A global scheme for controlling the mean and standard deviation of a process based on the optimal design of gauges.

Authors

Summary

Control chart for variables are the reference tool for statistical monitoring of a quantitative variable. However, there are processes where the exact measurement is a task of high complexity or cost. In these cases, it is possible the monitoring employing a gage instrument designed to classify items that are above, below, or between a pair of reference limits. Using this approach, the process control has the economy and agility of a control scheme by attributes. In the literature, it is possible to find different alternatives to design such schemes. In this paper a global scheme is introduced that englobes many of the previous proposals, and includes a new control scheme, thanks to employing an additional parameter. Through an optimization procedure designed specifically for this new scheme, it is possible to obtain the best parameters that maximize the performance for detecting a process shift.

Keywords: Statistical Process Control, Control Chart, Gauge, Grouped Data

1. Introducción

In statistical process control (SPC), control charts allow monitoring the stability of a quality characteristic of a production process over time. This control is based on periodically sampling the process and analyzing a sample size n of randomly selected items. This sampling information is summarized employing one or several statistics, which are commonly plotted on a control chart. If the statistic exceeds the control limits of the chart it should be accepted that there are a shift in the quality that is being monitored, see Montgomery[1].

When the quality characteristic X is a quantitative variable, the control usually consists of monitoring its mean (μ) and standard deviation (σ), to check whether they remain stable on their in-control values (μ_0, σ_0). For this purpose, control charts for variables have been developed. The real implementation of variable control chart requires accurate measurements of the variable, for which is necessary to have accurate measuring instruments and trained personnel. However, there are situations in industry where accurate measurement of a variable is complex, because one or more of the following drawbacks exist:

i. The measurement is complex.
ii. The measurement is expensive.
iii. The measurement is prone to measurement errors by the operator.
iv. The measurement is destructive, or may affect the functionality of the piece.

For example, the accurate measurement of the diameter of a hole in a unit can be complex. Aparisi *et al.*[2] show an example in which the amount of adherence of a tape is evaluated by measuring the travel distance of a metal ball falling from an inclined plane, Figure 1a. In this example, obtaining the exact traveled path length is not easy, but it is simple to determine whether that distance is between a pair of reference limits, limits that work like a gage.

Figure1b shows another example, measuring the diameter of a snack. The exact thickness measurement is very difficult, as it is very easy to damage the snack during the measurement, given its fluffiness. Therefore, it is quite common to obtain a measured thickness that does not represent the real value. Using a gage is a solution, an option that is actually employed in the real monitoring of this diameter.

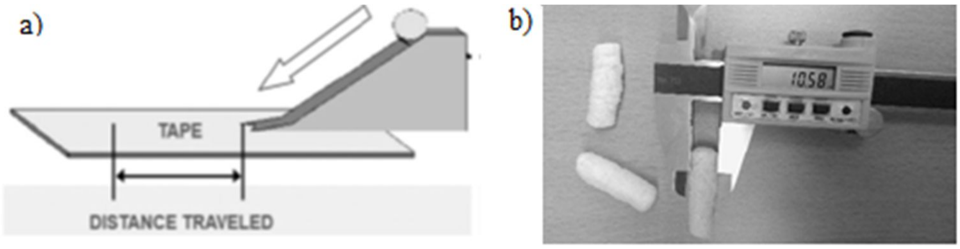

Figure 1. a) Gage instrument for monitoring the adhesive capacity of a tape. b) Difficulty on measuring the diameter of a snack.

Tippett[3], Stevens[4] and Mace[5] were the first authors to develop methods for controlling a quantitative variable with gages. Further developments are the papers of Ott and Mundel[6], Beja and Ladany[7] and Ladany[8]. In some processes, the application of these tools is the only alternative to employ, and in other cases, gages are an option that allows better operational costs and speed in the inspection of the process.

Gages many times are designed to have two dimensions, *S* and *L*, which divides the probability distribution of variable X in three intervals ($X < S$, $S < X < L$, $X > L$) as illustrated in Figure 2. Hence, in this case, the process control consist of randomly sampling *n* units, which are checked with the gage, recording the following results: number of units that are below *S* (Y_S), above *L* (Y_L), and between *S* and *L* (Y_C). These values are random variables with discrete probability distribution, having statistical information about the status of the process.

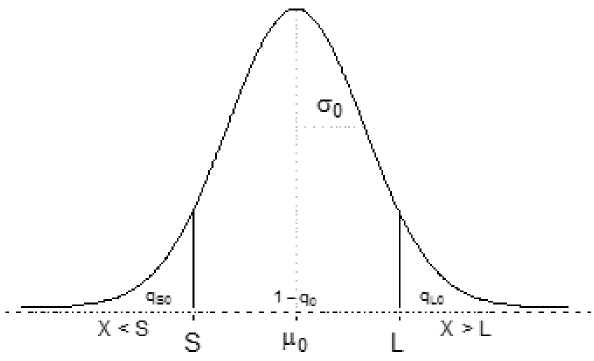

Figure 2. Probability density delimitation according with the gage's dimensions

For example, if the dimensions of the gage matches the product specifications and the count $Y_S + Y_L$ is the statistic to be plotted, the monitoring is equivalent to that performed in the attributes *np* control chart to control the percentage nonconforming. In many processes, employing product specifications may be inconvenient, since the processes nowadays reach high levels of capacity, and observing nonconforming units during inspection is normally not possible, even when some shifts occur.

Stevens[4] suggested that the optimal location for dimensions S and L it is the one that permits getting the best quality of information on counts Y_S, Y_C, Y_L in order to estimate the parameters μ_0, σ_0. He proposes to maximize the Expected Fisher's Information in a single observation to achieve this goal. Steiner *et al.*[9] developed these calculations for a characteristic with Normal distribution, and supplied a table of suggested values for dimensions S and L. These dimensions are equally spaced to the mean μ_0 ($S = \mu_0 - k\sigma_0$; $L = \mu_0 + k\sigma_0$) with an amplitude factor k, which changes depending the parameter to be controlled (mean, standard deviation, mean and standard deviation). Recommended values for k do not depend on the magnitude of the shift to be detected, or the sample size to be employed for process control.

Stevens[4] suggests using the difference of counts Y_S - Y_L as statistic to control the process, when the objective is detecting shifts in the mean. Or the sum $Y_S + Y_L$ if the objective is detecting shifts in the variability. On the other hand, Steiner *et al.*[9] assigned to each observation a pair of weights (Z_i^+, Z_i^-) that depend on the interval in which its measurement has been classified by the gage, and they propose using the maximum of averages $\left(\overline{Z_i^+}, \overline{Z_i^-}\right)$ as statistic for process control.

In the proposals of Stevens[4] and Steiner *et al.*[9] the optimal location of the gage dimensions, and the selection of the statistic for process control, are problems that faced independently. Therefore, after obtaining the optimal dimensions and selecting the statistic for process control, the problem of finding sample size n and control limit CL is of a discrete nature. Because of this discrete nature, it is normally impossible to find a control chart that meets a required probability of false alarm α. In the real application, the actual probability of false alarm may be quite different.

However, Aparisi *et al.*[2] proposed a new control scheme where the optimum dimensions of the gage, and the parameters of the control scheme, are treated as a whole. There are two proposals to monitor the process: the scheme Y_T, similar but not equal to the proposal np_x chart of Wu *et al.*[10] and the scheme $Y_S Y_L$. The scheme Y_T employs as statistic to be plotted the total number of classified units outside gage's dimensions, $Y_T = Y_S + Y_L$. The scheme $Y_S Y_L$ consists of two control charts, which show the counts Y_S and Y_L with a common control limit *CL*. In both proposals, an optimization to find the best scheme parameters and gage limits is carried out using friendly software. As the dimensions of the gauge are continuous variables, the optimization algorithm finds a solution that matches the required probability of false alarm, what normally does not occur in attribute control charts.

Aparisi *et al.*[2] include a comparison of performance versus the scheme proposed by Steiner *et al.*[9]. This comparison reveals that the scheme Y_T with optimized gauges dimensions is always the best option to detect shifts in process variability. To detect shifts in the process mean the scheme $Y_S Y_L$ was the best choice when the sample size is small (n = 3, 5) and the scheme of Steiner *et al.*[9] was the best alternative for detecting small shifts when the sample size was large (n = 15). For intermediate sample sizes (n = 7, 10) the performance of the scheme $Y_S Y_L$ and the proposal of Steiner *et al.*[9] are similar, and both outperforms the scheme Y_T. When simultaneous shifts in mean and variance are present, one of the schemes Y_T ,or $Y_S Y_L$, show the best performance.

As it has been shown, there is no a control scheme based on gauges that shows the best overall performance. Each proposal has strengths to detect certain types of shifts in the process. Hence, the final user must evaluate the performance of each of these proposals and select one that is best for a particular case.

This article introduces an evolution of the $Y_S Y_L$ scheme, named $wY_S Y_L$, where a new parameter, the weight w, is defined. The previous proposals of Stevens[4], Steiner *et al.*[9], Wu *et al.*[10] and Aparisi *et al.*[2] are obtained given to w a determined value. On the other hand, as the value of w is optimized, it is possible to find a new control scheme that can show better performance than all the previous proposals. As in the paper Aparisi *et al.*[2] the dimensions of the gage are included as decision variables in the optimization procedure, guarantying that the optimized scheme has the desired probability of false alarm.

Section 2 of this paper introduces the global scheme $wY_S Y_L$ and identifies cases where it is equivalent to the previous proposals. Section 3 develops the formulas for calculating the ARL, and the geometric interpretation of the parameters of the $wY_S Y_L$ scheme. In Section 4, the problem of the optimal design of this new scheme is introduced and solved. An evaluation of the performance is carried out in Section 5. Section 6 provides a brief analysis of sensitivity and, finally, Section 7 summarizes the article.

2. The $wY_S Y_L$ scheme.

In this Section, we introduce the new $wY_S Y_L$ control chart. It is assumed that the quality characteristic to be monitored follows a Normal distribution and that the parameters mean, μ_0, and standard deviation, σ_0, are known. The monitoring consists of taking randomly n units from the process, and applying a gage designed with dimensions S and L. During the inspection, the following values are recorded: Y_S number of units that are lower than S. Y_L number of units that are larger than L. We accept that the process is in-control state if:

$$\max(wY_S + Y_L, Y_S + wY_L) < CL \tag{1}$$

with $w \in [-n, 1]$

The $wY_S Y_L$ scheme is defined if the following parameters are known or fixed: sample size, n; control limit, CL; weight, w, and gage's dimensions, S and L. We have to take into account that each time that the equation (1) is applied, it is equivalent to control simultaneously these two statistics: $wY_S + Y_L$ and $wY_L + Y_S$. We could employ a control chart for each one. However, when using the maximum only a control chart is needed, what facilitates the monitoring without losing performance.

The wY_SY_L scheme may adapt its performance to detect different kind of shifts (only shifts in the mean, only shifts in the variability, or simultaneous shifts in means and variability) thanks to the weight *w*. For example, when $w = 1$ the equation (1) corresponds to monitoring the sum $Y_S + Y_L$, what is equivalent to the control charts np_x, Wu *et al.*[10] and Y_T, Aparisi *et al.*[2]. This approach is very effective to detect shifts in the process variability.

When $w = 0$, the wY_SY_L chart is equivalent to monitor $\max(Y_S , Y_L)$, or employing two control chats for Y_S and Y_L, with the same control limit, *CL*. In this case we obtain the control chart Y_SY_L proposed by Aparisi *et al.*[2], a chart with good performance to detect mean shifts and simultaneous shifts in the mean and in the variance.

When $w = -1$, the statistic that it is monitored is $\max(Y_S - Y_L , Y_L - Y_S)$ and two charting alternatives can be used. A chart for the difference $Y_S - Y_L$ with two control limit (-*CL*, *CL*) or a chart for the absolute difference $|Y_S - Y_L|$ with only one control limit *CL*. In this case, the wY_SY_L scheme reflects the idea shown on Stevens[4] to monitor mean shifts.

The cases where the wY_SY_L scheme is equivalent to the proposal found on Steiner *et al.*[9] are deduced on Section 2.1.

Figure 3 shows several examples of real use of the wY_SY_L control chart: a) general case. b) $w = 1$, c) $w = 0$ and d) $w = -1$.

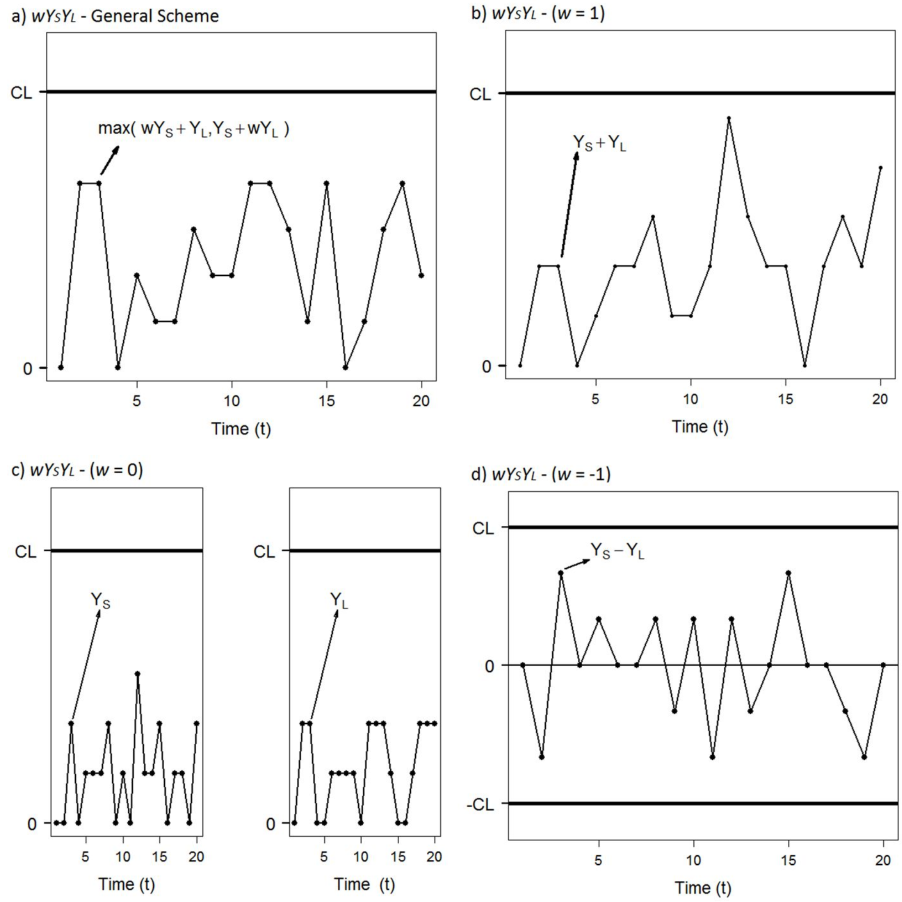

Figure 3. Control scheme wY_SY_L for a) the general case , $w \neq (1, 0, -1)$, b) $w = 1$, c) $w = 0$ and d) $w = -1$

2.1 Equivalence between wY_SY_L and Steiner *et al.*[9] scheme.

In their proposal for a control chart Steiner *et al.* [9] deduced the decision rule considering the following composite two-sided hypothesis test.

$$H_0\colon\ \theta = \theta_0 \qquad H_1\colon\ \theta = \theta_{1+} \quad or \qquad \theta = \theta_{1-}$$

Where $\theta = (\mu , \sigma)$ is the vector of parameters to monitoring, being $\theta_0 = (\mu_0 , \sigma_0)$ the value of these parameters when the process is in a in-control state. $\theta_{1+} = (\mu_{1+} , \sigma_{1+})$ and $\theta_{1-} = (\mu_{1-} , \sigma_{1-})$ are values that represent significant departures of the parameters from the null hypothesis in upward and downward directions, respectively. Using the composite likelihood ratio (CLR) as a test statistic, the following decision rule is derived to accept H_0:

$$\max\left(\sum_{i=1}^{n}\frac{Z_i^+}{n},\sum_{i=1}^{n}\frac{Z_i^-}{n}\right) < CL' \tag{2}$$

Here, Z_i^+ and Z_i^- are weights assigned to each observation classified in the j^{th} interval of the gauge. For a gage designed for classifying a measurement in one of with three intervals, the weights can be calculated as shown on Table I.

Table I. Z_i^+ and Z_i^- weights for a three intervals gauge. [9]

	Interval		
Weight	$X_i < S$	$S < X_i < L$	$X_i > L$
Z_i^+	$\ln\left(\frac{q_{S+}}{q_{S0}}\right)$	$\ln\left(\frac{1-q_{S+}-q_{L+}}{1-q_{S0}-q_{L0}}\right)$	$\ln\left(\frac{q_{L+}}{q_{L0}}\right)$
Z_i^-	$\ln\left(\frac{q_{S-}}{q_{S0}}\right)$	$\ln\left(\frac{1-q_{S-}-q_{L-}}{1-q_{S0}-q_{L0}}\right)$	$\ln\left(\frac{q_{L-}}{q_{L0}}\right)$

Here, (q_{j0} , q_{j+} , q_{j-}) represent the probability that an observation will be to classified in the j^{th} interval when the parameter vector values are (θ_0 , θ_{1+} , θ_{1-}), respectively.

The decision rule defined in (2) is satisfied when simultaneously:

$$\left(\frac{Z_1^+}{n}\right)Y_s + \left(\frac{Z_2^+}{n}\right)Y_C + \left(\frac{Z_3^+}{n}\right)Y_L < CL' \tag{3}$$

$$\left(\frac{Z_1^-}{n}\right)Y_s + \left(\frac{Z_2^-}{n}\right)Y_C + \left(\frac{Z_3^-}{n}\right)Y_L < CL' \tag{4}$$

Since $Y_C = n - Y_S - Y_L$, (3) and (4) can be simplified:

$$(Z_1^+ - Z_2^+)Y_s + (Z_3^+ - Z_2^+)Y_L < n(CL' - Z_2^+) \tag{5}$$

$$(Z_1^- - Z_2^-)Y_s + (Z_3^- - Z_2^-)Y_L < n(CL' - Z_2^-) \tag{6}$$

If the probability distribution of X is symmetrical and a gage with equidistant dimensions about the mean is used, then $q_{S0} = q_{L0} = q_0/2$. If Additionally (θ_{1+}, θ_{1-}) are equidistant from the null hypothesis ($\theta_{1+} - \theta_0 = \theta_0 - \theta_{1-}$), ($q_{S+} = q_{L-}$, $q_{L+} = q_{L-}$) and ($Z_1^+ = Z_3^-$, $Z_2^+ = Z_2^-$, $Z_3^+ = Z_1^-$).

For this case, $w_1 = (Z_1^+ - Z_2^+) = (Z_3^- - Z_2^+)$ and $w_2 = (Z_3^+ - Z_2^+) = (Z_1^- - Z_2^+)$ and Equations (5) and (6) can be rewritten as:

$$w_1 Y_s + w_2 Y_L < n(CL' - Z_2^+) \tag{7}$$

$$w_2 Y_s + w_1 Y_L < n(CL' - Z_2^-) \tag{8}$$

Dividing Equations (7) and (8) by w_2 and set $w = \frac{w_1}{w_2}$, $CL = \frac{n(CL' - Z_2^+)}{w_2}$ is obtain:

$$wY_s + Y_L < CL \tag{9}$$

$$Y_s + wY_L < CL \tag{10}$$

The equations (9) and (10) coincides with the decision rule proposal for the scheme wY_SY_L. From the above it follows that the proposal Steiner *et al.*[9] is included in the overall scheme wY_SY_L, as a specific case when the design parameters (w, CL) take the values:

$$w = \frac{w_1}{w_2} = \frac{\ln\left(\frac{q_{S+}}{q_{S0}}\right) - \ln\left(\frac{1 - q_{S+} - q_{L+}}{1 - q_{S0} - q_{L0}}\right)}{\ln\left(\frac{q_{L+}}{q_{L0}}\right) - \ln\left(\frac{1 - q_{S+} - q_{L+}}{1 - q_{S0} - q_{L0}}\right)} \quad , \quad CL = \frac{n\left(CL' - \ln\left(\frac{1 - q_{S+} - q_{L+}}{1 - q_{S0} - q_{L0}}\right)\right)}{\ln\left(\frac{q_{L+}}{q_{L0}}\right) - \ln\left(\frac{1 - q_{S+} - q_{L+}}{1 - q_{S0} - q_{L0}}\right)}$$

3. ARL calculation for the *wY*S*Y*L scheme.

The most common measurement for evaluating the performance of a control chart is the ARL (Average Run Length), which is the expected number of plotted points until an out of control signal is obtained. As the points in the wY_SY_L scheme are independent, the run lengths follows a geometric distribution and its expected value (ARL) can be calculated as the reciprocal of the probability of signal (point outside control limits).

We are assuming that the quality characteristic X follows a Normal distribution with parameters μ_0 and σ_0 when the process is in-control, and that the gage dimensions are equidistant to the process mean, $S = \mu_0 - k\sigma_0$ and $L = \mu_0 + k\sigma_0$. However, this scheme is adaptable for other types of probability distributions, and for gages with asymmetric dimensions.

Under these assumptions, when the process is in control, the probability of a unit being outside the limits of the gauge is $q_0 = 2\Phi(-k)$, with $q_{S0} = q_{L0} = q_0/2$, and the counts Y_S, and Y_L follow a Multinomial (n, $q_0/2$, $q_0/2$). Where $\Phi(-k)$ is the standard Normal cumulative distribution.

When the process mean shifts to $\mu_1 = \mu_0 + \delta\sigma_0$, and the standard deviation to $\sigma_1 = r\sigma_0$, finding a unit outside the dimensions of the gauge has a probability:

$$q_{S1} = \Phi\left(\frac{-k-\delta}{r}\right) \quad , \quad q_{L1} = \Phi\left(\frac{-k+\delta}{r}\right) \tag{11}$$

where the counts Y_S and Y_L follow a Multinomial (n, q_{S1} q_{L1}).

Let us define Ω as the set of values that can take Y_S and Y_L when a sample size n is used, and Ω^{IC} as the subset of Ω that meet the decision rule defined in equation (1) for diagnosis. Rewriting equation (1) as $\left(Y_L < CL - wY_S,\ Y_L < \frac{CL}{w} - \frac{1}{w}Y_S\right)$, then it is clear that Ω^{IC} is conformed by the set of values Y_S and Y_L found between a pair of straight lines with intercepts $\left(CL, \frac{CL}{w}\right)$ and scopes $\left(-w, \frac{-1}{w}\right)$ respectively, as shown in Figure 4, for specific cases with a) $w = -1$, b) $w = 0$, c) $w = 1$ and d) $w > 1$.

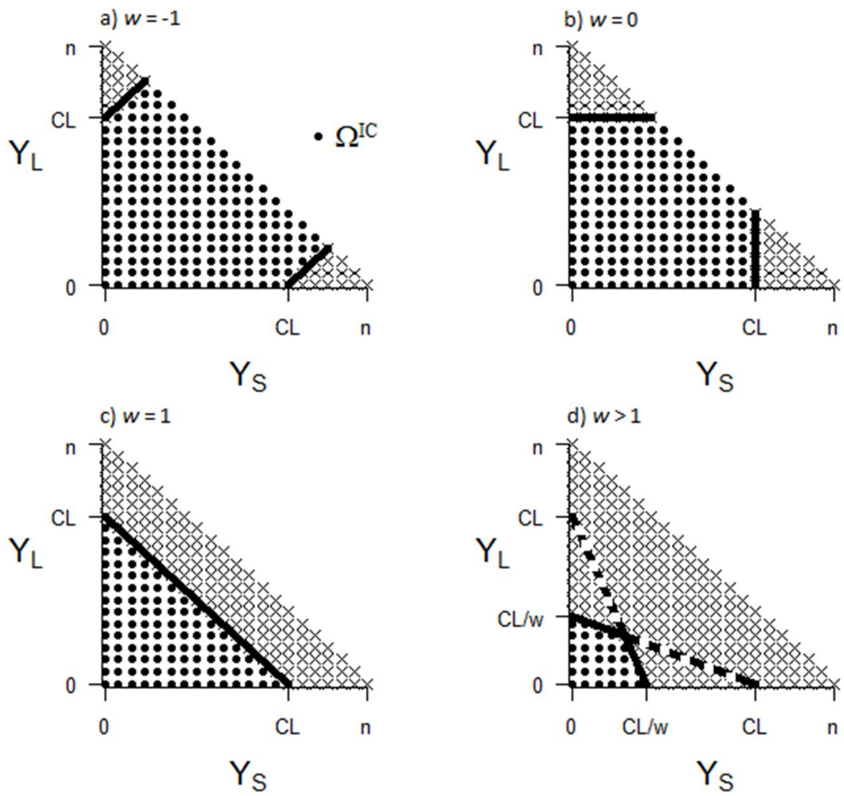

Figure 4. Graphical representation of the Ω and Ω^{IC} sets

Note that when $w > 1$ (Figure 4d), solutions (CL, w) and (CL/w, $1/w$) generate the same subset Ω^{IC}, therefore they are dual configurations for the problem. Additionally, it is easy to deduce that the maximum slope of the straight lines is obtained when $w = -n$. Therefore, the parameters of the control chart can be dimensioned in the range $-n < w \leq 1$ and $0 < CL \leq n$.

Once defined the dimensions S and L and the parameters of the control chart (n, CL, w), the probability of a false alarm is:

$$\propto = 1 - \sum_{Y_S, Y_L} \sum_{\in \Omega^{IC}} \frac{n!}{Y_S!\, Y_L!\, (n - Y_S - Y_L)!} \left(\frac{q_0}{2}\right)^{Y_S + Y_L} (1 - q_0)^{(n - Y_S - Y_L)} \tag{12}$$

When there is a process shift ($\delta \neq 0$ and/or $r > 1$) the probability of having a signal in the control chart is:

$$1-\beta = 1 - \sum_{Y_S, Y_L \in \Omega^{IC}} \sum \frac{n!}{Y_S!\,Y_L!\,(n-Y_S-Y_L)!} (q_{S1})^{Y_S} (q_{L1})^{Y_L} (1-q_{S1}-q_{L1})^{(n-Y_S-Y_L)} \tag{13}$$

Recalling that the ARL is the reciprocal of the probability signal, when the process is in-control state $ARL(\delta = 0, r = 1) = \frac{1}{\alpha}$ and when the process is out of control, then $ARL(\delta \neq 0 \ o \ r > 1) = \frac{1}{1-\beta}$.

4. Scheme Optimization and example of application.

The optimization of the wY_SY_L scheme has the objective of finding the best chart's parameters (w, CL) and gauge's dimensions (S, L) to detect with maximum performance, i.e. to minimize the out-of-control ARL, a process shift of magnitude ($\delta = \delta^*$, $r = r^*$), given a sample size, n. In addition, the optimum scheme must have the required in-control ARL (ARL_0). The values δ^* and r^* define a critical process shift, and they must be specified for the final user, taking into account, for example, the quality loss that this shift produces.

Assuming that symmetric gages are employed, then $k = -\Phi^{-1}\left(\frac{q_0}{2}\right)$ and the problem of finding the best gage dimensions S and L is reduced to find the percentage of units classified by the gauge when the process is in control (q_0).

Formally, the optimization problem is posed as:

Given:

- In-control process mean: μ_0
- In-control process variability: σ_0
- Sample size: n
- Required in-control ARL: ARL_0

- Process shift: ($\delta = \delta^*$, $r = r^*$)

Find:

- Control limit: CL
- Weight: w
- Percentage of units classified by the gauge when the process is in control: q_0

Minimize: $ARL_1 = ARL(\delta = \delta^*,\ r = r^*)$

Subject to: $ARL(\delta = 0,\ r = 1) = ARL_0$

An optimization algorithm for the wY_SY_L has been developed. The algorithm makes an exhaustive search for the couple (w, CL), and for this couple a Newton-Raphson search is done to find the value of q_0 that satisfies the restriction $ARL(\delta = 0, r = 1) = ARL_0$. A summary of the optimization algorithm follows:

1. A ordered sequence of values for the search of the weight w is defined: $w = \{w_{(1)}, w_{(2)}, \dots, w_{(i)}, \dots, w_{(I)}\}$ with $-n < w_{(i)} \leq 1$.
2. A value $w_{(i)}$ is selected from step 1, applying the statistic $\max(w_{(i)}Y_S + Y_L, Y_S + w_{(i)}Y_L)$ over the element of the set Ω. The result is a ordered set of values for control limit, CL. $CL_{(i)} = \{CL_{(i1)} < CL_{(i2)} < \dots < CL_{(ij)} < \dots < CL_{(iJ)}\}$
   - 2.1 A value $CL_{(ij)}$ is selected from step 2. With $w_{(i)}$ and $CL_{(ij)}$ Newton-Raphson method is employed to find the root q_{ij} that satisfies $ARL(\delta = 0, r = 1) = ARL_0$.
   - 2.2 The values $w_{(i)}$, $CL_{(ij)}$ and q_{ij} are used to calculate $ARL_1 = ARL(\delta = \delta^*, r = r^*)$
   - 2.3 Steps 2.1 and 2.2 are repeated until trying all values of the set $CL_{(i)}$ obtained in step 2 or until find the first $CL_{(ij)}$ without solution for $0 < q_{ij} < 1$
3. Step 2 is repeated with new values for $w_{(i)}$ until ending the exploration of the set w defined in step 1.
4. The minimum value for ARL_1 and its associated parameters ($w_{(i)}$, $CL_{(ij)}$, q_{ij}) are the optimum solution.

If in Step 1 the value of w is unique and fixed as w = 1, 0, or -1, then the optimization is carried out for the control charts Y_T, $Y_S Y_L$ or $Y_S - Y_L$ respectively.

This optimization algorithm has been implemented in a friendly Windows® software, in order to facilitate the use of this new control scheme by the final user. It is available from the authors.

4.1 Example of application.

In a filling process of a carbonated liquid, the distance between the bottle stopper and the level of filling is monitored. When bottles are filled too much, there is a potential risk of breaking due to excessive pressure. On the other hand, too low filling levels produce a degradation of the product quality. When the process is in-control, the distance follows a Normal distribution with mean μ_0 = 4 cm and standard deviation σ_0 = 0.3 cm. In this process the objective is detecting a mean shift to μ_1 = 4.15 (δ^* = 0.5) and standard deviation increase of 20% (r^* = 1.2).

The current process monitoring consists of employing a $\bar{X} - S$ scheme with in-control ARL = 370 and sample size n = 6. It is desired to make easier the sampling procedure employing a gauge inspection (Figure 5) and the $wY_S Y_L$ scheme.

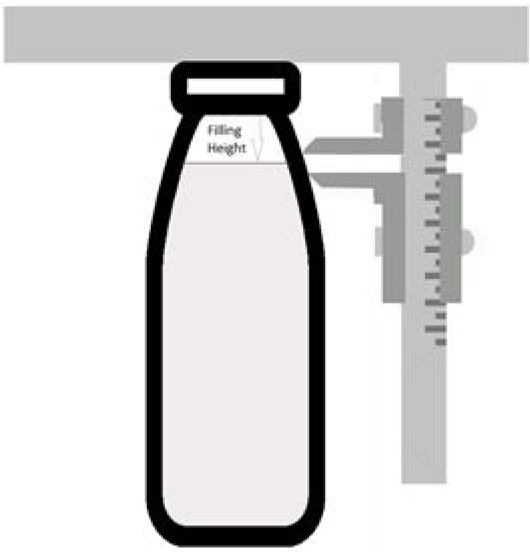

Figure 5. Gauge design for monitoring filling height.

Figure 6 shows the software for optimization of the wY_SY_L scheme. The software can be employed for calculating the ARL for a given set of chart's parameters and a given shift size, or for the optimization of the parameters, selecting the desired option in the "Operation" group. As Figure 6 shows, the final user has inputted the optimization problem previously stated for the filling process. The best parameters are shown in the results memo. In addition, the software also shows the performance of the control charts $\bar{X},\ \bar{X}-S,\ S$ that are equivalent, i.e., with the same in-control ARL and sample size.

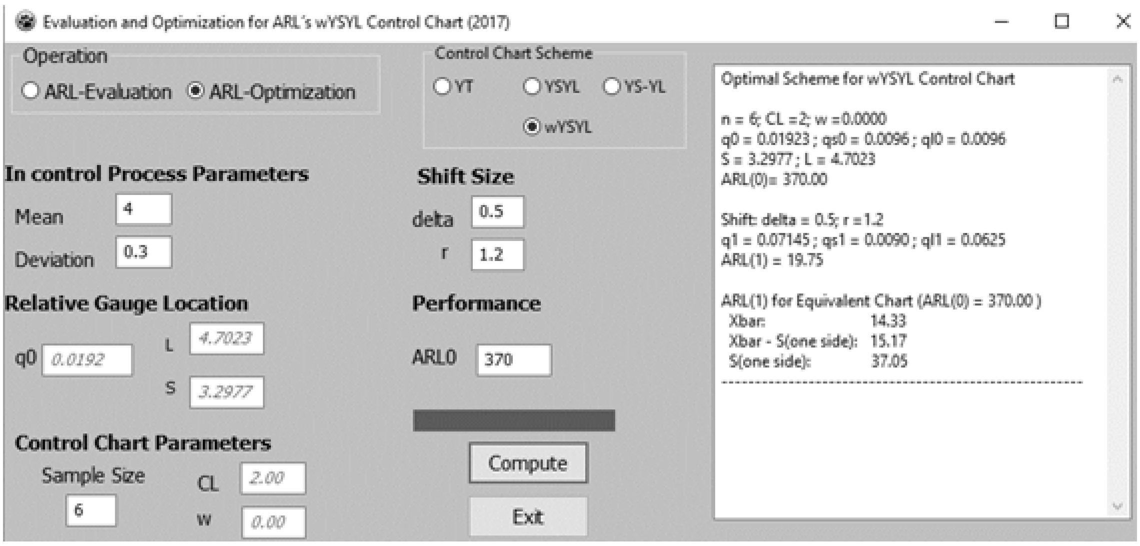

Figure 6. Optimization software solving the example of application.

The optimized parameters are: $w = 0$ and $CL = 2$ with gauge dimensions low $S = 3.2977$ cm and high $L = 4.7023$ cm. The out-of-control ARL, ARL_1, for these parameters is 19.75, slightly larger than the $\bar{X}-S$ scheme that shows an out-of-control ARL equals to = 15.17. However, the monitoring of the bottles is now much easier and economical than measuring the exact distance. Moreover, if this value of ARL_1 was not acceptable, taking into account the easiness of the sampling procedure when gages are employed, it is possible to increase a bit sample size to match the performance of the $\bar{X}-S$ scheme. Following this idea, Table II shows the optimal parameters when sample size is $n = 7$ and $n = 8$. When $n = 8$, the ARL_1 of the wY_SY_L scheme is better than $\bar{X}-S$ with $n = 6$, and possibly having still much lower operating costs.

Table II. Optimal parameters of wY_SY_L scheme when sample size is increased.

n	(w, CL, q_0)	(S, L)	ARL_1
6	(0, 2, 0.01925)	(3.298, 4.702)	19.75
7	(0, 3, 0.07003)	(3.457, 4.544)	16.82
8	(0, 3, 0.06003)	(3.436, 4.564)	14.71

5.- Performance analysis.

In this Section a performance analysis of the wY_SY_L scheme is carried out for three different scenarios: i. Shifts only in the process mean ($\delta^* > 0$, $r^* = 1$). ii. Shifts only in the variability ($\delta^* = 0$, $r^* > 1$). iii. Simultaneous shifts in mean and variability ($\delta^* > 0$, $r^* > 1$). For simplicity, the results for $ARL_0 = 370$ are the only shown. For other in-control ARLs the conclusions are equivalent.

Table II shows the out-of-control ARLs for wY_SY_L schemes optimized for different mean shifts (δ^*) and for different sample sizes (n). In the comparison the out-of-control ARLs (ARL_1) of the optimized Y_SY_L scheme Aparisi *et al.*[2] are shown as well. Last column displays $\%ARL_1$, the percentage of reduction in the out-of-control ARL when the wY_SY_L is employed. The comparison includes the out-of-control ARLs of the variable schemes $\bar{X}$ and $\bar{X} - S$. To perform a fair comparison, all the schemes have the same in-control ARL, $ARL_0 = 370$. The probability of false alarm in the $\bar{X} - S$ scheme is equal for each control chart.

The analysis of Table II returns that in all cases the optimum weight is $w = -1$. Therefore, to detect mean shifts the best chart is the difference $Y_S - Y_L$. Stevens[4] already suggested this difference of statistics for controlling the process mean. However, its optimized performance never was analyzed, and our optimization includes the gage's dimensions as parameters to be optimized as well.

Data on Table II show that for mean shifts the wY_SY_L scheme always outperforms the Y_SY_L scheme. For example, for $n = 10$ the optimized Y_SY_L scheme for a shift $\delta^* = 0.25$

has an out-of-control ARL_1 equals to 104.20, whilst wY_SY_L has ARL_1 = 94.85, i. e., a reduction of 9%, without any additional cost. For the same shift size (δ^* = 0.25), with simple size n = 50, the percentage of reduction in the value of ARL_1 is 26.2%. As it is possible to check, the percentage of reduction is larger when sample size increases, and for small shifts in the mean.

For a shift δ^* = 0.25, and with sample size n = 5 and 10, the ARL's of wY_SY_L and Y_SY_L schemes are lower than the ARL's of the $\bar{X} - S$ scheme. This result does not hold for the Y_SY_L scheme for $n \geq 15$, although it still true for the wY_SY_L scheme for all evaluated sample sizes. Therefore, the wY_SY_L scheme is a good alternative with lower operational cost and better performance for detecting small mean process shifts than the $\bar{X} - S$ scheme.

Table III. Optimum ARLs for the wY_SY_L scheme. Shifts in the mean (δ^*). ARL_0 =370

		ARL_1				% ARL_1
	δ^*	wY_SY_L†	Y_SY_L	$\bar{X}$	$\bar{X}-S$	(Y_SY_L)
n = 5	0.25	159.6	164.2	133.0	181.6	2.8
	0.50	46.7	49.0	33.4	50.5	4.6
	0.75	16.4	17.3	10.8	15.5	5.1
	1.00	7.1	7.4	4.5	6.0	4.9
	1.25	3.7	3.8	2.4	2.9	4.2
	1.50	2.3	2.3	1.6	1.8	3.3
	1.75	1.6	1.7	1.2	1.3	2.4
	2.00	1.3	1.3	1.1	1.1	1.4
n = 10	0.25	94.9	104.2	73.2	108.0	9.0
	0.50	19.3	22.3	12.8	18.7	13.5
	0.75	5.8	6.8	3.8	4.9	14.4
	1.00	2.6	2.9	1.8	2.1	12.7
	1.25	1.5	1.7	1.2	1.3	9.0
	1.50	1.2	1.2	1.0	1.1	5.0
n = 15	0.25	62.8	73.2	47.3	71.4	14.2
	0.50	10.4	12.9	7.0	9.7	19.5
	0.75	3.1	3.8	2.2	2.6	18.4
	1.00	1.6	1.8	1.2	1.3	13.0
	1.25	1.1	1.2	1.0	1.1	6.0
n = 30	0.25	26.8	34.4	19.4	29.0	22.0
	0.50	3.5	4.7	2.5	3.1	24.2
	0.75	1.4	1.6	1.2	1.2	14.9
	1.00	1.0	1.1	1.0	1.0	4.0
n = 50	0.25	12.9	17.5	9.2	13.1	26.2
	0.50	1.8	2.3	1.4	1.6	21.8
	0.75	1.0	1.1	1.0	1.0	6.4

† Always optimum was w = -1

When only shifts in the variability are considered, in all cases analyzed the optimum wY_SY_L scheme always has an optimum weight $w = 1$. Hence, the Y_T control chart proposed by Aparisi *et al.*[2] is the best option for monitoring only variability shifts. For that reason we do not include in this paper a performance analysis of the wY_SY_L with w = 1. The reader can consult Aparisi *et al.*[2].

Table IV shows the ARL´s of the wY_SY_L scheme optimized for simultaneous shifts in the mean and in the variability (δ^*, r^*). Table IV also includes the optimized weights, *w*. The majority of times the values of *w* are *w* = (1, 0, -1). For the rest of cases, which have optimized weight $w \neq (1, 0, -1)$, the $\%ARL_1$ is evaluated taking as reference scheme the wY_SY_L optimized with w fixed to closest integer value. Additionally, the ARL´s of $\bar{X} - S$ control chart is included.

When the shift considered is one with large value for the variability shift, and with small mean shift, the optimum weight tends to be positive with limit value equals to $w = 1$. Therefore, the optimum wY_SY_L scheme tends to be in these cases the Y_T control chart, the best chart for detecting variability shifts.

In the cases that the mean shift δ^* is large, and the variability shift r^* is small, the optimum weight tends to be negative with limit value $w = -1$. As the main shift is produced for the mean, the wY_SY_L scheme approaches to the $Y_S - Y_L$ chart, with has the best performance for detecting mean shifts. In the other cases, where the magnitudes of δ^* and r^* are similar, the optimized wY_SY_L schemes shows low weight values, adapting a configuration close to the Y_SY_L scheme.

For a shift $\delta^* = 0.25$, and with sample size $n = 5$ and 10, the ARL's of wY_SY_L and Y_SY_L schemes are lower than the ARL's of the $\bar{X} - S$ scheme. This result does not hold for the Y_SY_L scheme for $n \geq 15$, although it still true for the wY_SY_L scheme for all evaluated sample sizes. Therefore, the wY_SY_L scheme is a good alternative with lower operational cost and better performance for detecting small mean process shifts than the $\bar{X} - S$ scheme.

6.- Sensitivity analysis of the optimization

The user has to choose the shifts δ^* and r^* for the optimization of the wY_SY_L scheme. In many charts the optimized parameters depend on the magnitude of the shift selected for optimization, as occurs, for example, in the EWMA or MEWMA control charts Aparisi and García-Díaz[11]. In this Section the sensitivity of the wY_SY_L scheme's parameters, (w, CL, q_0), for different sizes of shift is studied.

Table IV. Optimal weight w and ARLs for the wY_SY_L scheme. Simultaneous shifts in the mean and in the variability (δ^*, r^*). ARL_0 = 370

					***ARL*1**						**%*ARL*1**		
		w			wY_SY_L			$\bar{X}-S$			(wY_SY_L†)		
		r*			***r****			***r****			***r****		
	δ^*	**1.2**	**1.4**	**1.6**	**1.2**	**1.4**	**1.6**	**1.2**	**1.4**	**1.6**	**1.2**	**1.4**	**1.6**
n = 5	**0.2**	1	1	1	45.7	14.6	7.0	37.5	11.6	5.4			
	0.6	0	0	1	17.1	8.9	5.3	13.5	6.9	4.1			
	1.0	-1	0	0	5.9	4.4	3.4	4.5	3.5	2.7			
n = 10	**0.2**	1	1	1	30.5	8.4	3.8	23.2	6.2	2.9			
	0.6	0	0	1	8.2	4.7	2.9	6.2	3.6	2.2			
	1.0	-1	0	0	2.7	2.2	1.9	2.0	1.8	1.5			
n = 15	**0.2**	0.1	1	1	22.2	5.7	2.6	16.4	4.1	2.0	0.9		
	0.6	0	0	1	5.1	3.1	2.1	3.8	2.4	1.6			
	1.0	-1	0	0	1.7	1.6	1.4	1.4	1.3	1.2			
n = 30	**0.2**	0.4	1	1	11.0	2.8	1.5	8.0	2.0	1.2	5.3		
	0.4	0	0.4	1	4.7	2.2	1.4	3.7	1.7	1.2		5.4	
	0.6	0	0	1	2.2	1.6	1.2	1.8	1.4	1.1			
n = 50	**0.2**	0.3	0.6	1	6.0	1.7	1.1	4.4	1.3	1.0	9.9	1.2	
	0.4	0	0.1	1	2.5	1.4	1.1	2.1	1.2	1.0		5.4	
	0.6	-0.4	0	1	1.4	1.2	1.1	1.2	1.1	1.0	1.3		

† Optimized for the closest integer w value.

In Section 5 it was already mentioned that for mean shifts the optimum weight was always w = -1, and that for variability shifts the optimum is w = 1. Therefore, in these cases, the parameter weight w is very robust for a given type of shift. If the parameters CL and q_0 are considered, they show variability that depends on the type of shift to detect (only mean, only variability), sample size, and in-control ARL. However, the magnitude of the shift does not apparently affect to the values of these parameters.

Table V shows the optimum parameters CL and q_0 obtained for shifts only in the mean and only in the variability, with ARL_0 = 370. For other in-control ARLs and sample sizes, the reader may employ the software for optimization.

Table V. Optimum parameters for the wY_SY_L scheme. Only shifts in the mean (δ^*), only shifts in the variability (r^*). ARL_0 = 370

	δ^* **0.25 – 2.0**	r^* **1.2 - 2.0**
n	All w = -1 (CL , q_0)	All w = 1 (CL , q_0)
5	(4 , 0.2754)	(2 , 0.01672)
10	(6 , 0.3470)	(2 , 0.00792)
15	(8 , 0.4325)	(3 , 0.01919)
20	(10 , 0.5200)	(3 , 0.02219)
25	(11 , 0.5039)	(5 , 0.03974)
30	(12 , 0.5012)	(5 , 0.03274)
50	(16 , 0.5420)	(7 , 0.03796)

When the wY_SY_L scheme is optimized for simultaneous shifts in mean and variability, the optimum weight w depends on the equilibrium of the magnitudes of δ^* and r^*. With sample sizes n = 5, 10, and 15 the optimum w is a value that is close to one of the three values, w = (-1 ,0 ,1). When sample size is larger, n = 30 and 50 there are cases with alternatives values for w. Table VI summarizes the optimum combination of parameters w, CL, q_0 for simultaneous shifts in mean and variability.

As commented before, when the chart is optimized for large r^* shits and small δ^* shifts the optimum weight is w = 1. It is important to note that in these optimizations, all the optimum parameters have the same value if the scheme was optimized only for variance shifts (see Table V). Therefore, the same gage can be employed for both type of cases, and the same control chart can be employed in both cases as well.

In other cases, employing the same gage and chart is not possible, as the optimum parameters are not equal. In order to evaluate the impact of shift sizes in the performance of the wY_SY_L scheme, Table VII shows a comparison among charts that have been optimized for several couples of (δ^*, r^*) = (0.4, 1.2), (0.4, 1.6), (0.8, 1.2), (0.8, 1.6) with sample sizes n = 5, 10, 15.

Points (0.4, 1.2) and (0.8, 1.6) represent equilibrated shifts in mean and variance. For these two couples, the differences in the performance of the optimized chart are not important. Although the shift sizes are different, both are in the same direction. However, when schemes optimized for cases (0.4, 1.6) and (0.8, 1.2) are compared, the result is different. The combination (0.4, 1.6) is a case where there is a small shift in the mean and a large shit in the variability. Whilst the (0.8, 1.2) case shows a medium size shift in both parameters. As Table VII shows, the performance of the two optimized scheme is quite different.

As a conclusion, the new parameter w of the wY_SY_L scheme lets to have a very flexible chart to monitor the mean and/or variability of a process. However, this flexibility produces a chart with different performance if it is optimized for different scenarios. Nevertheless, this is a common issue of control charts, as occurs, for example, with the EWMA control chart, as mentioned before.

Table VI. Optimum parameters for the wY_SY_L scheme. Simultaneous shifts in the mean and in the variability (δ^*, r^*). ARL_0 = 370

		w					*CL*					q_0				
		r^*					r^*					r^*				
	δ^*	**1.2**	**1.4**	**1.6**	**1.8**	**2.0**	**1.2**	**1.4**	**1.6**	**1.8**	**2.0**	**1.2**	**1.4**	**1.6**	**1.8**	**2.0**
n = 5	**0.20**	1	1	1	1	1	2	2	2	2	2	0.0167	0.0167	0.0167	0.0167	0.0167
	0.40	0	1	1	1	1	2	2	2	2	2	0.0235	0.0167	0.0167	0.0167	0.0167
	0.60	0	0	1	1	1	2	2	2	2	2	0.0235	0.0235	0.0167	0.0167	0.0167
	0.80	-1	0	0	1	1	3	2	2	2	2	0.1097	0.0235	0.0235	0.0167	0.0167
	1.00	-1	0	0	1	1	3	2	2	2	2	0.1097	0.0235	0.0235	0.0167	0.0167
n = 10	**0.20**	1	1	1	1	1	3	2	2	2	2	0.0298	0.0079	0.0079	0.0079	0.0079
	0.40	0	1	1	1	1	3	2	2	2	2	0.0467	0.0079	0.0079	0.0079	0.0079
	0.60	0	0	1	1	1	4	3	2	2	2	0.1076	0.0467	0.0079	0.0079	0.0079
	0.80	0	0	0	1	1	4	3	2	2	2	0.1076	0.0467	0.0111	0.0079	0.0079
	1.00	-1	0	0	0	1	4	3	3	2	2	0.1185	0.0467	0.0467	0.0111	0.0079
n = 15	**0.20**	0.1	1	1	1	1	3.2	3	3	3	3	0.0611	0.0192	0.0192	0.0192	0.0192
	0.40	0	0.1	1	1	1	4	2.2	3	3	3	0.0680	0.0290	0.0192	0.0192	0.0192
	0.60	0	0	1	1	1	5	3	3	3	3	0.1190	0.0301	0.0192	0.0192	0.0192
	0.80	-0.9	0	0	1	1	5.1	3	3	3	3	0.1899	0.0301	0.0301	0.0192	0.0192
	1.00	-1	0	0	1	1	6	3	3	3	3	0.2189	0.0301	0.0301	0.0192	0.0192
n = 30	**0.20**	0.4	1	1	1		5.2	5	5	5		0.0587	0.0327	0.0327	0.0327	
	0.40	0	0.4	1	1		7	5.2	5	5		0.1157	0.0587	0.0327	0.0327	
	0.60	0	0	1	1		9	5	5	5		0.1889	0.0587	0.0327	0.0327	
n = 50	**0.20**	0.3	0.6	1			8.5	7.4	8			0.0898	0.0586	0.0487		
	0.40	0	0.1	1			11	5.2	4			0.1554	0.0415	0.0116		
	0.60	-0.4	0	1			12	9	4			0.2334	0.1086	0.0116		

Table VII. ARLs profiles of wY_SY_L scheme optimized for simultaneous shift in the mean and in the variability (δ^*, r^*). ARL_0 = 370

		δ, r			
	δ^*, r^*	**0.4, 1.2**	**0.4, 1.6**	**0.8, 1.2**	**0.8, 1.6**
n = 5	**0.4, 1.2**	**29.72**	7.38	10.15	4.37
	0.4, 1.6	33.05	**6.28**	13.18	4.39
	0.8, 1.2	34.88	14.58	**9.91**	6.66
	0.8, 1.6	29.72	7.38	10.15	**4.37**
n = 10	**0.4, 1.2**	**16.79**	4.33	4.79	2.41
	0.4, 1.6	21.28	**3.42**	7.83	2.42
	0.8, 1.2	17.38	6.36	**4.43**	2.86
	0.8, 1.6	18.66	3.81	6.05	**2.37**
n = 15	**0.4, 1.2**	**10.95**	3.06	2.95	1.72
	0.4, 1.6	14.97	**2.38**	5.02	1.75
	0.8, 1.2	12.08	6.05	**2.70**	2.30
	0.8, 1.6	11.88	2.53	3.50	**1.66**

7.- Conclusions.

This paper introduces a global proposal for the monitoring of the mean and variability of a process in cases where the measurement is complex, and the use of a gage is adequate. This new scheme is named wY_SY_L, whose optimum design takes into account the dimensions of the gage and the parameters of the control chart.

Though varying the weight w, the performance of the wY_SY_L scheme is adapted to monitoring shifts in the mean, in the variability, or both at the same time. The idea of including the gage dimensions in the optimization procedure permits to obtain a very good performance, including the fact of achieving the desired value of in-control ARL, what it is not a common result in attributes control charts. In order to facilitate the real use of this new scheme, a friendly-use software or Windows® has been developed.

The wY_SY_L scheme is a global proposal that under some restrictions becomes some of the previous proposals Stevenes[4], Steiner *et al.*[9], Wu *et al.*[10] y Aparisi *et al.*[2]. For that reason, the optimization of its parameters guarantees to obtain the best possible performance of this family of control charts.

The evaluation of the performance of the optimized wY_SY_L scheme shows that the best statistic for controlling only the variability is $Y_T = Y_S + Y_L$, where Ys is the number of items of the sample that are classified as small by the gage, and Y_L the number of items that are classified as large. If the objective is detecting shifts in the process mean, the best monitoring is based on the difference $Y_S - Y_L$. In addition, the chart Y_SY_L is the best option for detecting simultaneous shifts in mean and variability, is the shifts are about the same magnitude.

A sensitivity analysis has been carried out, showing that the optimum parameters are very robust for different shift sizes, although its values depends on sample size employed, and on type of shift that the user want to detect.

The paper assumes that the quality characteristics follows a Normal distribution. A future work may consist of adapting the chart for different distributions.

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