Optimal repetitive reliability inspection of manufactured lots for lifetime models using prior information

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Abstract

Repetitive group inspection of production lots is considered to develop the failure-censored plan with minimal expected sampling effort using prior information. Optimal reliability test plans are derived for the family of log-location and scale lifetime distributions, whereas a generalized beta distribution is assumed to model the nonconforming proportion, p. A highly efficient and quick step-by-step algorithm implemented in an R library is determined in order to solve the underlying mixed nonlinear programming problem. Conventional repetitive group plans are often very effective in reducing the average sample number with respect other inspection schemes, but sample sizes may increase under certain conditions such as high censoring. The inclusion of previous knowledge from past empirical results contributes to reduce drastically the amount of sampling required in life testing. Moreover, the use of expected sampling risks improves significantly the assessment of the real producer and consumer sampling risks. These aspects are crucial in saving sampling and time inspection costs of reliability plans. Several tables and figures are presented to analyze the effect of the available prior evidence about p. The results show that the proposed lot inspection scheme clearly outperforms the standard repetitive group plans obtained under the traditional approach based on conventional risks. Finally, an application to the manufacture of integrated circuits is included for illustrative purposes.

Keywords: Repetitive inspection; Expected producer and consumer risks; Expected average sample number; Log-location and scale lifetime distribution; Generalized beta prior model

1. Introduction

The methods of acceptance sampling in quality control are very important, not only in industrial manufacturing, but also in other fields as reliability analysis, pattern recognition, and so on. In this

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context, the full inspection of large batches of raw materials or production processes are usually unaffordable due to time and economical costs reasons.

The sampling methods are widely extended in acceptance control and, in the case of reliability demonstration and acceptance tests, are used to study the acceptability of a product regarding its lifetime. These plans are characterized by the censoring schemes implemented in the inspection testing and, generally, can be obtained by solving constrained optimization problems using computational procedures. There is extensive literature about the acceptance sampling in reliability and many authors have studied and proposed sampling plans under different practical cases. Among others, the papers by Chen, Li and Lam (2007), Arizono, Kawamura and Takemoto (2008), Tsai, Lu and Wu (2008), Lee, Wu and Hong (2009), Lu and Tsai (2009), Fernández (2011), Fernández, Pérez-González, Aslam and Jun (2011), Fernández and Pérez-González (2012a) and Fernández and Pérez-González (2012b) are just a sample.

The conventional inspection procedure considered in the design of sampling plans is the single sampling. However, this scheme can be generalized to other complex schemes based on the sampling repetition, as the double and sequential sampling. In the design of censored sampling plans, the repetition of lot sampling could lead to save inspection costs, specially in the case of high-quality products. Therefore, the schemes that allow the lot reinspection are more efficient in terms of the average sample number (ASN) than the single and double sampling. The repetitive group sampling plans, that considers a repeated inspection scheme, was proposed by Sherman (1965) and studied by several authors as Balamurali, Park, Jun, Kim and Lee (2005), Balamurali and Jun (2006), Aslam, Azam and Jun (2013), Aslam, Niaki, Rasool and Fallahnezhad (2012), Wu, Wu, and Chen (2015), Arizono, Okada, Tomohiro and Takemoto (2016), Lee, Wu and Chen (2016) and Nezhad and Seifi (2015). Repetitive group sampling plans are employed as reference plans in designing other complex inspection schemes as the skip-lot plans, developed by Suresh and Umamaheswari (2016), Wu, Lee and Huang (2020) and Wu, Lee and Huang (2021), or the multiple state repetitive group sampling plans, which are analysed by Aslam, Yen, Chang and Jun (2013), Yan, Liu and Azam (2017) and, recently, Aslam, Yen, Chang, Al-Marshadi and Jun (2019), Khan, Aslam, Ahmad and Jun (2019) and Kannan, Jeyadurga and Balamurali (2021). Furthermore, Pérez-González, Fernández, Kohansal and Asgharzadeh (2019) and Pérez-González, Fernández and Kohansal (2020) have developed attributes sampling plans using a truncated scheme that are based on the repetitive group inspection. However, the repetitive censored plans could be inefficient for high censoring levels because the required sample sizes of life tests tend to increase.

This paper presents an optimal approximate design of failure-censored (or type II censored) reliability sampling plans using a repetitive group inspection. In the rest of the paper, for simplicity, the repetitive group scheme is named as repetitive inspection. The design of the plans assume a log-location and scale distribution of the lifetime variable. These distributions are known as the most extensively used family of parametric lifetime models. The plans are determined using the asymptotic approximation to the operating characteristic function of the proportion p of nonconforming units in the lot or the process. The designs proposed are determined when prior information on p is available. Traditionally, the sampling inspection usually assumes a constant fraction defective but there are many production processes where this assumption does not hold, in which case it is convenient to adopt an adequate prior model that summarizes uncertainty about p. Several authors have considered the use of prior models into the inference procedure in acceptance sampling. We can cite the articles by Fallahnezhad and Aslam (2013), Hsieh and Lu (2013), Tsai, Chiang, Liang and Yang (2014) and, more recently, Fernández (2015a), Fernández and Pérez-González (2016), Pérez-González, Fernández and Kohansal (2020) and Fernández, Pérez-González and Suárez-Rancel (2021), who have proposed several designs of sampling plans using different prior distributions.

The use of a prior model to include subjective knowledge about the fraction defective in the design of sampling plans can imply a significant reduction of the sampling effort in life testing. In this case, the expected producer and consumer risks provide better coverage against the acceptation and rejection of unacceptable or satisfactory lots, respectively. The proposed optimal approximate censored sampling plans using the repetitive inspection scheme are obtained after solving the mixed nonlinear optimization problem subject to several risks constraints. To describe the available information about p, a generalized beta distribution model is considered. This model represents a very appropriate prior distribution to incorporate requirements such as the prior impartiality between the producer and the consumer or the restriction in the parameter interval domain. A methodology is presented to quickly determine the accurate approximate solution of the underlying problem that minimizes the expected average sample size ($EASN$). The results obtained illustrate that the inclusion of previous data about the fraction defective into the design of censored repetitive sampling plans reduces significantly the sampling effort required with respect to the standard repetitive plans. This represents a crucial aspect in the reliability acceptance control where the censoring applied in the sampling plans to reduce the inspection time can imply a noteworthy increase in the sample size. Consequently, the conventional sampling plans could be very inefficient in inspecting lots when the product is highly expensive or under limiting conditions on the production (e.g., an eventual shortage of certain components due to lack of raw materials).

The paper is structured as follows. Section 2 shows the log-location and scale lifetime models and the decision criteria used in the construction of the acceptance sampling plan. The repetitive sampling scheme is presented in Section 3 as well as the operating characteristic function. Next, Section 4 defines the expected producer and consumer risks associated with the repetitive lot inspection scheme. The prior model about the defective rate per unit used in defining the sampling risks is described by a generalized beta distribution in SubSection 4.1. Section 5 shows the mixed nonlinear program that must be solved to determine the censored repetitive plan using the prior information and the $EASN$ minimization as the optimality criterion. In this Section is proposed the computational procedures to obtain the solution of the constrained optimization problem by using, in turn, the ASN optimal repetitive plans. Next, Section 6 gives a numerical example to illustrate the results and, finally, Section 7 presents a brief discussion and conclusions.

2. Log-location–scale reliability model

The most used models in reliability analysis belong to the log-location–scale family of distributions. Assume that the lifetime T of a product in a testing process has a distribution of this family, then the log-lifetime variable $X = \log(T)$ has a location–scale distribution with unknown parameters of location μ and scale $\sigma > 0$. Therefore, the cumulative distribution function (cdf) and probability density function (pdf) of X can be expressed, respectively, as

$$F(x; \mu, \sigma) = F_0((x - \mu)/\sigma) \quad \text{and} \quad f(x; \mu, \sigma) = \sigma^{-1} f_0((x - \mu)/\sigma), \quad -\infty < x < \infty,$$

where $F_0(\cdot)$ and $f_0(\cdot)$ are the standardized cdf and pdf of the pivotal quantity $Y = (X - \mu)/\sigma$.

The product in considered conforming or nondefective if its lifetime T is, at least, a given lower specification limit t_ω, i.e. t_ω is the smallest lifetime of a nondefective item. Moreover, $p = \Pr(T < t_\omega)$ represents the proportion of nonconforming items. Observe, therefore, that $\log(t_\omega)$ is the p-quantile of the distribution of X denoted by $\xi_p = \mu + \sigma F_0^{-1}(p)$. A life test experiment is applied to determine whether there is a high probability that the current lifetime of the product is at least t_ω. The producer would be interested in checking whether the lot or batch is assumed to be good, i.e. the proportion defective is enough low to be considered acceptable and it should not exceed an acceptable quality level p_α. Instead, the consumer would consider that bad lots must be rejected, in which case a larger value of the proportion defective should be deemed unacceptable or rejectable, concretely above a given rejectable quality level p_β. In this paper it is assumed that $0 < p_\alpha < p_\beta < 0.5$ and the life test consists in deciding between the null hypothesis of lot acceptance, $H_0 : p \leq p_\alpha$, or the alternative hypothesis of lot rejection, $H_1 : p \geq p_\beta$.

An appropriate inspection scheme must be applied to determine a test statistic to make the lot decision. A commonly used pivotal quantity is given by $Q(p) \equiv Q(p, n, r) = (\hat{\mu} - \xi_p)/\hat{\sigma}$, where $\hat{\mu}$ and $\hat{\sigma}$ are the maximum likelihood estimators, provided they exist, of μ and σ, respectively. Then, the null hypothesis is accepted if $Q(p)$ is large enough when the nonconforming proportion is close to the acceptable quality level, i.e., $p = p_\alpha$.

In a type II or failure censored single sampling scheme by variables, only one sample of n units is randomly selected from a large submitted lot of items and placed on life test simultaneously. The reliability experiment is terminated at the time of the rth failure, where $r \leq n$, and the censoring level is given by $q = 1 - r/n$. Consider that $\mathbf{Y} = \{Y_{1:n}, ..., Y_{r:n}\}$ represents the sample of ordered observed standardized log-lifetimes of the product, i.e. $T_i = \exp(\mu + \sigma Y_i)$ denotes the lifetime of the sample unit corresponding to the ith failure, with $i = 1, ...r$. Thus, given this sample, the likelihood function for (μ, σ) is defined by

$$\mathcal{L}(\mu, \sigma \mid \mathbf{Y}) \propto \textstyle\prod_{i=1}^{r} f_0(Y_{i:n})\{1 - F_0(Y_{i:n})\}^{n-r}.$$

Then, the decision criteria consists in accepting the lot whenever $Q(p) \geq k$, and rejecting, otherwise, where k represents an acceptance constant to be determined. It is obvious that, in this single sampling scheme, the decision about the lot is taken based on the inspection of only one sample.

In this paper, a repetitive scheme is considered in order to make the decision by using possible additional samples.

3. Censored repetitive lot inspection

The censored repetitive sampling scheme allows for the reinspection of lots by considering both acceptance and rejection limits, k_r and k_a, in the decision criteria, where k_r, k_a are real numbers. An initial sample of size n is first withdrawn from the lot and $Q(p)$ is evaluated. If $Q(p) \geq k_a$ the lot is accepted, and if $Q(p) < k_r$, then the lot is rejected, where $k_r \leq k_a$. In the case where $k_r \leq Q(p) < k_a$, then an additional sample is taken from the lot to repeat the procedure until a final decision is reached. Therefore, the failure-censored repetitive sampling plan is described by the sample size n, the censoring level q (or, equivalently, the number of observed failures r), and the acceptance and rejection constants k_a and k_r. This q-censored repetitive plan is denoted as $S_q = (n, k_r, k_a)$ and the decision criteria can be described as

- *Step 1.* Initialize $i = 1$.
- *Step 2.* Select the ith sample of size n at random from the lot or batch. Observe the lifetimes and compute $\{Y_{1:n}^{(i)}, ..., Y_{r:n}^{(i)}\}$, where $Y_{j:n}^{(i)}$ represents the standardized log-lifetime of the j-th item of this i-th sample, with $j = 1, ..., r$.
- *Step 3.* Compute $Q^{(i)}(p) = (\hat{\mu}^{(i)} - \xi_p)/\hat{\sigma}^{(i)}$, where $\hat{\mu}^{(i)}$ and $\hat{\sigma}^{(i)}$ are the corresponding maximum likelihood estimators of μ and σ for the ith sample, and
  - accept the lot if $Q^{(i)}(p) \geq k_a$,
  - reject the lot if $Q^{(i)}(p) < k_r$,
  - otherwise, when the decision cannot be made, repeat step 2 with $i = i + 1$.

Note that if $k_a = k_r$, then the failure censored repetitive plan coincides with the conventional censored single sampling plan by variables; see Balamurali, Park, Jun, Kim and Lee (2005), Balamurali and Jun (2006) and Jun, Lee, Lee and Balamurali (2010). The operating characteristic (OC) function for the censored repetitive plan can be defined by obtaining the probabilities $P_{acc}(p)$ and $P_{rej}(p)$ of accepting and rejecting a lot based on a single sample, respectively. Consider that $V(k) = \hat{\mu} - k\hat{\sigma} = U_1 - kU_2$, where $U_1 = (\hat{\mu} - \mu)/\sigma$ and $U_2 = \hat{\sigma}/\sigma$ are pivotal quantities. Then

$$\begin{aligned} P_{acc}(p) \equiv P_{acc}(p; n, k_a, q) &= \Pr(Q(p) \geq k_a \mid p) = \Pr(V(k_a) > \xi_p) \\ P_{rej}(p) \equiv P_{rej}(p; n, k_r, q) &= \Pr(Q(p) < k_r \mid p) = \Pr(V(k_r) < \xi_p), \end{aligned}$$

where $0 < p < 1$. Therefore, the probability of repeating the sampling is

$$P_{rpt}(p) \equiv P_{rpt}(p; n, k_a, k_r, q) = \Pr(k_r \leq Q(p) < k_a \mid p) = 1 - P_{acc}(p) - P_{rej}(p),$$

and the OC curve is given by

$$L(p) \equiv L(p; n, k_a, k_r, q) = P_{acc}(p) \sum_{i=0}^{\infty} P_{rpt}(p)^i = \frac{P_{acc}(p)}{P_{acc}(p) + P_{rej}(p)}.$$

In this situation, the probabilities functions $P_{acc}(p)$ and $P_{rej}(p)$ cannot be expressed in closed forms and, consequently, the OC curve needs to be approximated. According to Bhattacharyya (1985), the asymptotic distribution of $\sqrt{n}(U_1, U_2 - 1)$ is $N(\mathbf{0}, I^{-1}(\hat{\mu}, \hat{\sigma}))$, where $I(\hat{\mu}, \hat{\sigma})$ is the observed information matrix

$$I(\hat{\mu}, \hat{\sigma}) = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix} = -\frac{\sigma^2}{n} \begin{pmatrix} E\left[\frac{\partial^2 \log \mathcal{L}(\mu,\sigma|\mathbf{Y})}{\partial \mu^2}\right] & E\left[\frac{\partial^2 \log \mathcal{L}(\mu,\sigma|\mathbf{Y})}{\partial \mu \partial \sigma}\right] \\ E\left[\frac{\partial^2 \log \mathcal{L}(\mu,\sigma|\mathbf{Y})}{\partial \mu \partial \sigma}\right] & E\left[\frac{\partial^2 \log \mathcal{L}(\mu,\sigma|\mathbf{Y})}{\partial \sigma^2}\right] \end{pmatrix}$$

and the expressions of γ_{ij} have been derived by Fernández and Pérez-González (2012a) as

$$\begin{aligned} \gamma_{11} &= g_0(t; q) + \frac{f_0(u_q)^2}{q}, \\ \gamma_{12} &= g_1(t; q) + f_0(u_q) + \frac{u_q f_0(u_q)^2}{q}, \\ \gamma_{22} &= g_2(t; q) + \frac{\{q + u_q f_0(u_q)\}^2}{q} - 1, \end{aligned}$$

where $g_i(t; q) = \int_{-\infty}^{u_q} \frac{f_0'(t)^2}{f_0(t)} t^i dt$ for $i = 0, 1, 2$, and $u_q = F_0^{-1}(1 - q)$ for $q \in (0, 1)$.

Therefore, the lot acceptance and rejection probabilities are asymptotically given by $P_{acc}(p) \equiv \Pr(\Delta(n, k_a) > \delta(n, k_a, p, q))$ and $P_{rej}(p) \equiv \Pr(\Delta(n, k_r) < \delta(n, k_r, p, q))$, where

$$\Delta(n, k) = \frac{\sqrt{n}(U_1 - kU_2 + k)}{\sqrt{\gamma_{11} - 2k\gamma_{12} + k^2\gamma_{22}}}$$

and

$$\delta(n, k, p, q) = \frac{\sqrt{n}(F_0^{-1}(p) + k)}{\sqrt{\gamma_{11} - 2k\gamma_{12} + k^2\gamma_{22}}}.$$

Then, denoting $\Phi[\cdot]$ as the standard normal cdf, it follows that

$$\begin{aligned} P_{acc}(p) &\simeq 1 - \Phi[\delta(n, k_a, p, q)] \\ P_{rej}(p) &\simeq \Phi[\delta(n, k_r, p, q)]. \end{aligned}$$

This approximation is precise enough for the majority of practical purposes, excepting for small sample sizes and large values of the censoring level.

4. Optimal reliability sampling plans using prior information

In defining classical acceptance sampling plans, it is common to consider two prefixed levels of p under an agreement of the producer and the consumer; the producer considers an acceptable quality level (AQL), that is usually denoted as p_α, whereas the rejectable quality level (RQL) proposed by the consumer is given by p_β. Then, the conventional producer risk is the probability of rejecting an AQL lot and is given by $CPR(p_\alpha) \equiv CPR(p_\alpha; n, k_r, k_a, q) = 1 - L(p_\alpha)$, whereas the conventional consumer risk represents the probability of accepting an RQL lot, i.e. $CCR(p_\beta) \equiv CCR(p_\beta; n, k_r, k_a, q) = L(p_\beta)$. However, in some situations, a prior knowledge on the fraction defective can be available and the producer and the consumer can agree for prior model on p. The expected sampling risks can be defined in order to incorporate these prior models in obtaining the reliability sampling plan.

The expected producer risk (EPR) is the probability of rejecting a good lot, i.e. a lot of products with a maximum nonconforming proportion under the AQL, whereas the expected consumer risk (ECR) represents the probability of accepting a bad lot, i.e. a lot of products with a minimum nonconforming proportion exceeding the RQL. Therefore, the expressions of EPR and ECR are

$$EPR(p_\alpha; n, k_r, k_a, q) = E_{h_1}[1 - L(p) \mid p \leq p_\alpha] = \int_{p \leq p_\alpha} \{1 - L(p)\} \frac{h_1(p)}{H_1(p_\alpha)} dp$$

and

$$ECR(p_\beta; n, k_r, k_a, q) = E_{h_2}[L(p) \mid p \geq p_\beta] = \int_{p \geq p_\beta} L(p) \frac{h_2(p)}{1 - H_2(p_\beta)} dp,$$

respectively, where $E_h[\cdot]$ represents the mathematical expectation under the prior $h(\cdot)$ that can be defined over a given domain of p. This paper considers the case in which both producer and consumer want to ensure the neutrality of the prior model and they agree with the selected prior distribution, $h(p) = h_1(p) = h_2(p)$. Anyway, under certain situations, the producer could specify a different prior model to that selected by the consumer.

4.1. Prior distributions on proportion nonconforming

The beta distribution is frequently used as a prior knowledge model to describe the stochastic variations of the fraction defective. More precisely, the generalized beta (GB) distribution with parameters a, b, l and u ($a > 0, b > 0, 0 \leq l < u \leq 1$) represents a more flexible version of beta distribution that has been studied by Fernández and Pérez-González (2012a). The GB distribution of p, denoted as $p \sim GB(a, b, l, u)$, has a density function defined by

$$h(p) \equiv h(p; a, b, l, u) = \frac{(p - l)^{a-1}(u - p)^{b-1}}{B(a, b)(u - l)^{a+b-1}}, \quad l \leq p \leq u,$$

and the cumulative distribution function is given by

$$H(p) = \frac{B_{inc}[(p - l)/(u - l); a, b]}{B(a, b)}, \quad l < p < u,$$

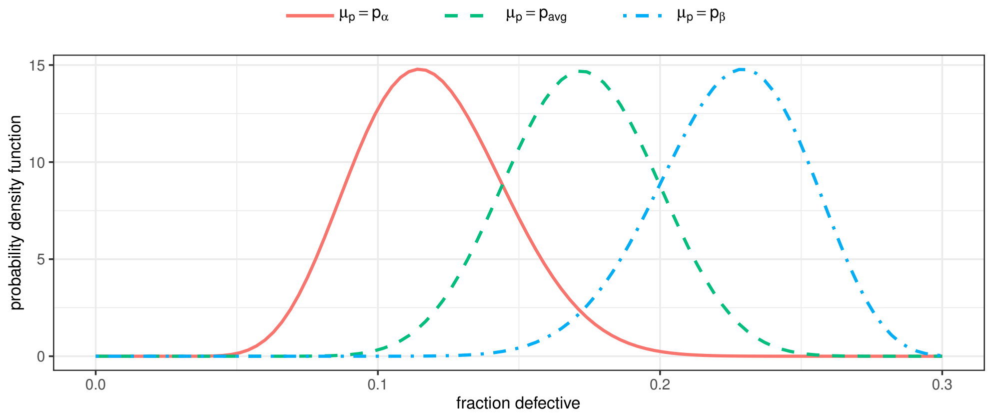

Figure 1: Generalized beta prior pdfs of p when μ_p is equal to p_α, p_β and p_{avg} for σ_p^2 fixed.

where $B(a, b)$ denotes the beta function and

$$B_{inc}[z; a, b] = \int_0^z p^{a-1}(1-p)^{b-1} dp, \quad z \in [0, 1],$$

is the incomplete beta function. Then, the prior mean and variance of the nonconforming proportion p are given, respectively, by

$$\mu_p \equiv E[p] = l + \frac{a(u-l)}{a+b} \quad \text{and} \quad \sigma_p^2 \equiv V[p] = \frac{ab(u-l)^2}{(a+b)^2(a+b+1)}.$$

Obviously, l and u are the minimum and maximum nonconforming proportion, in which case l and $u-l$ represent location and scale parameters, respectively. Therefore, as was explained before, if there exists prior knowledge about the range of the p, then GB priors allow the possibility to incorporate that information. In the case of lack of information to that extent, then $l = 0$ and $u = 1$. The GB family of densities also admits different symmetrical and asymmetrical shapes because the shape and skewness of the distribution are controlled by the parameters a and b. Observe that the standard $Beta(a, b)$ distribution is a particular case of GB distribution when $l = 0$ and $u = 1$. When $a = b = 1$, then the GB distribution is the uniform $U(l, u)$.

There exists other possibilities to incorporate the available information into the GB prior. Based on the opinions of the producer and the consumer, the prior mean μ_p can be considered to be equal to the AQL or RQL, i.e. $\mu_p = p_\alpha$ or $\mu_p = p_\beta$. Sometimes, in reliability sampling plans, it can be required to assume a position of neutrality between the producer and the consumer. Under an agreement, a consensus value of μ_p could be the average of both quality levels, i.e. $p_{avg} = (p_\alpha + p_\beta)/2$. In this case, the variance can be approximated using the range rule of thumb $\sigma_p = (p_\beta - p_\alpha)/4$. As a graphical illustration, when $p_\alpha = 0.119$ and $p_\beta = 0.225$, Figure 1 displays the GB prior densities for the different values of the mean proportion defective, assuming that the variance σ_p^2 is kept fixed.

Fernández and Pérez-González (2011) have also considered the impartiality of prior distributions to establish the neutral position between producers and consumers in which both hypotheses test, $p \leq p_\alpha$ and $p \geq p_\beta$, should have the same weight. Therefore, given $0 < \varepsilon \leq 1/2$, the ε-impartial prior distributions are those satisfying that $\Pr(p \leq p_\alpha) = \Pr(p \geq p_\beta) = \varepsilon$.

5. Design of optimal censored repetitive sampling plans

The q-censored sampling plan $S_q = (n, k_r, k_a)$ must ensure certain protection to manufacturers against the rejection of good lots produced with a quality under the AQL and, simultaneously, provide the assurance to consumers against the acceptance of bad lots that have quality over the RQL. This means that both producer and consumer specify the maximum tolerable expected risks that they are willing to accept. The producer wants that $EPR(p_\alpha; n, k, q) \leq \alpha$ whereas the consumer desires $ECR(p_\beta; n, k, q) \leq \beta$, where $\alpha, \beta < 0.5$ represent the corresponding maximum risks.

In defining the sampling plan it is also required to optimize the sampling effort in order to achieve the efficiency of the inspection saving economical costs and testing time. The expected value of the average sample number, $EASN$, under a prior distribution of p represents an objective measure of the inspection effort of $S_q = (n, k_r, k_a)$ and is expressed as

$$EASN(n, k_r, k_a, q) = E_h[ASN(p; n, k_r, k_a, q)] = \int_l^u ASN(p; n, k_r, k_a, q)h(p)dp,$$

where, assuming a nonconforming proportion $l < p < u$, $ASN(p; n, k_r, k_a, q)$ is the average number of units observed in the repetitive inspection before to make a final decision about the lot acceptance. Therefore,

$$ASN(p; n, k_r, k_a, q) = \frac{nP_{acc}(p)}{P_{acc}(p) + P_{rej}(p)}.$$

The optimal q-censored repetitive plan (n^*, k_r^*, k_a^*), denoted as S_q^*, corresponds to the inspection plan that minimizes the $EASN$ satisfying the requirements of producer and consumer about the quality levels AQL and LQL and the maximum tolerable expected risks, and assuming a prior distribution $h(\cdot)$ of p defined over $[l, u] \subseteq [0, 1]$. Given $q \in [0, 1)$, the corresponding constrained optimization problem can be formulated as the following mixed nonlinear program

$$\begin{array}{ll} \text{minimize} & EASN(n, k_r, k_a, q) \\ \text{subject to} & EPR(p_\alpha; n, k_r, k_a, q) \leq \alpha, \\ & ECR(p_\beta; n, k_r, k_a, q) \leq \beta, \\ & k_r < k_a \end{array} \tag{1}$$

where $n \in \mathbb{N}$ and $k_r, k_a \in \mathbb{R}$, with $\mathbb{N}$ and $\mathbb{R}$ denoting the sets of natural and real numbers, respectively. The problem can be expressed in a compact way as

$$\min_{k_r<k_a} \{EASN(S_q) : EPR(p_\alpha; S_q) \leq \alpha, ECR(p_\beta; S_q) \leq \beta\}.$$

A procedure to obtain the optimal solution of this optimization problem can be designed by obtaining, firstly, the designs of standard censored repetitive plans under conventional risks. These designs can serve as a starting point in the iterative search method used in solving the problem.

The following procedures have been implemented in several computational programs developed with R language that are included in an open source library called *RepetPlan* available in the Zenodo repository on Pérez-González (2021).

5.1. ASN_{avg}-optimal design of repetitive censored sampling plans using conventional risks

The design of variables repetitive group sampling plans have been studied by different authors, as Balamurali, Park, Jun, Kim and Lee (2005) or Balamurali and Jun (2006). In the following, an efficient procedure is defined in order to obtain, for $0 \leq q \leq 1$, the designs $S'_q = (n', k'_r, k'_a)$ of the q-censored repetitive sampling plans under the traditional approach, i.e. based on the specification of quality levels, AQL and LQL, and conventional allowable sampling risks α and β. The problem consists in determine the set of feasible plans $W_q \equiv W_q(p_\alpha, p_\beta, \alpha, \beta)$ given by

$$\begin{aligned} W_q = \{S'_q : \quad & CPR(p_\alpha; S'_q) = \alpha,\ CCR(p_\beta; S'_q) = \beta, \\ & n' \in \mathbb{N},\ k'_r, k'_a \in \mathbb{R},\ k'_r < k'_a\} \end{aligned} \tag{2}$$

A step-by-step computational procedure to obtain the plans of W_q using an iterative full Newton method is described as follows

- *Step 1.* Choose the maximum allowable values of the producer and consumer risks, α and β, the corresponding quality levels, p_α and p_β, and the censoring level q.
- *Step 2.* In order to obtain an initial plan to be used in the procedure, the repetitive sampling plan when $k'_a = k'_r$ is computed. This plan corresponds to the well known single censored plan (n_0, k_0, q) satisfying the following risks

  $$1 - P_{acc}(p_\alpha) = \Phi[\delta(n_0, k_0, p_\alpha, q)] = \alpha \quad \text{and} \quad P_{acc}(p_\beta) = \Phi[\delta(n_0, k_0, p_\beta, q)] = \beta, \tag{3}$$

  in which case

  $$k_0 = \frac{z_\alpha F_0^{-1}(p_\beta) + z_\beta F_0^{-1}(p_\alpha)}{-(z_\alpha + z_\beta)} \quad \text{and} \quad n_0 = \left\lceil \frac{(\gamma_{11} - 2k_0\gamma_{12} + k_0^2\gamma_{22})(z_\alpha + z_\beta)^2}{(F_0^{-1}(p_\beta) - F_0^{-1}(p_\alpha))^2} \right\rceil,$$

  where z_ν is the ν-quantile of the standard normal distribution and $\lceil x \rceil$ denotes the smallest integer greater than x.
- *Step 3.* Initialize $n' = n_0$.

- *Step 4.* The start values for k_r' and k_a' for the iterative method can be obtained by approximating $CPR(p_\alpha)$ and $CPR(p_\beta)$ using the following first-order Taylor polynomial approximation in (3)

$$\Phi[\delta(n,k,p,q)] \approx \Phi[\delta(n,k_0',p,q)] + \phi[\delta(n,k_0',p,q)]\frac{\partial}{\partial k}\delta(n,k_0',p,q)(k-k_0')$$

  where $\phi[\cdot]$ denotes the density of the standard normal distribution and $(\partial/\partial k)\delta(n,k,p,q)$ the first derivative of $\delta(n,k,p,q)$ with respect to k. The value $k_0' = \max\{k_0, \gamma_{12}/\gamma_{22}\}$ is considered to ensure the existence of this derivative. Then, after some calculations, the following expressions are derived

$$k_{0;r} = k_0' - \frac{\alpha\beta}{\beta-(1-\alpha)}(w_\beta - w_\alpha) - w_\alpha \quad \text{and} \quad k_{0;a} = k_0' + \frac{(1-\alpha)\beta}{\beta-(1-\alpha)}(w_\beta - w_\alpha),$$

  where

$$w_\beta = \frac{\Phi[\delta(n,k_0',p_\beta,q)] - (1-\beta)}{\beta\phi[\delta(n,k_0',p_\beta,q)]\frac{\partial}{\partial k}\delta(n,k_0',p_\beta,q)}$$

  and

$$w_\alpha = \frac{\Phi[\delta(n,k_0',p_\alpha,q)] - \alpha}{(1-\alpha)\phi[\delta(n,k_0',p_\alpha,q)]\frac{\partial}{\partial k}\delta(n,k_0',p_\alpha,q)}.$$

- *Step 5.* Applying a full Newton method using the start values $k_{0;a}$ and $k_{0;r}$, compute k_a' and k_r' from the nonlinear equations $CPR(p_\alpha; n', k_r', k_a', q) = \alpha$ and $CCR(p_\beta; n', k_r', k_a', q) = \beta$. Then, if $k_r' \leq k_a'$, select $S_q' = (n', k_r', k_a')$ as a feasible plan of W_q.

- *Step 6.* If $n > 1$, set $n' = n' - 1$ and go to the Step 4. Otherwise, the iterative search is finished and W_q is completely determined.

The above method is an efficient procedure that allow to quickly determine, in the case of the repetitive censored scheme, the plan of W_q satisfying a given optimality criterion. For example, the ASN at both AQL and LQL can be computed for the feasible plans over the plan in W_q and the optimal repetitive design minimizing $ASN_{avg} = \{ASN(p_\alpha) + ASN(p_\beta)\}/2$ should be easily determined.

For comparison purposes, Table 1 collects repetitive sampling plans with minimum ASN_{avg} for selected values of p_α, p_β and q when X follows both a normal and an extreme-value distribution. Observe that this kind of designs generally requires a greater sampling effort when the degree of censoring increases. Both n and ASN also increments for larger discrepancies between the AQL and LQL, which suppose a waste of sample units in the case of destructive life testing.

5.2. EASN-optimal design of repetitive censored sampling plans using expected risks

The censored repetitive plans obtained in the above procedure can be also employed to solve the problem (1) of the repetitive plan minimizing the $EASN$, under given requirements of maximum expected risks at both AQL and LQL, when a prior distribution $h(\cdot)$ of p is assumed.

Table 1: Repetitive plans with minimum ASN_{avg} for selected levels on censoring q when $\alpha = 0.05$ and $\beta = 0.10$.

		$q = 0.0$				$q = 0.4$				$q = 0.8$			
p_α	p_β	n	k_r	k_a	ASN_{avg}	n	k_r	k_a	ASN_{avg}	n	k_r	k_a	ASN_{avg}
Normal distribution													
0.00041	0.0184	11	2.278	3.184	16.77	15	2.272	3.266	23.95	30	2.376	3.179	43.33
0.00284	0.0311	16	1.962	2.672	25.56	22	1.987	2.670	34.39	36	2.042	2.662	53.69
0.00654	0.0426	19	1.783	2.416	31.28	25	1.803	2.420	40.54	39	1.867	2.380	57.65
0.0319	0.0942	27	1.354	1.794	44.04	32	1.367	1.795	51.34	38	1.396	1.780	57.65
0.0615	0.1420	29	1.101	1.488	47.43	32	1.109	1.492	51.99	38	1.134	1.449	56.07
0.1190	0.2250	28	0.776	1.132	46.43	30	0.789	1.122	47.91	44	0.787	1.073	65.49
Extreme value distribution													
0.00041	0.0184	7	4.834	7.229	10.04	10	4.890	7.252	14.27	20	5.025	7.221	27.80
0.00284	0.0311	10	3.783	5.770	16.34	14	3.871	5.692	21.85	26	4.030	5.531	37.20
0.00654	0.0426	13	3.376	4.902	20.99	17	3.420	4.904	27.03	29	3.551	4.783	42.24
0.0319	0.0942	21	2.423	3.339	34.28	25	2.458	3.324	39.57	33	2.509	3.290	49.62
0.0615	0.1420	25	1.962	2.669	40.31	27	1.974	2.682	43.56	35	2.014	2.608	51.75
0.1190	0.2250	27	1.430	1.992	43.59	27	1.436	2.002	43.77	41	1.426	1.924	62.43

Again, a computational procedure can also be defined to obtain the optimal solution $S_q^* = (n^*, k_r^*, k_a^*, q)$ of the programming problem. In this case, it is very useful to bound above and below the optimal sample size n^*. Observe that a lower bound is given by $n_{inf} = \text{minimize}\{n' : S'_q \in W_q(l, u, \alpha, \beta)\}$ whereas an upper bound can be obtained as $n_{sup} = \text{maximize}\{n' : S'_q \in W_q(p_\alpha, p_\beta, \alpha, \beta)\}$.

The corresponding procedure to determine the $EASN$-optimal repetitive plan can be summarized in the following way:

- *Step 1.* Select the values of maximum acceptable risks at the corresponding quality levels as well as the censoring level, i.e. $\alpha, \beta, p_\alpha, p_\beta$ and q. Choose the prior distribution $h(p)$ defined for $p \in [l, u]$.
- *Step 2.* Compute n_{inf} and n_{sup} from the feasible set W_q.
- *Step 3.* Initialize $n = n_{sup} = EASN_{opt}$.
- *Step 4.* Check if there exists a feasible plan $(n, k'_r, k'_a) \in W_q$ corresponding to the current value of the sample size. If there is no plan for this value of n, go to Step 6. Otherwise, consider k'_r and k'_a as starting values in applying the full Newton method to compute the solution of the nonlinear equations $EPR(p_\alpha; n, k''_r, k''_a, q) = \alpha$ and $ECR(p_\beta; n, k''_r, k''_a, q) = \beta$.
- *Step 5.* If $k''_r \leq k''_a$ and $EASN(n, k''_r, k''_a) \leq EASN_{opt}$, select $S_q^* = (n, k''_r, k''_a)$.
- *Step 6.* If $n > n_{inf}$, set $n = n - 1$, and go to the Step 4. Otherwise, finish the iterative search and S_q^* is the optimal repetitive plan.

The above procedure is applied to compute the optimal repetitive sampling plans with minimum expected $EASN$ for arbitrary values of q assuming a GB prior of p. The plans are shown in Tables 2 and 3 for the normal and extreme value distribution of X, respectively.

It is assumed that $p \in [l, u]$, where $l = p_\alpha/5$ and $u = p_\beta + (p_\alpha - l)$. In this case, the designs are determined when the prior mean μ_p is given by the quality levels, p_α and p_β, and the average p_{avg} of these values. As was indicated earlier, the standard deviation is $\sigma_p = (p_\beta - p_\alpha)/4$. We can appreciate, both the normal and the extreme value distribution of X, that the sample size n is quite similar to that of the standard repetitive plans of Table 1 when μ_p is assumed to be the average of AQL and LQL. The ASN_{avg} of the corresponding repetitive plans under conventional risks is slightly lower that the $EASN$ of the repetitive plans using expected risks, for the different degrees of censoring, but it cannot be considered that there is a significant difference between both kind of plans. However, the reduction in the sampling effort is more apparent when μ_p is equal to AQL or LQL. The decrease percentage in the ASN_{avg} of the standard repetitive plans with respect to the $EASN$ of the optimal plans shown in these Tables is, approximately, between 20% and 30%, although becomes larger (above 40%) in the case of high quality levels.

Tables 4 and 5 present the design of $EASN$-optimal repetitive plans in the case of impartial GB prior of p. These plans are determined, using the computational procedure described before, for several values of the impartiality parameter values, $\epsilon = 0.2, 0.3, 0.4$. Note that $EASN$ of the optimal designs reduces significantly in comparison with the ASN_{avg} of the corresponding plans in 1. A percentage of reduction between 20% and 40% is observed for the smaller value of the impartiality parameter, although the reduction raises above 70% for larger values. In this case, the $EASN$-optimal plans provide a greater sampling effort saving for the less restrictive quality levels.

6. Illustrative example

For illustrative and comparative purposes, we can consider the reliability analysis of complementary metal-oxide-semiconductor (CMOS) integrated circuits. According to Kharchenko (2015), many failures of these components are caused by oxide film defects. These defects produce electrical charge accumulation during the circuit operation, which eventually can lead to an electrical failure. Suppose that X is the random variable that represents the time until the failure and assume that X follows a normal distribution. We also assume that the producer and the consumer agree that the probabilities of rejecting an integrated circuit with good quality and accepting a circuit with bad quality must be at most $\alpha = 0.05$ and $\beta = 0.10$, respectively. The respective AQL and LQL are given by $p_\alpha = 0.119$ and $p_\beta = 0.225$. Then, the standard optimal repetitive sampling plan with these requirements when $q = 0.5$ is $S'_{0.5} = (30, 0.787, 1.124)$ and $ASN_{avg} = 48.17$. Figure 2 shows the classical sampling risks, CPR and CCR, and expected sampling risks, EPR and ECR, of this plan when one of the quality levels is kept fixed and the other varies. A generalized beta family of priors with domain $[l, u]$ is used to model the available prior information on p, where $l = p_\alpha/5$ and $u = p_\beta + (p_\alpha - l)$. Then, the expected sampling risks are computed for different values of μ_p with

Table 2: Repetitive plans with minimum $EASN$ for selected levels on censoring q under a normal distribution of X when $\alpha = 0.05$, $\beta = 0.10$, and p follows a GB prior with $\mu_p = p_\alpha, p_{avg}, p_\beta$ and $\sigma_p = (p_\beta - p_\alpha)/4$.

			$\mu_p = p_\alpha$				$\mu_p = p_{avg}$				$\mu_p = p_\beta$			
q	p_α	p_β	n	k_r	k_a	$EASN$	n	k_r	k_a	$EASN$	n	k_r	k_a	$EASN$
0.0	0.00041	0.0184	-	-	-	-	9	2.246	3.380	16.68	-	-	-	-
	0.00284	0.0311	9	2.124	2.948	13.36	15	2.028	2.657	25.40	8	1.847	3.232	14.66
	0.00654	0.0426	11	1.912	2.619	16.79	18	1.851	2.382	30.86	12	1.714	2.624	21.51
	0.0319	0.0942	19	1.412	1.850	29.97	25	1.400	1.758	42.56	18	1.292	1.860	32.85
	0.0615	0.1420	21	1.142	1.528	33.85	27	1.144	1.448	45.43	20	1.053	1.517	35.78
	0.1190	0.2250	21	0.810	1.157	33.95	27	0.823	1.080	44.09	20	0.739	1.136	35.29
0.4	0.00041	0.0184	-	-	-	-	13	2.275	3.408	23.46	-	-	-	-
	0.00284	0.0311	13	2.167	2.927	18.92	20	2.035	2.683	33.91	11	1.878	3.308	19.45
	0.00654	0.0426	15	1.939	2.619	22.75	23	1.855	2.406	39.79	15	1.711	2.715	27.27
	0.0319	0.0942	23	1.429	1.847	35.86	30	1.415	1.753	49.46	21	1.304	1.876	37.39
	0.0615	0.1420	24	1.158	1.521	37.90	30	1.152	1.448	49.65	22	1.063	1.525	38.46
	0.1190	0.2250	23	0.829	1.139	35.42	28	0.827	1.080	45.33	21	0.747	1.132	36.10
0.8	0.00041	0.0184	-	-	-	-	25	2.357	3.419	41.19	-	-	-	-
	0.00284	0.0311	26	2.291	2.768	32.93	32	2.074	2.710	51.94	19	1.963	3.469	30.16
	0.00654	0.0426	27	2.046	2.496	35.62	35	1.904	2.391	55.69	23	1.778	2.777	37.37
	0.0319	0.0942	30	1.484	1.790	41.77	36	1.445	1.730	54.79	26	1.350	1.842	41.18
	0.0615	0.1420	28	1.184	1.481	40.39	35	1.175	1.412	52.69	28	1.109	1.443	42.49
	0.1190	0.2250	30	0.802	1.114	43.92	41	0.835	1.036	60.56	37	0.795	1.024	53.59

Table 3: Repetitive plans with minimum $EASN$ for selected levels on censoring q under an extreme value distribution of X when $\alpha = 0.05$, $\beta = 0.10$, and p follows a GB prior with $\mu_p = p_\alpha, p_{avg}, p_\beta$ and $\sigma_p = (p_\beta - p_\alpha)/4$.

			$\mu_p = p_\alpha$				$\mu_p = p_{avg}$				$\mu_p = p_\beta$			
q	p_α	p_β	n	k_r	k_a	$EASN$	n	k_r	k_a	$EASN$	n	k_r	k_a	$EASN$
0.0	0.00041	0.0184	-	-	-	-	5	4.548	8.770	9.14	-	-	-	-
	0.00284	0.0311	6	4.440	6.528	8.73	9	3.914	5.846	15.42	5	3.565	7.955	8.62
	0.00654	0.0426	8	3.841	5.372	11.34	12	3.520	4.867	19.98	7	3.055	6.175	13.48
	0.0319	0.0942	15	2.565	3.465	23.71	19	2.506	3.290	32.65	13	2.260	3.639	24.43
	0.0615	0.1420	19	2.071	2.717	29.33	23	2.035	2.604	38.28	16	1.843	2.827	29.22
	0.1190	0.2250	21	1.507	2.017	32.67	25	1.490	1.928	41.23	18	1.349	2.063	31.88
0.4	0.00041	0.0184	-	-	-	-	7	4.569	9.177	12.87	-	-	-	-
	0.00284	0.0311	9	4.592	6.290	12.18	12	3.940	5.925	20.47	7	3.659	8.121	11.54
	0.00654	0.0426	11	3.910	5.321	15.22	15	3.515	4.967	25.55	9	3.089	6.458	17.04
	0.0319	0.0942	19	2.632	3.392	28.04	22	2.521	3.305	37.53	15	2.283	3.698	27.54
	0.0615	0.1420	21	2.089	2.720	32.35	25	2.048	2.610	41.21	17	1.850	2.880	30.92
	0.1190	0.2250	22	1.529	2.000	33.25	25	1.495	1.935	41.23	18	1.356	2.081	31.61
0.8	0.00041	0.0184	-	-	-	-	15	4.826	8.988	24.68	-	-	-	-
	0.00284	0.0311	19	4.840	5.863	22.67	21	4.028	6.021	34.48	13	3.804	8.949	20.11
	0.00654	0.0426	20	4.056	5.160	25.85	25	3.619	4.922	39.49	16	3.288	6.467	26.18
	0.0319	0.0942	26	2.696	3.334	36.27	30	2.589	3.235	46.53	21	2.378	3.610	34.19
	0.0615	0.1420	27	2.135	2.646	38.17	32	2.086	2.545	48.39	24	1.935	2.690	37.76
	0.1190	0.2250	29	1.480	1.987	43.55	38	1.501	1.862	57.98	33	1.423	1.866	49.53

Table 4: Repetitive plans with minimum $EASN$ for selected levels on censoring q under a normal distribution of X when $\alpha = 0.05$, $\beta = 0.10$, and p follows a ε-impartial GB distribution with $\varepsilon = 0.2, 0.3, 0.4$.

			$\varepsilon = 0.2$				$\varepsilon = 0.3$				$\varepsilon = 0.4$			
q	p_α	p_β	n	k_r	k_a	$EASN$	n	k_r	k_a	$EASN$	n	k_r	k_a	$EASN$
0.0	0.00041	0.0184	8	2.3420	3.4406	12.90	7	2.2912	3.6325	12.01	7	2.3371	3.6011	11.22
	0.00284	0.0311	11	2.0306	2.8342	17.90	9	1.9861	3.0170	15.54	9	2.0695	2.9629	13.56
	0.00654	0.0426	12	1.8161	2.5845	20.84	11	1.8584	2.6063	17.27	9	1.8518	2.7482	14.19
	0.0319	0.0942	16	1.3703	1.8629	26.40	12	1.3459	1.9661	19.75	9	1.3677	2.0725	13.65
	0.0615	0.1420	16	1.0928	1.5538	27.29	12	1.0765	1.6212	19.55	8	1.0668	1.7601	12.34
	0.1190	0.2250	16	0.7785	1.1516	25.79	11	0.7373	1.2297	17.76	7	0.7293	1.3236	10.22
0.4	0.00041	0.0184	12	2.3893	3.4231	18.62	11	2.3758	3.5413	17.41	11	2.4184	3.5143	16.42
	0.00284	0.0311	15	2.0465	2.8629	24.35	13	2.0374	2.9916	21.33	12	2.0676	3.0479	18.79
	0.00654	0.0426	17	1.8681	2.5395	27.31	14	1.8534	2.6777	22.86	12	1.8704	2.7912	19.06
	0.0319	0.0942	19	1.3834	1.8716	31.03	15	1.3808	1.9479	23.47	11	1.3909	2.0905	16.52
	0.0615	0.1420	19	1.1259	1.5259	30.10	14	1.1059	1.6057	21.78	9	1.0785	1.8004	14.06
	0.1190	0.2250	17	0.7908	1.1444	26.70	12	0.7608	1.2108	18.53	7	0.7079	1.3961	10.88
0.8	0.00041	0.0184	23	2.4526	3.4711	34.63	23	2.4937	3.4437	32.90	23	2.5311	3.4215	31.49
	0.00284	0.0311	26	2.1169	2.8498	39.31	24	2.1403	2.9163	35.23	23	2.1868	2.9324	31.85
	0.00654	0.0426	27	1.9314	2.5178	40.12	24	1.9524	2.5943	34.51	21	1.9786	2.7058	29.76
	0.0319	0.0942	24	1.4337	1.8289	35.44	20	1.4527	1.8721	27.53	15	1.4729	2.0198	20.27
	0.0615	0.1420	22	1.1486	1.4856	32.45	17	1.1484	1.5321	23.79	12	1.1730	1.6301	15.70
	0.1190	0.2250	25	0.7934	1.0775	36.08	18	0.7670	1.1102	25.19	11	0.7468	1.1781	14.55

Table 5: Repetitive plans with minimum $EASN$ for selected levels on censoring q under an extreme value distribution of X when $\alpha = 0.05$, $\beta = 0.10$, and p follows a ε-impartial GB distribution with $\varepsilon = 0.2, 0.3, 0.4$.

			$\varepsilon = 0.2$				$\varepsilon = 0.3$				$\varepsilon = 0.4$			
q	p_α	p_β	n	k_r	k_a	$EASN$	n	k_r	k_a	$EASN$	n	k_r	k_a	$EASN$
0.0	0.00041	0.0184	5	5.0928	8.3061	7.48	5	5.2579	8.2063	7.05	5	5.4114	8.1253	6.71
	0.00284	0.0311	7	4.0280	6.3102	11.14	6	4.0204	6.7585	9.74	6	4.2690	6.5829	8.56
	0.00654	0.0426	8	3.4644	5.5163	13.73	7	3.5251	5.7490	11.42	6	3.6175	6.0727	9.42
	0.0319	0.0942	12	2.4458	3.5920	20.49	10	2.4921	3.6930	15.44	7	2.5054	4.1262	10.65
	0.0615	0.1420	14	1.9721	2.8171	23.22	10	1.9164	3.0521	16.80	7	1.9537	3.3327	10.68
	0.1190	0.2250	15	1.4336	2.0700	24.33	10	1.3572	2.2804	16.95	7	1.4145	2.3923	9.94
0.4	0.00041	0.0184	7	5.0835	8.6407	10.80	7	5.2415	8.5259	10.22	7	5.3893	8.4328	9.76
	0.00284	0.0311	10	4.1299	6.2109	15.15	9	4.1931	6.4722	13.35	8	4.2592	6.8296	11.87
	0.00654	0.0426	11	3.5533	5.4447	17.93	10	3.6569	5.5691	15.09	9	3.8111	5.7248	12.65
	0.0319	0.0942	15	2.5143	3.5250	23.91	12	2.5274	3.7060	18.24	9	2.5983	4.0324	12.91
	0.0615	0.1420	16	2.0136	2.7881	25.33	12	1.9939	2.9630	18.54	8	1.9907	3.3764	12.13
	0.1190	0.2250	15	1.4374	2.0919	24.58	11	1.4134	2.2135	17.29	7	1.4024	2.5051	10.37
0.8	0.00041	0.0184	16	5.4099	8.0215	21.75	16	5.5562	7.9410	20.84	15	5.5773	8.3121	20.09
	0.00284	0.0311	19	4.2882	6.0872	26.79	17	4.3347	6.4446	24.14	17	4.5527	6.3021	21.93
	0.00654	0.0426	20	3.7323	5.2358	29.06	18	3.8151	5.4340	25.12	16	3.9392	5.7080	21.71
	0.0319	0.0942	21	2.6017	3.4346	30.60	17	2.6211	3.6245	24.01	14	2.7654	3.7820	17.78
	0.0615	0.1420	21	2.0718	2.6816	30.17	16	2.0667	2.8238	22.40	12	2.1708	2.9771	15.05
	0.1190	0.2250	24	1.4689	1.9342	34.42	17	1.4294	2.0256	24.04	11	1.4764	2.1403	13.98

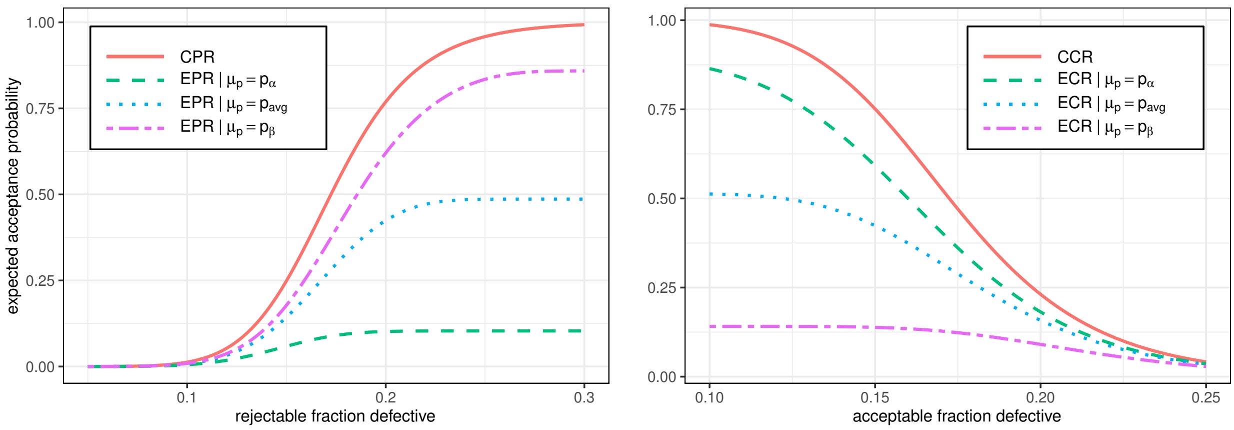

Figure 2: Conventional risks and expected risks of the standard optimal censored repetitive plan $S'_{0.5} = (30, 0.787, 1.124)$ corresponding to a normal distribution of X versus p_β when $p_\alpha = 0.119$ (left) and p_α when $p_\beta = 0.225$ (right).

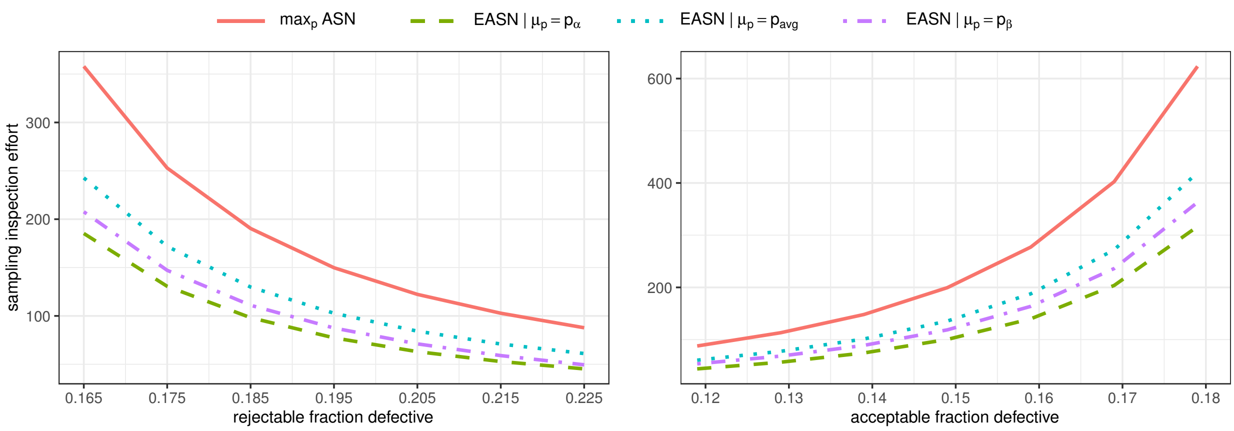

Figure 3: Maximum and expected average sample number of the respective ASN_{avg} and $EASN$-optimal censored repetitive plans corresponding to a normal distribution of X with $q = 0.8$ versus p_β when $p_\alpha = 0.119$ (left) and p_α when $p_\beta = 0.225$ (right).

a constant standard deviation given by $\sigma_p = (p_\beta - p_\alpha)/4$.

In Figure 3 is presented a comparison, when the censoring degree is $q = 0.8$ and X follows a normal distribution, between the maximum ASN of the standard ASN_{avg}-optimal censored repetitive plans, S'_q, and the expected sampling effort of the $EASN$-optimal censored repetitive designs, S^*_q obtained for different values of μ_p. These plans are determined varying one of the quality levels and keeping fixed the other one.

Under the above requirements of maximum risks and quality levels, we observe in Table 1 that the ASN_{avg}-optimal repetitive q-censored plans S'_q are $(28, 0.776, 1.132)$, $(30, 0.789, 1.122)$ and $(44, 0.787, 1.073)$ for $q = 0.0, 0.4$ and 0.8, respectively, when X follows a normal distribution. The corresponding ASN_{avg} of these plans are equal to 46.43, 47.91 and 65.49. In the case of assuming an extreme-value distribution of X, the respective optimal repetitive q-censored plans

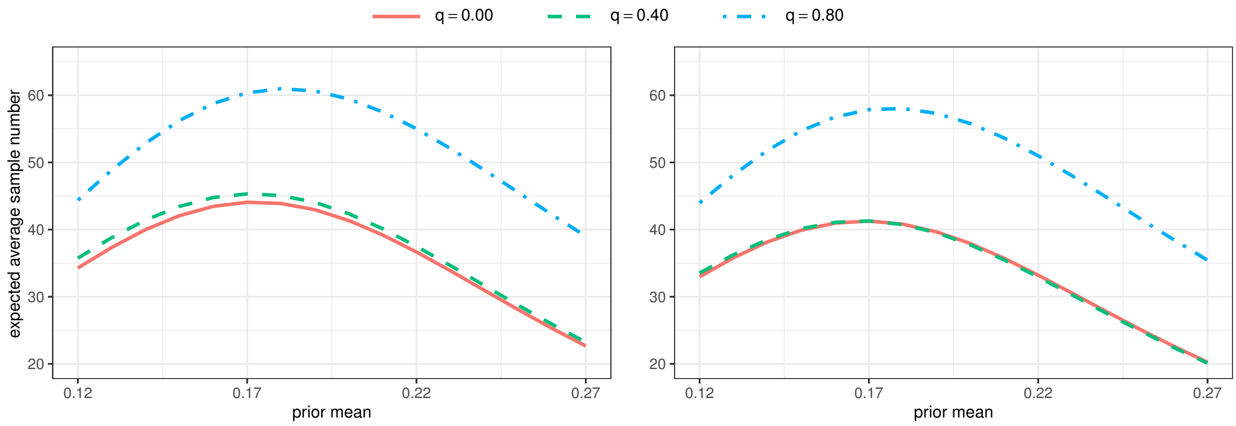

Figure 4: Expected average sample number of $EASN$-optimal q-censored repetitive plans versus μ_p when $q = 0.0, 0.5, 0.8$ when X follows a normal distribution (left) and an extreme-value distribution (right).

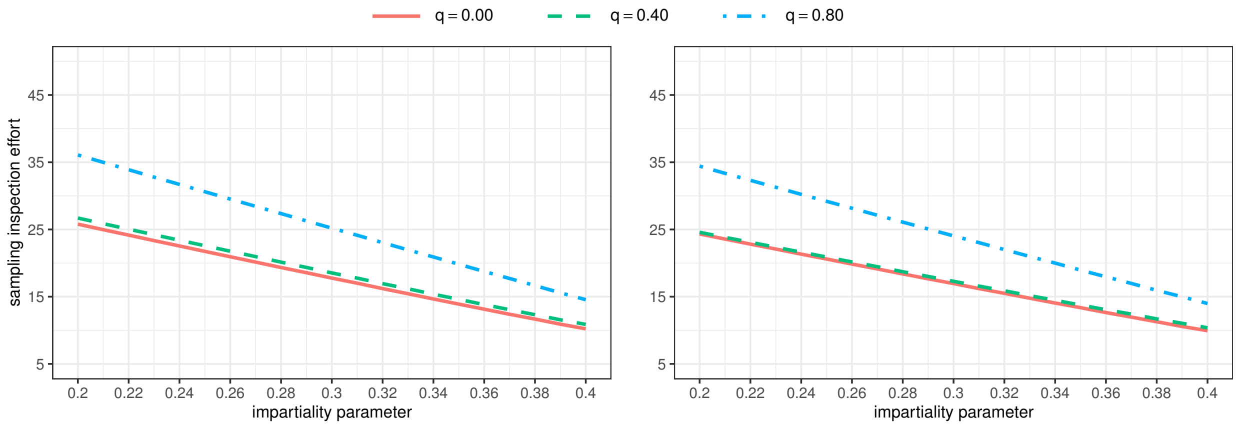

Figure 5: Expected average sample number of $EASN$-optimal q-censored repetitive plans versus ε when $q = 0.0, 0.5, 0.8$ when X follows a normal distribution (left) and an extreme-value distribution (right).

are $(27, 1.430, 1.992)$, $(27, 1.436, 2.002)$ and $(41, 1.426, 1.924)$ for $q = 0.0, 0.4$ and 0.8, with ASN_{avg} are 43.59, 43.77 and 62.43.

Figure 4 illustrates, for both normal and extreme value distributions, the sampling effort of the $EASN$-optimal censored repetitive plans for different values of μ_p. We can observe that the proposed repetitive censored plans have a expected average sample number that reduces significantly the corresponding ASN_{avg} of the standard censored repetitive plans. Note that the percentage reduction becomes to $50\% - 55\%$ in some designs with respect to the ASN_{avg} for both distributions. The reduction is more apparent when the prior mean is equal to one of the quality levels. Likewise, when an impartial GB ϵ-prior is considered by the producer and the consumer under a mutual agreement, the impartiality parameter also affects to the sampling effort of the optimal truncated plans, as it can be observed in Figure 5. In this case, a noticeable decreasing trend is observed in $EASN$ of the repetitive plans when ϵ increases for different degrees of censoring q. In this case, we appreciate that the percentages of reduction in $EASN$ that can be achieved are between 70%

and 80%, approximately.

7. Concluding remarks

Repetitive sampling plans by variables represent widely used inspection schemes in quality control for the acceptance of products and manufacturing processes. However, repetitive plans often require larger sample sizes to decide on the acceptability or rejection of production batches. This paper presents a methodology to determine the censored repetitive plan in reliability testing, based on expected sampling risks, that considers the prior information about the nonconforming proportion in lots. The proposed designs optimize the expected number of inspected product items and are particularly appropriate in case of destructive testing of expensive products as well as for moderately or highly censored lifetime data. In these circumstances, the sample size of the plans can increase dramatically in such a way that the amount of testing should be excessive in terms of economical costs or test time reasons. The inclusion of prior knowledge about the nonconforming proportion in designing the repetitive sampling plans can suppose an essential aspect to improve the inspection under these conditions. This previous information can proceed from the past experimental evidence, but can also represent subjective and objective information incorporated into the decision procedure.

The optimal censored repetitive plans are characterized by the sample size, n^*, and the rejection and acceptance constants, k_r^* and k_a^*, of the sampling plan. These plans are determined for log-location and scale lifetime distributions, such as the Weibull and lognormal models, and considering a generalized beta distribution to model the available knowledge about the nonconforming proportion, p. The use of this family of distributions results very adequate when there exists enough evidence to define the range of p in a restricted interval $[l, u]$, with $0 \leq l < u \leq 1$, especially when the product reliability standard is quite high, and it is available some precise information about lower and upper bounds on the fraction defective. Likewise, the generalized beta prior enables to reflect constraints about the mean and variance of the nonconforming proportion, which gives a better description of the random behaviour of the fraction defective from the production process. The producer and the consumer can specify their own prior distributions arbitrarily but, when a neutrality is required in the choice of the model, an impartial prior should be selected. The impartiality criterion represents a reasonable way to solve possible conflicts between the producer and the consumer because assigns the same probability to the acceptance of defective lots and the rejection of satisfactory lots.

This paper shows several efficient and quick computational procedures to determine the censored repetitive plans by solving a constrained nonlinear optimization problem. Firstly, a step-by-step procedure has been presented to determine the optimal censored repetitive designs under the traditional approach of conventional risks. Next, these designs are used in defining a simple and efficient procedure to obtain the solution of the optimization problem because they provide with lower and upper bounds of the sample size that can narrow the search of the optimal plan minimizing the sampling effort. These computational procedures have been developed in an R library distributed under a MIT license that is available for the users in Pérez-González (2021) and `https://github.com/ULL-STAT/RepetPlan`.

The design of censored inspection schemes using expected risks developed in this paper supposes a very convenient approach to improve the evaluation of real producer and consumer sampling risks. Additionally, the inclusion of prior information of p into the reliability sampling plans supposes a significant saving in the sample size of the optimal designs, whereas the traditional repetitive plans can increase substantially the number of test units and, consequently, the economical and time costs, specially when the censoring is high. This is a crucial aspect to be considered when the sample units to be tested are particularly expensive or the production is restricted or limited. In this situation, the available prior knowledge is not required to be very accurate to get a noticeable reduction of the sampling effort that is needed for lot sentencing in the repetitive inspection.

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