Row or column completion of polynomial matrices of given degree *

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Abstract

We solve the problem of characterizing the existence of a polynomial matrix of fixed degree when its eigenstructure (or part of it) and some of its rows (columns) are prescribed. More specifically, we present a solution to the row (column) completion problem of a polynomial matrix of given degree under different prescribed invariants: the whole eigenstructure, all of it but the row (column) minimal indices, and the finite and/or infinite structures. Moreover, we characterize the existence of a polynomial matrix with prescribed degree and eigenstructure over an arbitrary field.

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1 Introduction

Polynomial matrices are very often used to study the dynamical behavior of systems of differential or difference equations [3, 20, 25, 28, 33, 36, 37]. These equations arise in many different scientific areas such as engineering, physics, economics and biology. When the polynomial matrix associated with the system is regular, a closed formula for its solution can be given in terms of the finite and infinite elementary divisors of the matrix; when it is singular, the solution also depends on the left and right null spaces of the matrix, which are related to its minimal indices.

On the other hand, a very important problem in applications is the matrix completion problem. A matrix completion problem consists in characterizing all or some of the invariants of a matrix with respect to a given equivalence relation, when some entries of the matrix have been prescribed. Different types of matrices, different equivalence relations, or different positions of the prescribed components lead to different type of problems. For example, a variety of pencil completion problems appear in the design of linear control systems to modify the structure of the system (see, for instance, [4, 18, 29] and the references therein.

The research on matrix completion problems has been very active for a long time. Since the 1950s numerous problems in the field have been studied for constant and polynomial matrices. The literature in the area is vast, therefore the references given in this section do not intend at all to be exhaustive, but a few examples shall be mentioned. Results have been obtained for the problem of obtaining a square constant matrix when a submatrix and similarity invariants are prescribed

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[6, 7, 8, 19, 24, 27, 30, 31, 34, 35, 38], and also for rectangular constant matrices when invariants for the feedback equivalence relation and some of its rows or its columns are prescribed [2, 11, 39].

Some results concerning matrix pencil completion problems for the strict equivalence were obtained in [1, 5, 22]. The general matrix pencil completion problem was finally stated in [29], and since then enormous progresses have been made, most of them by the same authors (see [12, 13, 14, 15, 16, 18]).

Especially related to our work are the references [11, 14, 30, 35, 38]. A fundamental result for polynomial matrices was obtained in [30, 35]. The result is known as the *interlacing inequalities* for invariant factors and underlies many of the above mentioned papers. It characterizes the invariants for the unimodular equivalence of a polynomial matrix (i.e., the invariant factors) when a submatrix is prescribed and there is no restriction on the degree of the completion. The authors also characterized the invariants for the similarity of square constant matrices when a principal submatrix is prescribed. The other three references are devoted to row (column) completion problems. In [38], the problem of prescribing the similarity invariants of a square constant matrix when some of its rows are prescribed, is solved. A solution to the analogous problem for rectangular constant matrices and feedback invariants of pairs of matrices was given in [11]. For matrix pencils and prescribed invariants for the strict equivalence, a solution was obtained in [12], where an implicit solution was provided. In [13] the same problem was solved in terms of a (complex) explicit solution. Later, in [14], the solution was simplified (see also [18]).

The invariants for the strict equivalence of matrix pencils are the invariant factors (finite structure), the infinite elementary divisors (infinite structure) and the row and column minimal indices; i.e., the left and right minimal indices as polynomial matrices (singular structure). For polynomial matrices of arbitrary degree, these 4 types of invariants are known as the *eigenstructure* of the matrix. The eigenstructure of a polynomial matrix is invariant for the strict equivalence, but it does not form a complete system of invariants for such relation (see [10]). For polynomial matrices the finite and infinite structures can be given in terms of the homogeneous invariant factors.

We are interested in the following problem:

Problem 1.1 *Let $P(s) \in \mathbb{F}[s]^{m\times n}$ be a polynomial matrix of $\deg(P(s)) = d$. Find necessary and sufficient conditions for the existence of a polynomial matrix $W(s) \in \mathbb{F}[s]^{z\times n}$ of $\deg(W(s)) \leq d$ such that $\begin{bmatrix} P(s) \\ W(s) \end{bmatrix}$ has prescribed eigenstructure, or part of it.*

Since the eigenstructure consists of four types of invariants, depending on the invariant(s) prescribed we get different problems; Problem 1.1 leads actually to 15 problems. In this paper we solve Problem 1.1 when the whole eigenstructure (homogeneous invariant factors and row and column minimal indices) is prescribed, therefore generalizing the result of [14]. To achieve it, we turn a row completion problem of a polynomial matrix into a row completion problem of a matrix pencil, using a suitable linearization (the first Frobenious companion form) of the polynomial matrix. This allows us to use the result on row completion of pencils in [14, 18]. We also solve some of the particular cases appearing when part of the eigenstructure is prescribed.

Results are obtained for row completion problems, but it is a matter of transposition to obtain solutions to the corresponding column completion problems.

When no row is prescribed, i.e., when we want to characterize the existence of a poynomial matrix with prescribed eigenstructure, the problem was solved in [9] for infinite fields. Here we remove this restriction and provide a solution to the problem for arbitrary fields.

The paper is organized as follows. Section 2 is a preliminary section. It is structured in three subsections. Subsection 2.1 contains notation and some basic results. In Subsection 2.2 we present a known result of row completion of matrix pencils with the corresponding notation and definitions. Subsection 2.3 is devoted to the Frobenius companion form of a polynomial matrix. In Section 3 we characterize the existence of a polynomial matrix with prescribed eigenstructure in an arbitrary field. In Section 4 we prove that the row completion problem of a polynomial matrix is equivalent to the row completion problem of a pencil, via the Frobenius companion form. In different subsections we present the solution to Problem 1.1 when prescribing the whole eigenstructure (Subsection 4.1), the whole eigenstructure but the row (column) minimal indices (Subsection 4.2 (4.3)), and the finite and/or infinite elementary divisors (Subsection 4.4). We also include a section of conclusions and future work (Section 5).

2 Preliminaries

2.1 Notation and previous results

Let $\mathbb{F}$ be a field and $\overline{\mathbb{F}}$ its algebraic closure. $\mathbb{F}[s]$ denotes the ring of polynomials in the indeterminate s with coefficients in $\mathbb{F}$, $\mathbb{F}(s)$ is the field of fractions of $\mathbb{F}[s]$, i.e., the field of rational functions over $\mathbb{F}$, and $\mathbb{F}[s,t]$ denotes the ring of polynomials in two variables s,t with coefficients in $\mathbb{F}$. A polynomial in $\mathbb{F}[s]$ is *monic* if its leading coefficient is 1. We will say that a polynomial in $\mathbb{F}[s,t]$ is *monic* if it is monic with respect to the variable s. Given two polynomials p,q, by $p \mid q$ we mean that p is a divisor of q. By $\operatorname{lcm}(p,q)$ we mean the monic least common multiple of p and q.

We denote by $\mathbb{F}^{m\times n}$, $\mathbb{F}[s]^{m\times n}$ and $\mathbb{F}(s)^{m\times n}$ the vector spaces over $\mathbb{F}$ of $m\times n$ matrices with elements in $\mathbb{F}$, $\mathbb{F}[s]$ and $\mathbb{F}(s)$, respectively. $\mathrm{Gl}_n(\mathbb{F})$ will be the general linear group of invertible matrices in $\mathbb{F}^{n\times n}$.

Let $P(s)\in\mathbb{F}[s]^{m\times n}$ be a polynomial matrix. It can be written as $P(s)=P_ds^d+P_{d-1}s^{d-1}+\cdots+P_1s+P_0\in\mathbb{F}[s]^{m\times n}$, $P_i\in\mathbb{F}^{m\times n}$, $0\le i\le d$, with $P_d\neq 0$ for some integer d. Then, d is the *degree* of $P(s)$ (denoted by $\deg(P(s))$).

The *normal rank* of $P(s)$, denoted by $\operatorname{rank}(P(s))$, is the order of the largest non identically zero minor of $P(s)$, i.e., it is the rank of $P(s)$ considered as a matrix on $\mathbb{F}(s)$. We will refer to it as the *rank* of $P(s)$.

A polynomial matrix $P(s)\in\mathbb{F}[s]^{m\times n}$ is *regular* if $m=n$ and $\det(P(s))$ is not identically zero. Otherwise it is *singular*.

A polynomial matrix $P(s)\in\mathbb{F}[s]^{n\times n}$ is *unimodular* if $\det(P(s))$ is a non zero constant. Given a polynomial matrix $P(s)\in\mathbb{F}[s]^{m\times n}$, $r=\operatorname{rank}(P(s))$, there exist unimodular matrices $U(s)\in\mathbb{F}[s]^{m\times m}$ and $V(s)\in\mathbb{F}[s]^{n\times n}$ such that

$$U(s)P(s)V(s)=D(s)=\left[\begin{array}{ccc|c} \alpha_1(s) & & & \\ & \ddots & & 0 \\ & & \alpha_r(s) & \\ \hline & 0 & & 0 \end{array}\right],$$

where $\alpha_1(s)\mid\cdots\mid\alpha_r(s)$ are monic polynomials and are called the *invariant factors* of $P(s)$. The matrix $D(s)$ is the *Smith form* of $P(s)$ (see, for instance, [23, 25]).

It is said that $\lambda\in\overline{\mathbb{F}}\cup\{\infty\}$ is an *eigenvalue* of $P(s)$ if $\operatorname{rank}(P(\lambda))<\operatorname{rank}(P(s))$; here we understand that $P(\infty)=P_d$. The set of eigenvalues of $P(s)$ is the *spectrum* of the matrix, and we denote it by $\Lambda(P(s))$.

Given $P(s)=P_ds^d+P_{d-1}s^{d-1}+\cdots+P_1s+P_0\in\mathbb{F}[s]^{m\times n}$ with $P_d\neq 0$, we define the *reversal matrix polynomial* $\operatorname{rev}P(t)$ as

$$\operatorname{rev}P(t)=t^dP(1/t)=P_d+P_{d-1}t+\cdots+P_1t^{d-1}+P_0t^d. \tag{1}$$

This matrix and $P(s)$ have the same rank but $\operatorname{rev}P(t)$ may be of degree smaller than d. It turns out that ∞ is an eigenvalue of $P(s)$ if and only if 0 is an eigenvalue of $\operatorname{rev}P(t)$.

Notice that the finite eigenvalues of $P(s)$ are the roots in $\overline{\mathbb{F}}$ of the polynomial $\alpha_r(s)$ in the Smith form of $P(s)$. Factorizing the invariant factors as products of irreducible polynomials over $\overline{\mathbb{F}}$, we can write

$$\alpha_i(s)=\prod_{\lambda\in\Lambda(P(s))\setminus\{\infty\}}(s-\lambda)^{n_i(\lambda,P(s))},\quad 1\le i\le r.$$

The factors $(s-\lambda)^{n_i(\lambda,P(s))}$ with $n_i(\lambda,P(s))>0$ are the *elementary divisors* of $P(s)$ over $\overline{\mathbb{F}}$ corresponding to λ, and the integers $n_1(\lambda,P(s))\leq\cdots\leq n_r(\lambda,P(s))$ are the *partial multiplicities* of λ in $P(s)$.

The *infinite elementary divisors* of $P(s)$ and the *partial multiplicities of* ∞ in $P(s)$ are the elementary divisors of $\operatorname{rev}P(t)$ corresponding to 0 and the partial multiplicities of 0 in $\operatorname{rev}P(t)$, respectively. For simplicity, we will denote $e_i=n_i(0,\operatorname{rev}P(t))$ for $1\leq i\leq r$, where if $0\notin\Lambda(\operatorname{rev}P(t))$ (i.e., $\infty\notin\Lambda(P(s))$), we take $e_1=\cdots=e_r=0$.

We recall now the singular structure of a polynomial matrix. Denote by $\mathcal{N}_\ell(P(s))$ and $\mathcal{N}_r(P(s))$ the *left* and *right null-spaces* over $\mathbb{F}(s)$ of $P(s)$, respectively, i.e., if $P(s)\in\mathbb{F}[s]^{m\times n}$,

$$\begin{aligned}\mathcal{N}_\ell(P(s))&=\{x(s)\in\mathbb{F}(s)^{m\times 1}:x(s)^TP(s)=0\},\\ \mathcal{N}_r(P(s))&=\{x(s)\in\mathbb{F}(s)^{n\times 1}:P(s)x(s)=0\}.\end{aligned}$$

These sets are vector subspaces of $\mathbb{F}(s)^{m\times 1}$ and $\mathbb{F}(s)^{n\times 1}$, respectively. For a subspace $\mathcal{V}$ of $\mathbb{F}(s)^{m\times 1}$ it is possible to find a basis consisting of vector polynomials; it is enough to take an arbitrary basis and multiply each vector by a least common multiple of the denominators of its entries. The *order* of a polynomial basis is defined as the sum of the degrees of its vectors (see [20]). A *minimal basis* of $\mathcal{V}$ is a polynomial basis with least order among all polynomial bases of $\mathcal{V}$. The increasing ordered list of degrees of the vector polynomials of a minimal basis is always the same (see [20]). These degrees are called the *minimal indices* of $\mathcal{V}$.

A *right (left) minimal basis* of a polynomial matrix $P(s)$ is a minimal basis of $\mathcal{N}_r(P(s))$ ($\mathcal{N}_\ell(P(s))$). The *right (left) minimal indices* of $P(s)$ are the minimal indices of $\mathcal{N}_r(P(s))$ ($\mathcal{N}_\ell(P(s))$). From now on in this paper, we will work with the right (left) minimal indices ordered decreasingly, and we will refer to them as the *column (row) minimal indices* of $P(s)$. Notice that a polynomial matrix $P(s)\in\mathbb{F}[s]^{m\times n}$ of $\operatorname{rank}(P(s))=r$ has $m-r$ row and $n-r$ column minimal indices. The invariant factors, the infinite elementary divisors and the column and row minimal indices form the *eigenstructure* of a polynomial matrix $P(s)$ (see, for instance, [9]).

A *matrix pencil* is a polynomial matrix $P(s)\in\mathbb{F}[s]^{m\times n}$ of degree at most 1. Two matrix pencils $A(s)=sA_1+A_0, B(s)=sB_1+B_0\in\mathbb{F}[s]^{m\times n}$ are *strictly equivalent* ($A(s)\overset{s.e.}{\sim}B(s)$) if there exist invertible matrices $Q\in\mathrm{Gl}_m(\mathbb{F})$, $R\in\mathrm{Gl}_n(\mathbb{F})$ such that $A(s)=QB(s)R$. A complete system of invariants for the strict equivalence of matrix pencils is formed by the invariant factors, the infinite elementary divisors (or, equivalently, the partial multiplicities of ∞), and the column and row minimal indices. They are known as the *Kronecker invariants* of the pencil. The associated canonical form is the Kronecker canonical form. For details see [21, Ch. 2] or [23, Ch. 12] for infinite fields, and [32, Ch. 2] for arbitrary fields.

Remark 2.1 In the above references, and as far as we know, the definition of the reversal of a constant pencil is different from the definition given in (1) for polynomial matrices, which leads to different infinite structure for constant matrices. More specifically, given a pencil $A(s)=sA_1+A_0$, the reversal pencil is generally defined as $\operatorname{rev}A(t)=A_1+tA_0$. According to this definition, the reversal of a constant pencil A_0 is tA_0, therefore the partial multiplicities of ∞ in a constant pencil of rank r are $e_1=\cdots=e_r=1$. However, the common definition of the reversal of a polynomial matrix is that given in (1). Hence, the reversal of a constant matrix A_0 is A_0, therefore the partial multiplicities of ∞ in a constant matrix of rank r are $e_1=\cdots=e_r=0$. In this paper we adopt the second option; i.e., we will work with the reversal of a polynomial matrix according to the definition given in (1).

In the matrix pencil literature, it is also common to join together the invariant factors and the partial multiplicities of ∞ in the so called homogeneous invariant factors (see, for instance, [21, Ch. 2] or [23, Ch. 12]). This concept can also be extended to polynomial matrices (see [21, Ch. 2], [40]). In fact, if $\alpha_1(s)\mid\cdots\mid\alpha_r(s)$ are the invariant factors of a polynomial matrix $P(s)$ and $e_1\leq\cdots\leq e_r$ its partial multiplicities of ∞, they can be summarized in a chain of monic homogeneous polynomials $\gamma_1(s,t) \mid \cdots \mid \gamma_r(s,t)$, $\gamma_i(s,t) \in \mathbb{F}[s,t]$, $1 \leq i \leq r$, called the *homogeneous invariant factors* of $P(s)$, of the form

$$\gamma_i(s,t) = t^{e_i} t^{\deg(\alpha_i)} \alpha_i \left(\frac{s}{t}\right), \quad 1 \leq i \leq r. \tag{2}$$

In turn, the homogeneous invariant factors of a polynomial matrix determine the invariant factors $\alpha_i(s) = \gamma_i(s,1)$, $1 \leq i \leq r$ and the partial multiplicities of ∞, $e_1 \leq \cdots \leq e_r$.

We will take $\gamma_i(s,t) = 1$ whenever $i < 1$ and $\gamma_i(s,t) = 0$ when $i > r$. As a consequence, $\alpha_i(s) = 1$ and $e_i = 0$ for $i < 1$, and $\alpha_i(s) = 0$ for $i > r$. We also agree that $e_i = +\infty$ for $i > r$.

Remark 2.2 Notice that for a polynomial matrix $P(s)$ of degree d and leading coefficient P_d, $\operatorname{rev} P(0) = P_d \neq 0$, which means that $e_1 = n_1(\infty, P(s)) = n_1(0, \operatorname{rev} P(t)) = 0$ (see [9, Lemma 2.7]), equivalently,

$$\gamma_1(s,0) \neq 0. \tag{3}$$

In what follows we will work with the homogeneous invariant factors instead of working with the invariant factors and the infinite elementary divisors separately, for it significantly simplifies expressions. Indeed, observe that given $\alpha(s), \beta(s) \in \mathbb{F}[s]$ and two non negative integers k, ℓ, if $\phi(s,t) = t^k t^{\deg(\alpha)} \alpha\left(\frac{s}{t}\right)$ and $\psi(s,t) = t^\ell t^{\deg(\beta)} \beta\left(\frac{s}{t}\right)$ then

$$\left.\begin{matrix} \alpha(s) \mid \beta(s), \\ k \leq \ell \end{matrix}\right\} \quad \text{if and only if} \quad \phi(s,t) \mid \psi(s,t).$$

Unlike the case of matrix pencils, the homogeneous invariant factors and the minimal indices are not a complete set of invariants for the strict equivalence of matrix polynomials (as for matrix pencils, $P(s) \overset{s.e.}{\sim} \bar{P}(s)$ if $\bar{P}(s) = QP(s)R$, Q, R non singular) (see [10, Subsection 3.1]). In the lack of better structure, we shall use this one, that is, the eigenstructure. Given $P(s), \bar{P}(s) \in \mathbb{F}[s]^{m \times n}$ we will write $P(s) \approx \bar{P}(s)$ if they have the same eigenstructure.

For non constant matrix pencils, the sum of the degrees of the homogeneous invariant factors plus the sum of the minimal indices is equal to the rank. The following is an extension of that result to polynomial matrices given in [10]; we state it here in terms of the homogeneous invariant factors.

Lemma 2.3 (Index Sum Theorem for Matrix Polynomials [10, Theorem 6.5]) *Let $P(s) \in \mathbb{F}[s]^{m \times n}$, $\deg(P(s)) = d$, $\operatorname{rank}(P(s)) = r$. Let $\gamma_1(s,t) \mid \cdots \mid \gamma_r(s,t)$, $d_1 \geq \cdots \geq d_{n-r}$ and $v_1 \geq \cdots \geq v_{m-r}$ be the homogeneous invariant factors, column minimal indices and row minimal indices of $P(s)$, respectively. Then,*

$$\sum_{i=1}^{r} \deg(\gamma_i) + \sum_{i=1}^{n-r} d_i + \sum_{i=1}^{m-r} v_i = rd. \tag{4}$$

Notice that if $P(s) \approx \bar{P}(s)$ then $\operatorname{rank}(P(s)) = \operatorname{rank}(\bar{P}(s))$ (they have the same number of invariants), and therefore, by Lemma 2.3, $\deg(P(s)) = \deg(\bar{P}(s))$. Notice also that if $P(s), \bar{P}(s) \in \mathbb{F}[s]^{m \times n}$ are matrix pencils, $P(s) \approx \bar{P}(s)$ if and only if $P(s) \overset{s.e.}{\sim} \bar{P}(s)$.

Later in this paper we will frequently use Lemma 2.3 without making any further reference to it.

2.2 Row completion of matrix pencils

In [14] (see also [18]) a solution to Problem 1.1 is given when the eigenstructure is prescribed and the polynomial matrix is of degree one. We bring here the result (see Theorem 2.6 below); it will be used later to solve the general case. The theorem provides a solution of the row completion problem; by transposition, and interchanging the row and column minimal indices, the result applies for the column completion problem. To state it we need some notation and definitions.

Given two integers n and m, whenever $n > m$ we take $\sum_{i=n}^{m} = 0$. In the same way, if a condition is stated for $n \leq i \leq m$ with $n > m$, we understand that the condition disappears.

Let $a_1, \dots, a_m$ be a sequence of integers. Whenever we write $\mathbf{a} = (a_1, \dots, a_m)$, we will understand that $a_1 \geq \cdots \geq a_m$, and we will take $a_i = \infty$ for $i < 1$ and $a_i = -\infty$ for $i > m$. If $a_m \geq 0$, the sequence $\mathbf{a} = (a_1, \dots, a_m)$ is called a *partition*.

Let $\mathbf{a} = (a_1, \dots, a_m)$ and $\mathbf{b} = (b_1, \dots, b_m)$ be two sequences of integers. It is said that $\mathbf{a}$ is *majorized* by $\mathbf{b}$ (denoted by $\mathbf{a} \prec \mathbf{b}$) if $\sum_{i=1}^{k} a_i \leq \sum_{i=1}^{k} b_i$ for $1 \leq k \leq m-1$ and $\sum_{i=1}^{m} a_i = \sum_{i=1}^{m} b_i$ (this is an extension to sequences of integers of the definition of majorization given for partitions in [26]).

We introduce next the concept of generalized majorization.

Definition 2.4 [17, Definition 2] *Let $\mathbf{d} = (d_1, \dots, d_m)$, $\mathbf{a} = (a_1, \dots, a_s)$ and $\mathbf{g} = (g_1, \dots, g_{m+s})$ be sequences of integers. We say that $\mathbf{g}$ is majorized by $\mathbf{d}$ and $\mathbf{a}$ ($\mathbf{g} \prec' (\mathbf{d}, \mathbf{a})$) if*

$$d_i \geq g_{i+s}, \quad 1 \leq i \leq m, \tag{5}$$

$$\sum_{i=1}^{h_j} g_i - \sum_{i=1}^{h_j - j} d_i \leq \sum_{i=1}^{j} a_i, \quad 1 \leq j \leq s, \tag{6}$$

where $h_j = \min\{i \,:\, d_{i-j+1} < g_i\}$, $1 \leq j \leq s$ ($d_{m+1} = -\infty$),

$$\sum_{i=1}^{m+s} g_i = \sum_{i=1}^{m} d_i + \sum_{i=1}^{s} a_i. \tag{7}$$

Remark 2.5

1. The definition of h_j implies that $j \leq h_j \leq m + j, 1 \leq j \leq s$.

2. In the case that $s = 0$, condition (6) disappears, and conditions (5) and (7) are equivalent to $\mathbf{d} = \mathbf{g}$. On the other hand, if $m = 0$ then $\mathbf{g} \prec' (\mathbf{d}, \mathbf{a})$ is equivalent to $\mathbf{g} \prec \mathbf{a}$.

3. Let k be an integer, $\bar{\mathbf{d}} = (d_1 + k, \dots, d_m + k)$, $\bar{\mathbf{a}} = (a_1 + k, \dots, a_s + k)$ and $\bar{\mathbf{g}} = (g_1 + k, \dots, g_{m+s} + k)$. Then $\mathbf{g} \prec' (\mathbf{d}, \mathbf{a})$ if and only if $\bar{\mathbf{g}} \prec' (\bar{\mathbf{d}}, \bar{\mathbf{a}})$.

The next result is in [18]; we state it for non constant pencils.

Theorem 2.6 [18, Theorem 4.3] *Let $C(s) \in \mathbb{F}[s]^{(\bar{r}+p)\times(\bar{r}+q)}$ be a non constant matrix pencil,* $\operatorname{rank}(C(s)) = \bar{r}$. *Let $\bar{\phi}_1(s,t) \mid \cdots \mid \bar{\phi}_{\bar{r}}(s,t)$ be its homogeneous invariant factors, $\bar{\mathbf{c}} = (\bar{c}_1, \dots, \bar{c}_q)$ its column minimal indices, and $\bar{\mathbf{u}} = (\bar{u}_1, \dots, \bar{u}_p)$ its row minimal indices, where $\bar{u}_1 \geq \cdots \geq \bar{u}_\theta > \bar{u}_{\theta+1} = \cdots = \bar{u}_p = 0$. Let x and y be non negative integers. Let $D(s) \in \mathbb{F}[s]^{(\bar{r}+p+x+y)\times(\bar{r}+q)}$ be a matrix pencil,* $\operatorname{rank}(D(s)) = \bar{r} + x$. *Let $\bar{\gamma}_1(s,t) \mid \cdots \mid \bar{\gamma}_{\bar{r}+x}(s,t)$ be its homogeneous invariant factors, $\bar{\mathbf{d}} = (\bar{d}_1, \dots, \bar{d}_{q-x})$ its column minimal indices, and $\bar{\mathbf{v}} = (\bar{v}_1, \dots, \bar{v}_{p+y})$ its row minimal indices, where $\bar{v}_1 \geq \cdots \geq \bar{v}_{\bar{\theta}} > \bar{v}_{\bar{\theta}+1} = \cdots = \bar{v}_{p+y} = 0$. There exists a pencil $A(s)$ such that $\begin{bmatrix} C(s) \\ A(s) \end{bmatrix} \overset{s.e.}{\sim} D(s)$ if and only if*

$$\bar{\gamma}_i(s,t) \mid \bar{\phi}_i(s,t) \mid \bar{\gamma}_{i+x+y}(s,t), \quad 1 \leq i \leq \bar{r}, \tag{8}$$

$$\bar{\theta} \geq \theta, \tag{9}$$

$$\bar{\mathbf{c}} \prec' (\bar{\mathbf{d}}, \bar{\mathbf{a}}), \tag{10}$$

$$\bar{\mathbf{v}} \prec' (\bar{\mathbf{u}}, \bar{\mathbf{b}}), \tag{11}$$

$$\sum_{i=1}^{\bar{r}+x} \deg(\operatorname{lcm}(\bar{\phi}_{i-x}, \bar{\gamma}_i)) \leq \sum_{i=1}^{p+y} \bar{v}_i - \sum_{i=1}^{p} \bar{u}_i + \sum_{i=1}^{\bar{r}+x} \deg(\bar{\gamma}_i), \tag{12}$$

where $\bar{\mathbf{a}} = (\bar{a}_1, \ldots, \bar{a}_x)$ *and* $\bar{\mathbf{b}} = (\bar{b}_1, \ldots, \bar{b}_y)$ *are*

$$\begin{array}{rl} \bar{a}_1 = & \sum_{i=1}^{p+y} \bar{v}_i - \sum_{i=1}^{p} \bar{u}_i + \sum_{i=1}^{\bar{r}+x} \deg(\bar{\gamma}_i) - \sum_{i=1}^{\bar{r}+x-1} \deg(\mathrm{lcm}(\bar{\phi}_{i-x+1}, \bar{\gamma}_i)) - 1, \\ \bar{a}_j = & \sum_{i=1}^{\bar{r}+x-j+1} \deg(\mathrm{lcm}(\bar{\phi}_{i-x+j-1}, \bar{\gamma}_i)) - \sum_{i=1}^{\bar{r}+x-j} \deg(\mathrm{lcm}(\bar{\phi}_{i-x+j}, \bar{\gamma}_i)) - 1, \\ & \hfill 2 \leq j \leq x, \end{array} \tag{13}$$

$$\begin{array}{rl} \bar{b}_1 = & \sum_{i=1}^{p+y} \bar{v}_i - \sum_{i=1}^{p} \bar{u}_i + \sum_{i=1}^{\bar{r}+x} \deg(\bar{\gamma}_i) - \sum_{i=1}^{\bar{r}+x} \deg(\mathrm{lcm}(\bar{\phi}_{i-x-1}, \bar{\gamma}_i)), \\ \bar{b}_j = & \sum_{i=1}^{\bar{r}+x} \deg(\mathrm{lcm}(\bar{\phi}_{i-x-j+1}, \bar{\gamma}_i)) - \sum_{i=1}^{\bar{r}+x} \deg(\mathrm{lcm}(\bar{\phi}_{i-x-j}, \bar{\gamma}_i)), \ 2 \leq j \leq y. \end{array} \tag{14}$$

Remark 2.7

1. Applying [17, Lemmas 1 and 2], in Theorem 2.6 we obtain that $\bar{a}_1 \geq \cdots \geq \bar{a}_x$ and $\bar{b}_1 \geq \cdots \geq \bar{b}_y \geq 0$.

2. If $C(s)$ has $\bar{\alpha}_1(s) \mid \cdots \mid \bar{\alpha}_{\bar{r}}(s)$ and $\bar{e}_1 \leq \cdots \leq \bar{e}_{\bar{r}}$ as invariant factors and partial multiplicities of ∞, respectively, and $\bar{\beta}_1(s) \mid \cdots \mid \bar{\beta}_{\bar{r}+x}(s)$ are the invariant factors of $D(s)$ and $\bar{f}_1 \leq \cdots \leq \bar{f}_{\bar{r}+x}$ are its partial multiplicities of ∞, then the condition (8) is equivalent to

$$\bar{\beta}_i(s) \mid \bar{\alpha}_i(s) \mid \bar{\beta}_{i+x+y}(s), \quad 1 \leq i \leq \bar{r},$$

$$\bar{f}_i \leq \bar{e}_i \leq \bar{f}_{i+x+y}, \quad 1 \leq i \leq \bar{r}.$$

2.3 Frobenius companion form

In this subsection we state that two polynomial matrices of the same degree have the same eigenstructure if and only if their first Frobenius companion forms are strictly equivalent.

Definition 2.8 *Let* $P(s) = P_d s^d + P_{d-1} s^{d-1} + \cdots + P_1 s + P_0 \in \mathbb{F}[s]^{m \times n}$, $P_d \neq 0$, $d \geq 1$. *The* first Frobenius companion form *of* $P(s)$ *is the* $(m + (d-1)n) \times dn$ *pencil* $C_P(s) = sX_1 + Y_1$ *with*

$$X_1 = \begin{bmatrix} P_d & & & \\ & I_n & & \\ & & \ddots & \\ & & & I_n \end{bmatrix} \quad \text{and} \quad Y_1 = \begin{bmatrix} P_{d-1} & P_{d-2} & \cdots & P_0 \\ -I_n & 0 & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & -I_n & 0 \end{bmatrix}.$$

Notice that when $d = 1$, $C_P(s) = P(s)$. The following lemma is a consequence of [10, Theorem 5.3 and Theorem 4.1].

Lemma 2.9 *Let* $P(s) \in \mathbb{F}[s]^{m \times n}$ *be a polynomial matrix of degree* $d \geq 1$ *and let* $C_P(s)$ *be its first Frobenius companion form. Then,*

1. *If* $\phi_1(s,t), \ldots, \phi_r(s,t)$ *are the homogeneous invariant factors of* $P(s)$ *then the homogeneous invariant factors of the pencil* $C_P(s)$ *are* $1, \overset{(d-1)n}{\ldots}, 1, \phi_1(s,t), \ldots, \phi_r(s,t)$.

2. *If* $c_1 \geq \cdots \geq c_{n-r}$ *are the column minimal indices of* $P(s)$ *then* $c_1 + d - 1 \geq \cdots \geq c_{n-r} + d - 1$ *are the column minimal indices of* $C_P(s)$.

3. *If* $u_1 \geq \cdots \geq u_{m-r}$ *are the row minimal indices of* $P(s)$ *then* $u_1 \geq \cdots \geq u_{m-r}$ *are the row minimal indices of* $C_P(s)$.

As an immediate consequence of Lemma 2.9 we obtain the next corollary.

Corollary 2.10 *Let* $P(s), \bar{P}(s) \in \mathbb{F}[s]^{m \times n}$ *be polynomial matrices satisfying that* $\deg(P(s)) = \deg(\bar{P}(s)) \geq 1$, *and let* $C_P(s), C_{\bar{P}}(s)$ *be their first Frobenius companion forms, respectively. Then,* $P(s) \approx \bar{P}(s)$ *if and only if* $C_P(s) \overset{s.e.}{\sim} C_{\bar{P}}(s)$.

3 Polynomial matrices with prescribed eigenstructure

The aim of this section is to show a characterization of the existence of a polynomial matrix of degree d with prescribed eigenstructure over arbitrary fields. For $d \geq 1$, the result was obtained in [9, Theorem 3.3] but in that paper the proof of the sufficiency is valid only when $\mathbb{F}$ is an infinite field. In Theorem 3.1 we remove this restriction. We also include the case $d = 0$.

For matrix pencils the sufficiency follows from the Kronecker canonical form. For $d > 1$ we use the first Frobenius companion form for transforming the problem into a row completion problem of the subpencil formed by the last $(d-1)n$ rows of this companion form (see the pencil $C(s)$ in (3.1)). With this idea in mind, by Theorem 2.6, the conditions of [9, Theorem 3.3] allow us to complete $C(s)$ up to a pencil with the desired eigenstructure.

Theorem 3.1 *Let m, n, $r \leq \min\{m,n\}$ be positive integers and d a non negative integer. Let $\gamma_1(s,t) \mid \cdots \mid \gamma_r(s,t)$ be monic homogeneous polynomials with coefficients in an arbitrary field $\mathbb{F}$. Let $(d_1,\dots,d_{n-r})$, $(v_1,\dots,v_{m-r})$ be partitions. Then there exists $P(s) \in \mathbb{F}[s]^{m\times n}$, $\operatorname{rank}(P(s)) = r$, $\deg(P(s)) = d$, with homogeneous invariant factors $\gamma_1(s,t),\dots,\gamma_r(s,t)$ and column and row minimal indices $d_1,\dots,d_{n-r}$ and $v_1,\dots,v_{m-r}$, respectively, if and only if (3) and (4) hold.*

Proof. The proof of the necessity in [9, Theorem 3.3] is valid for arbitrary fields. Observe that it also holds in the case that $d = 0$. Therefore, we only need to prove the sufficiency. Assume that (3) and (4) are satisfied.

For $d = 0$, from (4) we obtain $\gamma_1(s,t) = \cdots = \gamma_r(s,t) = 1$ and $d_1 = \cdots = d_{n-r} = v_1 = \cdots = v_{m-r} = 0$. Then, any matrix $P_0 \in \mathbb{F}^{m\times n}$ of rank r has the desired invariants.

For $d = 1$, using the Kronecker canonical form of matrix pencils, we can build a matrix polynomial $P(s) \in \mathbb{F}[s]^{m\times n}$ of degree 1 and rank r having $\gamma_1(s,t),\dots,\gamma_r(s,t)$ as homogeneous invariant factors and $d_1,\dots,d_{n-r}$ and $v_1,\dots,v_{m-r}$ as column and row minimal indices, respectively. Indeed, let $\alpha_i(s) = \gamma_i(s,1)$, $1 \leq i \leq r$, and let $e_1 \leq \cdots \leq e_r$ be defined by (2). Assume that $\alpha_1(s) = \cdots = \alpha_w(s) = 1 \neq \alpha_{w+1}(s)$ $(0 \leq w \leq r)$, $e_1 = \cdots = e_q = 0 < e_{q+1}$ $(1 \leq q \leq r)$, $d_1 \geq \cdots \geq d_\rho > 0$ $(0 \leq \rho \leq n-r)$, $v_1 \geq \cdots \geq v_\theta > 0$ $(0 \leq \theta \leq m-r)$. Notice that because of (3), $q \geq 1$. Let

$$P(s) = \begin{bmatrix} \operatorname{diag}(C(s), N(s), L(s), R(s)) & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{F}[s]^{m\times n},$$

where

$$C(s) = \operatorname{diag}(C_{\alpha_{w+1}}(s),\dots,C_{\alpha_r}(s)), \quad N(s) = \operatorname{diag}(N_{e_{q+1}}(s),\dots,N_{e_r}(s)),$$
$$L(s) = \operatorname{diag}(L_{d_1}(s),\dots,L_{d_\rho}(s)), \text{ and } R(s) = \operatorname{diag}(R_{v_1}(s),\dots,R_{v_\theta}(s)).$$

Here the block matrix $C_\alpha(s) \in \mathbb{F}[s]^{\deg(\alpha)\times\deg(\alpha)}$ is the first Frobenius companion form of $\alpha(s)$,

$$N_k(s) = \begin{bmatrix} 1 & s & & \\ & \ddots & \ddots & \\ & & \ddots & s \\ & & & 1 \end{bmatrix} \in \mathbb{F}[s]^{k\times k}, \quad L_k(s) = \begin{bmatrix} s & 1 & & \\ & \ddots & \ddots & \\ & & s & 1 \end{bmatrix} \in \mathbb{F}[s]^{k\times(k+1)},$$

and $R_k(s) = L_k(s)^T \in \mathbb{F}[s]^{(k+1)\times k}$. If $\deg(\alpha_r(s)) > 0$ or $d_1 > 0$ or $v_1 > 0$ then $\deg(P(s)) = 1$. Otherwise, by (3) and (4), $\sum_{i=2}^r e_i = r$. Thus, $e_r > 1$ and $\deg(P(s)) = 1$. Now, by (4), $\operatorname{rank}(P(s)) = r$, and $P(s)$ is the desired pencil.

Let $d > 1$ and

$$C(s) = \begin{bmatrix} -I_n & sI_n & 0 & \cdots & 0 & 0 \\ 0 & -I_n & sI_n & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & sI_n & 0 \\ 0 & 0 & 0 & \dots & -I_n & sI_n \end{bmatrix} \in \mathbb{F}[s]^{(d-1)n\times dn}. \tag{15}$$

Then, $\bar{r} = \operatorname{rank}(C(s)) = (d-1)n$, the homogeneous invariant factors of $C(s)$ are $\bar{\phi}_1(s,t) = \cdots = \bar{\phi}_{\bar{r}}(s,t) = 1$, and the columns of the matrix $\begin{bmatrix} s^{d-1}I_n \\ s^{d-2}I_n \\ \vdots \\ sI_n \\ I_n \end{bmatrix}$ form a right minimal basis for $C(s)$; hence, denoting the column minimal indices of $C(s)$ by $\bar{\mathbf{c}} = (\bar{c}_1, \dots, \bar{c}_n)$, we have $\bar{c}_i = d-1$, $1 \leq i \leq n$. As $C(s)$ does not have row minimal indices, we put $\bar{\mathbf{u}} = \emptyset$. We also take $\theta = 0$.

We introduce now a collection of homogeneous polynomials and two partitions of integers intended to be the Kronecker invariants of an $(m + (d-1)n) \times dn$ pencil. Define

$$\begin{aligned} \bar{\gamma}_i(s,t) = &\quad 1, \quad 1 \leq i \leq \bar{r}, \\ \bar{\gamma}_{\bar{r}+i}(s,t) = &\quad \gamma_i(s,t), \quad 1 \leq i \leq r, \end{aligned}$$

$$\bar{\mathbf{d}} = (\bar{d}_1, \dots, \bar{d}_{n-r}), \quad \bar{\mathbf{v}} = (\bar{v}_1, \dots, \bar{v}_{m-r}),$$

where

$$\bar{d}_i = d_i + d - 1, \quad 1 \leq i \leq n - r,$$

$$\bar{v}_i = v_i, \quad 1 \leq i \leq m - r.$$

Observe that $n - r = dn - (\bar{r} + r)$ and $m - r = m + (d-1)n - (\bar{r} + r)$, and taking into account (4),

$$\sum_{i=1}^{\bar{r}+r} \deg(\bar{\gamma}_i) + \sum_{i=1}^{n-r} \bar{d}_i + \sum_{i=1}^{m-r} \bar{v}_i = rd + (n-r)(d-1) = \bar{r} + r.$$

Applying the result for $d = 1$, there is a pencil $D(s) \in \mathbb{F}[s]^{(m+(d-1)n)\times dn}$, $\deg(D(s)) = 1$, $\operatorname{rank}(D(s)) = \bar{r} + r$, with homogeneous invariant factors $\bar{\gamma}_1(s,t), \dots, \bar{\gamma}_{\bar{r}+r}(s,t)$, column minimal indices $\bar{d}_1, \dots, \bar{d}_{n-r}$, and row minimal indices $\bar{v}_1, \dots, \bar{v}_{m-r}$.

Let us see that the invariants of the pencils $C(s)$ and $D(s)$ satisfy the conditions (8)-(12) of Theorem 2.6. Take $x = r$, $y = m - r$, and $\bar{\theta} = \#\{i \;:\; \bar{v}_i > 0\}$. Then (8) and (9) hold. Since $\sum_{i=1}^{\bar{r}+r} \deg(\operatorname{lcm}(\bar{\phi}_{i-r}, \bar{\gamma}_i)) = \sum_{i=1}^{\bar{r}+r} \deg(\bar{\gamma}_i)$ and $\sum_{i=1}^{m-r} \bar{v}_i \geq 0$, condition (12) holds.

Let $\bar{\mathbf{a}} = (\bar{a}_1, \dots, \bar{a}_r)$ and $\bar{\mathbf{b}} = (\bar{b}_1, \dots, \bar{b}_{m-r})$ be defined as in (13) and (14), respectively. Trivially $\bar{\mathbf{v}} \prec \bar{\mathbf{b}}$, which is equivalent to (11) (see Remark 2.5.2). Finally,

$$\bar{d}_i = d_i + d - 1 \geq d - 1 = \bar{c}_{i+r}, \quad 1 \leq i \leq n - r,$$

$$\begin{aligned} \textstyle\sum_{i=1}^{n-r} \bar{d}_i + \sum_{i=1}^{r} \bar{a}_i = &\quad \textstyle\sum_{i=1}^{n-r} d_i + (n-r)(d-1) + \sum_{i=1}^{m-r} v_i + \sum_{i=1}^{r} \deg(\gamma_i) - r \\ = &\quad \textstyle rd + (n-r)(d-1) - r = n(d-1) = \sum_{i=1}^{n} \bar{c}_i. \end{aligned}$$

Let $j \in \{1, \dots, r\}$. Let $h_j = \min\{i \;:\; \bar{d}_{i-j+1} < \bar{c}_i\}$. By Remark 2.5.1, $j \leq h_j \leq n - r + j$. For $j \leq i \leq (n-r) + j - 1$, we have

$$\bar{d}_{i-j+1} = d_{i-j+1} + d - 1 \geq d - 1 = \bar{c}_i;$$

hence, $h_j = n - r + j$ and

$$\begin{aligned} &\textstyle\sum_{i=1}^{j} \bar{a}_i + \sum_{i=1}^{h_j - j} \bar{d}_i - \sum_{i=1}^{h_j} \bar{c}_i = \sum_{i=1}^{m-r} v_i + \sum_{i=r-j+1}^{r} \deg(\gamma_i) - j \\ &\textstyle+ \sum_{i=1}^{n-r} d_i + (n-r)(d-1) - (n-r+j)(d-1) \\ &\textstyle= \sum_{i=1}^{r} \deg(\gamma_i) + \sum_{i=1}^{n-r} d_i + \sum_{i=1}^{m-r} v_i - \sum_{i=1}^{r-j} \deg(\gamma_i) - jd. \end{aligned}$$

From (4), we obtain $\sum_{i=1}^{j} \bar{a}_i + \sum_{i=1}^{h_j - j} \bar{d}_i - \sum_{i=1}^{h_j} \bar{c}_i = (r-j)d - \sum_{i=1}^{r-j} \deg(\gamma_i)$ and $\sum_{i=1}^{r} \deg(\gamma_i) \leq rd$. As $\deg(\gamma_1) \leq \cdots \leq \deg(\gamma_r)$, it follows that $\sum_{i=1}^{r-j} \deg(\gamma_i) \leq (r-j)d$. Therefore, $(r-j)d - \sum_{i=1}^{r-j} \deg(\gamma_i) \geq 0$, and condition (10) holds. By Theorem 2.6, there exists a pencil $W(s) \in$

$\mathbb{F}[s]^{m\times dn}$ such that $\begin{bmatrix} W(s)\\ C(s)\end{bmatrix}$ has $\bar\gamma_1(s,t),\dots,\bar\gamma_{\bar r+r}(s,t)$ as homogeneous invariant factors, the integers $\bar d_1,\dots,\bar d_{n-r}$ as column minimal indices, and $\bar v_1,\dots,\bar v_{m-r}$ as row minimal indices. Let

$$\begin{bmatrix} W(s)\\ C(s)\end{bmatrix} = \begin{bmatrix} W_{d-1}(s) & W_{d-2}(s) & W_{d-3}(s) & \cdots & W_1(s) & W_0(s)\\ -I_n & sI_n & 0 & \cdots & 0 & 0\\ 0 & -I_n & sI_n & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & sI_n & 0\\ 0 & 0 & 0 & \dots & -I_n & sI_n \end{bmatrix},$$

and let $W_i(s)=sW_{i,1}+W_{i,0}\in\mathbb{F}[s]^{m\times n}$, $0\le i\le d-1$. There exists $U\in \mathrm{Gl}_{m+(d-1)n}(\mathbb{F})$ such that

$$\begin{bmatrix} W(s)\\ C(s)\end{bmatrix} \overset{s.e.}{\sim} U\begin{bmatrix} W(s)\\ C(s)\end{bmatrix} = \begin{bmatrix} sP_d+P_{d-1} & P_{d-2} & P_{d-3} & \cdots & P_1 & P_0\\ -I_n & sI_n & 0 & \cdots & 0 & 0\\ 0 & -I_n & sI_n & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & sI_n & 0\\ 0 & 0 & 0 & \dots & -I_n & sI_n \end{bmatrix},$$

where $P_d=W_{d-1,1}$, $P_0=W_{0,0}$, and $P_i=W_{i,0}+W_{i-1,1}$, $1\le i\le d-1$.

Let us denote $K(s)=U\begin{bmatrix} W(s)\\ C(s)\end{bmatrix}$. From (3) we derive $\bar\gamma_{\bar r+1}(s,0)\neq 0$. Thus, $n_i(\infty,K(s))=n_i(0,\operatorname{rev}K(t))=0$, $1\le i\le \bar r+1$. Observe that

$$\operatorname{rev}K(t)=\begin{bmatrix} P_d & 0\\ 0 & I_{\bar r}\end{bmatrix} + t\begin{bmatrix} P_{d-1} & \cdots & P_1 & P_0\\ & -I_{\bar r} & & 0\end{bmatrix},$$

and $\operatorname{rev}K(0)=\begin{bmatrix} P_d & 0\\ 0 & I_{\bar r}\end{bmatrix}$. As $n_{\bar r+1}(0,\operatorname{rev}K(t))=0$, the $(\bar r+1)$-th invariant factor of $\operatorname{rev}K(t)$ is not a multiple of t, therefore, $\operatorname{rank}(\operatorname{rev}K(0))\ge \bar r+1$ and $P_d\neq 0$.

Let $P(s)=P_ds^d+P_{d-1}s^{d-1}+\cdots+P_1s+P_0\in\mathbb{F}[s]^{m\times n}$. By Lemma 2.9, $P(s)$ has homogeneous invariant factors $\gamma_1(s,t),\dots,\gamma_r(s,t)$, column minimal indices $d_1,\dots,d_{n-r}$, and row minimal indices $v_1,\dots,v_{m-r}$. □

4 Row (column) completion of polynomial matrices of given degree

The next proposition turns a row completion problem of polynomial matrices into a row completion problem of matrix pencils. As a solution to the latter is known, out of the solution of the problem for matrix pencils we are able to find a solution to the problem for polynomial matrices.

Proposition 4.1 *Let $P(s)\in\mathbb{F}[s]^{m\times n}$ and $Q(s)\in\mathbb{F}[s]^{(m+z)\times n}$ be polynomial matrices such that $\deg(P(s))=\deg(Q(s))=d\ge 1$, and let $C_P(s),C_Q(s)$ be their first Frobenius companion forms, respectively. Then, there exists $W(s)\in\mathbb{F}[s]^{z\times n}$ such that $\deg(W(s))\le d$ and $\begin{bmatrix} P(s)\\ W(s)\end{bmatrix}\approx Q(s)$ if and only if there exists a matrix pencil $A(s)\in\mathbb{F}[s]^{z\times dn}$ such that $\begin{bmatrix} C_P(s)\\ A(s)\end{bmatrix}\overset{s.e.}{\sim} C_Q(s)$.*

Proof. Let $P(s)=P_ds^d+P_{d-1}s^{d-1}+\cdots+P_1s+P_0$, $P_d\neq 0$. Assume that there exists $W(s)=W_ds^d+W_{d-1}s^{d-1}+\cdots+W_1s+W_0\in\mathbb{F}[s]^{z\times n}$, $\deg(W(s))\le d$, such that $\bar P(s)=\begin{bmatrix} P(s)\\ W(s)\end{bmatrix}\approx Q(s)$.

Observe that $\deg(\bar{P}(s)) = d$, and let

$$C_{\bar{P}}(s) = \begin{bmatrix} sP_d+P_{d-1} & P_{d-2} & \dots & P_1 & P_0 \\ sW_d+W_{d-1} & W_{d-2} & \dots & W_1 & W_0 \\ -I_n & sI_n & \ddots & 0 & 0 \\ 0 & -I_n & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -I_n & sI_n \end{bmatrix} \in \mathbb{F}[s]^{(m+z+(d-1)n)\times dn},$$

be the first Frobenius companion form of $\bar{P}(s)$. By Corollary 2.10, $C_{\bar{P}}(s) \overset{s.e.}{\sim} C_Q(s)$. Let $A(s) = \begin{bmatrix} sW_d + W_{d-1} & W_{d-2} & \dots & W_1 & W_0 \end{bmatrix} \in \mathbb{F}[s]^{z\times dn}$. Then, $C_Q(s) \overset{s.e.}{\sim} C_{\bar{P}}(s) \overset{s.e.}{\sim} \begin{bmatrix} C_P(s) \\ A(s) \end{bmatrix}$.

Conversely, assume that there exists a matrix pencil

$$A(s) = \begin{bmatrix} sA_{d-1,1} + A_{d-1,0} & sA_{d-2,1} + A_{d-2,0} & \dots & sA_{0,1} + A_{0,0} \end{bmatrix} \in \mathbb{F}[s]^{z\times dn},$$

such that $\begin{bmatrix} C_P(s) \\ A(s) \end{bmatrix} \overset{s.e.}{\sim} C_Q(s) \in \mathbb{F}[s]^{(m+(d-1)n+z)\times dn}$. Let

$$\hat{C}(s) = \begin{bmatrix} C_P(s) \\ A(s) \end{bmatrix} = \begin{bmatrix} sP_d+P_{d-1} & P_{d-2} & \dots & P_1 & P_0 \\ -I_n & sI_n & \ddots & 0 & 0 \\ 0 & -I_n & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -I_n & sI_n \\ sA_{d-1,1}+A_{d-1,0} & sA_{d-2,1}+A_{d-2,0} & \dots & sA_{1,1}+A_{1,0} & sA_{0,1}+A_{0,0} \end{bmatrix}.$$

There exists $U \in \mathrm{Gl}_{m+z+(d-1)n}(\mathbb{F})$ such that

$$U\hat{C}(s) = \begin{bmatrix} sP_d+P_{d-1} & P_{d-2} & \dots & P_1 & P_0 \\ sA_{d-1,1}+A'_{d-1} & A'_{d-2} & \dots & A'_1 & A'_0 \\ -I_n & sI_n & \ddots & 0 & 0 \\ 0 & -I_n & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -I_n & sI_n \end{bmatrix} \in \mathbb{F}[s]^{(m+z+(d-1)n)\times dn},$$

where $A'_i = A_{i,0} + A_{i-1,1}$, $0 \le i \le d-1$ ($A_{-1,1} = 0$). Let $W(s) = s^d A_{d-1,1} + s^{d-1}A'_{d-1} + \dots + sA'_1 + A'_0 \in \mathbb{F}[s]^{z\times n}$ and $\bar{P}(s) = \begin{bmatrix} P(s) \\ W(s) \end{bmatrix}$. Then $C_{\bar{P}}(s) = U\hat{C}(s) \overset{s.e.}{\sim} C_Q(s)$. By Corollary 2.10, $\bar{P}(s) \approx Q(s)$. $\square$

4.1 Prescription of the whole eigenstructure

The next theorem generalizes Theorem 2.6 to polynomial matrices. It contains a solution to Problem 1.1 when the whole eigenstructure is prescribed. In subsequent subsections different particular cases of the problem are solved when only some of the invariants are prescribed. Our target is to analyze all of the possible cases, but as there are many possibilities, we will present here only some of them. In a future paper we will accomplish the study of the remaining cases.

Theorem 4.2 *Let $P(s) \in \mathbb{F}[s]^{m\times n}$ be a polynomial matrix, $\deg(P(s)) = d$, $\mathrm{rank}(P(s)) = r$. Let $\phi_1(s,t) \mid \dots \mid \phi_r(s,t)$ be its homogeneous invariant factors, $\mathbf{c} = (c_1, \dots, c_{n-r})$ its column minimal indices, and $\mathbf{u} = (u_1, \dots, u_{m-r})$ its row minimal indices, where $u_1 \ge \dots \ge u_\eta > u_{\eta+1} = \dots = u_{m-r} = 0$.*

Let z and x be integers such that $0 \le x \le \min\{z, n-r\}$. Let $\gamma_1(s,t) \mid \dots \mid \gamma_{r+x}(s,t)$ be monic homogeneous polynomials, and $\mathbf{d} = (d_1, \dots, d_{n-r-x})$ and $\mathbf{v} = (v_1, \dots, v_{m+z-r-x})$ two partitions, *where* $v_1 \geq \cdots \geq v_{\bar\eta} > v_{\bar\eta+1} = \cdots = v_{m+z-r-x} = 0$. *There exists a polynomial matrix* $W(s) \in \mathbb{F}[s]^{z\times n}$ *such that* $\deg(W(s)) \leq d$, $\operatorname{rank}\left(\begin{bmatrix} P(s) \\ W(s)\end{bmatrix}\right) = r+x$, *and* $\begin{bmatrix} P(s) \\ W(s)\end{bmatrix}$ *has* $\gamma_1(s,t) \mid \cdots \mid \gamma_{r+x}(s,t)$ *as homogeneous invariant factors,* $d_1, \dots, d_{n-r-x}$ *as column minimal indices and* $v_1, \dots, v_{m+z-r-x}$ *as row minimal indices if and only if*

$$\gamma_i(s,t) \mid \phi_i(s,t) \mid \gamma_{i+z}(s,t), \quad 1 \leq i \leq r, \tag{16}$$

$$\bar\eta \geq \eta, \tag{17}$$

$$\mathbf{c} \prec' (\mathbf{d}, \mathbf{a}), \tag{18}$$

$$\mathbf{v} \prec' (\mathbf{u}, \mathbf{b}), \tag{19}$$

$$\sum_{i=1}^{r+x} \deg(\operatorname{lcm}(\phi_{i-x}, \gamma_i)) \leq \sum_{i=1}^{m+z-r-x} v_i - \sum_{i=1}^{m-r} u_i + \sum_{i=1}^{r+x} \deg(\gamma_i), \quad \textit{with equality when } x = 0, \tag{20}$$

where $\mathbf{a} = (a_1, \dots, a_x)$ *and* $\mathbf{b} = (b_1, \dots, b_{z-x})$ *are*

$$\begin{array}{rl} a_1 = & \sum_{i=1}^{m+z-r-x} v_i - \sum_{i=1}^{m-r} u_i \\ & + \sum_{i=1}^{r+x} \deg(\gamma_i) - \sum_{i=1}^{r+x-1} \deg(\operatorname{lcm}(\phi_{i-x+1}, \gamma_i)) - d, \\ a_j = & \sum_{i=1}^{r+x-j+1} \deg(\operatorname{lcm}(\phi_{i-x+j-1}, \gamma_i)) - \sum_{i=1}^{r+x-j} \deg(\operatorname{lcm}(\phi_{i-x+j}, \gamma_i)) - d, \\ & \hfill 2 \leq j \leq x, \end{array} \tag{21}$$

$$\begin{array}{rl} b_1 = & \sum_{i=1}^{m+z-r-x} v_i - \sum_{i=1}^{m-r} u_i + \sum_{i=1}^{r+x} \deg(\gamma_i) - \sum_{i=1}^{r+x} \deg(\operatorname{lcm}(\phi_{i-x-1}, \gamma_i)), \\ b_j = & \sum_{i=1}^{r+x} \deg(\operatorname{lcm}(\phi_{i-x-j+1}, \gamma_i) - \sum_{i=1}^{r+x} \deg(\operatorname{lcm}(\phi_{i-x-j}, \gamma_i)), \\ & \hfill 2 \leq j \leq z-x. \end{array} \tag{22}$$

Proof.

First notice that $\mathbf{a}$ and $\mathbf{b}$ are well defined (see Remark 2.7.1).

If $d = 0$, then $\phi_1(s,t) = \cdots = \phi_r(s,t) = 1$, $c_1 = \cdots = c_{n-r} = 0$, $u_1 = \cdots = u_{m-r} = 0$.

If $d \geq 1$, take $\bar r = (d-1)n + r$, $y = z - x$, $p = m - r = m + (d-1)n - \bar r$, $q = n - r = dn - \bar r$, and let $C_P(s)$ be the first Frobenius companion form of $P(s)$. Then $C_P(s) \in \mathbb{F}[s]^{(\bar r+p)\times(\bar r+q)}$ and $\operatorname{rank}(C_P(s)) = \bar r$. Let $\bar\phi_1(s,t) \mid \cdots \mid \bar\phi_{\bar r}(s,t)$, $\bar c_1 \geq \cdots \geq \bar c_q$ and $\bar u_1 \geq \cdots \geq \bar u_\theta > \bar u_{\theta+1} = \cdots = \bar u_p = 0$ be the homogeneous invariant factors, column minimal indices and row minimal indices of $C_P(s)$, respectively, and let $\bar{\mathbf{c}} = (\bar c_1, \dots, \bar c_q)$, and $\bar{\mathbf{u}} = (\bar u_1, \dots, \bar u_p)$. By Lemma 2.9,

$$\begin{array}{l} \bar\phi_i(s,t) = 1, \quad 1 \leq i \leq (d-1)n, \\ \bar\phi_{i+(d-1)n}(s,t) = \phi_i(s,t), \quad 1 \leq i \leq r, \\ \bar c_i = c_i + d - 1, \quad 1 \leq i \leq n - r = q, \\ \theta = \eta, \\ \bar u_i = u_i, \quad 1 \leq i \leq m - r = p. \end{array}$$

Assume that there exists $W(s) \in \mathbb{F}[s]^{z\times n}$ such that $\deg(W(s)) \leq d$, $\operatorname{rank}\left(\begin{bmatrix} P(s) \\ W(s)\end{bmatrix}\right) = r + x$, and $\begin{bmatrix} P(s) \\ W(s)\end{bmatrix}$ has $\gamma_1(s,t) \mid \cdots \mid \gamma_{r+x}(s,t)$ as homogeneous invariant factors, $d_1, \dots, d_{n-r-x}$ as column minimal indices and $v_1, \dots, v_{m+z-r-x}$ as row minimal indices. Put $Q(s) = \begin{bmatrix} P(s) \\ W(s)\end{bmatrix}$, $\mathbf{d} = (d_1, \dots, d_{q-x})$ and $\mathbf{v} = (v_1, \dots, v_{p+y})$.

If $d = 0$, then $v_1, \dots, v_{m+z-r-x} = 0$; hence $a_1 = \cdots = a_x = 0$, $b_1 = \cdots = b_{z-x} = 0$ and (16)-(20) trivially hold.

If $d \geq 1$, let $C_Q(s) \in \mathbb{F}[s]^{(\bar{r}+p+x+y)\times(\bar{r}+q)}$ be the first Frobenius companion form of $Q(s)$. Then $\operatorname{rank}(C_Q(s)) = \bar{r} + x$. By Proposition 4.1 there exists a matrix pencil $A(s) \in \mathbb{F}[s]^{z\times n}$ such that $\begin{bmatrix} C_P(s) \\ A(s) \end{bmatrix} \overset{s.e.}{\sim} C_Q(s)$.

Let $\bar{\gamma}_1(s,t) \mid \cdots \mid \bar{\gamma}_{\bar{r}+x}(s,t)$, $\bar{d}_1 \geq \cdots \geq \bar{d}_{q-x}$ and $\bar{v}_1 \geq \cdots \geq \bar{v}_{\bar{\theta}} > \bar{v}_{\bar{\theta}+1} = \cdots = \bar{v}_{p+y} = 0$ be the homogeneous invariant factors, column minimal indices and row minimal indices of $C_Q(s)$, respectively, and let $\bar{\mathbf{d}} = (\bar{d}_1, \dots, \bar{d}_{q-x})$ and $\bar{\mathbf{v}} = (\bar{v}_1, \dots, \bar{v}_{p+y})$. By Lemma 2.9,

$$\begin{array}{l} \bar{\gamma}_i(s,t) = 1, \quad 1 \leq i \leq (d-1)n, \\ \bar{\gamma}_{i+(d-1)n}(s,t) = \gamma_i(s,t), \quad 1 \leq i \leq r+x, \\ \bar{d}_i = d_i + d - 1, \quad 1 \leq i \leq n-r-x = q-x, \\ \bar{\theta} = \bar{\eta}, \\ \bar{v}_i = v_i, \quad 1 \leq i \leq m+z-r-x = p+y. \end{array}$$

By Theorem 2.6, (8)-(12) hold, where $\bar{\mathbf{a}} = (\bar{a}_1, \dots, \bar{a}_x)$ and $\bar{\mathbf{b}} = (\bar{b}_1, \dots, \bar{b}_y)$ are defined as in (13) and (14), respectively. It is easy to see that $\bar{a}_j = a_j + (d-1)$ for $1 \leq j \leq x$ and $\bar{b}_j = b_j$ for $1 \leq j \leq y$. Then (8)-(12) are equivalent to (16)-(20) (see Remark 2.5.3 for the equivalence between (10) and (18)).

Assume now that (16)-(20) hold. Notice that from (21), or from (20) for $x = 0$, we obtain

$$\sum_{i=1}^{x} a_i = \sum_{i=1}^{m+z-r-x} v_i - \sum_{i=1}^{m-r} u_i + \sum_{i=1}^{r+x} \deg(\gamma_i) - \sum_{i=1}^{r} \deg(\operatorname{lcm}(\phi_i, \gamma_i)) - xd.$$

From (16) we have $\sum_{i=1}^{r} \deg(\operatorname{lcm}(\phi_i, \gamma_i)) = \sum_{i=1}^{r} \deg(\phi_i)$ and, taking into account (18) we get

$$\sum_{i=1}^{r+x} \deg(\gamma_i) + \sum_{i=1}^{n-r-x} d_i + \sum_{i=1}^{m+z-r-x} v_i = \sum_{i=1}^{n-r} c_i + \sum_{i=1}^{m-r} u_i + \sum_{i=1}^{r} \deg(\phi_i) + xd = (r+x)d.$$

Moreover, because of $\phi_1(s,0) \neq 0$ and (16), we obtain $\gamma_1(s,0) \neq 0$. By Theorem 3.1, there exists $Q(s) \in \mathbb{F}[s]^{(m+z)\times n}$, $\operatorname{rank}(Q(s)) = r + x$, $\deg(Q(s)) = d$, with homogeneous invariant factors $\gamma_1(s,t), \dots, \gamma_{r+x}(s,t)$ and column and row minimal indices $d_1, \dots, d_{n-r-x}$ and $v_1, \dots, v_{m+z-r-x}$, respectively.

If $d = 0$, the $\gamma_1(s,t) = \cdots = \gamma_{r+x}(s,t) = 1$, $d_1 = \cdots = d_{n-r-x} = 0$ and $v_1, \dots, v_{m+z-r-x} = 0$; hence $a_1 = \cdots = a_x = 0$, $b_1 = \cdots = b_{z-x} = 0$. Choosing $W \in \mathbb{F}^{z\times n}$ such that $\operatorname{rank} \begin{bmatrix} P \\ W \end{bmatrix} = r + x$, the matrix $\begin{bmatrix} P \\ W \end{bmatrix}$ has the prescribed invariants.

If $d \geq 1$, let $C_Q(s) \in \mathbb{F}[s]^{(\bar{r}+p+x+y)\times(\bar{r}+q)}$ be the first Frobenius companion form of $Q(s)$. Then $\operatorname{rank}(C_Q(s)) = \bar{r} + x$. Let $\bar{\gamma}_1(s,t) \mid \cdots \mid \bar{\gamma}_{\bar{r}+x}(s,t)$, $\bar{d}_1 \geq \cdots \geq \bar{d}_{q-x}$ and $\bar{v}_1 \geq \cdots \geq \bar{v}_{\bar{\theta}} > \bar{v}_{\bar{\theta}+1} = \cdots = \bar{v}_{p+y} = 0$ be the homogeneous invariant factors, column minimal indices and row minimal indices of $C_Q(s)$, respectively, and let $\bar{\mathbf{d}} = (\bar{d}_1, \dots, \bar{d}_{q-x})$ and $\bar{\mathbf{v}} = (\bar{v}_1, \dots, \bar{v}_{p+y})$. As in the proof of the necessity, (16)-(20) are equivalent to (8)-(12). By Theorem 2.6, there exists a matrix pencil $A(s) \in \mathbb{F}[s]^{z\times dn}$ such that that $\begin{bmatrix} C_P(s) \\ A(s) \end{bmatrix} \overset{s.e.}{\sim} Q(s)$. By Proposition 4.1 there exists a polynomial matrix $W(s) \in \mathbb{F}[s]^{z\times n}$ such that $\deg(W(s)) \leq d$ and $\begin{bmatrix} P(s) \\ W(s) \end{bmatrix} \approx Q(s)$. □

Remark 4.3 In Theorem 4.2 some of the conditions are expressed in terms of the row minimal indices. Occasionally, in the paper it is necessary to express them in terms of the column minimal indices.

Assume that (16)-(20) hold. From (16), (18), and (20) for $x = 0$, we obtain

$$\sum_{i=1}^{m+z-r-x} v_i - \sum_{i=1}^{m-r} u_i + \sum_{i=1}^{r+x} \deg(\gamma_i) = \sum_{i=1}^{n-r} c_i - \sum_{i=1}^{n-r-x} d_i + \sum_{i=1}^{r} \deg(\phi_i) + xd,$$

therefore, from (20), and taking into account (19) for $x = z$, we get

$$\sum_{i=1}^{r+x} \deg(\operatorname{lcm}(\phi_{i-x}, \gamma_i)) \leq \sum_{i=1}^{n-r} c_i - \sum_{i=1}^{n-r-x} d_i + \sum_{i=1}^{r} \deg(\phi_i) + xd, \quad \text{with equality when } x = z. \tag{23}$$

Moreover, (18) and (19) hold for $\mathbf{a} = (a_1, \dots, a_x)$ and $\mathbf{b} = (b_1, \dots, b_{z-x})$ defined as

$$\begin{array}{rl} a_1 = & \sum_{i=1}^{n-r} c_i - \sum_{i=1}^{n-r-x} d_i \\ & + \sum_{i=1}^{r} \deg(\phi_i) - \sum_{i=1}^{r+x-1} \deg(\operatorname{lcm}(\phi_{i-x+1}, \gamma_i)) + (x-1)d, \\ a_j = & \sum_{i=1}^{r+x-j+1} \deg(\operatorname{lcm}(\phi_{i-x+j-1}, \gamma_i)) - \sum_{i=1}^{r+x-j} \deg(\operatorname{lcm}(\phi_{i-x+j}, \gamma_i)) - d, \\ & \hfill 2 \leq j \leq x, \end{array} \tag{24}$$

$$\begin{array}{rl} b_1 = & \sum_{i=1}^{n-r} c_i - \sum_{i=1}^{n-r-x} d_i + \sum_{i=1}^{r} \deg(\phi_i) - \sum_{i=1}^{r+x} \deg(\operatorname{lcm}(\phi_{i-x-1}, \gamma_i)) + xd, \\ b_j = & \sum_{i=1}^{r+x} \deg(\operatorname{lcm}(\phi_{i-x-j+1}, \gamma_i) - \sum_{i=1}^{r+x} \deg(\operatorname{lcm}(\phi_{i-x-j}, \gamma_i)), \\ & \hfill 2 \leq j \leq z-x. \end{array} \tag{25}$$

Conversely, (16)-(19) and (23) with $\mathbf{a}$ and $\mathbf{b}$ defined as in (24) and (25), respectively, imply (16)-(20), with $\mathbf{a}$ and $\mathbf{b}$ defined as in (21) and (22).

4.2 Prescription of finite and infinite structures and column minimal indices

In this subsection we solve Problem 1.1 when the invariants to be achieved are the invariant factors, the infinite elementary divisors and the column minimal indices (equivalently, the homogeneous invariant factors and the column minimal indices). Previously, we need a technical lemma.

Given two sequences of integers $\mathbf{u} = (u_1, \dots, u_p)$ and $\mathbf{b} = (b_1, \dots, b_y)$ the union, $\mathbf{u} \cup \mathbf{b}$, is the decreasingly ordered sequence of the $p + y$ integers of $\mathbf{u}$ and $\mathbf{b}$.

Lemma 4.4 *Let $\mathbf{u} = (u_1, \dots, u_p)$ and $\mathbf{b} = (b_1, \dots, b_y)$ be sequences of integers. Then*

$$\mathbf{u} \cup \mathbf{b} \prec' (\mathbf{u}, \mathbf{b}).$$

Proof. Let $\mathbf{v} = \mathbf{u} \cup \mathbf{b}$ and let $k_j = \min\{i \ : \ b_j > u_i\}$, $1 \leq j \leq y$. We have $1 \leq k_1 \leq k_2 \leq \cdots \leq k_y \leq p + 1$, and (recall that if a condition is stated for $a \leq i \leq b$ with $a > b$, we understand that the condition disappears)

$$\begin{array}{lll} v_i = u_i \geq b_1, & 1 \leq i \leq k_1 - 1, & v_{k_1} = b_1 > u_{k_1}, \\ v_i = u_{i-1} \geq b_2, & k_1 + 1 \leq i \leq k_2, & v_{k_2+1} = b_2 > u_{k_2}, \\ \vdots & \vdots & \vdots \\ v_i = u_{i-y+1} \geq b_y, & k_{y-1} + y - 1 \leq i \leq k_y + y - 2, & v_{k_y+y-1} = b_y > u_{k_y}, \\ v_i = u_{i-y} \geq b_{y+1} = -\infty, & k_y + y \leq i \leq p + y. & \end{array}$$

In summary, putting $k_0 = 1$ and $k_{y+1} = p + 1$,

$$\begin{array}{lll} v_i = u_{i-j+1} \geq b_j, & k_{j-1} + j - 1 \leq i \leq k_j + j - 2, & 1 \leq j \leq y + 1, \\ v_{k_j+j-1} = b_j > u_{k_j}, & 1 \leq j \leq y. & \end{array}$$

It is clear that $u_i \geq v_{i+y}$, $1 \leq i \leq p$.

For $1 \leq j \leq y$, let $h_j = \min\{i \ : \ u_{i-j+1} < v_i\}$. Let us see that $h_j = k_j + j - 1$. For $i = k_j + j - 1$ we obtain $u_{i-j+1} = u_{k_j} < b_j = v_{k_j+j-1}$; hence $h_j \leq k_j + j - 1$. Moreover, if $i \leq k_j + j - 2$, then

$i-j+1 \leq k_j - 1$. Let $\ell = \min\{r \; : \; i-j+1 \leq k_r - 1\}$. Then $\ell \leq j$, $k_{\ell-1} \leq i-j+1 \leq k_\ell - 1$, and $u_{i-j+1} = v_{i+\ell-j} \geq v_i$; hence, $h_j = k_j + j - 1$. Furthermore, for $1 \leq j \leq y$,

$$\begin{array}{rl}\sum_{i=1}^{h_j} v_i = & \sum_{i=1}^{k_j+j-1} v_i = \sum_{r=1}^{j} \sum_{i=k_{r-1}+r-1}^{k_r+r-1} v_i \\ = & \sum_{r=1}^{j} b_r + \sum_{r=1}^{j} \sum_{i=k_{r-1}+r-1}^{k_r+r-2} u_{i-r+1} = \sum_{r=1}^{j} b_r + \sum_{r=1}^{j} \sum_{i=k_{r-1}}^{k_r-1} u_i \\ = & \sum_{i=1}^{j} b_i + \sum_{i=1}^{k_j-1} u_i = \sum_{i=1}^{j} b_i + \sum_{i=1}^{h_j-j} u_i.\end{array}$$

Finally, notice that $\sum_{i=1}^{p+y} v_i = \sum_{i=1}^{p} u_i + \sum_{i=1}^{y} b_i$. □

Theorem 4.5 *Let $P(s) \in \mathbb{F}[s]^{m\times n}$ be a polynomial matrix, $\deg(P(s)) = d$, $\operatorname{rank}(P(s)) = r$. Let $\phi_1(s,t) \mid \cdots \mid \phi_r(s,t)$ be its homogeneous invariant factors and $\mathbf{c} = (c_1, \ldots, c_{n-r})$ its column minimal indices.*

Let z and x be integers such that $0 \leq x \leq \min\{z, n-r\}$ and let $\gamma_1(s,t) \mid \cdots \mid \gamma_{r+x}(s,t)$ be monic homogeneous polynomials and $d_1 \geq \cdots \geq d_{n-r-x}$ non negative integers. There exists a polynomial matrix $W(s) \in \mathbb{F}[s]^{z\times n}$ such that $\deg(W(s)) \leq d$, $\operatorname{rank}\left(\begin{bmatrix} P(s) \\ W(s) \end{bmatrix}\right) = r+x$, and $\begin{bmatrix} P(s) \\ W(s) \end{bmatrix}$ has $\gamma_1(s,t) \mid \cdots \mid \gamma_{r+x}(s,t)$ as homogeneous invariant factors and $\mathbf{d} = (d_1, \ldots, d_{n-r-x})$ as column minimal indices if and only if (16), (18) and (23) hold, where $\mathbf{a} = (a_1, \ldots, a_x)$ is defined as in (24).

Proof. The necessity follows directly from Theorem 4.2 (see Remark 4.3). For the converse, let us assume that (16), (18) and (23) hold. Let $\mathbf{u} = (u_1, \ldots, u_{m-r})$ be the row minimal indices of $P(s)$, where $u_1 \geq \cdots \geq u_\eta > u_{\eta+1} = \cdots = u_{m-r} = 0$. Define $\mathbf{b} = (b_1, \ldots, b_{z-x})$ as in (25) and $\mathbf{v} = (v_1, \ldots, v_{m+z-r-x}) = \mathbf{u} \cup \mathbf{b}$. Then, by Lemma 4.4, condition (19) holds.

Denoting $\bar\eta = \#\{i \; : \; v_i > 0\}$, clearly (17) holds. By Theorem 4.2 and Remark 4.3, the result follows. □

4.3 Prescription of finite and infinite structures and row minimal indices

Now we solve Problem 1.1 when the prescribed invariants are the homogeneous invariant factors and the row minimal indices. We use the following technical lemma.

Lemma 4.6 *Let $\mathbf{c} = (c_1, \ldots, c_q)$ and $\mathbf{a} = (a_1, \ldots, a_x)$ be sequences of integers such that $q > x \geq 0$. Let $\ell = \min\{j \; : \; \sum_{i=1}^{j} c_i > \sum_{i=1}^{j} a_i\}$.*

1. *If there exists a sequence of integers $\mathbf{d} = (d_1, \ldots, d_{q-x})$ such that $\mathbf{c} \prec' (\mathbf{d}, \mathbf{a})$ then*

$$\sum_{i=1}^{x+1} c_i - c_\ell \geq \sum_{i=1}^{x} a_i, \tag{26}$$

$$\sum_{i=j+2}^{x+1} c_i \geq \sum_{i=j+1}^{x} a_i, \quad \ell \leq j \leq x-1. \tag{27}$$

2. *If (26) and (27) hold, let*

$$d_1 = \sum_{i=1}^{x+1} c_i - \sum_{i=1}^{x} a_i, \quad d_i = c_{i+x}, \quad 2 \leq i \leq q-x. \tag{28}$$

Then $d_1 \geq \cdots \geq d_{q-x}$ and $\mathbf{c} \prec' (\mathbf{d}, \mathbf{a})$, where $\mathbf{d} = (d_1, \ldots, d_{q-x})$.

Remark 4.7 Bearing in mind that $q \geq x+1$ and $a_{x+1} = -\infty$, we have $\sum_{i=1}^{x+1} c_i > \sum_{i=1}^{x+1} a_i$; hence, ℓ is well defined, $1 \leq \ell \leq x+1$, and $c_\ell \geq c_{x+1}$. If condition (26) is satisfied, then $\sum_{i=1}^{x} c_i \geq \sum_{i=1}^{x} a_i$.

Moreover, if $\ell = x$ or $\ell = x+1$, then (27) vanishes, therefore it is trivially fulfilled.

Proof.[Proof of Lemma 4.6]

1. Let us assume that there exists a sequence of integers $\mathbf{d} = (d_1, \ldots, d_{q-x})$ such that $\mathbf{c} \prec' (\mathbf{d}, \mathbf{a})$. It implies $\sum_{i=1}^{q} c_i = \sum_{i=1}^{q-x} d_i + \sum_{i=1}^{x} a_i$ and $d_i \geq c_{i+x}$, $1 \leq i \leq q-x$; hence

$$\sum_{i=1}^{j} d_i = \sum_{i=1}^{q} c_i - \sum_{i=1}^{x} a_i - \sum_{i=j+1}^{q-x} d_i \leq \sum_{i=1}^{x+j} c_i - \sum_{i=1}^{x} a_i, \quad 0 \leq j \leq q-x. \tag{29}$$

If $\ell = x+1$, then (27) vanishes and, from (29) for $j = 0$, $\sum_{i=1}^{x+1} c_i - c_\ell = \sum_{i=1}^{x} c_i \geq \sum_{i=1}^{x} a_i$; i.e. (26) holds.

If $\ell \leq x$, let $h_j = \min\{i \ : \ d_{i-j+1} < c_i\}$, $1 \leq j \leq x$. We know that $\sum_{i=1}^{h_j} c_i \leq \sum_{i=1}^{h_j-j} d_i + \sum_{i=1}^{j} a_i$, and $j \leq h_j \leq q-x+j$, $1 \leq j \leq x$ (see Remark 2.5.1). As $\sum_{i=1}^{\ell} c_i > \sum_{i=1}^{\ell} a_i$, it follows that $\ell < h_\ell$; hence $d_1 \geq c_\ell$. From (29), $c_\ell \leq d_1 \leq \sum_{i=1}^{x+1} c_i - \sum_{i=1}^{x} a_i$; i.e., (26) also holds. Moreover, for $\ell \leq j \leq x-1$, $d_1 \geq c_\ell \geq c_j$; therefore $h_j > j$, and from (29)

$$\sum_{i=1}^{h_j} c_i \leq \sum_{i=1}^{h_j-j} d_i + \sum_{i=1}^{j} a_i \leq \sum_{i=1}^{x+h_j-j} c_i - \sum_{i=1}^{x} a_i + \sum_{i=1}^{j} a_i = \sum_{i=1}^{x+h_j-j} c_i - \sum_{i=j+1}^{x} a_i;$$

hence $\sum_{i=1}^{x-j} c_{h_j+i} = \sum_{i=h_j+1}^{h_j+x-j} c_i \geq \sum_{i=j+1}^{x} a_i$. As $h_j \geq j+1$, we have $c_{h_j+i} \leq c_{j+i+1}$, $1 \leq i \leq x-j$. Thus, $\sum_{i=1}^{x-j} c_{h_j+i} \leq \sum_{i=1}^{x-j} c_{j+i+1} = \sum_{i=j+2}^{x+1} c_i$, from where we obtain (27).

2. Let us assume that (26) and (27) hold and let $d_1, \ldots, d_{q-x}$ be defined as in (28). From (26), $d_1 \geq c_\ell \geq c_{1+x}$; thus, $d_1 \geq \cdots \geq d_{q-x}$ and $d_i \geq c_{i+x}$, $1 \leq i \leq q-x$. Moreover, $\sum_{i=1}^{q-x} d_i = \sum_{i=1}^{q} c_i - \sum_{i=1}^{x} a_i$. Let $h_j = \min\{i \ : \ d_{i-j+1} < c_i\}$, $1 \leq j \leq x$. It only remains to prove that, for $1 \leq j \leq x$,

$$\sum_{i=1}^{h_j} c_i \leq \sum_{i=1}^{h_j-j} d_i + \sum_{i=1}^{j} a_i. \tag{30}$$

We have $d_{i-j+1} \geq c_i$, $j \leq i \leq h_j - 1$ and, since $h_j \geq j$, it follows that $c_{h_j} \leq c_j$.

Let $j \in \{1, \ldots, \ell-1\}$. By the definition of ℓ, $\sum_{i=1}^{j} c_i \leq \sum_{i=1}^{j} a_i$, and

$$\sum_{i=1}^{h_j} c_i = c_{h_j} + \sum_{i=1}^{j-1} c_i + \sum_{i=j}^{h_j-1} c_i \leq \sum_{i=1}^{j} c_i + \sum_{i=j}^{h_j-1} d_{i-j+1} \leq \sum_{i=1}^{j} a_i + \sum_{i=1}^{h_j-j} d_i;$$

i.e., (30) holds for $1 \leq j \leq \ell - 1$.

Let $j \in \{\ell, \ldots, x\}$. Since $d_1 \geq c_\ell \geq c_j$, we have $h_j > j$. Let $h_j - j = k$. Then $1 \leq k \leq q-x$ and

$$\begin{aligned} & \textstyle\sum_{i=1}^{h_j-j} d_i + \sum_{i=1}^{j} a_i - \sum_{i=1}^{h_j} c_i \\ = \ & \textstyle\sum_{i=1}^{x+1} c_i - \sum_{i=1}^{x} a_i + \sum_{i=2}^{h_j-j} c_{i+x} + \sum_{i=1}^{j} a_i - \sum_{i=1}^{h_j} c_i \\ = \ & \textstyle\sum_{i=1}^{k+x} c_i - \sum_{i=1}^{k+j} c_i - \sum_{i=j+1}^{x} a_i = \sum_{i=k+j+1}^{k+x} c_i - \sum_{i=j+1}^{x} a_i. \end{aligned}$$

If $k = 1$ then, from (27), we obtain $\sum_{i=k+j+1}^{k+x} c_i - \sum_{i=j+1}^{x} a_i \geq 0$; i.e. (30) holds. If $k > 1$, then by the definition of h_j, $d_{i+1} \geq c_{j+i}$ for $0 \leq i \leq k-1$. Then $c_{x+i+1} \geq c_{j+i}$ for $1 \leq i \leq k-1$; hence $c_{j+i} = c_{j+i+1} = \cdots = c_{x+i+1}$ for $1 \leq i \leq k-1$. For $2 \leq i \leq k-1$, we have $j+i-1 \leq j+i \leq x+i$, which means that these sequences overlap, therefore

$$c_{j+1} = c_{j+2} = \cdots = c_{j+k-1} = \cdots = c_{x+k}.$$

As a consequence, $\sum_{i=j+2}^{x+1} c_i = (x-j)c_{x+k} = \sum_{i=k+j+1}^{k+x} c_i$; hence

$$\sum_{i=1}^{h_j-j} d_i + \sum_{i=1}^{j} a_i - \sum_{i=1}^{h_j} c_i = \sum_{i=j+2}^{x+1} c_i - \sum_{i=j+1}^{x} a_i.$$

From (27) we derive (30).

□

Theorem 4.8 *Let $P(s) \in \mathbb{F}[s]^{m\times n}$ be a polynomial matrix,* $\deg(P(s)) = d$, $\operatorname{rank}(P(s)) = r$. *Let $\phi_1(s,t) \mid \cdots \mid \phi_r(s,t)$ be its homogeneous invariant factors, $\mathbf{c} = (c_1, \ldots, c_{n-r})$ its column minimal indices, and $\mathbf{u} = (u_1, \ldots, u_{m-r})$ its row minimal indices, where $u_1 \geq \cdots \geq u_\eta > u_{\eta+1} = \cdots = u_{m-r} = 0$.*

Let z and x be integers such that $0 \leq x \leq \min\{z, n-r\}$ and let $\gamma_1(s,t) \mid \cdots \mid \gamma_{r+x}(s,t)$ be monic homogeneous polynomials and $v_1 \geq \cdots \geq v_{\bar\eta} > v_{\bar\eta+1} = \cdots = v_{m+z-r-x} = 0$ be non negative integers. Let $\mathbf{a} = (a_1, \ldots, a_x)$ and $\mathbf{b} = (b_1, \ldots, b_{z-x})$ be as in (21) and (22), respectively.

1. *If $x = n-r$, there exists a polynomial matrix $W(s) \in \mathbb{F}[s]^{z\times n}$ such that* $\deg(W(s)) \leq d$, $\operatorname{rank}\left(\begin{bmatrix} P(s) \\ W(s) \end{bmatrix}\right) = r+x$, *and $\begin{bmatrix} P(s) \\ W(s) \end{bmatrix}$ has $\gamma_1(s,t) \mid \cdots \mid \gamma_{r+x}(s,t)$ as homogeneous invariant factors and $\mathbf{v} = (v_1, \ldots, v_{m+z-r-x})$ as row minimal indices if and only if (16), (17), (19), (20) and*
$$\mathbf{c} \prec \mathbf{a}. \tag{31}$$

2. *If $x < n-r$, there exists a polynomial matrix $W(s) \in \mathbb{F}[s]^{z\times n}$ such that* $\deg(W(s)) \leq d$, $\operatorname{rank}\left(\begin{bmatrix} P(s) \\ W(s) \end{bmatrix}\right) = r+x$, *and $\begin{bmatrix} P(s) \\ W(s) \end{bmatrix}$ has $\gamma_1(s,t) \mid \cdots \mid \gamma_{r+x}(s,t)$ as homogeneous invariant factors and $\mathbf{v} = (v_1, \ldots, v_{m+z-r-x})$ as row minimal indices if and only if (16), (17), (19), (20), (26) and (27) hold.*

Proof.

1. Assume that $x = n-r$. The result follows from Theorem 4.2 taking into account that if $\mathbf{d} = \emptyset$ then (18) and (31) are equivalent (see Remark 2.5.2).

2. Assume that $x < n-r$. The necessity follows directly from Theorem 4.2 and Lemma 4.6.1. Conversely, let us assume that (16), (17), 19), (20), (26) and (27) hold. From (26) and (27), by Lemma 4.6.2 there exists a sequence of integers $\mathbf{d} = (d_1, \ldots, d_{n-r-x})$ satisfying (18). We have $d_{n-r-x} \geq c_{n-r} \geq 0$. By Theorem 4.2, the result follows.

□

4.4 Prescription of finite and/or infinite structures

This subsection is devoted to solve Problem 1.1 when the invariant factors and/or the infinite elementary divisors are prescribed. In Theorem 4.10, both the finite and infinite structures are prescribed, in Theorem 4.12 we only prescribe the finite structure, and in Theorem 4.13 we only prescribe the infinite structure. We also start with a technical lemma.

Lemma 4.9 *Let $\mathbf{c} = (c_1, \ldots, c_q)$, $\mathbf{d} = (d_1, \ldots, d_{q-x})$ and $\mathbf{a} = (a_1, \ldots, a_x)$ be sequences of integers. If $\mathbf{c} \prec' (\mathbf{d}, \mathbf{a})$ then $(c_1, \ldots, c_x) \prec (a_1 + K, a_2, \ldots, a_x)$, where $K = \sum_{i=1}^{q-x}(d_i - c_{i+x})$.*

Proof. For $1 \le j \le x$, let $h_j = \min\{i \ : \ d_{i-j+1} < c_i\}$. We know that $j \le h_j \le q - x + j$ (see Remark 2.5.1) and $\sum_{i=1}^{h_j} c_i \le \sum_{i=1}^{h_j - j} d_i + \sum_{i=1}^{j} a_i$; hence

$$\begin{array}{rl}\sum_{i=1}^{j} c_i \le & \sum_{i=1}^{j} a_i + \sum_{i=1}^{h_j-j} d_i - \sum_{i=j+1}^{h_j} c_i = \sum_{i=1}^{j} a_i + \sum_{i=1}^{h_j-j} d_i - \sum_{i=1}^{h_j-j} c_{i+j} \\ \le & \sum_{i=1}^{j} a_i + \sum_{i=1}^{h_j-j} d_i - \sum_{i=1}^{h_j-j} c_{i+x}, \quad 1 \le j \le x. \end{array}$$

Moreover, as $d_i \ge c_{i+x}$ for $1 \le i \le q-x$, we conclude that $\sum_{i=1}^{h_j-j}(d_i - c_{i+x}) \le \sum_{i=1}^{q-x}(d_i - c_{i+x}) = K$. Therefore, we obtain

$$\sum_{i=1}^{j} c_i \le \sum_{i=1}^{j} a_i + K, \quad 1 \le j \le x.$$

Finally, notice that

$$\sum_{i=1}^{x} c_i = \sum_{i=1}^{x} a_i + \sum_{i=1}^{q-x} d_i - \sum_{i=1}^{q-x} c_{i+x} = \sum_{i=1}^{x} a_i + K,$$

is also satisfied. $\square$

Theorem 4.10 *Let $P(s) \in \mathbb{F}[s]^{m\times n}$, $\deg(P(s)) = d$, $\operatorname{rank}(P(s)) = r$. Let $\phi_1(s,t) \mid \cdots \mid \phi_r(s,t)$ be its homogeneous invariant factors, $\mathbf{c} = (c_1, \ldots, c_{n-r})$ its column minimal indices, and $\mathbf{u} = (u_1, \ldots, u_{m-r})$ its row minimal indices, where $u_1 \ge \cdots \ge u_\eta > u_{\eta+1} = \cdots = u_{m-r} = 0$.*

Let z and x be integers such that $0 \le x \le \min\{z, n-r\}$ and let $\gamma_1(s,t) \mid \cdots \mid \gamma_{r+x}(s,t)$ be monic homogeneous polynomials.

1. *If $x < z$ or $x = z = n-r$, then there exists a polynomial matrix $W(s) \in \mathbb{F}[s]^{z\times n}$ such that $\deg(W(s)) \le d$, $\operatorname{rank}\left(\begin{bmatrix} P(s) \\ W(s)\end{bmatrix}\right) = r + x$ and $\begin{bmatrix} P(s) \\ W(s)\end{bmatrix}$ has $\gamma_1(s,t) \mid \cdots \mid \gamma_{r+x}(s,t)$ as homogeneous invariant factors if and only if (16) and*

$$\begin{array}{l}\sum_{i=1}^{r+x-j} \deg(\operatorname{lcm}(\phi_{i-x+j}, \gamma_i)) + \sum_{i=1}^{m-r} u_i + \sum_{i=1}^{j} c_i + \sum_{i=x+1}^{n-r} c_i \\ \le (r+x-j)d, \quad 0 \le j \le x-1, \\ \qquad\qquad\qquad \textit{with equality for } j = 0 \textit{ when } x = z = n-r. \end{array} \tag{32}$$

2. *If $x = z < n-r$, then there exists a polynomial matrix $W(s) \in \mathbb{F}[s]^{z\times n}$ such that $\deg(W(s)) \le d$, $\operatorname{rank}\left(\begin{bmatrix} P(s) \\ W(s)\end{bmatrix}\right) = r+x$ and $\begin{bmatrix} P(s) \\ W(s)\end{bmatrix}$ has $\gamma_1(s,t) \mid \cdots \mid \gamma_{r+x}(s,t)$ as homogeneous invariant factors if and only if (16),*

$$\sum_{i=1}^{x+1} c_i - c_\ell \ge \sum_{i=1}^{r+x} \deg(\gamma_i) - \sum_{i=1}^{r} \deg(\phi_i) - xd, \tag{33}$$

$$\begin{array}{r}\sum_{i=j+2}^{x+1} c_i \ge \sum_{i=1}^{r+x-j} \deg(\operatorname{lcm}(\phi_{i-x+j}, \gamma_i)) - \sum_{i=1}^{r} \deg(\phi_i) - (x-j)d, \\ \ell \le j \le x-1, \end{array} \tag{34}$$

where $\ell = \min\{j \ : \ \sum_{i=1}^{j} c_i > \sum_{i=1}^{r+x} \deg(\gamma_i) - \sum_{i=1}^{r+x-j} \deg(\operatorname{lcm}(\phi_{i-x+j}, \gamma_i)) - jd\}$.

Proof.

1. Case $x < z$ or $x = z = n-r$. Assume that there exists $W(s) \in \mathbb{F}[s]^{z\times n}$ such that $\deg(W(s)) \le d$, $\operatorname{rank}\left(\begin{bmatrix} P(s) \\ W(s)\end{bmatrix}\right) = r+x$ and $\begin{bmatrix} P(s) \\ W(s)\end{bmatrix}$ has $\gamma_1(s,t) \mid \cdots \mid \gamma_{r+x}(s,t)$ as homogeneous invariant factors. Let $\mathbf{d} = (d_1, \ldots, d_{n-r-x})$ be the column minimal indices of $\begin{bmatrix} P(s) \\ W(s)\end{bmatrix}$. By Theorem

4.5, (16), (18) and (23) hold, where $\mathbf{a} = (a_1, \dots, a_x)$ is defined as in (24). From (23) we obtain

$$\sum_{i=1}^{r+x} \deg(\operatorname{lcm}(\phi_{i-x}, \gamma_i)) + \sum_{i=1}^{m-r} u_i + \sum_{i=1}^{n-r-x} d_i \le (r+x)d,$$

with equality if $x = z$. From (18) we get $\sum_{i=1}^{n-r-x} d_i \ge \sum_{i=1}^{n-r-x} c_{i+x} = \sum_{i=x+1}^{n-r} c_i$. Therefore, (32) holds for $j = 0$.

For $1 \le j \le x-1$, from (18), Lemma 4.9, and the definition of a_j, we obtain

$$\begin{aligned}
\textstyle\sum_{i=1}^{j} c_i \le & \textstyle\sum_{i=1}^{j} a_i + \sum_{i=1}^{n-r-x} d_i - \sum_{i=x+1}^{n-r} c_i \\
= & \textstyle\sum_{i=1}^{n-r} c_i - \sum_{i=1}^{n-r-x} d_i + \sum_{i=1}^{r} \deg(\phi_i) + xd \\
& \textstyle - \sum_{i=1}^{r+x-j} \deg(\operatorname{lcm}(\phi_{i-x+j}, \gamma_i)) - jd + \sum_{i=1}^{n-r-x} d_i \\
& \textstyle - \sum_{i=x+1}^{n-r} c_i \\
= & \textstyle (r+x-j)d - \sum_{i=1}^{m-r} u_i - \sum_{i=1}^{r+x-j} \deg(\operatorname{lcm}(\phi_{i-x+j}, \gamma_i)) \\
& \textstyle - \sum_{i=x+1}^{n-r} c_i.
\end{aligned}$$

Thus, (32) holds.

Conversely, assume that (16) and (32) hold. Define

$$\begin{aligned}
\hat{a}_1 = & \textstyle\sum_{i=1}^{x} c_i + \sum_{i=1}^{r} \deg(\phi_i) - \sum_{i=1}^{r+x-1} \deg(\operatorname{lcm}(\phi_{i-x+1}, \gamma_i)) + (x-1)d, \\
a_j = & \textstyle\sum_{i=1}^{r+x-j+1} \deg(\operatorname{lcm}(\phi_{i-x+j-1}, \gamma_i)) - \sum_{i=1}^{r+x-j} \deg(\operatorname{lcm}(\phi_{i-x+j}, \gamma_i)) \\
& -d, \qquad\qquad 2 \le j \le x.
\end{aligned}$$

By condition (32) for $j = 0$,

$$(r+x)d \ge \sum_{i=1}^{r+x} \deg(\operatorname{lcm}(\phi_{i-x}, \gamma_i)) + rd - \sum_{i=1}^{r} \deg(\phi_i) - \sum_{i=1}^{x} c_i;$$

hence

$$\hat{a}_1 \ge \sum_{i=1}^{r+x} \deg(\operatorname{lcm}(\phi_{i-x}, \gamma_i)) - \sum_{i=1}^{r+x-1} \deg(\operatorname{lcm}(\phi_{i-x+1}, \gamma_i)) - d.$$

By Remark 2.7.1, we have $\hat{a}_1 \ge a_2 \ge \dots \ge a_x$. Let $\hat{\mathbf{a}} = (\hat{a}_1, a_2, \dots, a_x)$, then for $1 \le j \le x$,

$$\begin{aligned}
& \textstyle\hat{a}_1 + \sum_{i=2}^{j} a_i \\
= & \textstyle\sum_{i=1}^{x} c_i + \sum_{i=1}^{r} \deg(\phi_i) - \sum_{i=1}^{r+x-j} \deg(\operatorname{lcm}(\phi_{i-x+j}, \gamma_i)) + (x-j)d \\
= & \textstyle(r+x-j)d - \sum_{i=x+1}^{n-r} c_i - \sum_{i=1}^{m-r} u_i - \sum_{i=1}^{r+x-j} \deg(\operatorname{lcm}(\phi_{i-x+j}, \gamma_i)).
\end{aligned}$$

From (32) we obtain

$$\hat{a}_1 + \sum_{i=2}^{j} a_i \ge \sum_{i=1}^{j} c_i, \quad 1 \le j \le x-1.$$

Moreover, $\hat{a}_1 + \sum_{i=2}^{x} a_i = \sum_{i=1}^{x} c_i$.

If $n = r + x$ then $\mathbf{c} \prec \hat{\mathbf{a}}$, and let $\mathbf{d} = \emptyset$ so that $\mathbf{c} \prec' (\mathbf{d}, \hat{\mathbf{a}})$ holds. Otherwise, if $n > r + x$ by Lemma 4.6 there exists a sequence of integers $\mathbf{d} = (d_1, \dots, d_{n-r-x})$ such that $\mathbf{c} \prec' (\mathbf{d}, \hat{\mathbf{a}})$, $d_i = c_{i+x}$ for $2 \le i \le n-r-x$, and $d_1 = \sum_{i=1}^{x+1} c_i - \hat{a}_1 - \sum_{i=2}^{x} a_i = c_{x+1}$.

Let $\mathbf{a} = (a_1, \dots, a_x)$ be defined as in (24). Then $a_1 = \hat{a}_1$, therefore (18) holds. From (32) for $j = 0$ we obtain

$$\begin{aligned}
\textstyle\sum_{i=1}^{r+x} \deg(\operatorname{lcm}(\phi_{i-x}, \gamma_i)) \le & \textstyle\sum_{i=1}^{r} \deg(\phi_i) + \sum_{i=1}^{n-r} c_i - \sum_{i=1}^{n-r-x} c_{i+x} + xd \\
= & \textstyle\sum_{i=1}^{r} \deg(\phi_i) + \sum_{i=1}^{n-r} c_i - \sum_{i=1}^{n-r-x} d_i + xd,
\end{aligned}$$

with equality if $x = z = n - r$, i.e., (23) is satisfied in this case. By Theorem 4.5, the result follows.

2. Case $x = z < n - r$. As $x = z$, observe that if there exits $W(s) \in \mathbb{F}[s]^{z \times n}$ such that $\operatorname{rank}\left(\begin{bmatrix} P(s) \\ W(s) \end{bmatrix}\right) = r + x$, then the row minimal indices of $\begin{bmatrix} P(s) \\ W(s) \end{bmatrix}$ are the row minimal indices of $P(s)$, i.e., $\mathbf{v} = \mathbf{u}$. For the sufficiency we prescribe $\mathbf{v} = \mathbf{u}$. The result follows from Theorem 4.8.

□

Remark 4.11 We would like to remark that if $x = z < n - r$, conditions (16), (33) and (34) imply (32). Recall that $1 \leq \ell \leq x + 1$ (see Remark 4.7).

For $0 \leq j < \ell$, from the definition of ℓ and (33),

$$\begin{array}{l} \sum_{i=1}^{r+x-j} \deg(\operatorname{lcm}(\phi_{i-x+j}, \gamma_i)) + \sum_{i=1}^{m-r} u_i + \sum_{i=1}^{j} c_i + \sum_{i=x+1}^{n-r} c_i \\ \leq \sum_{i=1}^{r+x} \deg(\gamma_i) + \sum_{i=1}^{m-r} u_i + \sum_{i=x+1}^{n-r} c_i - jd \\ \leq (r + x - j)d - c_\ell + c_{x+1} \leq (r + x - j)d. \end{array}$$

For $\ell \leq j \leq x - 1$, from (34),

$$\begin{array}{l} \sum_{i=1}^{r+x-j} \deg(\operatorname{lcm}(\phi_{i-x+j}, \gamma_i)) + \sum_{i=1}^{m-r} u_i + \sum_{i=1}^{j} c_i + \sum_{i=x+1}^{n-r} c_i \\ \leq (r + x - j)d - c_{j+1} + c_{x+1} \leq (r + x - j)d. \end{array}$$

When we only prescribe the invariant factors, we obtain the next theorem. The solution to the row completion is now characterized in terms of two conditions: the interlacing inequalities (35), and an additional condition (36). In fact, condition (35) can be derived from [30, 35], where the problem of prescription of the invariant factors of a polynomial matrix when a submatrix is fixed was solved. The appearance of condition (36) is due to degree condition on the completion matrix $W(s)$, restriction that was not taken into account in [30, 35].

Theorem 4.12 *Let $P(s) \in \mathbb{F}[s]^{m \times n}$, $\deg(P(s)) = d$, $\operatorname{rank}(P(s)) = r$. Let $\alpha_1(s) \mid \cdots \mid \alpha_r(s)$ be its invariant factors, $e_1 \leq \cdots \leq e_r$ its partial multiplicities of ∞, $\mathbf{c} = (c_1, \ldots, c_{n-r})$ its column minimal indices, and $\mathbf{u} = (u_1, \ldots, u_{m-r})$ its row minimal indices.*

Let z and x be integers such that $0 \leq x \leq \min\{z, n - r\}$ and let $\beta_1(s) \mid \cdots \mid \beta_{r+x}(s)$ be monic polynomials. There exists a polynomial matrix $W(s) \in \mathbb{F}[s]^{z \times n}$ such that $\deg(W(s)) \leq d$, $\operatorname{rank}\left(\begin{bmatrix} P(s) \\ W(s) \end{bmatrix}\right) = r + x$ and $\begin{bmatrix} P(s) \\ W(s) \end{bmatrix}$ has $\beta_1(s) \mid \cdots \mid \beta_{r+x}(s)$ as invariant factors if and only if

$$\beta_i(s) \mid \alpha_i(s) \mid \beta_{i+z}(s), \quad 1 \leq i \leq r, \tag{35}$$

$$\begin{array}{l} \sum_{i=1}^{r+x-j} \deg(\operatorname{lcm}(\alpha_{i-x+j}, \beta_i)) + \sum_{i=1}^{r} e_i + \sum_{i=1}^{m-r} u_i + \sum_{i=1}^{j} c_i + \sum_{i=x+1}^{n-r} c_i \\ \leq (r + x - j)d, \quad 0 \leq j \leq x - 1. \end{array} \tag{36}$$

Proof. Let $\phi_1(s,t) \mid \cdots \mid \phi_r(s,t)$ be the homogeneous invariant factors of $P(s)$. Assume that there exists $W(s) \in \mathbb{F}[s]^{z \times n}$ such that $\deg(W(s)) \leq d$, $\operatorname{rank}\left(\begin{bmatrix} P(s) \\ W(s) \end{bmatrix}\right) = r + x$, and $\begin{bmatrix} P(s) \\ W(s) \end{bmatrix}$ has $\beta_1(s) \mid \cdots \mid \beta_{r+x}(s)$ as invariant factors. Let $f_1 \leq \cdots \leq f_{r+x}$ be the partial multiplicities of ∞ and $\gamma_1(s,t) \mid \cdots \mid \gamma_{r+x}(s,t)$ the homogeneous invariant factors of $\begin{bmatrix} P(s) \\ W(s) \end{bmatrix}$. By Theorem 4.10 and Remark 4.11, (16) and (32) hold. Condition (16) implies (35). For $0 \leq j \leq x - 1$ we have

$$\begin{array}{rl} \sum_{i=1}^{r+x-j} \deg(\operatorname{lcm}(\phi_{i-x+j}, \gamma_i)) \geq & \sum_{i=1}^{r+x-j} \deg(\operatorname{lcm}(\alpha_{i-x+j}, \beta_i)) + \sum_{i=1}^{r+x-j} e_{i-x+j} \\ = & \sum_{i=1}^{r+x-j} \deg(\operatorname{lcm}(\alpha_{i-x+j}, \beta_i)) + \sum_{i=1}^{r} e_i; \end{array}$$

hence (32) implies (36).

Conversely, assume that (35) and (36) hold. Let

$$q=(r+x)d-\sum_{i=1}^{r+x}\deg(\operatorname{lcm}(\alpha_{i-x},\beta_i))-\sum_{i=1}^{r}e_i-\sum_{i=1}^{m-r}u_i-\sum_{i=x+1}^{n-r}c_i.$$

If $x=0$ then $q=0$ and if $x>0$, from (36) we obtain $q\geq 0$. Define

$$\begin{array}{l} f_i=0, \quad 1\leq i\leq x,\\ f_{i+x}=e_i, \quad 1\leq i\leq r-1,\\ f_{r+x}=e_r+q, \end{array}$$

and

$$\gamma_i(s,t)=t^{f_i}t^{\deg(\beta_i)}\beta_i\left(\frac{s}{t}\right), \quad 1\leq i\leq r+x.$$

We have $f_1\leq\cdots\leq f_{r+x}$, and $f_i\leq e_i\leq f_{i+z}, 1\leq i\leq r$. Thus, $\gamma_1(s,t)\mid\cdots\mid\gamma_{r+x}(s,t)$ and from (35) we derive (16). If $x>1$, for $1\leq j\leq x$, we have $f_{i+x-j}\leq f_{i+x-1}=e_{i-1}\leq e_i$, $1\leq i\leq r$, therefore $\sum_{i=1}^{r+x-j}\max\{e_{i-x+j},f_i\}=\sum_{i=1}^{r}e_i$, $1\leq j\leq r$. For $j=0$ we obtain $\sum_{i=1}^{r+x}\max\{e_{i-x},f_i\}=\sum_{i=1}^{r}e_i+q$. Thus,

$$\begin{array}{rcl} \sum_{i=1}^{r+x}\deg(\operatorname{lcm}(\phi_{i-x},\gamma_i)) &=& \sum_{i=1}^{r+x}\deg(\operatorname{lcm}(\alpha_{i-x},\beta_i))+\sum_{i=1}^{r}e_i+q\\ &=& (r+x)d-\sum_{i=1}^{m-r}u_i-\sum_{i=x+1}^{n-r}c_i,\\ \sum_{i=1}^{r+x-j}\deg(\operatorname{lcm}(\phi_{i-x+j},\gamma_i)) &=& \sum_{i=1}^{r+x-j}\deg(\operatorname{lcm}(\alpha_{i-x+j},\beta_i))+\sum_{i=1}^{r}e_i,\\ && \hfill 1\leq j\leq x-1, \end{array}$$

hence from (36) we obtain (32).

If $x<z$ or $x=z=n-r$ the result follows from Theorem 4.10 (item 1).

If $x=z<n-r$, from (16) we have

$$\sum_{i=1}^{r+x}\deg(\gamma_i)=\sum_{i=1}^{r+x}\deg(\operatorname{lcm}(\phi_{i-x},\gamma_i))=(r+x)d-\sum_{i=1}^{m-r}u_i-\sum_{i=x+1}^{n-r}c_i.$$

Therefore,

$$\sum_{i=1}^{r+x}\deg(\gamma_i)-\sum_{i=1}^{r}\deg(\operatorname{lcm}(\phi_i,\gamma_i))-xd=\sum_{i=1}^{x}c_i,$$

and from (32) we obtain

$$\begin{array}{l} \sum_{i=1}^{r+x}\deg(\gamma_i)-\sum_{i=1}^{r+x-j}\deg(\operatorname{lcm}(\phi_{i-x+j},\gamma_i))-jd\\ =(r+x-j)d-\sum_{i=1}^{m-r}u_i-\sum_{i=x+1}^{n-r}c_i-\sum_{i=1}^{r+x-j}\deg(\operatorname{lcm}(\phi_{i-x+j},\gamma_i))\\ \geq\sum_{i=1}^{j}c_i, \hfill 1\leq j\leq x-1. \end{array}$$

Thus, (33) holds and (34) vanishes. The result follows from Theorem 4.10 (item 2). □

If we only prescribe the infinite elementary divisors we obtain the following theorem.

Theorem 4.13 *Let $P(s)\in\mathbb{F}[s]^{m\times n}$, $\deg(P(s))=d$, $\operatorname{rank}(P(s))=r$. Let $\alpha_1(s)\mid\cdots\mid\alpha_r(s)$ be its invariant factors, $e_1\leq\cdots\leq e_r$ its partial multiplicities of ∞, $\mathbf{c}=(c_1,\ldots,c_{n-r})$ its column minimal indices, and $\mathbf{u}=(u_1,\ldots,u_{m-r})$ its row minimal indices.*

Let z and x be integers such that $0\leq x\leq\min\{z,n-r\}$ and let $f_1\leq\cdots\leq f_{r+x}$ be non negative integers. There exists a polynomial matrix $W(s)\in\mathbb{F}[s]^{z\times n}$ such that $\deg(W(s))\leq d$, $\operatorname{rank}\left(\begin{bmatrix}P(s)\\W(s)\end{bmatrix}\right)=r+x$ and $\begin{bmatrix}P(s)\\W(s)\end{bmatrix}$ has $f_1\leq\cdots\leq f_{r+x}$ as partial multiplicities of ∞ if and only if

$$\begin{array}{c} f_i\leq e_i\leq f_{i+z}, \quad 1\leq i\leq r,\\ \sum_{i=1}^{r+x-j}\max\{e_{i-x+j},f_i\}+\sum_{i=1}^{r}\deg(\alpha_i)+\sum_{i=1}^{m-r}u_i+\sum_{i=1}^{j}c_i+\sum_{i=x+1}^{n-r}c_i\\ \leq(r+x-j)d, \quad 0\leq j\leq x-1. \end{array}$$

Proof. The proof is analogous to that of the Theorem 4.12. To prove the sufficiency, define $\beta_i(s) = 1, 1 \leq i \leq x$, $\beta_{i+x}(s) = \alpha_i(s)$, $1 \leq i \leq r-1$ and $\beta_{r+x}(s) = \alpha_r(s)\tau(s)$, where $\tau(s)$ is a monic polynomial of $\deg(\tau) = (r+x)d - \sum_{i=1}^{r} \deg(\alpha_i) - \sum_{i=1}^{r+x} \max\{e_{i-x}, f_i\} - \sum_{i=1}^{m-r} u_i - \sum_{i=x+1}^{n-r} c_i$. □

5 Conclusions and future work

In this work we have solved the problem of row (column) completion of a polynomial matrix of bounded degree when the eigenstructure has been prescribed (Subsection 4.1). We also have presented the particular cases where we prescribe the finite and infinite structures and the column (row) minimal indices (Subsections 4.2 and 4.3), and the finite and/or infinite structures (Subsection 4.4). In a future paper we will accomplish the study of the remaining cases, i.e., the prescription of the column and/or row minimal indices, of both the column and row minimal indices and the finite (respectively, infinite) structure, of the column minimal indices and the finite (respectively, infinite) structure and, finally, of the row minimal indices and the finite (respectively, infinite) structure.

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