*$ with $(i \times i)^{-1}(\mathcal{U}^*) \in \mu(B)$. Hence we can find $z \in B$ with $(i \times i)^{-1}(\mathcal{U}^*) \in \mu(\{z\})$. But $z \neq \infty$ implies $\mathcal{U}^* \in \mu^*(\{z\})$. Evidently, $i : (X, \mathcal{B}^X) \longrightarrow (X^*, \mathcal{B}^{X^*})$ is bounded, and furthermore for $\mathcal{U} \in \mu(B)$, $B \in \mathcal{B}^X \backslash \{\emptyset\}$, $(i \times i)(\mathcal{U}) \in \mu^*(i[B]) = \mu^*(B)$ is valid, since $R^* \in (i \times i)(\mathcal{U})$ implies $R^* \supset (i \times i)[R]$ for some $R \in \mathcal{U}$. Hence $(i \times i)^{-1}[R^*] \supset (i \times i)^{-1}[(i \times i)[R]] = R \neq \emptyset$. Thus the trace of $(i \times i)(\mathcal{U})$ exists and coincides with $\mathcal{U}$. Consequently, i is beq-map. Now, let $(\mathcal{A}^X, \eta)$ be crossconvergence on X such that $i : (X, \mathcal{A}^X, \eta) \longrightarrow (X^*, \mathcal{B}^{X^*}, \mu^*)$ is beq-map. We show that $(\mathcal{A}^X, \eta) \leq (\mathcal{B}^X, \mu)$, which means that $\mathcal{A}^X \subset \mathcal{B}^X$, and for any $A \in \mathcal{A}^X \backslash \{\emptyset\}$, $\mathcal{U} \in \eta(A)$ implying $\mathcal{U} \in \mu(A)$. Thus, $1_X : (X, \mathcal{A}^X, \eta) \longrightarrow (X, \mathcal{B}^X, \mu)$ is beq-map. Now, $A \in \mathcal{A}^X$ implies $i(A) = A \in \mathcal{B}^{X^*}$, hence $A \in \mathcal{B}^X$ follows , since $A = \{\infty\}$ contradicts.
Let for $A \in \mathcal{A}^X \backslash \{\emptyset\}$, $\mathcal{U} \in \eta(A)$. Consequently, $(i \times i)(\mathcal{U}) \in \mu^*(i[A]) = \mu^*(A)$ follows by the hypothesis. And since $A \in \mathcal{B}^{X^*} \backslash \{\emptyset, \{\infty\}\}$ we have $\{(\infty,\infty)\}^* \in (i \times i)(\mathcal{U})$ or $\{(\infty,\infty)\}^* \notin (i \times i)(\mathcal{U})$ with $(i \times i)^{-1}((i \times i)(\mathcal{U})) \in \mu(A)$. But $\{(\infty,\infty)\}^* \in (i \times i)(\mathcal{U})$ contradicts by the definition of X^*. In the other case, the trace of $(i \times i)(\mathcal{U})$ exists, s.t. $\mathcal{U} \in \mu(A)$ follows, since i is injective, and the claim results.
Now, let $(Y, \mathcal{B}^Y, \mu^Y)$ be a crossconvergence space and $f : (A^Y, \mathcal{B}^Y_j, \mu^Y_j) \longrightarrow (X, \mathcal{B}^X, \mu)$ be a beq-map from a subspace $(A^Y, \mathcal{B}^Y_j, \mu^Y_j)$ of $(Y, \mathcal{B}^Y, \mu^Y)$, where $j : A^Y \longrightarrow Y$ denotes the corresponding inclusion. We have to show that $f^* : (Y, \mathcal{B}^Y, \mu^Y) \longrightarrow (X^*, \mathcal{B}^{X^*}, \mu^*)$ is beq-map too, where $f^* : Y \longrightarrow X^*$ is defined by setting:

$$f^*(y) := \begin{cases} f(y) & \text{if} \quad y \in A^Y \\ \infty & \text{if} \quad y \notin A^Y \end{cases}$$

The following diagram illustrates the above situation.

$$\begin{tikzcd}
(A^Y, \mathcal{B}^Y_j, \mu^Y_j) \arrow[r, "j"] \arrow[d, "f"'] & (Y, \mathcal{B}^Y, \mu^Y) \arrow[d, "f^*"] \\
(X, \mathcal{B}^X, \mu) \arrow[r, "i"] & (X^*, \mathcal{B}^{X^*}, \mu^*)
\end{tikzcd}$$

At the first we show that $f^*(Y, \mathcal{B}^Y) \longrightarrow (X^*, \mathcal{B}^{X^*})$ is bounded. So let $D \in \mathcal{B}^Y$. If $f^*[D] = \{\infty\}$, then there is nothing to show. In the case of $\infty \notin f^*[D]$, $D \cap A^Y \in \mathcal{B}^Y_j$ implying $f[D \cap A^Y] \in \mathcal{B}^X$ by the hypothesis. We show that $f^*[D] \subset f[D \cap A^Y]$ is valid. $z \in f^*[D]$ implies $z = f^*(y)$ for some $y \in D$. If assuming $y \in Y\backslash A^Y$, $f^*(y) = \infty$ follows which contradicts. But $y \in A^Y$ shows the claim.
Now, let $\mathcal{U} \in \mu^Y(D)$, $D \in \mathcal{B}^Y\backslash\{\varnothing\}$, our goal is $(f^* \times f^*)(\mathcal{U}) \in \mu^*(f^*[D])$. Choose $y \in D$ s.t. $\mathcal{U} \in \mu^Y(\{y\})$. Then we discuss the alternation $y \in Y\backslash A^Y$ respectively $y \in A^Y$.
$y \in Y\backslash A^Y$ implies $f^*(y) = \infty$ and thus $\mu^*(\{\infty\}) = \mu^*(\{f^*(y)\})$ with $(f^* \times f^*)(\mathcal{U}) \in FIL(X^* \times X^*) = \mu^*(\{f^*(y)\}) \subset \mu^*(f^*[D])$.
$y \in A^Y$ implies $f^*(y) = f(y)$. And $(j \times j)^{-1}(\mathcal{U})$ exists iff $(Y \times Y)\backslash(A^Y \times A^Y) \notin \mathcal{U}$. Choose $R \in \mathcal{U}$ with $R \not\subseteq (Y \times Y)\backslash(A^Y \times A^Y)$. Then there exists $(y_1, y_2) \in R$ with $(y_1, y_2) \in A^Y \times A^Y$, hence the claim results. In this case, $(j \times j)^{-1}(\mathcal{U}) \in \mu^Y_j(\{y\})$, since $(j \times j)^{-1}(\mathcal{U}) \supset \mathcal{U}$. But f is equiform, hence $(f \times f)((j \times j)^{-1}(\mathcal{U})) \in \mu(\{f(y)\})$. Thus $((f \times f)((j \times j)^{-1}(\mathcal{U}))^* \in \mu^*(\{f(y)\})$ (see the begin of the proof 4.7.). But $((f \times f)((j \times j)^{-1}(\mathcal{U})))^* \subset (f^* \times f^*)(\mathcal{U})$ implies $(f^* \times f^*)(\mathcal{U}) \in \mu^*(\{f(y)\})$, and the claim follows. (Note, that for $R \in \mathcal{U}((f \times f)[R \cap (A^Y \times A^Y)])^* = (f \times f)[R \cap (A^Y \times A^Y)] \cup (\{\infty\} \times X^*) \cup (X^* \times \{\infty\}) \supset (f^* \times f^*)[R \cap (A^Y \times A^Y)] \cup (f^* \times f^*)[R \cap (Y \times Y\backslash A^Y \times A^Y)] = (f^* \times f^*)[R]$).
If $(j \times j)^{-1}(\mathcal{U})$ does not exists, $((Y \times Y)\backslash(A^Y \times A^Y) \in \mathcal{U}$. Then $(f^* \times f^*)[(Y \times Y)\backslash(A^Y \times A^Y)] = (f^* \times f^*)[((Y\backslash A^Y) \times (Y\backslash A^Y)) \cup ((A^Y \times (Y\backslash A^Y)) \cup ((Y\backslash A^Y) \times A^Y)] = (f^*[Y\backslash A^Y] \times f^*[Y\backslash A^Y]) \cup (f^*[A^Y] \times f^*[Y\backslash A^Y]) \cup (f^*[Y\backslash A^Y] \times f^*[A^Y]) = (\{\infty\} \times \{\infty\}) \cup (f[A^Y] \times \{\infty\}) \cup (\{\infty\} \times f[A^Y]) \subset \{(\infty, \infty)\}^*$, i.e. $\{(\infty, \infty)\}^* \in (f^* \times f^*)(\mathcal{U})$. Thus $(f^* \times f^*)(\mathcal{U}) \in \mu^*(\{f(y)\})$, and the claim follows. $\square$

Remark 6.7. **TOP** is *not* extensional, since in **TOP** quotients are *not* hereditary. Furthermore, as Preuss, [27] has shown, **UNIF**, the category of uniform spaces and uniformly continuous maps and **ULIM**, the category of uniform limit spaces and uniformly continuous maps are *not* extensional. On the other hand **SUCONV**, the category of semiuniform convergence spaces and related maps, [26] respectively **GCONV**, the category of generalized point-convergence spaces and continuous maps are *both* extensional. And, at this end, we still note that **b-UFIL**, the category of b-uniform filter spaces and bounded uniformly continuous maps, [17] and **ORDC**, the category of orderconvergence spaces and bounded continuous maps, [21] possesses also the above mentioned property.
Hence, **CCONV** takes one's place into the concept of convenient topological constructs, because as the

next, in addition, we show that **CCONV** has natural function space structures, with is leading to the property of being cartesian closed. And, in this context, **CCONV** fulfils the following laws:

(1) *First* exponential law, $X^{Y\times Z}$ is isomorphic to $(X^Y)^Z$;
(2) *Second* exponential law, $(\prod_{i\in I} X_i)^Y$ is isomorphic to $\prod_{i\in I}(X_i^Y)$;
(3) *Third* exponential law, $X^{\coprod^{i\in I} Y_i}$ is isomorphic to $\prod_{i\in I}(X^{Y_i})$;
(4) *Distributive* law, (see also [27]), $X \times \coprod^{i\in I} Y_i$ is isomorphic to $\coprod^{i\in I}(X \times Y)$.

Proposition 6.8. *For two crossconvergence spaces $(X, \mathcal{B}^X, \mu^X)$, $(Y, \mathcal{B}^Y, \mu^Y)$ we consider the set $Y^X := \{f | f : X \longrightarrow Y$ is beq-map$\}$ and define a crossconvergence $(\mathcal{B}^{Y^X}, \mu^{Y^X})$ on Y^X by setting $\mathcal{B}^{Y^X} : \{B^* \subset Y^X : \forall\ B \in \mathcal{B}^X,\ B^*(B) \in \mathcal{B}^Y$, where $B^*(B) := \{f(x) : f \in B^*,\ x \in B\}$, $\mu^{Y^X}(\emptyset) = \{(\underline{P}Y^X \times \underline{P}Y^X)\}$ and for $B^* \in \mathcal{B}^{Y^X}\backslash\emptyset\}$ we put: $\mu^{Y^X}(B^*) := \{\mathcal{U}^* \in FIL(Y^X \times Y^X) : \exists\ f \in B^* \forall B \in \mathcal{B}^X\backslash\{\emptyset\}\ \forall\ \mathcal{U} \in FIL(X \times X)(\mathcal{U} \in \mu^X(B)$ implies $\mathcal{U}^*(\mathcal{U}) \in \mu^Y(f[B]))\}$, where $\mathcal{U}^*(\mathcal{U})$ denotes the filter generated by the set $R^*(R)$ with $R^*(R) := \{(f(x), g(z)) : (f, g) \in R^*, (x, z) \in R\}$. $(\mathcal{B}^{Y^X}, \mu^{Y^X})$ is called the bounded equiform function crossconvergence on Y^X (in short beq-function crossconvergence), such that the evaluation map $e : (X \times Y^X, \mathcal{B}^{X\times Y^X}, \mu^{X\times Y^X}) \longrightarrow (Y, \mathcal{B}^Y, \mu^Y)$ is beq- map, where $e(x, f) := f(x)$ for any $x \in X$ and $f \in Y^X$.*

Proof. Obviously, $(\mathcal{B}^{Y^X}, \mu^{Y^X})$ is crossconvergence on Y^X, since especially for $f \in Y^X$, and $\mathcal{U} \in \mu(B)$, $B \in \mathcal{B}^X\backslash\{\emptyset\}$, $\dot{f} \times \dot{f}\ (\mathcal{U}) = f \times f(\mathcal{U}) \in \mu^Y(f[B])$. And furthermore for $B^* \in \mathcal{B}^{Y^X}\backslash\{\emptyset\}$ and $\mathcal{U}^* \in \mu^{Y^X}(B^*)$ we can find $f \in B^*$ with the corresponding defined property. But $\mathcal{U} \in \mu(B)$ implies $\mathcal{U} \in \mu(\{x\})$ for some $X \in B$ and thus $\mathcal{U}^*(\mathcal{U}) \in \mu^Y(\{f(x)\})$ implying $\mathcal{U}^* \in \mu^{Y^X}(\{f\})$. The converse is evident. The evaluation map $e : X \times Y^X \longrightarrow Y$, $e(x, f) := f(x)$, $e : (X \times Y^X, \mathcal{B}^{X\times Y^X}, \mu^{X\times Y^X}) \longrightarrow (Y, \mathcal{B}^Y, \mu^Y)$ is beq-map, where $(\mathcal{B}^{X\times Y^X}, \mu^{X\times Y^X})$ denotes the *initial* crossconvergence on $X \times Y^X$ with respect to $((X \times Y^X), (p \times p_Y), (\mathcal{B}^X, \mu^X), (\mathcal{B}^Y, \mu^Y))$ and $p_X : X \times Y^X \longrightarrow X$, $p_{Y^X} : X \times Y^X \longrightarrow Y^X$ as the corresponding projections, see 4.2.

At the first, e is bounded because of $B \in \mathcal{B}^{X\times Y^X}$ implies $p_X[B] \in \mathcal{B}^X$ and $p_{Y^X}[B] \in \mathcal{B}^{Y^X}$. By the definition of $\mathcal{B}^{Y^X}$, we are getting $p_{Y^X}[B](p_X[B]) \in \mathcal{B}^Y$. The inclusion $e[B] \subset p_{Y^X}[B](p_X[B])$ is holding, since $y \in e[B]$ implies $y = f(x)$ for some $(x, f) \in B$. Hence $p_X(x, f) = x$ and $p_{Y^X}(x, f) = f$ follow, showing that $y = f(x) \in p_{Y^X}[B](p_X[B])$ is valid. Now, let $B \in \mathcal{B}^{X\times Y^X}\backslash\{\emptyset\}$ with $\mathcal{U} \in \mu^{X\times Y^X}(B)$. We have to verify that $(e \times e)(\mathcal{U}) \in \mu^Y(e[B])$ is valid. Since $(\mathcal{B}^{X\times Y^X}, \mu^{X\times Y^X})$ is the initial crossconvergence on $X \times Y^X$ we are getting $\mathcal{U} \in \mu^{X\times Y^X}(\{x, f\})$ for some $(x, f) \in B$, $(p_X, \times p_X)(\mathcal{U}) \in \mu^X(p_X[\{x, f)\}]$ and $(p_Y^X \times p_Y^X)(\mathcal{U}) \in \mu^{Y^X}(p_{Y^X}[\{x, f\}])$. Thus $p_X[\{(x, f)\}] = x$ and $p_{Y^X}[\{(x, f)\}] = f$ implying $(p_X \times p_X)(\mathcal{U}) \in \mu^X(\{x\})$ and $(p_{Y^X} \times p_{Y^X})(\mathcal{U}) \in \mu^{Y^X}(\{f\})$. By the definition of μ^{Y^X} we obtain $(p_{Y^X} \times p_{Y^X})(\mathcal{U})((p_X \times p_X)(\mathcal{U})) \in \mu^Y(\{f(x)\}) = \mu^Y(\{e(x, f)\}) \subset \mu^Y(e[B])$. It remains to show that $(p_{Y^X}) \times p_{Y^X})(\mathcal{U})((p_X \times p_X)(\mathcal{U})) \subset (e \times e)(\mathcal{U})$.

$V \in (p_{Y^X} \times p_{Y^X})(\mathcal{U})((p_X \times p_X)(\mathcal{U}))$ implies $V \supset R^*(R)$ for some $R^* \in (p_{Y^X} \times p_{Y^X})(\mathcal{U})$ and some

$R \in (p_X \times p_X)(\mathcal{U})$.

Hence $R^* \supset (p_{Y^X} \times p_{Y^X})[U_1]$ for some $U_1 \in \mathcal{U}_1$ and $R \supset (p_X \times p_X)[U_2]$ for some $U_2 \in \mathcal{U}_2$. Thus $R^* \supset (p_Y \times p_Y)[U_1 \cap U_2]$ and $R \supset (p_X \times p_X)[U_1 \cap U_2]$. Note also that $U_1 \cap U_2 \in \mathcal{U}$ is holding. Now let $z \in (e \times e)[U_1 \cap U_2]$, thus $z = (e \times e)(x, f)$ for some $(xf) \in U_1 \cap U_2$. Consequently, $p_X(x, f) = x$ and $p_{Y^X}(x, f) = f$ implying $(f, f) \in (p_{Y^X} \times p_{Y^X})[U_1 \cap U_2]$ and $(x, x) \in (p_X \times p_X)[U_1 \cap U_2]$ resulting into $z \in R^*(R)$, and the claim follows. □

Proposition 6.9. *Let $(X, \mathcal{B}^X, \mu^X)$, $(Y, \mathcal{B}^Y, \mu^Y)$ and $(Z, \mathcal{B}^Z, \mu^Z)$ be crossconvergence spaces and $f : (X \times Z, \mathcal{B}^{X\times Z}, \mu^{X\times Z}) \longrightarrow (Y, \mathcal{B}^Y, \mu^Y)$ any beq-map. Then the associated function $\hat{f} : (Z, \mathcal{B}^Z, \mu^Z) \longrightarrow (Y^X, \mathcal{B}^{Y^X}, \mu^{Y^X})$ defined by $\hat{f}(z)(x) := f(x, z)$ for each $z \in X$ and $x \in X$ is also beq-map.*

Proof. At the first, we prove that $\hat{f} : (Z, \mathcal{B}^Z) \longrightarrow (Y^X, \mathcal{B}^{Y^X})$ is bounded. In doing so, let $D \in \mathcal{B}^Z$, we must show $\hat{f}[D] \in \mathcal{B}^{Y^X}$. So let $B \in \mathcal{B}^X$, then we claim $\hat{f}[D](B) \in \mathcal{B}^Y$. Since $p_Z^{-1}[D] \cap p_X^{-1}[B] \in \mathcal{B}^{X\times Z}$, here $p_X : X \times Z \longrightarrow X$, $p_Z : X \times Z \longrightarrow Z$ denotes the corresponding projection, we are getting $f[p_Z^{-1}[D] \cap p_X^{-1}[B]] \in \mathcal{B}^Y$ by the hypothesis. It remains to verify that the inclusion $\hat{f}[D](B) \subset f[p_Z^{-1}[D] \cap p_X^{-1}[B]]$ is holding. $y \in \hat{f}[D](B)$ implies $y = \hat{f}(z)(x)$ for some $z \in D$ and some $x \in B$. Hence $y = f(x, z)$ follows. But $(x, z) \in p_z^{-1}[D] \cap p_X[B]$ implying $y \in f[p_z^{-1}[D] \cap p_X^{-1}[B]]$, and the claim holds. Further let for $D \in \mathcal{B}^Z \backslash \{\varnothing\}$ and for $\mathcal{U} \in FIL(Z \times Z)$, $\mathcal{U} \in \mu^Z(D)$. Our goal is to verify that $(\hat{f} \times \hat{f})(\mathcal{U}) \in \mu^{Y^X}(\hat{f}[D])$ is valid. Choose $z \in D$ such that $\mathcal{U} \in \mu^Z(\{z\})$, then $\hat{f}(z) \in \mathcal{B}^{Y^X}$ and $\hat{f}(z) \in \hat{f}[D]$ follows. Now, let $B \in \mathcal{B}^X \backslash \{\varnothing\}$ and $\mathcal{V} \in FIL(X \times X)$ such that $\mathcal{V} \in \mu^X(B)$ is holding, hence $\mathcal{V} \in \mu^X(\{x\})$ for some $x \in B$. So, it remains to prove that $(\hat{f} \times \hat{f})(\mathcal{U})(\mathcal{V}) \in \mu^Y(\hat{f}(z)(x)\}) = \mu^Y(\{f(x, z)\})$ is valid. We have $\{(x, z)\} \in \mathcal{B}^{X\times Z}$, and by setting $\mathcal{E} := \{R \subset (X \times Z) \times (X \times Z) : \exists U \in \mathcal{U}, \exists V \in \mathcal{V}, R \supset (p_z \times p_z)^{-1}[U] \cap (p_X \times p_X)^{-1}[V]\}$, $\mathcal{E} \in \mu^{X\times Z}(\{x, z\})$ follows, since $(p_z \times p_z)(\mathcal{U}) \supset \mathcal{U}$ and $(p_X \times p_X)(\mathcal{V}) \supset \mathcal{V}$ are implying $(p_z \times p_z)(\mathcal{U}) \in \mu^Z(\{z\})$ and $(p_X \times p_X)(\mathcal{V}) \in \mu^X(\{x\})$. Then by the hypothesis we obtain $(f \times f)(\mathcal{E}) \in \mu^Y(\{f(x, z)\})$. Furthermore we have $(\hat{f} \times \hat{f})(\mathcal{U})(\mathcal{V}) \supset (f \times f)(\mathcal{E})$, because of $D \in (f \times f)(\mathcal{E})$ implies $\mathcal{D} \supset (f \times f)[R]$ for some $R \in \mathcal{E}$. Then we can find some $U \in \mathcal{U}$ and some $V \in \mathcal{V}$ such that $(f \times f)[R] \supset (f \times f)[(p_Z \times p_Z)^{-1}[U] \cap (p_X \times p_X)^{-1}[V]] \supset (\hat{f} \times \hat{f})[U](V)$, since $y \in (\hat{f} \times \hat{f})[U](V)$ implies $y = (\hat{f}(z_1)(x_1), \hat{f}(z_2)(x_2))$ for some $(\hat{f}(z_1), \hat{f}(z_2)) \in (\hat{f} \times \hat{f})[U]$ with $(z_1, z_2) \in U$ and$(x_1, x_2) \in V$, hence $\hat{f}(z_1)(x_1) = f(x_1, z_1)$ and $\hat{f}(z_2)(x_2) = f(x_2, z_2)$ implying $(x_1, z_1) \in (p_Z \times p_Z)^{-1}[U] \cap (p_X \times p_X)^{-1}[V]$ and $(x_2, z_2) \in (p_z \times p_z)^{-1}[U] \cap (p_X \times p_X)^{-1}[V]$. Consequently $y = (\hat{f}(z_1)(x_1), \hat{f}(z_2)(x_2)) = (f(x_1, z_1), f(x_2, z_2)) = ((f \times f)(x_1, z_1), (f \times f)(x_2, z_2))$ results. □

Theorem 6.10. *The strong topological construct **CCONV** is cartesian closed.*

Proof. By applying 6.2., 6.4. and 6.9., respectively. □

Remark 6.11. Preuss, [27] pointed out that **TOP** and **UNIF** are *not* cartesian closed. On the other hand **SUCONV** and **GCONV** possess that *nice* property. Furthermore, **CHY**, the category of Cauchy spaces (and Cauchy continuous maps) is also cartesian closed

Corollary 6.12. *CCONV is a quasitopos respectively a topological universe in which quotients are productive, and thus it constitutes a strong topological universe, see above.*

Remark 6.13. **TOP**, **UNIF**, **CHY** are *not* strong topological universes, but **SUCONV**, **GCONV**, **BOUND**, **ORDC**, **b-UFIL** and **CCONV** possess these properties. Thus, **CCONV** also delivers a contribution to Convenient Topology in which convergence structures are available but now are more enlarged to a concept called Bounded Topology in which in addition **bounded** structures such as bornology, b-uniform filtration, b-topology, supertopology, set-convergence or precrossconvergence are involved by studying **PRE-CCONV** invariants, i.e. properties of precrossconvergences, which are preserved by isomorphisms in **PRE-CCONV**. This includes also the study of full and isomorphism- closed subcategories of **PRE-CCONV** as already formerly described

7. Discussion and proposals

The reader will observe that in the equivalent concept **FSAT-CCONV** of **PUCONV** we can renunciate on the properties of being fixed and saturated. It remains now to be a concept of spaces $(X, \mathcal{B}^X, \mu)$ which will be now *enriched* by the following additional postulates, i.e.

(f) $(\mathcal{B}^X, \mu)$ is *filtered*, meaning that $B \in \mathcal{B}^X$ and $\mathcal{U}_1, \mathcal{U}_2 \in \mu(B)$ implying $\mathcal{U}_1 \cap \mathcal{U}_2 \in \mu(B)$;

(s) $(\mathcal{B}^X, \mu)$ is *symmetric*, i.e. $B \in \mathcal{B}^X$ and $\mathcal{U} \in \mu(B)$ implying $\mathcal{U}^{-1} \in \mu(B)$;

(c) $(\mathcal{B}^X, \mu)$ is *connected* by $B \in \mathcal{B}^X$ and $\mathcal{U}_1, \mathcal{U}_2 \in \mu(B)$ implying $\mathcal{U}_1 \circ \mathcal{U}_2 \in \mu(B)$, where $\mathcal{U}_1 \circ \mathcal{U}_2$ is filter generated by $\{R_1 \circ R_2 : R_1 \in \mathcal{U}_1, R_2 \in \mathcal{U}_2\}$.

Let us call a crossconvergence $(\mathcal{B}^X, \mu)$ and the corresponding space $(X, \mathcal{B}^X, \mu)$ *epicrossconvergence*, respectively *epicrossconvergence* space, provided $(\mathcal{B}^X, \mu)$ satisfies the conditions (f), (s) and (c), respectively. By denoting **EPICCONV** the full subcategory of **CCONV**, it is now possible to analyses the fundamental full subcategory **ULIM** of **PUCONV**, whose objects are the uniform limit spaces, by using the more *general* concept of epicrossconvergence spaces. Here, a preuniform convergence structure J on X is called *uniform limit structure* and the space (X, J) *uniform limit space* provided it satisfies the following conditions: $\mathcal{U}_1, \mathcal{U}_2 \in \mu$ implying $\mathcal{U}_1 \cap \mathcal{U}_2 \in J$ and $\mathcal{U}_1 \circ \mathcal{U}_2 \in J$ if it exists, and $\mathcal{U} \in J$ implies $\mathcal{U}^{-1} \in J$.

By using 4.7. the categories **ULIM** and **FSAT-EPICCONV** are isomorphic. Consequently, the concept of epicrossconvergence is a more general one and contains **UNIF**, too.

Moreover, for an epicrossconvergence $(\mathcal{B}^X, \mu)$ we can *naturally* consider so-called μ-Cauchy filter (in short μ-Cfilter), where a filter $\mathcal{F} \in FIL(X)$ is said to be *μ-Cfilter* provided that $\mathcal{F} \times \mathcal{F} \in \mu(\{x\})$ for some $x \in X$. Furthermore, we say a filter $\mathcal{F} \in FIL(X)$ is *μ-convergent* provided the existence of an element $x \in X$ such that $\dot{x} \times \mathcal{F} \in \mu(\{x\})$. Then completeness can be introduced by combining these two conditions, i.e. , an epicrossconvergence $(\mathcal{B}^X, \mu)$ and the corresponding space $(X, \mathcal{B}^X, \mu)$ are called *complete*, provided that any μ-Cfilter is μ-convergent. The following statement justifies this definition: An uniform limit space is complete if and only if its corresponding epicrossconvergence space is complete. Now, it seems to be of interest to develop a "completion-theory" for epicrossconvergence spaces.

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References

[1] B. Averbukh, On topologies on the underlying set of a topological monoid induced by its unitary extensions, Topol. Algebra Appl. 10, no. 1 (2022), 25–35.

[2] M. Baran, S. Kula, T. M. Baran, and M. Quasim, Closure operators in semiuniform convergence spaces, Filomat 30, no. 1 (2016), 131–140.

[3] M. Baran, S. Kula and A. Erclyes, T_0 and T_1 semiuniform convergence spaces, Filomat 27, no. 4 (2013), 537–546.

[4] H. L. Bentley, H. Herrlich and E. Lowen-Colebunders, The category of Cauchy spaces is cartesian closed, Topology Appl. 27 (1987), 105–112.

[5] D. Doitchinov, Supertopological spaces and special classes of extensions of topological spaces, In: Faddeev, L. D., Mal'cev, A. A. (eds) Topology. Lecture Notes in Mathematics, vol 1060, Springer, Berlin, Heidelberg (1984), 17–25.

[6] V. A. Efremovic, The geometry of proximity, Mat. Sbornik 32 (73), (1952), 189–200.

[7] A. Eriyes, T. Baran, and M. Quasim, Closure operators in constant filter convergence spaces, Konuralp Journal Math. 8, no. 1 (2020), 185–191.

[8] B. G. C. Herrejón and F. Mynard, On connected subsets of a convergence space, Int. J. Topol. 2, no. 3 (2025), 13.

[9] H. Herrlich, Topology structures, Math. Centre Tracts 52 (1974), 59–122.

[10] H. Hogbe-Nlend, Theorié des Bornologies et applications, Lecture Notes in Math. n. 213, Springer Berlin (1971).

[11] V. M. Ivanova and A. A. Ivanov, Contiguity spaces and bicompact extensions, DOKL. Akad. Nauk SSSR 127 (1959), 20–22.

[12] G. Jäger, Non-symmetric convergence and regularity, Appl. Gen. Topol. 26, no. 1 (2025), 447–467.

[13] H. J. Kowalski, Topologische Räume, Basel-Stuttgart, Birkhaüser (1961).

[14] D. Leseberg, A new concept of convergence space, Math. Pann. 19, no. 2 (2008), 291–303.

[15] D. Leseberg, On topological induced b-convergences, Topol. Proc. 37 (2011), 293–313.

[16] D. Leseberg, Improved nearness research. Adv. Pure Math. 4, no. 11 (2014), 610–626.

[17] D. Leseberg and Z. Vaziry, The quasitopos of b-uniform filter spaces, Math. Appl. 7 (2018), 155–171.

[18] D. Leseberg and Z. Vaziry, Bounded Topology, LAP, Acad. Pub. (2019).

[19] D. Leseberg and Z. Vaziry, Pseudonearness as an important part of bounded topology, ICRAPAM (2021), 258–270.

[20] D. Leseberg and Z. Vaziry, The Cauchy completion of symmetric b-uniform filter space, Math. Appl. 10 (2021), 125–142.

[21] D. Leseberg and Z. Vaziry, Orderconvergence, a quasitopos for bounded topology BT, Researchgate (2025), 10.13140/RG.2.2.29777.85605.

[22] M. W. Lodato, On topological induced generalized proximity relations. I. Proc. Amer. Math. Soc. 15, no. 3 (1964), 417–422.

[23] F. Mynard and S. Dolecki, Convergence Foundation of Topology, World Scientific Pub (2016).

[24] D. Pallaschke and D. Pumplün, Banach limits revised, Adv. Pure Math. 6, no. 13 (2016), 1022–1036.

[25] M. J. Penon, Quasitopos, C. R. Acad. Sci. Paris, Sér A. 276 (1973), 237–240.

[26] G. Preuss, Semiuniform convergence spaces, Math. Jap. 41(1995), 465-491.

[27] G. Preuss, Foundation of Topology. An approach to Convenient Topology, Dordrecht, Kluwer Acad. Pub. (2002).

[28] G. Preuss, Prefilter spaces and a precompletion of preuniform convergence spaces related to some well-known completions, Topology Appl. 156, no 12 (2009), 2005–2012.

[29] D. Pumplün, A universal compactification of topological positively convex sets and moduls, Journal of Convex Analysis 41 (2011), 255–267.

[30] Y. M. Smirnov, On completeness of proximity spaces, Dokl. Acad. Nauk SSSR 88 (1953), 761–764.

[31] A. Weil, Sur les espaces à structures uniformes et sur la toplogie générale, Act. Sci. Ind. 551 (1937).

[32] O. Wyler, On convergence of filters and ultrafilters to subsets, Lect. Notes Comp. Sci. 393 (1988), 340–350.