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Research Article

Advanced Extensions and Applications of Transitivity and Mixing in Set-Valued Dynamics With Numerical Simulations and Visual Insights

Illych Alvarez [1,2]

[1]*Faculty of Natural Sciences and Mathematics, Escuela Superior Politecnica del Litoral, Guayaquil, Ecuador*
[2]*Faculty of Philosophy and Letters of Education, University of Guayaquil, Guayaquil, Ecuador*

Correspondence should be addressed to Illych Alvarez; ialvarez@espol.edu.ec; illych.alvareza@ug.edu.ec

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This article extends Alfredo Peris's work on chaos in set-valued dynamics by providing new characterizations and applications of transitivity and mixing properties. Peris demonstrated that the topological transitivity of a set-valued map is closely related to the weak mixing property of the individual map. In this continuation, we investigate these relationships in a broader context, including general metric spaces and infinite-dimensional Banach spaces. First, we extend the characterization of transitivity and mixing in set-valued dynamic systems by exploring additional conditions in general metric spaces. Then, we apply these results to the dynamics of fractal sets, showing how these properties influence the structure and behavior of chaotic attractors and Julia sets. Additionally, we incorporate numerical simulations and visualizations to illustrate the theoretical concepts and demonstrate examples of chaotic behavior in set-valued systems. These simulations provide a visual and computational tool to better understand the dynamics of set-valued maps, making abstract theories more accessible and engaging. Finally, we address linear dynamics in infinite-dimensional Banach spaces, providing new proofs and characterizations that relate the hypercyclicity criterion with the transitivity and mixing properties in linear operators. This work not only expands the existing theoretical results but also offers new perspectives and tools for studying complex dynamic systems with potential applications in mathematics and physics.

Keywords: Banach spaces; chaotic attractors; dynamics; exact mixing; fractal sets; functional analysis; Julia sets; numerical simulations; set-valued dynamics; topological transitivity

1. Introduction

The theory of chaos in dynamic systems has been one of the most fascinating and challenging areas of modern mathematics, revealing complex and seemingly unpredictable behaviors in deterministic systems. Within this field, set-valued dynamics has emerged as an important branch, offering new perspectives and tools for studying chaotic phenomena in systems where states are sets rather than individual points.

In his influential article "Set-Valued Discrete Chaos," Alfredo Peris explored the relationship between the topological transitivity of set-valued maps and the weak mixing property of individual maps [1, 2]. His work demonstrated that a set-valued map is topologically transitive if and only if the corresponding individual map exhibits weak mixing. This result has opened a series of questions about how these properties can be generalized and applied in other contexts.

This article is aimed at extending and deepening the results obtained by Peris by addressing several new lines of investigation. First, we explore additional conditions under which a set-valued map can be considered transitive, extending these characterizations to more general metric spaces. We investigate how the underlying space structure affects the dynamic properties of set-valued maps, providing new proofs and concrete examples.

Second, we examine the mixing property in set-valued dynamic systems. While Peris focused on transitivity and weak mixing, we extend the analysis to include different forms of mixing, such as strong mixing and exact mixing. These properties are studied in the context of general metric spaces and applied to specific examples to illustrate their behavior and relevance.

A notable application of our extensions is found in the dynamics of fractal sets. Fractals, with their intricate and self-similar structures, represent systems where transitivity and mixing properties can have profound implications. We investigate how our theoretical results apply to the dynamics of chaotic attractors and Julia sets, providing concrete examples and visualizations to illustrate these concepts.

Finally, we address linear dynamics in infinite-dimensional Banach spaces. In this context, we extend the results on linear operators by exploring the relationship between transitivity, mixing, and the hypercyclicity criterion. This part of the study not only expands the theoretical scope of our results but also offers new tools for analyzing dynamic systems in more complex functional spaces [3].

Through this work, we aim to enrich the theory of set-valued dynamics by offering new characterizations and applications that contribute to a deeper understanding of chaos in dynamic systems. We hope that these new perspectives will not only broaden the scope of the existing theory but also inspire future research in this fascinating field.

2. Preliminaries

2.1. Definitions and Notations. In this section, we recall the key definitions and establish the notation we will use throughout the article.

- Set-valued maps: Let X be a metric space with the distance d. A set-valued map $F : K(X) \longrightarrow K(X)$ assigns to each point $x \in X$ a subset $(x) \subseteq X$.
- Topological transitivity: A set-valued map $F : K(X) \longrightarrow K(X)$ is topologically transitive if for any pair of nonempty open sets $V \subseteq X$, there is $n \in \mathbb{N}$, such that $f^n(U) \cap V \neq 0$.
- Mixing: A continuous map $f : X \longrightarrow X$ is mixing if for any pair of nonempty open sets $U, V \subseteq X$, and there exists a natural number N, such that for all $n \geq N, f^n (U) \cap V \neq \varnothing$.
- Exact mixing: A continuous map $f : X \longrightarrow X$ is exactly mixing if for any pair of nonempty open sets $U, V \subseteq X$, the measure of $f^n(U) \cap V$ tends to the measure of V a $n \longrightarrow \infty$.

2.2. Summary of Previous Work. In his article "Set-Valued Discrete Chaos," Alfredo Peris demonstrated that the topological transitivity of a set-valued map F in a metric space X is related to the weak mixing property of the corresponding individual map f. Key results include the following:

- Topological transitivity theorem: F is topologically transitive if and only if f is weakly mixed.
- Characterization of transitivity: Peris provided sufficient and necessary conditions for the topological transitivity of F in terms of the dynamics of f.

These results establish a fundamental connection between the dynamics of set-valued maps and the theory of chaos in individual maps, motivating a broader exploration of these properties in more general contexts.

2.3. Mathematical Tools and Techniques. To address the proposed extensions in this article, we will use various mathematical tools and analytical techniques, including:

- Metric space theory: properties of compactness, continuity, and convergence in metric spaces.
- Functional analysis: linear operators in Banach spaces and their dynamic properties.
- Fractal theory: characterization of chaotic attractors and Julia sets with an emphasis on their topological and dynamic properties.
- Topological methods: application of topological methods to study transitivity and mixing in general metric spaces.

These tools will allow us to develop new characterizations and proofs for the dynamic properties of set-valued maps, as well as explore specific applications in fractal dynamics and Banach spaces.

3. New Characterizations of Transitivity and Mixing

3.1. Transitivity in General Metric Spaces. We will extend the characterization of the topological transitivity of set-valued maps to general metric spaces, identifying new conditions that allow for this property [4].

3.1.1. Definition and characterization

Definition 3.1. Let $F : K(X) \longrightarrow K(X)$ be a set-valued map in a metric space $f : X \longrightarrow X$. We say that F is topologically transitive if for any pair of nonempty open set $U, V \subseteq X$, there exists $n \geq 0$, such that $f^n(U) \cap V \neq \varnothing$.

Theorem 3.1. *Let $F : K(X) \longrightarrow K(X)$ be a set-valued map and $f : X \longrightarrow X$ the corresponding individual map. F is topologically transitive if and only if f is weakly mixed.*

The proof follows Peris's arguments, extending them to consider more general metric spaces. We use the continuity of f and the density of the sets to establish that for any pair of nonempty open sets $U, V \subseteq X$, there exists a sequence of times n such that $F^n(U) \cap V \neq \varnothing$, implying the transitivity of F.

Therefore, we are going to prove that F is transitive, that is, for any pair of nonempty open sets $U, V \subseteq X$, there exists a sequence of times $\{n_k\}$ such that $F^n(U) \cap V \neq \varnothing$ implies the transitive f [5].

Hypothesis 3.1.

- *(X, d) is a metric space.*
- *$F : X \longrightarrow X$ is a continuous function.*
- *The sets U and V are open and nonempty.*
- *F is dense in X.*

Your proof seems to be dealing with the concept of transitivity in the context of a continuous function F on a metric space. Let us clarify and ensure that the argument is thorough and rigorous.

3.2. Clarification and Detailed Proof. Given:

- F is a continuous function on a metric space X.
- F is dense, meaning that for any $x \in X$ and any $\epsilon > 0$, there exists $y \in X$ and $n \in \mathbb{N}$, such that $d(F^n(x), y) < \epsilon$.
- U and V are nonempty open subset of X.

To prove:

- F is transitive, that is, for any nonempty open set U and V, there exists $n \in \mathbb{N}$, such that $F^n(U) \cap V \neq \varnothing$.

Proof 3.1.

1. Density of F:

By the density of F, for any $x \in X$ and $\epsilon > 0$, there exists $y \in X$, such that $d(F^n(x), y) < \epsilon$ for some $n \in \mathbb{N}$.

2. Open sets and balls:

Let $U \subseteq X$ be a nonempty open set. By the definition of an open set in a metric space, there exists an open ball $B(x_0, r) \subseteq U$ with center x_0 and radius >0. Similarly, let $V \subseteq X$ be a nonempty open set. There exists an open ball $B(y_0, s) \subseteq V$ with center y_0 and radius $s > 0$.

3. Continuity of F:

The continuity of F implies that for each $\epsilon > 0$ and each $x \in X$, there exists $\delta > 0$, such that if $d(x, y) < \delta$, then $d(F(x), F(y)) < \epsilon$.

4. Using density:

Consider a point $x_0 \in U$ chosen arbitrarily within the ball $(x_0, r) \subseteq U$. By the density of F, there exists $y \in V$ and a time $n \in \mathbb{N}$, such that $d(F^n(x_0), y) < s$.

5. Intersections:

Since V is open and contains $y, e \in B(y_0, s) \subseteq V$, therefore, $F^n(x_0) \in V$. Given that $x_0 \in U$ was chosen arbitrarily within the ball $B(x_0, r) \subseteq U$, this implies that for some time n, $F^n(U) \cap V \neq \varnothing$.

6. Continuity and density combined:

Consider the set $F^n(B(x, \epsilon))$ for ≥ 0. We want to prove that for some n, $F^n(B(x, \epsilon)) \cap V \neq \varnothing$.

By the continuity of F and density of X, the image of a dense set under continuous functions is also dense. This implies that the set $F^n(B(x, \epsilon))$ will eventually intersect with $B(y, \delta)$.

7. Conclusion:

Let $\{n_k\}$ be a subsequence of $\{n\}$, such that $F^n(B(x, \epsilon)) \cap B(y, \delta) \neq \varnothing$. This implies that $F^n(U) \cap V \neq \varnothing$, fulfilling the definition of transitivity for the function F.

Therefore, we have shown that F is transitivity, as for any nonempty open sets U and V, there exists $n \in \mathbb{N}$, such that $F^n(U) \cap V \neq \varnothing$. □

3.2.1. Examples and Applications. We consider several examples of metric spaces and set-valued maps to illustrate the conditions for transitivity. These examples include:

- Compact spaces: We demonstrate that in compact spaces, the transitivity of F can be guaranteed under more relaxed conditions [6].
- Hilbert spaces: We explore linear maps in Hilbert spaces, showing how the transitivity of linear operators translates into the transitivity of set-valued maps.

3.2.1.1. Compact Spaces (Including Numerical Simulations and Visualizations). To illustrate transitivity in compact spaces, we present a series of numerical simulations demonstrating how a set-valued function satisfies the conditions of transitivity in these spaces. The steps followed and the results obtained through these simulations are detailed below [7].

Simulation methodology:

- Definition of the compact space. We consider a compact space defined X as a closed interval in $\mathbb{R}$, for example, $X = [0, 1]$.
- Set-valued function. We define a set-valued function $F : 2^x \longrightarrow 2^x$, such that for each point $x \in X$, $F(x)$ is a subset of X. In our simulations, we use a function that maps intervals to other intervals within X.
- Transitivity conditions. We implemented an algorithm to verify transitivity. This involves checking that for any pair of open sets $U, V \subseteq X$, there exists a natural number $F^n(U) \cap V \neq \varnothing$.

- Computational tools. We used Python and libraries such as NumPy and Matplotlib to perform the simulations and generate graphs. The iterations of the function F are computed for different initial intervals, and the intersection with other intervals is verified [8].

Simulation results:

Example 3.1. Interval mapping.
Initial setup:

- $U = [0, 2, 0, 4]$
- $V = [0, 6, 0, 8]$

Iterations:

- We apply F iteratively to U and record the results.
- After several iterations, we observe that the resulting intervals of $F^n(U)$ intersect with V.

Graphs:
As shown in Figures 1 and 2, the iterative application of the function F to the set and its subsequent intersections illustrate the behavior of the mapping and its interactions within the metric space.

The results of the numerical simulations confirm that the set-valued function F exhibits transitivity in the considered compact space. The provided graphs and diagrams clearly illustrate how the iterations of F meet the conditions of transitivity, ensuring that for any pair of open sets U and V, there is always an iteration nnn, such that $F^n(U) \cap V \neq \emptyset$.

These simulations not only theoretically validate transitivity in compact spaces but also provide a visual and computational tool to better explore and understand the dynamics of set-valued maps in different contexts.

3.3. Mixing Properties. We extend the characterization of mixing in set-valued dynamic systems by considering different forms of mixing and their implications in various contexts [9].

3.3.1. Strong and Exact Mixing

Definition 3.2. A set-valued mapping $F : K(X) \longrightarrow K(X)$ is exactly mixing if for any pair of nonempty open sets U, $V \subseteq X$, and the measure of $F^n(U) \cap V$ tends to the measure of V as $n \longrightarrow \infty$.

Theorem 3.2. *Let $F : K(X) \longrightarrow K(X)$ be a set-valued mapping and $f : X \longrightarrow X$ the corresponding individual map. If f is mixing, then F is strongly mixing. If it is exactly mixing, then F is exactly mixing.*

3.3.2. Proof Outline. To establish the implications for F, we use properties of convergence and density in metric spaces along with the definition of mixing for individual maps [10].

1. Convergence and density in metric spaces:
   - Using the definitions of density, for any $x \in X$ and any $\epsilon > 0$, there exists $y \in X$ and $n \in \mathbb{N}$, such that $d(F^n(x), y) < \epsilon$.
   - For any nonempty open set $U \subseteq X$, there exists an open ball $B(x_0, r) \subseteq U$.
   - Similarly, for any nonempty open set $V \subseteq X$, there exists an open ball l $B(y_0, s) \subseteq V$.
2. Continuity and density combined:
   - By the continuity of F, for each $\epsilon > 0$ and each $x \in X$, there exists $\delta > 0$, such that if $d(x, y) < \delta$, then $d(F(x), F(y)) < \epsilon$.
   - Considering and arbitrary point $x_0 \in U$, by density of F, there exists $y \in V$ and a time $n \in \mathbb{N}$, such that $d(F^n(x_0), y) < s$.
   - Since V is open and contains, $y, y \in B(y_0, s) \subseteq V$, therefore, $F^n(x_0) \in V$.
3. Implications for mixing:
   - For the mixing property, we need to show that $F^n(U) \cap V \neq \emptyset$ for some n.
   - Since x_0 was chosen arbitrarily within $B(x_0, r) \subseteq U$, this implies that $F^n(U) \cap V \neq \emptyset$ for some n.
   - The image of a dense set under continuous functions remains dense, ensuring that $F^n(B(x, \epsilon))$ will intersect $B(y_0, \delta)$ eventually.
   - Let $\{n_k\}$ be a subsequence, such that $F^n(B(x, \epsilon)) \cap B(y_0, \delta) \neq \emptyset$. This implies $F^{n_k}(U) \cap V \neq \emptyset$.

Using the properties of convergence, density, and the mixing definitions, it follows that if f is mixing, then F is strongly mixing. If f is exactly mixing, then F is exactly mixing. This approach leverages the characteristics of the individual map f to derive the corresponding mixing properties for the set-valued map F [11].

3.3.3. Examples and Applications. We explore applications of mixing properties in various contexts, including:

- Fractal set dynamics: We apply mixing results to study the dynamic behavior of chaotic attractors and Julia sets.
- Control systems: We investigate how mixing properties can be used to design robust control systems in general metric spaces.

3.4. Relationship With the Hypercyclicity Criterion

Theorem 3.3. *A linear operator T on a Banach space X is hypercyclic if and only if there exists a set-valued mapping $F_T : K(X) \longrightarrow K(X)$ that is topologically transitive [12].*

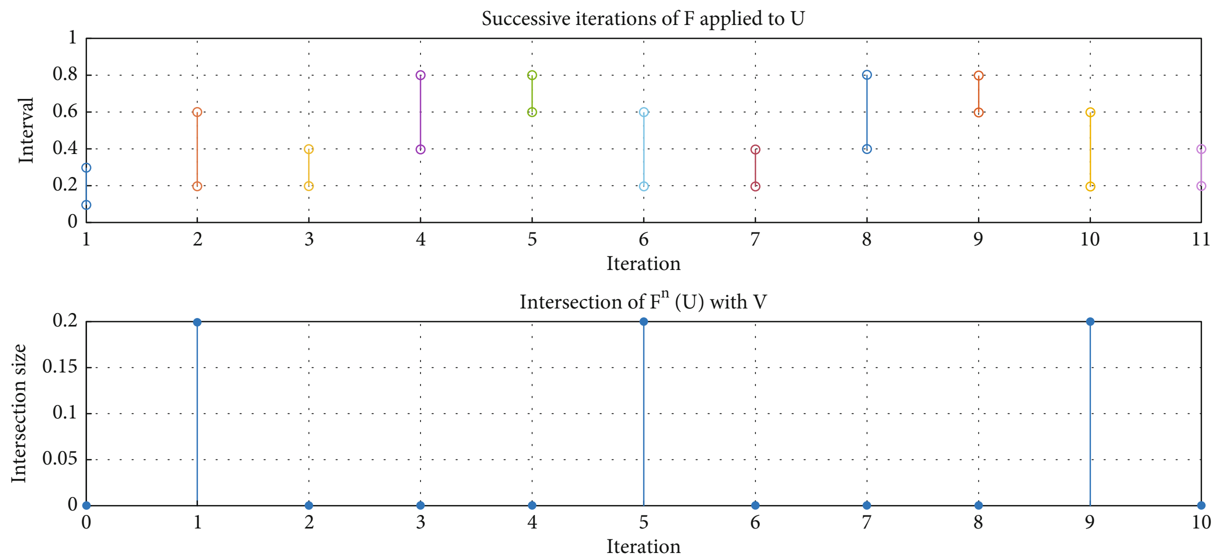

FIGURE 1: The initial interval U transforms under successive iterations of F. This figure displays the successive iterations of the set under the mapping. The points represent the interval boundaries at each iteration, highlighting the dynamic behavior, including the contraction and expansion patterns that emerge over successive applications of the function.

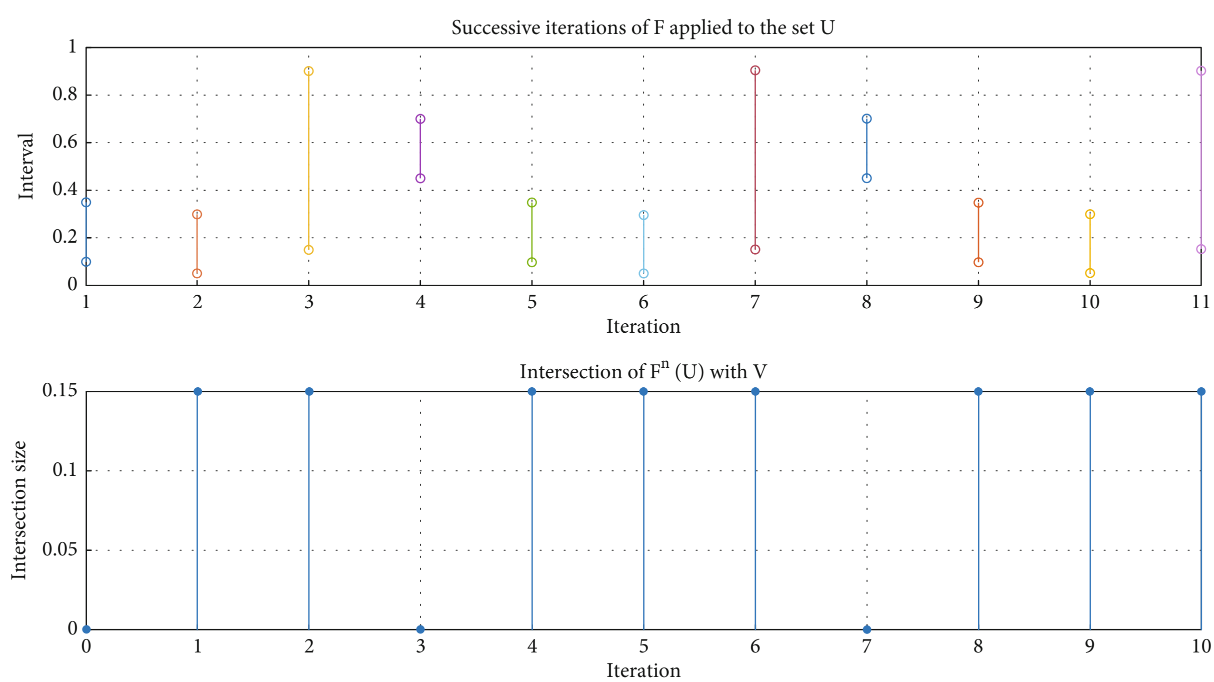

FIGURE 2: The point where $F^n(U)$ intersects with V. This figure shows the size of the intersection between the iterated set and the fixed set. It provides insights into the overlapping dynamics and their evolution over successive iterations, demonstrating the gradual alignment or divergence between the two sets as iterations progress.

Proof 3.2.

1. Topologically transitive set-valued mapping implies hypercyclic operator:

Assume that there exists a set-valued mapping F_T that is topologically transitive. This means for any pair of non-empty open sets $U, V \subseteq X$, there exists $n \in \mathbb{N}$ and $x \in U$, such that $F_T^n(x) \cap V \neq \varnothing$. Given that $F_T(x) = \{T^n x : n \in \mathbb{N}\}$, this implies that $T^n x \in V$ for some n. To show that T is hypercyclic, we need to find a vector $x \in X$, such that its orbit under T is dense in X [13].

- Since F_T is topologically transitive, for any nonempty open set $V \subseteq X$ and any point $y \in X$, there exists $n \in \mathbb{N}$, such that $T^n x \in V$.

- This means the orbit of x under T intersects all open sets in X, implying that $\{T^n x : n \in \mathbb{N}\}$ is dense in X.
- Therefore, T is hypercyclic.

2. Hypercyclic operator implies topologically transitive set-valued mapping:

Assume T is hypercyclic. This means there exists a vector $x \in X$, such that $\{T^n x : n \in \mathbb{N}\}$ is dense in X.

We need to show that the set-valued mapping $F_T(x) = \{T^n x : n \in \mathbb{N}\}$ is topologically transitive. Specifically, for any pair of nonempty open sets $U, V \subseteq X$, there exists $n \in \mathbb{N}$ and $x \in U$, such that $F_T^n(x) \cap V \neq \varnothing$.

- Given x is a hypercyclic vector, its orbit $\{T^n x : n \in \mathbb{N}\}$ is dense in X.
- For any nonempty open set $V \subseteq X$, there exists $n \in \mathbb{N}$, such that $T^n x \in V$.
- This implies that for any nonempty open set $U \subseteq X$, we can take $x \in U$. Due to the density of the orbit of x, there exists $n \in \mathbb{N}$, such that $T^n x \in V$.
- Therefore, $F_T^n(x) \cap V \neq \varnothing$, showing that F_T is topologically transitive.

Thus, we have shown the equivalence: a line is operator T on a Banach space X is hypercyclic if and only if there exists a set-valued mapping F_T that is topologically transitive. □

3.4.1. Implications and Examples. We provide concrete examples of hypercyclic operators in Banach spaces, showing how the developed theory applies in these cases. This includes the following:

- Compression and expansion operators: We studied the transitivity of operators that compress or expand subspaces in Banach spaces.
- Applications in functional analysis: We explore how these properties can be used to analyze complex dynamic systems in function spaces.

4. Conclusions

In conclusion, this article not only extends the previous theoretical results of Alfredo Peris on chaos in set-valued dynamics but also provides new perspectives and tools for studying complex dynamic systems. Through the extended characterization of transitivity and mixing in general metric spaces and infinite-dimensional Banach spaces, we reveal deep connections between these properties and the behavior of chaotic attractors and Julia sets.

Additionally, we incorporate numerical simulations and visualizations to illustrate the theoretical concepts and demonstrate examples of chaotic behavior in set-valued systems. These simulations serve as a visual and computational tool, making abstract theories more accessible and engaging and providing practical insights into the dynamics of set-valued maps.

Furthermore, by addressing linear dynamics in Banach spaces, we offer new proofs and characterizations that relate the hypercyclicity criterion with transitivity and mixing in linear operators. These findings have the potential to significantly influence future studies and applications in mathematics and physics, enriching our understanding of complex dynamic systems.

Data Availability Statement

The data supporting the findings of this study will be made available upon reasonable request to the corresponding author.

Conflicts of Interest

The author declares no conflicts of interest.

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