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New fixed point theorems in metric spaces with w-distance

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Abstract

This paper presents some new fixed point results for a class of weakly contractive maps in a metric space. We use the concept of w-distance in the framework of metric spaces. As far as we know, the concept of w-distance is being utilized for the first time to study Suzuki-type antecedent conditions with weakly contractive maps. An illustrative example demonstrates the generality of the main result.

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1. Introduction and Preliminaries

Let (X, d) be a metric space. A self-map T of X is a contraction if there exists $k \in [0, 1)$ such that for each $x, y \in X$,

$$d(Tx, Ty) \leq kd(x, y). \tag{1.1}$$

The classical Banach contraction principle (in short BCP) states that a self-map T of a complete metric space admits a unique fixed point if T is a contraction.

Alber and Guerre-Delabriere [1] gave a new idea to generalize the BCP, they coined the concept of weakly contractive maps to obtain their result.

Definition 1.1 ([1])**.** A self-map T of X is weakly contractive if for each $x, y \in X$,

$$d(Tx, Ty) \leq d(x, y) - \phi(d(x, y)), \tag{1.2}$$

where $\phi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that $\phi(0) = 0$ and $\phi(t) > 0$ for all $t > 0$.

The following existence result for weakly contractive maps in the framework of metric space is due to Rhoades [17].

Theorem 1.2 ([17])**.** *A self-map T of a complete metric space has a unique fixed point if T is a weakly contractive.*

The above result was generalized by among others, Dutta and Chaudhary [7], Doric [6], Popescu [14], Singh et al. [19] and Moradi and Farajzadeh [13].

We remark that weakly contractive maps significantly generalize contraction maps. If $\phi(t) = (1 - k)t$, for $0 < k < 1$, in (1.2) then it reduces to (1.1). Further, let ϕ be a lower semi-continuous map from the right then $\varphi(t) = t - \phi(t)$ is upper semi-continuous from the right and (1.2) becomes

$$d(Tx, Ty) \leq \varphi(d(x, y)). \tag{1.3}$$

Therefore, weakly contractive maps for which ϕ is lower semi-continuous from the right are Boyd-Wong [3] type. Condition (1.2) reduces to Reich [16] type contractive condition if $k(t) = 1 - \frac{\phi(t)}{t}$ for $t > 0$ and $k(0) = 0$.

The following fixed point theorem was obtained by Doric [6].

Theorem 1.3 ([6]). *Let T be a self-map of a complete metric space X such that for every $x, y \in X$,*

$$\psi(d(Tx, Ty)) \leq \psi(m(x, y)) - \phi(m(x, y)), \tag{1.4}$$

where

$$m(x, y) \quad = \quad \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\}.$$

and ψ and ϕ are defined as:

(i) *$\psi : [0, \infty) \to [0, \infty)$ is a continuous and monotone nondecreasing function with $\psi(t) = 0$ iff $t = 0$.*

(ii) *$\phi : [0, \infty) \to [0, \infty)$ is lower semi-continuous function with $\phi(t) = 0$ iff $t = 0$.*

Then T has a unique fixed point.

Popescu [14] obtained the following result by weakening the conditions on ϕ and ψ functions.

Theorem 1.4 ([14]). *Let T be a self-map of a complete metric space X such that for every $x, y \in X$, condition (1.4) is satisfied where ψ and ϕ are defined as follows*

(i) *$\psi : [0, \infty) \to [0, \infty)$ is a monotone nondecreasing function with $\psi(t) = 0$ if $t = 0$.*

(ii) *$\phi : [0, \infty) \to [0, \infty)$ is a function with $\phi(t) = 0$ if $t = 0$, and $\liminf_{n\to\infty} \phi(a_n) > 0$ if $\lim_{n\to\infty} a_n = a > 0$.*

(iii) *$\phi(a) > \psi(a) - \psi(a^-)$ for any $a > 0$, where $\psi(a^-)$ is the left limit of ψ at a.*

Then T has a unique fixed point.

Suzuki [21] used an entirely different approach to generalize the BCPs.

Theorem 1.5. *Let T be a self-map of a complete metric space X. Define a nonincreasing function θ from $[0, 1)$ onto $(1/2, 1]$ by*

$$\theta(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq (\sqrt{5} - 1)/2, \\ (1 - r)r^{-2} & \text{if } (\sqrt{5} - 1)/2 \leq r \leq 2^{-\frac{1}{2}}, \\ (1 + r)^{-1} & \text{if } 2^{-\frac{1}{2}} \leq r < 1. \end{cases}$$

Assume that there exists $r \in [0, 1)$ such that for all $x, y \in X$,

$$\theta(r)d(x, Tx) \leq d(x, y) \text{ implies } d(Tx, Ty) \leq rd(x, y).$$

Then T has a unique fixed point z in X. Moreover $\lim_n T^n x = z$ for all $x \in X$.

Singh et al. [20], among others, used the same approach taking weakly contractive maps and obtained the following result.

Theorem 1.6 ([20]). *Let T be a self-map of a complete metric space X such that for every $x, y \in X$,*

$$\frac{1}{2}d(x,Tx) \leq d(x,y) \implies \psi(d(Tx,Ty)) \leq \psi(m(x,y)) - \phi(m(x,y)), \tag{1.5}$$

where ψ and ϕ are defined as in Theorem 1.3. Then T has a unique fixed point.

Kada et. al. [9] formulated a new distance function on a metric space and called it w-distance. They demonstrated the fruitfulness of w-distance by using it to generalize the famous Caristi's fixed point theorem [4], nonconvex minimization theorem of Takahashi [22] and Ekeland's ϵ-variational principle [8]. Later the notion of w-distance was widely used by several researchers to obtain their results, see for instance, [2, 5, 10, 11, 12, 15, 23, 24] and references thereof.

Definition 1.7 ([9]). Let X be a metric space endowed with a metric d. A function $p : X \times X \to [0, \infty)$ is called a w-distance on X if it satisfies the following conditions:

(1) $p(x, z) \leq p(x, y) + p(y, z)$ for any $x, y \in X$,
(2) p is lower semi-continuous in its second variable, i.e., if $x \in X$ and $y_n \to y$ in X, then $p(x, y) \leq \liminf_{n\to\infty} p(x, y_n)$,
(3) For each $\varepsilon > 0$, there exists a $\delta > 0$ such that $p(x, y) \leq \delta$ and $p(x, z) \leq \delta$ imply $d(y, z) \leq \varepsilon$.

The following lemma is useful and will be used in the sequel.

Lemma 1.8 ([9]). *Let (X, d) be a metric space and p be a w-distance on X.*

(1) *If $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} p(x_n, x) = \lim_{n\to\infty} p(x_n, y) = 0$ then $x = y$. In particular, if $p(z, x) = p(z, y) = 0$ then $x = y$.*
(2) *If $p(x_n, y_n) \leq \alpha_n p(x_n, y) \leq \beta_n$ for any $n \in \mathbb{N}$ ($\mathbb{N}$ is the set of naturals), where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, \infty)$ converging to 0, then $\{y_n\}$ converges to y.*
(3) *Let p be a w-distance on a metric space (X, d) and $\{x_n\}$ be a sequence in X such that for each $\varepsilon > 0$ there exists a $N_\varepsilon \in \mathbb{N}$ such that $m > n > N_\varepsilon$ implies $p(x_n, x_m) < \varepsilon$ (or $\lim_{n,m\to\infty} p(x_n, x_m) = 0$) then $\{x_n\}$ is a Cauchy sequence.*

The following theorem is proved by Lakzian and Rhoades [12].

Theorem 1.9 ([12]). *Let T be a self-map of a complete metric space (X, d) and p be a w-distance on X such that $p(x, x) = 0$ and for every $x, y \in X$,*

$$p(Tx, Ty) \leq M(x, y) - \phi(M(x, y)), \tag{1.6}$$

where

$$M(x,y) \quad = \quad \max\left\{p(x,y), p(x,Tx), p(y,Ty), \frac{p(x,Ty)+p(Tx,y)}{2}\right\}.$$

and $\phi : [0,\infty) \to [0,\infty)$ *is a non-decreasing and continuous function satisfying the conditions:*

(1) $\phi(t) > 0$ *for* $t > 0$ *and* $\phi(0) = 0$,

(2) *for each* $\{t_n\} \subseteq [0,\infty)$, $\lim_{n\to\infty} t_n = 0$ *if and only if* $\lim_{n\to\infty} \phi(t_n) = 0$.

Suppose that either $\inf\{p(x,w) + p(x,Tx) : x \in X\} > 0$ *for every* $w \in X$ *with* $w \neq Tw$ *or* T *is continuous. Then* T *has a unique fixed point.*

In the present paper, we use the concept of w-distance in the framework of metric spaces. In our formulation, Suzuki type antecedent condition is applied on a weakly contractive map. Notice that the significance of Suzuki type condition lies in the fact that it does not force the map to be continuous. Further, the classes of control functions φ and ϕ used are wider than those studied by Dutta and Choudhury [7], Doric [6], Lakzian and Rhoades [11], Lakzian et al. [10], Wongyat and Sintunavarat [24], and references therein. We also note that Romaguera and Tirado [18], obtained a w-distance extension of Suzuki's theorem from a very different approach. The results presented herein extend and generalize the main result of Singh et al. [20, Theorem 2.1] (see Theorem 1.4 above)and some other known results in [2, 10, 11, 23, 24, 6, 7, 13, 14]. The generality of main result is demonstrated by an example.

2. Main Results

Theorem 2.1. *Let* T *be a self-map of a complete metric space* (X,d) *and* p *be a* w*-distance on* X *such that* $p(x,x) = 0$ *for all* $x \in X$ *and for every* $x, y \in X$,

$$\frac{1}{2}p(x,Tx) \leq p(x,y) \implies \psi(p(Tx,Ty)) \leq \psi(M(x,y)) - \phi(M(x,y)), \tag{2.1}$$

where ψ *and* ϕ *are defined as*

(i) $\psi : [0,\infty) \to [0,\infty)$ *is a monotone nondecreasing function with* $\psi(t) = 0$ *if* $t = 0$,

(ii) $\phi : [0,\infty) \to [0,\infty)$ *is a function with* $\phi(t) = 0$ *if* $t = 0$, *and* $\liminf_{n\to\infty} \phi(a_n) > 0$ *if* $\lim_{n\to\infty} a_n = a > 0$,

(iii) $\phi(a) > \psi(a) - \psi(a^-)$ *for any* $a > 0$, *where* $\psi(a^-)$ *is a left limit of* ψ *at* a.

Suppose that $\inf\{p(x,w) + p(x,Tx) : x \in X\} > 0$ *for every* $w \in X$ *with* $w \neq Tw$. *Then* T *has a unique fixed point.*

Proof. Pick $x_0 \in X$ and construct a sequence $\{x_n\}$ in X such that

$$x_{n+1} = Tx_n \quad n = 0, 1, 2, \ldots.$$

For any n,

$$\frac{1}{2}p(x_{n-1}, x_n) < p(x_{n-1}, x_n).$$

Therefore by (2.1), we have

$$\psi(p(Tx_{n-1}, Tx_n)) = \psi(p(x_n, x_{n+1})) \leq \psi(M(x_{n-1}, x_n)) - \phi(M(x_{n-1}, x_n)). \tag{2.2}$$

So, $\psi(p(x_n, x_{n+1})) \leq \psi(M(x_{n-1}, x_n))$. By the monotone property of the ψ function, we get

$$\begin{aligned} p(x_n, x_{n+1}) &\leq M(x_{n-1}, x_n) \qquad (2.3)\\ &= \max\{p(x_{n-1}, x_n), p(x_n, x_{n+1}), \frac{1}{2}[p(x_{n-1}, x_{n+1}) + p(x_n, x_n)]\}\\ &= \max\{p(x_{n-1}, x_n), p(x_{n-1}, x_n), p(x_n, x_{n+1}), \frac{1}{2}[p(x_{n-1}, x_{n+1})\}\\ &\leq \max\{p(x_{n-1}, x_n), p(x_n, x_{n+1}), \frac{1}{2}[p(x_{n-1}, x_n) + p(x_n, x_{n+1})]\}\\ &= \max\{p(x_{n-1}, x_n), p(x_n, x_{n+1})\}. \end{aligned}$$

If $\max\{p(x_{n-1}, x_n), p(x_n, x_{n+1})\} = p(x_n, x_{n+1})$, then by (2.2)

$$\psi(p(x_n, x_{n+1})) \leq \psi(p(x_n, x_{n+1})) - \phi(p(x_n, x_{n+1})) \text{ and}$$

$$\psi(p(x_n, x_{n+1})) \leq \psi(p(x_n, x_{n+1})), \text{ a contradiction.}$$

Therefore, $\max\{p(x_{n-1}, x_n), p(x_n, x_{n+1})\} = p(x_{n-1}, x_n)$. Hence by (2.3),

$$p(x_n, x_{n+1}) \leq M(x_{n-1}, x_n) \leq p(x_{n-1}, x_n). \tag{2.4}$$

This is true for any n. Hence the sequence $\{p(x_n, x_{n+1})\}$ is monotone non-increasing and bounded below. So, there exists $r \geq 0$ such that

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = r = \lim_{n\to\infty} M(x_{n-1}, x_n). \tag{2.5}$$

Let $r > 0$. If there exists n such that $p(x_{n-1}, x_n) = r$, then using (2.4), we have

$$p(x_n, x_{n+1}) = M(x_{n-1}, x_n) = r, \text{ and using (2.2) we get}$$

$$\psi(r) \leq \psi(r) - \phi(r). \text{ This is a contradiction.}$$

Now if $p(x_{n-1}, x_n) > r$ for all $n \geq 1$, then by (2.2) and (2.5), letting $n \to \infty$ we obtain

$$\psi(r^+) \leq \psi(r^+) - \lim_{n\to\infty} \inf \phi(M(x_{n-1}, x_n)), \text{ again a contradiction.}$$

Therefore,

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0. \tag{2.6}$$

Similarly, it can be shown that

$$\lim_{n\to\infty} p(x_{n+1}, x_n) = 0. \tag{2.7}$$

Next, we show that $\{x_n\}$ is a Cauchy sequence. By Lemma 1.8 (iii), it is sufficient to prove that

$$\lim_{n\to\infty, m\to\infty} p(x_n, x_m) = 0. \tag{2.8}$$

If (2.8) is not true, then there exists $\epsilon > 0$ for which one can find the subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ such that $n(k)$ is the smallest integer with $n(k) > m(k) > k$ such that

$$p(x_{m(k)}, x_{n(k)}) \geq \epsilon \text{ and } p(x_{m(k)}, x_{n(k)-1}) < \epsilon.$$

Then we have

$$\begin{aligned} \epsilon &\leq p(x_{m(k)}, x_{n(k)}) \leq p(x_{m(k)}, x_{n(k)-1}) + p(x_{n(k)-1}, x_{n(k)}) \\ &< \epsilon + p(x_{n(k)-1}, x_{n(k)}). \text{ Using (2.6), we have } \epsilon \leq p(x_{m(k)}, x_{n(k)}) \\ &\leq \epsilon. \end{aligned}$$

So,

$$\lim_{k\to\infty} p(x_{m(k)}, x_{n(k)}) = \epsilon. \tag{2.9}$$

By (2.6), (2.7) and the inequality

$$p(x_{m(k)}, x_{n(k)}) \leq p(x_{m(k)}, x_{n(k)-1}) + p(x_{n(k)-1}, x_{n(k)}),$$

we get

$$\epsilon \leq \lim_{k\to\infty} p(x_{m(k)}, x_{n(k)-1}). \tag{2.10}$$

In a similar manner, equations (2.6), (2.7) and the inequality

$$p(x_{m(k)}, x_{n(k)-1}) \leq p(x_{m(k)}, x_{n(k)}) + p(x_{n(k)}, x_{n(k)-1})$$

yields

$$\lim_{k\to\infty} p(x_{m(k)}, x_{n(k)-1}) \leq \epsilon. \tag{2.11}$$

From equations (2.10) and (2.11), we have

$$\lim_{k\to\infty} p(x_{m(k)}, x_{n(k)-1}) = \epsilon. \tag{2.12}$$

Similarly, it can be shown that

$$\begin{aligned}\lim_{k\to\infty} p(x_{m(k)-1}, x_{n(k)}) &= \lim_{k\to\infty} p(x_{m(k)-1}, x_{n(k)-1}) \\ &= \lim_{k\to\infty} p(x_{m(k)+1}, x_{n(k)}) \\ &= \lim_{k\to\infty} p(x_{m(k)}, x_{n(k)+1}) = \epsilon. \end{aligned} \tag{2.13}$$

From (2.12) and (2.13), it follows that

$$\lim_{k\to\infty} M(x_{m(k)-1}, x_{n(k)-1}) = \epsilon. \tag{2.14}$$

Let there exist a subsequence $\{k(q)\}$ of $\{k\}$ with $n(k(q)) > m(k(q))$ such that $\epsilon < p(x_{m(k(q))}, x_{n(k(q))})$ for any q. Then,

$$\begin{aligned}\frac{1}{2}p(x_{m(k(q))-1}, Tx_{m(k(q))-1}) &= \frac{1}{2}p(x_{m(k(q))-1}, x_{m(k(q))}) \\ &\leq p(x_{m(k(q))-1}, x_{m(k(q))}) \\ &\leq p(x_{m(k(q))-1}, x_{n(k(q))-1}). \end{aligned}$$

Therefore, by (2.1),

$$\begin{aligned}\psi(p(Tx_{m(k(q))-1}, Tx_{n(k(q))-1})) &= \psi(p(x_{m(k(q))}, x_{n(k(q))})) \\ &\leq \psi(M(x_{m(k(q))-1}, x_{n(k(q))-1})) \\ &\quad -\phi(M(x_{m(k(q))-1}, x_{n(k(q))-1})) \text{ for any } q. \end{aligned}$$

By (2.9) and (2.14), letting $q \to \infty$, we get

$$\psi(\epsilon^+) \leq \psi(\epsilon^+) - \lim_{q\to\infty} \inf \phi(M(x_{m(k(q))-1}, x_{n(k(q))-1})),$$

a contradiction. We repeat the procedure if there exists a subsequence $\{k(q)\}$ of $\{k\}$ such that $\epsilon < p(x_{m(k(q))}, x_{n(k(q))+1})$ for any q or $\epsilon < p(x_{m(k(q))+1}, x_{n(k(q))})$ for any q.Therefore, we can suppose that $p(x_{m(k)}, x_{n(k)}) \leq \epsilon$ $p(x_{m(k)}, x_{n(k)+1}) \leq \epsilon$ and $p(x_{m(k)+1}, x_{n(k)}) \leq \epsilon$ for any $k \geq k_1$. Then $M(x_{m(k)}, x_{n(k)}) = \epsilon$ for $k \geq k_3 = \max\{k_1, k_2\}$ where k_2 is such that $p(x_k, x_{k+1}) < \epsilon$ for all $k \geq k_2$.

Now

$$\frac{1}{2}p(x_{m(k)}, Tx_{m(k)}) = \frac{1}{2}p(x_{m(k)}, x_{m(k)+1}) \leq p(x_{m(k)}, x_{m(k)+1}) \leq p(x_{m(k)}, x_{n(k)})$$

Therefore, by (2.1), we have

$$\begin{aligned}\psi(p(Tx_{m(k)}, Tx_{n(k)})) &= \psi(p(x_{m(k)+1}, x_{n(k)+1})) \\ &\leq \psi(M(x_{m(k)}, x_{n(k)})) - \phi(M(x_{m(k)}, x_{n(k)})) \end{aligned}$$

or

$$\psi(p(x_{m(k)+1}, x_{n(k)+1})) \leq \psi(\epsilon) - \phi(\epsilon) \text{ for any } k \geq k_3. \tag{2.15}$$

Using the monotone property of ψ, we get $p(x_{m(k)+1}, x_{n(k)+1}) < \epsilon$ otherwise $\psi(\epsilon) = 0$. Making $k \to \infty$ in (2.15) we get $\psi(\epsilon^-) \leq \psi(\epsilon) - \phi(\epsilon)$, a contradiction to condition (iii). Therefore (2.8) is true. Hence by lemma 1.8, $\{x_n\}$ is a Cauchy sequence. By the completeness of X there exists a $z \in X$ such that $x_n \to z$ as $n \to \infty$.

Now we show that z is a fixed point of T. By (2.8), for each $\epsilon > 0$ there exists an $N_\epsilon \in \mathbb{N}$ such that $n > N_\epsilon$ implies $p(x_{N_\epsilon}, x_n) < \epsilon$. From the lower semi-continuity of $p(x, .)$ and $x_n \to z$, it follows that

$$p(x_{N_\epsilon}, z) \leq \lim_{n\to\infty} \inf p(x_{N_\epsilon}, x_n) \leq \epsilon$$

So $p(x_{N_\epsilon}, z) \leq \epsilon$. Now, if we set $\epsilon = 1/k$ and $N_\epsilon = n_k$ then letting $k \to \infty$ we have

$$\lim_{k\to\infty} p(x_{n_k}, z) = 0. \tag{2.16}$$

If we assume the contrary i.e. $z \neq Tz$. Then, we have

$$0 < \inf\{p(x, z) + p(x, Tx) : x \in X\} \leq \inf\{p(x_n, z) + p(x_n, x_{n+1}) : n \in \mathbb{N}\} = 0$$

by (2.6) and (2.16) letting $n \to \infty$. Which is a contradiction. Therefore, $z = Tz$.

Finally, we prove the uniqueness of the fixed point z. Let y be another fixed point of T. For any fixed point z of T, we claim $p(z, z) = 0$. If not then $p(z, z) > 0$.

$$\frac{1}{2}p(z, Tz) = \frac{1}{2}p(z, z) \leq p(z, z)$$

Therefore by (2.1), we have

$$\begin{aligned}\psi(p(z, z)) = \psi(p(Tz, Tz)) &\leq \psi(M(z, z)) - \phi(M(z, z))\\ &= \psi(p(z, z)) - \phi(p(z, z)), \text{a contradiction.}\end{aligned}$$

So,

$$p(z, z) = 0. \tag{2.17}$$

Further, let v be another fixed point of T. Then

$$\frac{1}{2}p(z, Tz) = \frac{1}{2}p(z, z) = 0 \leq p(z, v).$$

Condition (2.1), now implies that

$$\psi(p(z, v)) = \psi(p(Tz, Tv)) \leq \psi(M(z, v)) - \phi(M(z, v)) = \psi(p(z, v)) - \phi(p(z, v)).$$

Hence $\phi(p(z, v)) = 0$. So,

$$p(z, v) = 0. \tag{2.18}$$

It follows from the application of Lemma 1.8 (i) and conditions (2.17) and (2.18) that $z = v$. This completes the proof. $\square$

We remark (see, Moradi [13]) that for the function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(t) = 0$ if and only if $t = 0$ the following conditions are equivalent:

(i) If $\lim_{n\to\infty} a_n = a > 0$ then $\liminf_{n\to\infty} \phi(a_n) > 0$.
(ii) If $\lim_{n\to\infty} \phi(a_n) = 0$ then $\lim_{n\to\infty} a_n = 0$ for all bounded sequences $\{a_n\}$.

The following example demonstrates the generality of Theorem 2.1 over Theorem 1.6 ([13, Theorem 3.1]) and certain results in [2, 6, 7, 10, 12, 13, 14, 20, 23, 24].

Example 2.2. Let $X = \{0, 2\} \cup \{1, 3, 5, 7, \ldots\}$ with the metric d being a discrete metric defined as, for all $x, y \in X$,

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

Suppose the self-map T on X be such that

$$T(x) = \begin{cases} 0 & \text{if } x = 5 \\ 2 & \text{if } x = 7 \\ 1 & \text{otherwise} \end{cases}$$

Let the maps $\psi, \phi : [0, \infty) \to [0, \infty)$ be defined as

$$\psi(t) = \begin{cases} \frac{1}{2}t & \text{if } 0 \leq t \leq 1 \\ t^2 - \frac{1}{4} & \text{if } t > 1 \end{cases}$$

and

$$\phi(t) = \begin{cases} \frac{t^2}{4} & \text{if } 0 \leq t \leq 1 \\ \frac{1}{2} & \text{if } t > 1 \end{cases}$$

For all $x, y \in X$, let the w-distance $p : X \times X \to [0, \infty)$ be such that

$$p(x, y) = \begin{cases} \frac{3}{2} & \text{if } y = 0 \\ \frac{1}{2}(y - x) & \text{if } 0 \leq x \leq y \\ \frac{1}{4}(x - y) & \text{if } x > y \end{cases}$$

Then it is clear that (X, d) is a complete metric space with w-distance p. Function $\psi(t)$ is monotone nondecreasing and $\psi(t) = 0$ at $t = 0$. Also, $\phi(1) = \frac{1}{4} > 0 = \psi(1) - \psi(1^-)$ shows that the condition (iii) of Theorem 2.1 is also satisfied. To see that condition (ii) is also satisfied, let $a_n = \frac{1}{n}$ be a bounded sequence then

$$\lim_{n\to\infty} \phi(a_n) = \lim_{n\to\infty} \left(\frac{1}{4n^2}\right) = 0$$

as well as

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$$

for the bounded sequence $a_n = \frac{1}{n}$. It is easy to verify that condition (2.1) and all the other conditions of Theorem 2.1 are satisfied and $x = 1$ is the unique fixed point of T.

It is evident that condition (1.6) (see above) and condition (3.14) used by Lakzian and Rhoades [12, Corollary 2], are not satisfied for $x = 7$ and $y = 5$ since

$$p(Tx, Ty) = p(2, 0) = \frac{3}{2} \geq M(x, y) - \phi(M(x, y)).$$

Further notice that condition (3.2) of Wongyat and Sintunavarat [24, Theorem 3.6], condition (2.1) of Lakzian et al. [10, Theorem 2.1], condition (3.3) of Aydi et al. [2, Theorem 3.3], condition (2.1) of Ume [23, Theorem 3.1], are not satisfied for the same values $x = 7$ and $y = 5$.

Moreover, condition 1.6 (see Theorem 1.4 above) used by Singh et al. [20] is not satisfied for $x = 3$ and $y = 5$ with the discrete metric defined above. Finally, it can be easily verified that conditions used by Popescu [14], Doric [6], and Moradi and Farajzadeh [13] are also not satisfied for $x = 3$ and $y = 5$.

In Theorem 2.1, if we replace the condition $\inf\{p(x, w) + p(x, Tx) : x \in X\} > 0$ for every $w \in X$ with $w \neq Tw$ by the continuity of map T then we obtain the following result.

Theorem 2.3. *Let T be a continuous self-map of a complete metric space (X, d) and p be a w-distance on X such that $p(x, x) = 0$ and condition (2.1) of Theorem 2.1 is satisfied where ψ and ϕ are defined by conditions (i), (ii) and (iii) of Theorem 2.1. Then T has a unique fixed point.*

Proof. Following the proof of Theorem 2.1, it can be shown that $\{x_n\}$ is a Cauchy sequence. By the completeness of X, there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$. Using the continuity of T, we get $x_{n+1} = Tx_n \to Tz$ as $n \to \infty$. Therefore, $z = Tz$. Finally, the uniqueness of the fixed point can be proved easily by duplicating the steps used in the proof of Theorem 2.1. □

From the above results, we immediately obtain the following corollary.

Corollary 2.4. *Let T be a continuous self-map of a complete metric space (X, d) and p be a w-distance on X such that $p(x, x) = 0$ and for every $x, y \in X$,*

$$\psi(p(Tx, Ty)) \leq \psi(M(x, y)) - \phi(M(x, y)), \tag{2.19}$$

where functions $\psi, \phi : [0, \infty) \to [0, \infty)$ are the same as in Theorem 2.1. Then T has a unique fixed point.

Corollary 2.5. *Let T be a self-map of a complete metric space (X, d) and p be a w-distance on X such that condition (2.19) is satisfied. Suppose that*

$$\inf\{p(x, w) + p(x, Tx) : x \in X\} > 0 \text{ for every } w \in X \text{ with } w \neq Tw.$$

Then T has a unique fixed point.

Corollary 2.6 ([12]). *Let T be a self-map of a complete metric space (X, d) and p be a w-distance on X such that $p(x, x) = 0$ and for every $x, y \in X$,*

$$p(Tx, Ty) \leq M(x, y) - \phi(M(x, y)), \tag{2.20}$$

where $\phi : [0, \infty) \to [0, \infty)$ is a non-decreasing and continuous function satisfying:

(i) *$\phi(t) > 0$ for $t > 0$ and $\phi(0) = 0$,*

(ii) *for each $\{t_n\} \subseteq [0, \infty)$, $\lim_{n\to\infty} t_n = 0$ if and only if $\lim_{n\to\infty} \phi(t_n) = 0$.*

Suppose that

$$\inf\{p(x, w) + p(x, Tx) : x \in X\} > 0 \text{ for every } w \in X \text{ with } w \neq Tw.$$

Then T has a unique fixed point.

Corollary 2.7 ([10]). *Let T be a self-map of a complete metric space (X, d) and p be a w-distance on X such that for every $x, y \in X$,*

$$\psi(p(Tx, Ty)) \leq \psi(p(x, y)) - \phi(p(x, y)),$$

where functions $\psi, \phi : [0, \infty) \to [0, \infty)$ are the same as in Theorem 2.1. Then T has a unique fixed point.

By putting $p = d$ in Theorem 2.1, we derive the following corollary which generalizes Theorem 1.4 (see [20, Theorem 2.1]).

Corollary 2.8. *Let T be a self-map of a complete metric space (X, d) such that for every $x, y \in X$, condition (1.5) is satisfied where functions $\psi, \phi : [0, \infty) \to [0, \infty)$ are the same as in Theorem 2.1. Then T has a unique fixed point.*

Following corollary is immediate from Corollary 2.8.

Corollary 2.9. *Theorem 1.4 above.*

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