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Disentangling mappings defined on ICIS

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Abstract
We study germs of hypersurfaces $(Y,0)\subset(\mathbb{C}^{n+1},0)$ that can be described as the image of $\mathscr{A}$-finite mappings $f\colon(X,S)\to(\mathbb{C}^{n+1},0)$ defined on an ICIS (X,S) of dimension n. We extend the definition of the Jacobian module given by Fernández de Bobadilla, Nuño-Ballesteros and Peñafort-Sanchis when $X=\mathbb{C}^n$, which controls the image Milnor number $\mu_I(X,f)$. We apply these results to prove the case $n=2$ of the generalised Mond conjecture, which states that $\mu_I(X,f)\geq \operatorname{codim}_{\mathscr{A}_e}(X,f)$, with equality if $(Y,0)$ is weighted homogeneous.

Keywords Image Milnor number · The Mond conjecture · ICIS

Mathematics Subject Classification Primary 58K15; Secondary 32S30 · 58K40

1 Introduction

The main objects that are studied along this paper are hypersurface singularities $(Y,0)\subset(\mathbb{C}^{n+1},0)$. It is well known that if a hypersurface germ has isolated singularity then one can define two different invariants to encode information about the hypersurface, namely the Milnor and Tjurina numbers, μ and τ, respectively. The first of them, μ, has a more topological flavour, since it counts the number of spheres in the homotopy type of a Milnor fibre of Y. On the other hand, the Tjurina number has a much more rigid nature, and counts the miminum number of parameters in a $\mathscr{K}$-versal deformation of the equation of the hypersurface. An inspection of the algebraic

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formulas that define them makes clear that $\mu \geq \tau$, with equality in the case that $(Y, 0)$ is weighted homogeneous.

In [10], D. Mond introduced an invariant analogous to the Milnor number for hypersurfaces $(Y, 0)$ which are parameterised as the image of an $\mathscr{A}$-finite mapping $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$. This means that f has finite $\mathscr{A}_e$-codimension, or, equivalently, that it has isolated instability. If $(n, n+1)$ are Mather's nice dimensions or if f has corank one then f admits a stable perturbation f_s with image Y_s. Mond showed that $Y_s \cap B_\epsilon$ has the homotopy type of a wedge of n-spheres, where B_ϵ is a ball of radius $\epsilon > 0$ centered at the origin in $\mathbb{C}^{n+1}$ and he defined the *image Milnor number*, $\mu_I(f)$, as the number of such spheres.

On the other hand, the $\mathscr{A}_e$-codimension of f, denoted by $\operatorname{codim}_{\mathscr{A}_e}(f)$, is an analytical invariant that measures the minimum number of parameters required to have an $\mathscr{A}$-versal unfolding of f. Thus, $\operatorname{codim}_{\mathscr{A}_e}(f)$ is analogous to the Tjurina number in the context of $\mathscr{A}$-equivalence of parameterised hypersurfaces. Although the invariants are written and described in terms of the mapping f, they can be seen to depend only on the analytic class of the image $(Y, 0)$, and hence $\mu_I(f)$ and $\operatorname{codim}_{\mathscr{A}_e}(f)$ are the natural invariants that correspond to μ and τ in this different context. The question is therefore natural:

Conjecture 1.1 (Mond conjecture) *Let $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ be an $\mathscr{A}$-finite mapping, where $(n, n+1)$ are Mather's nice dimensions. Then, $\mu_I(f) \geq \operatorname{codim}_{\mathscr{A}_e}(f)$, with equality if f is weighted homogeneous.*

The only known cases are $n = 1, 2$, and it is still an open problem if $n \geq 3$. This conjecture is therefore the analogue of the $\mu \geq \tau$ statement for hypersurfaces that admit an $\mathscr{A}$-finite normalisation mapping with normal space $(\mathbb{C}^n, S)$. Nevertheless, the amount of hypersurfaces satisfying this property is limited. In particular, one would have interest in studying this statement if the normal space of $(Y, 0)$ is singular (when $n \geq 2$). More precisely, in this article we study the case of hypersurfaces whose normalisation is $\mathscr{A}$-finite and whose normal space is an isolated complete intersection singularity (ICIS).

In [11], D. Mond and J. Montaldi settled the deformation theory of mappings $f : (X, S) \to (\mathbb{C}^p, 0)$ with (X, S) being an ICIS of dimension n, and they took the first steps to study an extension of the Mond conjecture in this general setting. Here, we consider such theory only in the case $p = n + 1$.

It turns out that the notions of $\operatorname{codim}_{\mathscr{A}_e}(X, f)$ and $\mu_I(X, f)$ can be defined in this wider context, as Mond and Montaldi showed. The former invariant, namely the codimension of (X, f), equals the minimum number of parameters required to have a versal unfolding of (X, f), where now unfoldings are allowed to deform both the mapping f and the space X. On the other hand, the definition of $\mu_I(X, f)$ is given in the same way as it is done for mappings with smooth source. In order to properly define it, one requires the existence of *stabilisations*, which are deformations (X_t, f_t) of (X, f) with the property that (X_t, f_t) is stable for every $t \neq 0$ small enough. According to [6], these particular deformations do exist provided $(n, n+1)$ are nice dimensions or provided f has corank one (that is, f admits an extension on a smooth space with only corank one singularities). We consider the image $Y_t = f_t(X_t)$ and we call $Y_t \cap B_\epsilon$ the *disentanglement* of $(Y, 0)$. As in the case that X is smooth, $Y_t \cap B_\epsilon$ has

Table 1 Analogy between singularities of mappings on ICIS (left), isolated hypersurface singularities (centre) and mappings on smooth spaces (right)

$f\colon (X,S)\to(\mathbb{C}^{n+1},0)$	$(Y,0)\subset(\mathbb{C}^{n+1},0)$	$f\colon(\mathbb{C}^n,S)\to(\mathbb{C}^{n+1},0)$
$\mathscr{A}$-equivalence	Biholomorphism	$\mathscr{A}$-equivalence
Stable	Smooth	Stable
Isolated instability	Isolated singularity	Isolated instability
Stabilisation	Smoothing	Stabilisation
$\operatorname{codim}_{\mathscr{A}_e}(X,f)$	$\tau(Y,0)$	$\operatorname{Codim}_{\mathscr{A}_e}(f)$
Versal unfolding	Versal deformation	Versal unfolding
Disentanglement	Milnor fibre	Disentanglement
$\mu_I(X,f)$	$\mu(Y,0)$	$\mu_I(f)$

the homotopy type of a wedge of n-spheres. The number of such spheres is therefore defined as the image Milnor number $\mu_I(X,f)$.

It follows from the definitions that $\mu_I(X,f)$ and $\operatorname{codim}_{\mathscr{A}_e}(X,f)$ coincide with μ and τ, respectively, when $(Y,0)$ is a hypersurface with isolated singularity and we consider f as the inclusion mapping $(Y,0)\hookrightarrow(\mathbb{C}^{n+1},0)$. Moreover, $\mu_I(X,f)$ and $\operatorname{codim}_{\mathscr{A}_e}(X,f)$ are also equal to the invariants $\mu_I(f)$ and $\operatorname{codim}_{\mathscr{A}_e}(f)$, respectively, when $X=\mathbb{C}^n$. Thus, the theory of mappings on ICIS provides a unified approach that includes both, the hypersurfaces with isolated singularities and the hypersurfaces parameterised as the image of a map germ with isolated instability. We show in Table 1 how the notions and invariants defined for mappings on ICIS extend the corresponding notions and invariants for isolated hypersurface singularities and for mappings on smooth spaces.

This settles the framework to study in this general case whether $\mu_I(X,f)\geq\operatorname{codim}_{\mathscr{A}_e}(X,f)$, with equality in the weighted homogeneous case. This question is what we refer to as the generalised Mond conjecture. The only case that was known before this article is studied in [1] by D. Henrique and J.J. Nuño-Ballesteros, and provides a postive answer for the generalised Mond conjecture if $n=1$ and $(X,S)\subset(\mathbb{C}^2,S)$ is a plane curve.

Our main contribution in this article is that we prove the generalised Mond conjecture in the case that $n=2$ in its whole generality. Therefore, we show that the $\mu\geq\tau$ statement can be extended to surfaces $(Y,0)$ in $(\mathbb{C}^3,0)$ with $\mathscr{A}$-finite normalisation mapping on an ICIS. Formally, the main theorem of this paper is the following result:

Theorem 1.2 *Let $f\colon(X,S)\to(\mathbb{C}^3,0)$ be an $\mathscr{A}$-finite mapping where (X,S) is an* ICIS *of dimension 2 with image $(Y,0)$. Then,*

$$\mu_I(X,f)\geq\operatorname{codim}_{\mathscr{A}_e}(X,f),$$

with equality if $(Y,0)$ is weighted homogeneous.

Observe that when $f\colon(\mathbb{C}^n,S)\to(\mathbb{C}^{n+1},0)$ is weighted homogeneous, then its image $(Y,0)$ is also weighted homogeneous, so our statement of Theorem 1.2 is a bit more general than the Mond Conjecture in the case $n=2$ and $X=\mathbb{C}^n$.

In order to prove Theorem 1.2, we adapt the construction of the module $M(g)$ and its relative version for unfoldings $M_{\mathrm{rel}}(G)$ that were defined in [4] by J. Fernández de Bobadilla, J. J. Nuño-Ballesteros and G. Peñafort-Sanchis. We extend to this general framework their main result, which states that the length of $M(g)$ equals $\mu_I(X, f)$ if and only if $M_{\mathrm{rel}}(G)$ is a Cohen-Macaulay module, and that, if each of these cases hold, the generalised Mond conjecture holds for the mapping (X, f).

2 Mappings on ICIS

Singularities of smooth mappings between manifolds is a classical subject in Singularity Theory. The infinitesimal methods were developed by Thom and Mather in the late sixties and by this reason it is known as Thom-Mather theory. We refer to the book [12] for a modern presentation of the theory, which also includes the extension to holomorphic germs between complex manifolds.

In [11] Mond and Montaldi extended the Thom-Mather theory of singularities of mappings $f\colon (X, S) \to (\mathbb{C}^p, 0)$ defined on an ICIS (X, S). The crucial point here is that they consider deformations not only of the mapping f, but also of the ICIS (X, S). Here we summarise some of the basic definitions and properties, in order to make this paper more self-contained. For a more detailed account we refer to the original paper by Mond and Montaldi [11].

Along this section we work with holomorphic map germs $f\colon (X, S) \to (\mathbb{C}^p, 0)$, where (X, S) is an ICIS of dimension n. It is usual to denote such a map germ by a pair (X, f), although sometimes we may omit the base set of the germ if it does not provide relevant information or it is clear from the context. The two first definitions declare the type of equivalences and deformations we are dealing with in this theory.

Definition 2.1 Two holomorphic map germs $f, g : (X, S) \to (\mathbb{C}^p, 0)$ are called $\mathscr{A}$-*equivalent* if there exist biholomorphisms (i.e., isomorphisms of complex space germs) $\varphi\colon (X, S) \to (X, S)$ and $\psi\colon (\mathbb{C}^p, 0) \to (\mathbb{C}^p, 0)$ such that we have a commutative diagram

$$\begin{array}{ccc} (X, S) & \xrightarrow{f} & (\mathbb{C}^p, 0) \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\psi} \\ (X, S) & \xrightarrow{g} & (\mathbb{C}^p, 0) \end{array}$$

Definition 2.2 An *unfolding of the pair* (X, f) is a map germ $F\colon (\mathcal{X}, S') \to (\mathbb{C}^p \times \mathbb{C}^r, 0)$ together with a flat projection $\pi : (\mathcal{X}, S') \to (\mathbb{C}^r, 0)$ and an isomorphism $j\colon (X, S) \to \left(\pi^{-1}(0), S'\right)$ such that the following diagram commutes

$$\begin{tikzcd}
 & (X,S) \arrow[dl, "j"'] \arrow[dr, "f\times\{0\}"] & \\
(\pi^{-1}(0),S') \arrow[d, hook] & & \left(\mathbb{C}^p\times\{0\},0\right) \arrow[d, hook] \\
(\mathcal{X},S') \arrow[rr, "F"] \arrow[dr, "\pi"'] & & (\mathbb{C}^p\times\mathbb{C}^r,0) \arrow[dl, "\pi_2"] \\
 & (\mathbb{C}^r,0) &
\end{tikzcd}$$

where $\pi_2 : \mathbb{C}^p \times \mathbb{C}^r \to \mathbb{C}^r$ is the Cartesian projection.

In Definition 2.2, $\mathbb{C}^r$ is called the *parameter space of the unfolding*. It is common to denote the unfolding by $(\mathcal{X}, \pi, F, j)$. For each parameter $u \in \mathbb{C}^r$ in a neighbourhood of the origin, we have a mapping $f_u \colon X_u \to \mathbb{C}^p$, where $X_u := \pi^{-1}(u)$, denoted also by (X_u, f_u).

Definition 2.3 Two unfoldings $(\mathcal{X}, \pi, F, j)$ and $(\mathcal{X}', \pi', F', j')$ over $\mathbb{C}^r$ are *isomorphic* if the following diagram commutes:

$$\begin{tikzcd}
 & (\mathcal{X},j(S)) \arrow[rr, "F"] \arrow[dr, "\pi"] \arrow[dd, "\Phi"'] & & (\mathbb{C}^p\times\mathbb{C}^r,0) \arrow[dl, "\pi_2"'] \arrow[dd, "\Psi"] \\
(X,S) \arrow[ur, "j"] \arrow[dr, "j'"'] & & (\mathbb{C}^r,0) & \\
 & (\mathcal{X}',j'(S)) \arrow[rr, "F'"] \arrow[ur, "\pi'"] & & (\mathbb{C}^p\times\mathbb{C}^r,0) \arrow[ul, "\pi_2"']
\end{tikzcd},$$

where Φ and Ψ are biholomorphisms and also Ψ is an unfolding of the identity of $(\mathbb{C}^p, 0)$ over $\mathbb{C}^d$.

If $(\mathcal{X}, \pi, F, j)$ is an unfolding of (X, f) over $(\mathbb{C}^r, 0)$, a germ $\rho : (\mathbb{C}^s, 0) \to (\mathbb{C}^r, 0)$ induces and unfolding $(\mathcal{X}_\rho, \pi_\rho, F_\rho, j_\rho)$ of (X, f) by a *base change* or, in other words, by the fibre product of F and $\mathrm{id}_{\mathbb{C}^p} \times \rho$:

$$\begin{tikzcd}
\mathcal{X}_\rho := \mathcal{X} \times_{\mathbb{C}^p\times\mathbb{C}^s} (\mathbb{C}^p\times\mathbb{C}^s) \arrow[r, "F_\rho"] \arrow[d] & \mathbb{C}^p\times\mathbb{C}^s \arrow[d, "\mathrm{id}_{\mathbb{C}^p}\times\rho"] \\
\mathcal{X} \arrow[r, "F"] & \mathbb{C}^p\times\mathbb{C}^r
\end{tikzcd},$$

where we omit the points of the germs for simplicity.

The unfolding $(\mathcal{X}, \pi, F, j)$ is *versal* if every other unfolding is isomorphic to an unfolding induced from the former by a base change, $(\mathcal{X}_\rho, \pi_\rho, F_\rho, j_\rho)$. A versal unfolding is called *miniversal* if it has a parameter space with minimal dimension.

Definition 2.4 A germ (X, f) is *stable* if any unfolding is *trivial*, that is, isomorphic to the constant unfolding $(X \times \mathbb{C}^r, \pi_2, f \times \mathrm{id}_{\mathbb{C}^r}, i)$.

We say that (X, f) has *isolated instability* if there exists a representative $f : X \to \mathbb{C}^p$ such that the restriction $f : X \setminus f^{-1}(0) \to \mathbb{C}^p \setminus \{0\}$ has only stable singularities.

A *stabilisation* of (X, f) is a 1-parameter unfolding $(\mathcal{X}, \pi, F, j)$ with the property that for any small enough $s \in \mathbb{C} \setminus \{0\}$, (X_s, f_s) has only stable singularities. Such a mapping (X_s, f_s), with $s \neq 0$, is called a *stable perturbation* of (X, f).

Next, we recall the notion of $\mathscr{A}_e$-codimension of a germ (X, f). In order to do this, we introduce the following notation:

- $\mathscr{O}_p$ is the local ring of holomorphic functions $(\mathbb{C}^p, 0) \to \mathbb{C}$,
- $\mathscr{O}_{X,S}$ is the (semi-)local ring of holomorphic functions $(X, S) \to \mathbb{C}$,
- $f^* : \mathscr{O}_p \to \mathscr{O}_{X,S}$ is the induced ring morphism $f^*(h) = h \circ f$,
- $\theta_{\mathbb{C}^p,0}$ is the $\mathscr{O}_p$-module of germs of vector fields on $(\mathbb{C}^p, 0)$,
- $\theta_{X,S}$ is the $\mathscr{O}_{X,S}$-module of germs of vector fields on (X, S), that is, if $(X, S) \subset (\mathbb{C}^{n+k}, S)$ then

$$\theta_{X,S} = \operatorname{Derlog}(X, S) \otimes_{\mathscr{O}_{\mathbb{C}^{n+k},S}} \mathscr{O}_{X,S},$$

  where $\operatorname{Derlog}(X, S)$ is the $\mathscr{O}_{\mathbb{C}^{n+k},S}$-submodule of $\theta_{\mathbb{C}^{n+k},S}$ of vector fields ξ which are tangent to X at any regular point,
- $\theta(f)$ is the module of vector fields along f,
- $\omega f : \theta_{\mathbb{C}^p,0} \to \theta(f)$ is the mapping $\omega(\eta) = \eta \circ f$,
- $tf : \theta_{X,S} \to \theta(f)$ is the mapping $tf(\xi) = d\tilde{f} \circ \xi$, for some analytic extension $\tilde{f}$ of f (it is not difficult to see that this definition of tf does not depend on the choice of the extension $\tilde{f}$).

Definition 2.5 The $\mathscr{A}_e$*-codimension* of (X, f) is defined as

$$\operatorname{codim}_{\mathscr{A}_e}(X, f) = \dim_{\mathbb{C}} \frac{\theta(f)}{tf(\theta_{X,S}) + \omega f(\theta_p)} + \sum_{x \in S} \tau(X, x),$$

where $\tau(X, x)$ is the Tjurina number of (X, x). When $\operatorname{codim}_{\mathscr{A}_e}(X, f) < \infty$, the germ (X, f) is called $\mathscr{A}$-finite.

Then, with this extended definition the $\mathscr{A}$-finiteness has the similar characterization (and consequences) as in the classical case (i.e., when X is non-singular). Namely, the generalizations of the classical theorems hold for maps on ICIS, as it is proved in [11]:

Theorem 2.6 (Versality theorem) *A germ (X, f) is $\mathscr{A}$-finite if and only if it admits a versal unfolding. Moreover, if (X, f) is $\mathscr{A}$-finite then* $\operatorname{codim}_{\mathscr{A}_e}(X, f)$ *is equal to the number of parameters in a miniversal unfolding.*

As a consequence, (X, f) is stable if and only if (X, S) is smooth and f is stable in the usual sense. This also gives an extension of the Mather-Gaffney criterion for mappings on ICIS (see [12] for the classical version and [6] for the ICIS case):

Corollary 2.7 (Mather-Gaffney criterion) *A germ (X, f) is $\mathscr{A}$-finite if and only if it has isolated instability.*

Another crucial fact is that any germ (X, f) with isolated instability admits a stabilisation, provided that (n, p) are nice dimensions in the sense of Mather or f has only kernel rank one singularities (that is, f admits an extension whose differential has kernel of dimension ≤ 1 everywhere). A proof in the case $X = \mathbb{C}^n$ can be found in [12] and the extension to the case of mappings on ICIS appears in [6].

Finally, we will recall the definition of image Milnor number in the case $p = n+1$. The original definition when $X = \mathbb{C}^n$ is due to Mond (see [10, 12]), but it can be adapted quite easily to mappings on ICIS (see [6]). In the case $p \leq n$, the analogous invariant is called the discriminant Milnor number, considered for the first time by Damon and Mond in [2] in the case $X = \mathbb{C}^n$ and extended to mappings on ICIS by Mond and Montaldi in [11]. The definition is clearly inspired in the classical Milnor number and is motivated by the following theorem.

We denote by B_ϵ the closed ball in $\mathbb{C}^{n+1}$ of radius $\epsilon > 0$ centered at the origin. We assume $f\colon (X, S) \to (\mathbb{C}^{n+1}, 0)$ is $\mathscr{A}$-finite and that either $(n, n+1)$ are nice dimensions of Mather or f has only corank one singularities. We take a stabilisation of (X, f) with stable perturbation (X_s, f_s).

Theorem 2.8 [6, 10] *For all ϵ, η, with $0 < \eta \ll \epsilon \ll 1$ and for all $s \in \mathbb{C}$, with $0 < |s| < \eta$, $f_s(X_s) \cap B_\epsilon$ has the homotopy type of a bouquet of spheres of dimension n. Moreover, the number of such spheres, denoted by $\mu_I(X, f)$, is independent of the choice of the stabilisation, the parameter s and the numbers ϵ, η.*

Definition 2.9 With the notation of Theorem 2.8, $f_s(X_s) \cap B_\epsilon$ is called the *disentanglement* and $\mu_I(X, f)$ is called the *image Milnor number* of (X, f).

The proof of Theorem 2.8 is based on arguments by Lê [18] and by Siersma [16]. In fact, the original formulation in [16] gives a recipe of how to compute $\mu_I(X, f)$ in a more algebraic way:

Theorem 2.10 [16] *With the notation of Theorem 2.8, let $G\colon (\mathbb{C}^{n+1} \times \mathbb{C}, 0) \to (\mathbb{C}, 0)$ be a reduced equation of the image of the stabilisation. Then,*

$$\mu_I(X, f) = \sum_{y \in B_\epsilon \setminus f_s(X_s)} \mu(g_s; y),$$

where $g_s(y) = G(y, s)$ and $\mu(g_s; y)$ is the Milnor number of g_s at y.

For the proofs of Theorems 2.8 and 2.10 we refer the reader to the paper [10] or the book [16, Theorem 2.3] for the smooth case and the paper [6] for the case that X is an ICIS, instead of the original source of Siersma [16]. The original version of Siersma requires two conditions: (1) that G is a topologically trivial deformation of g_0 over the Milnor sphere (in the stratified sense), and (2) that the critical points of g_s on $B_\epsilon \setminus f_s(X_s)$ are isolated if $s \neq 0$. The proofs given in [6, 10, 12] carefully check that the two conditions are automatically satisfied in our situation. Since the precise statement of the two conditions is quite technical and they are not necessary to follow the reading of the paper, we will omit them. We remark there is a mistake in the statement of [16, Theorem 2.3], where the fibres should be written as $g_s^{-1}(t) \cap B_\epsilon$ instead of $g_s^{-1}(t) \cap \partial B_\epsilon$.

Remark 2.11 In order to compute the image Milnor number $\mu_I(X, f)$, sometimes is more convenient to consider a stable unfolding (i.e., an unfolding which is stable as a germ) instead of a stabilisation. In such a case, the bifurcation set $\mathcal{B}$ is the set of parameters $u \in \mathbb{C}^r$ in a neighbourhood of the origin, such that $f_u\colon X_u \to \mathbb{C}^{n+1}$ has only stable singularities. When $(n, n+1)$ are nice dimensions or when f has only corank one singularities, $\mathcal{B}$ is a proper closed analytic subset germ in a neighbourhood of 0 in $\mathbb{C}^r$ (see [12] or [6]). A stabilisation can be constructed easily by taking a line $L \subset \mathbb{C}^r$ with the property that $L \cap \mathcal{B} = \{0\}$. If $u \notin \mathcal{B}$, then $f_u(X_u) \cap B_\epsilon$ has the homotopy type of a bouquet of n-spheres and the number of such spheres is $\mu_I(X, f)$. Analogously, if $G\colon (\mathbb{C}^{n+1} \times \mathbb{C}^r, 0) \to (\mathbb{C}, 0)$ is a reduced equation of the image of the unfolding, then

$$\mu_I(X, f) = \sum_{y \in B_\epsilon \setminus f_u(X_u)} \mu(g_u; y),$$

where $g_u(y) = G(y, u)$ and $\mu(g_u; y)$ is the Milnor number of g_u at y. The main advantage to consider a stable unfolding is that it is a mapping defined on a smooth space, so we can make use of the results of [4] to compute the image Milnor number.

3 The Jacobian modules: mappings on smooth spaces

In this section we recall from [4] the construction of the Jacobian modules $M(g)$ and $M_{\mathrm{rel}}(G)$ for map germs $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$.

We first assume that $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ is finite and that it has degree one onto its image. This happens, for instance, when f is $\mathscr{A}$-finite, by the Mather-Gaffney criterion. In particular, its image is a hypersurface $(Y, 0)$ in $(\mathbb{C}^{n+1}, 0)$ and the restriction of f to its image $(\mathbb{C}^n, S) \to (Y, 0)$ is the normalisation mapping of $(Y, 0)$. Hence, it induces a monomorphism of rings $\mathscr{O}_{Y,0} \hookrightarrow \mathscr{O}_{\mathbb{C}^n,S}$ that allows us to consider $\mathscr{O}_{Y,0}$ as a subring of $\mathscr{O}_{\mathbb{C}^n,S}$. For simplicity, we write $\mathscr{O}_n := \mathscr{O}_{\mathbb{C}^n,S}$. In this case, the diagram

$$\begin{tikzcd} \mathscr{O}_{n+1} \arrow[r, "f^*"] \arrow[dr, two heads, "\pi"] & \mathscr{O}_n \\ & \mathscr{O}_{Y,0} \arrow[u, hook] \end{tikzcd}$$

commutes, where π is the epimorphism associated to the natural inclusion of $(Y, 0)$ in $(\mathbb{C}^{n+1}, 0)$. We consider both $\mathscr{O}_{Y,0}$ and $\mathscr{O}_n$ to be $\mathscr{O}_{n+1}$-modules via the corresponding morphisms π and f^*, respectively.

Let us consider $g \in \mathscr{O}_{n+1}$ to be a reduced equation of $(Y, 0)$, so π is the quotient by the ideal generated by g. The following result from R. Piene [15] relates the conductor ideal of $\mathscr{O}_{Y,0}$ in $\mathscr{O}_n$, given by

$$C(f) = \{a \in \mathscr{O}_{Y,0} : a \cdot \mathscr{O}_n \subset \mathscr{O}_{Y,0}\},$$

with the partial derivatives of g and with the minors of the Jacobian matrix df. We remark that $C(f)$ is an ideal in both rings $\mathscr{O}_{Y,0}$ and $\mathscr{O}_n$.

Theorem 3.1 [15] *There exists a unique $\lambda \in \mathscr{O}_n$ such that, for every $\ell \in \{1, \dots, n+1\}$,*

$$\frac{\partial g}{\partial y_\ell} \circ f = (-1)^\ell \cdot \lambda \cdot \det(df_1, \dots, df_{\ell-1}, df_{\ell+1}, \dots, df_{n+1}).$$

Furthermore, the ideal $C(f)$ is principal, and generated by the element λ in $\mathscr{O}_n$.

Let us denote by $J(g)$ to the Jacobian ideal of g, which is generated by the partial derivatives $\frac{\partial g}{\partial y_\ell}$. This result therefore shows that $J(g) \cdot \mathscr{O}_n \subset C(f)$.

On the other hand, since f is a finite mapping with degree 1 onto its image, an application of Theorem 3.4 of [13] of D. Mond and R. Pellikaan shows that $(f^*)^{-1}(C(f)) = \mathscr{F}_1(f)$, where $\mathscr{F}_1(f)$ is the first Fitting ideal of $\mathscr{O}_n$ as an $\mathscr{O}_{n+1}$-module via f^*. In particular, the restriction $f^* \colon \mathscr{F}_1(f) \to C(f)$ is an epimorphism. Moreover, we also have,

$$\mathscr{F}_1(f) = (f^*)^{-1}(C(f)) \supset (f^*)^{-1}(J(g) \cdot \mathscr{O}_n) \supset J(g).$$

Definition 3.2 The restriction of f^* to $\mathscr{F}_1(f)$ induces an epimorphism of $\mathscr{O}_{n+1}$-modules

$$\frac{\mathscr{F}_1(f)}{J(g)} \to \frac{C(f)}{J(g) \cdot \mathscr{O}_n}.$$

We define $M(g)$ to be the $\mathscr{O}_{n+1}$-module given by the kernel of this epimorphism.

Observe that by Piene's Theorem 3.1, we have an isomorphism

$$\frac{C(f)}{J(g) \cdot \mathscr{O}_n} \cong \frac{\mathscr{O}_n}{R(f)},$$

where $R(f)$ is the ramification ideal, that is, the ideal in $\mathscr{O}_n$ generated by the maximal minors of the Jacobian matrix of f and the isomorphism is induced by multiplication by λ, the generator of $C(f)$.

By construction, the module $M(g)$ is determined by the short exact sequence

$$0 \to M(g) \to \frac{\mathscr{F}_1(f)}{J(g)} \to \frac{C(f)}{J(g) \cdot \mathscr{O}_n} \to 0.$$

Moreover, $M(g)$ can be computed directly by the following formula (see [4, Proposition 5.1]):

$$M(g) = \frac{(f^*)^{-1}(J(g) \cdot \mathscr{O}_n)}{J(g)}, \tag{1}$$

In fact, this is a consequence of the inclusion $J(g) \subset (f^*)^{-1}(J(g) \cdot \mathscr{O}_n)$, which follows in two steps:

$$J(g) \subset (f^*)^{-1}(f^*(J(g))) \subset (f^*)^{-1}(J(g) \cdot \mathscr{O}_n),$$

since $J(g)\cdot\mathscr{O}_n$ is generated by $f^*(J(g))$. In general, none of these two inclusions are equalities.

The most important property of $M(g)$ is proved in [4, Corollary 3.5]:

Proposition 3.3 *Let $f\colon(\mathbb{C}^n,S)\to(\mathbb{C}^{n+1},0)$ be $\mathscr{A}$-finite, with $n\geq 2$. Then*

$$\dim_{\mathbb{C}} M(g)=\operatorname{codim}_{\mathscr{A}_e}(f)+\dim_{\mathbb{C}} K(g),$$

where $K(g):=(J(g)+\langle g\rangle)/J(g)$.

Several consequences are obtained. The first one is that $M(g)=0$ if and only if f is stable and $g\in J(g)$. If $(n,n+1)$ are nice dimensions or if f has corank one, then the stability of f implies that f is always weighted homogeneous (see [12, Exercise 7.1.1 and Theorem 7.6]). In particular, under such conditions we have that $M(g)=0$ if and only if f is stable and that $\dim_{\mathbb{C}} M(g)<\infty$, when f is $\mathscr{A}$-finite (by the Mather-Gaffney criterion, see [4, Remark 3.5]). Finally, we have

$$\dim_{\mathbb{C}} M(g)\geq\operatorname{codim}_{\mathscr{A}_e}(f),$$

with equality if f is weighted homogeneous. Thus, in order to prove the Mond Conjecture it suffices to show that $\dim_{\mathbb{C}} M(g)=\mu_I(f)$.

We now recall the relative version of the Jacobian module for unfoldings. Suppose that $F\colon(\mathbb{C}^n\times\mathbb{C}^r,S\times\{0\})\to(\mathbb{C}^{n+1}\times\mathbb{C}^r,0)$ is an unfolding of f. If f is finite and has degree one onto its image, then F is also finite and has degree one onto its image. We consider $G\in\mathscr{O}_{n+1+r}$ a reduced equation of the image of F such that $G(y,0)=g(y)$ for all $y\in\mathbb{C}^{n+1}$. We also consider the *relative* Jacobian ideal of G as the ideal $J_y(G)$ in $\mathscr{O}_{n+1+r}$ generated by the partial derivatives of G with respect only to the variables $y_1,\dots,y_{n+1}$.

Definition 3.4 We define $M_{\mathrm{rel}}(G)$ to be the $\mathscr{O}_{n+1+r}$-module given by the kernel of the morphism

$$\frac{\mathscr{F}_1(F)}{J_y(G)}\to\frac{C(F)}{J_y(G)\cdot\mathscr{O}_{n+r}}. \tag{2}$$

As above, $M_{\mathrm{rel}}(G)$ is determined by the short exact sequence

$$0\to M_{\mathrm{rel}}(G)\to\frac{\mathscr{F}_1(F)}{J_y(G)}\to\frac{C(F)}{J_y(G)\cdot\mathscr{O}_{n+r}}\to 0.$$

One important issue in this case is that

$$J_y(G)\cdot\mathscr{O}_{n+r}=J(G)\cdot\mathscr{O}_{n+r}, \tag{3}$$

(see [4, Lemma 4.5]). In particular, in the right hand side of the epimorphism (2) we can change $J_y(G)\cdot\mathscr{O}_{n+r}$ by $J(G)\cdot\mathscr{O}_{n+r}$. This implies that

$$\frac{C(F)}{J_y(G)\cdot\mathscr{O}_{n+r}}=\frac{C(F)}{J(G)\cdot\mathscr{O}_{n+r}}\cong\frac{\mathscr{O}_{n+r}}{R(F)}, \tag{4}$$

where $R(F)$ is the ramification ideal of F. In particular, the above module (4) is Cohen-Macaulay of dimension $n+r-2$. This is used to prove the main property of $M_{\mathrm{rel}}(G)$, namely, that it specializes to $M(g)$ when $u=0$ (see [4, Theorem 4.6]):

Theorem 3.5 *Let $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ be $\mathscr{A}$-finite, with $n \geq 2$. For any r-parameter unfolding F, we have*

$$M_{\mathrm{rel}}(G) \otimes_{\mathscr{O}_r} \frac{\mathscr{O}_r}{\mathfrak{m}_r} \cong M(g).$$

The module $M_{\mathrm{rel}}(G)$ is considered here as an $\mathscr{O}_r$-module via the ring extension $\mathscr{O}_r \hookrightarrow \mathscr{O}_{n+1+r}$. Moreover, the isomorphism above is induced by the inclusion $j\colon \mathbb{C}^{n+1} \to \mathbb{C}^{n+1} \times \mathbb{C}^r$ given by $j(y) = (y, 0)$. Hence, it is, in fact, an isomorphism of $\mathscr{O}_{n+1}$-modules.

Remark 3.6 It is well known that given an R-module M and any ideal $I \subset R$, we have a natural isomorphism of R-modules (see e.g. [8, Exercise 2.7.5]]:

$$M \otimes_R \frac{R}{I} \cong \frac{M}{I \cdot M}.$$

In our case, for $R = \mathscr{O}_r$ and $I = \mathfrak{m}_r$, this gives

$$M_{\mathrm{rel}}(G) \otimes_{\mathscr{O}_r} \frac{\mathscr{O}_r}{\mathfrak{m}_r} \cong \frac{M_{\mathrm{rel}}(G)}{\mathfrak{m}_r \cdot M_{\mathrm{rel}}(G)}.$$

But we could also take $R = \mathscr{O}_{n+1+r}$ and $I = \mathfrak{m}_r \cdot \mathscr{O}_{n+1+r}$:

$$M_{\mathrm{rel}}(G) \otimes_{\mathscr{O}_{n+1+r}} \frac{\mathscr{O}_{n+1+r}}{\mathfrak{m}_r \cdot \mathscr{O}_{n+1+r}} \cong \frac{M_{\mathrm{rel}}(G)}{(\mathfrak{m}_r \cdot \mathscr{O}_{n+1+r}) \cdot M_{\mathrm{rel}}(G)}.$$

It is obvious that $(\mathfrak{m}_r \cdot \mathscr{O}_{n+1+r}) \cdot M_{\mathrm{rel}}(G) = \mathfrak{m}_r \cdot M_{\mathrm{rel}}(G)$. This gives a natural isomorphism

$$M_{\mathrm{rel}}(G) \otimes_{\mathscr{O}_{n+1+r}} \frac{\mathscr{O}_{n+1+r}}{\mathfrak{m}_r \cdot \mathscr{O}_{n+1+r}} \cong M_{\mathrm{rel}}(G) \otimes_{\mathscr{O}_r} \frac{\mathscr{O}_r}{\mathfrak{m}_r}.$$

Hence, we can consider any of these two tensor products above indistinctly.

4 The generalised version of the Jacobian module $M(g)$

In this section we see how to adapt the definition of $M(g)$ to the case of mappings on ICIS .

Let $f : (X, S) \to (\mathbb{C}^{n+1}, 0)$ be an $\mathscr{A}$-finite mapping defined on an ICIS $(X, S) \subset (\mathbb{C}^{n+k}, S)$ of dimension n. We choose:

- a $\mathscr{K}$-finite mapping $h : (\mathbb{C}^{n+k}, S) \to (\mathbb{C}^k, 0)$ with $(X, S) = h^{-1}(0)$,

- an analytic extension $\tilde{f} : (\mathbb{C}^{n+k}, S) \to (\mathbb{C}^{n+1}, 0)$ of f.

Then, we write $\hat{f} = (\tilde{f}, h) : (\mathbb{C}^{n+k}, S) \to (\mathbb{C}^{n+1+k}, 0)$. It is therefore clear that the restriction of $\hat{f}$ to (X, S) is precisely the mapping $(f, 0)$. Moreover, the diagram

$$\begin{tikzcd}
 & (\mathbb{C}^k, 0) \\
(\mathbb{C}^{n+k}, S) \arrow[ur, two heads, "h"] \arrow[r, "\hat{f}"] & (\mathbb{C}^{n+1+k}, 0) \arrow[u, two heads] \\
(X, S) \arrow[u, hook, "i"] \arrow[r, "f"] & (\mathbb{C}^{n+1}, 0) \arrow[u, hook, "j"']
\end{tikzcd}$$

commutes, where i is the inclusion and j is the natural embedding. In particular, $(\mathbb{C}^{n+k}, \hat{f})$ is an unfolding of (X, f) in the sense of Definition 2.2 with parameter space $\mathbb{C}^k$. After taking representatives, the induced deformations of (X, f) are the mappings $(X_t, \hat{f}_t)$, where $\hat{f}_t : X_t \subset \mathbb{C}^{n+k} \to \mathbb{C}^{n+1}$ is defined as $\hat{f}_t(x) = \tilde{f}(x, t)$ and $X_t = h^{-1}(t)$ for $t \in \mathbb{C}^k$ small enough. Notice that X_t will be smooth for generic values of t, but they can be singular, in general.

The key idea is that $\hat{f}$ is the simplest unfolding of f with smooth source. Hence, the definition of the module for the mapping f will be performed through a specialisation of the one from $\hat{f}$.

Since (X, f) has isolated instability, by the Mather-Gaffney geometric criterion, f is finite and has degree one onto its image, and the same property is inherited by the unfolding $\hat{f}$. We denote by $(Y, 0)$ the image of f and by $(\hat{Y}, 0)$ the image of $\hat{f}$. They are hypersurfaces in $(\mathbb{C}^{n+1}, 0)$ and $(\mathbb{C}^{n+1+k}, 0)$, respectively. We choose $g \in \mathscr{O}_{n+1}$ and $\hat{g} \in \mathbb{C}^{n+1+k}$ reduced equations of $(Y, 0)$ and $(\hat{Y}, 0)$, respectively, such that $g = \hat{g} \circ j$. We can proceed as in Definition 3.2, but we take a slightly different version:

Definition 4.1 We define $N(\hat{g})$ as the kernel of the epimorphism induced by $\hat{f}^*$:

$$\frac{\mathscr{F}_1(\hat{f})}{J_y(\hat{g})} \to \frac{C(\hat{f})}{J(\hat{g}) \cdot \mathscr{O}_{n+k}} \tag{5}$$

Then, we define the *Jacobian module* $M(g)$ as

$$M(g) = N(\hat{g}) \otimes_{\mathscr{O}_k} \frac{\mathscr{O}_k}{\mathfrak{m}_k}.$$

Obviously, when $X = \mathbb{C}^n$ and $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ is $\mathscr{A}$-finite then we take $\hat{f} = f$ and $\hat{g} = g$, hence $M(g)$ coincides with the Jacobian module of Definition 3.2.

The following example shows why $M(g)$ deserves the name of Jacobian module. It also shows that the choices of the ideals $J_y(\hat{g})$ and $J(\hat{g})$ in the source and the target of the epimorphism (5), respectively, are not arbitrary.

Example 4.2 Suppose $(Y,0)$ is a hypersurface with isolated singularity in $(\mathbb{C}^{n+1},0)$ and consider f as the inclusion $(Y,0)\hookrightarrow(\mathbb{C}^{n+1},0)$. Let $g\in\mathscr{O}_{n+1}$ be a reduced equation of $(Y,0)$. We choose $\hat{f}:(\mathbb{C}^{n+1},0)\to(\mathbb{C}^{n+1}\times\mathbb{C},0)$ as the mapping $\hat{f}=(\mathrm{id},g)$ and $\hat{g}(y,z)=g(y)-z$. Since $\hat{f}$ is an embedding, $\mathscr{F}_1(\hat{f})=\mathscr{O}_{n+2}$ and also $C(\hat{f})=\mathscr{O}_{n+1}$. Moreover, $J(\hat{g})=\mathscr{O}_{n+2}$, hence

$$N(\hat{g})=\frac{\mathscr{O}_{n+2}}{J_y(\hat{g})}$$

and thus,

$$M(g)=\frac{\mathscr{O}_{n+2}}{J_y(\hat{g})}\otimes_{\mathscr{O}_1}\frac{\mathscr{O}_1}{\mathfrak{m}_1}\cong\frac{\mathscr{O}_{n+1}}{J(g)}.$$

That is, $M(g)$ coincides with the classical Jacobian algebra of $(Y,0)$. We also see $J_y(\hat{g})\cdot\mathscr{O}_{n+1}\neq J(\hat{g})\cdot\mathscr{O}_{n+1}$ (compare with (3)).

Remark 4.3 The module $N(\hat{g})$ is determined by the short exact sequence

$$0\to N(\hat{g})\to\frac{\mathscr{F}_1(\hat{f})}{J_y(\hat{g})}\to\frac{C(\hat{f})}{J(\hat{g})\cdot\mathscr{O}_{n+k}}\to 0.$$

We finish this section with a formula which extends (1) for maps on ICIS and allows us to compute directly the module $N(\hat{g})$:

Proposition 4.4 *The following formula holds:*

$$N(\hat{g})=\frac{(\hat{f}^*)^{-1}(J(\hat{g})\cdot\mathscr{O}_{n+k})}{J_y(\hat{g})}.$$

Proof By definition, it is clear that

$$N(\hat{g})=\frac{(\hat{f}^*)^{-1}(J(\hat{g})\cdot\mathscr{O}_{n+k})\cap\mathscr{F}_1(\hat{f})}{J_y(\hat{g})}.$$

Since $J(\hat{g})\cdot\mathscr{O}_{n+k}\subset C(\hat{f})$, then $(\hat{f}^*)^{-1}(J(\hat{g})\cdot\mathscr{O}_{n+k})\subset\mathscr{F}_1(\hat{f})$, and the formula follows. □

5 A relative version for the generalised Jacobian module

In this section, a relative version of the module for unfoldings of $\hat{f}$ is defined. For the sake of simplicity, we consider unfoldings

$$F:(\mathbb{C}^{n+k}\times\mathbb{C}^r,S\times 0)\to(\mathbb{C}^{n+1}\times\mathbb{C}^k\times\mathbb{C}^r,0)$$

of the mapping $\hat{f}$ instead of general unfoldings of f with possibly non-smooth source. Let $G \in \mathscr{O}_{n+1+k+r}$ be an equation for the image of F such that $G(y, z, 0) = \hat{g}(y, z)$, where (y, z, u) are the coordinates of $(\mathbb{C}^{n+1+k+r}, 0)$, with $y \in \mathbb{C}^{n+1}$, $z \in \mathbb{C}^k$ and $u \in \mathbb{C}^r$. We then have that the diagram

$$\begin{tikzcd}
(\mathbb{C}^{n+k+r}, S \times 0) \arrow[r, "F"] & (\mathbb{C}^{n+1+k+r}, 0) \arrow[dr, "G"] & \\
(\mathbb{C}^{n+k}, S) \arrow[u, hook, "\hat{i}"] \arrow[r, "\hat{f}"] & (\mathbb{C}^{n+1+k}, 0) \arrow[u, hook, "\hat{j}"'] \arrow[r, "\hat{g}"] & (\mathbb{C}, 0) \\
(X, S) \arrow[u, hook, "i"] \arrow[r, "f"] & (\mathbb{C}^{n+1}, 0) \arrow[u, hook, "j"'] \arrow[ur, "g"] &
\end{tikzcd}$$

commutes, where $\hat{i}$, $\hat{j}$ are the natural immersions. Let us denote by

$$J_y(G) = \left\langle \frac{\partial G}{\partial y_1}, \ldots, \frac{\partial G}{\partial y_{n+1}} \right\rangle, \ J_z(G) = \left\langle \frac{\partial G}{\partial z_1}, \ldots, \frac{\partial G}{\partial z_k} \right\rangle,$$

and $J_{y,z}(G) = J_y(G) + J_z(G)$.

Definition 5.1 We define $M_{\mathrm{rel}}(G)$ as the kernel of the module epimorphism induced by F^*

$$\frac{\mathscr{F}_1(F)}{J_y(G)} \to \frac{C(F)}{J_{y,z}(G) \cdot \mathscr{O}_{n+k+r}}.$$

Hence, $M_{\mathrm{rel}}(G)$ fits into the short exact sequence

$$0 \to M_{\mathrm{rel}}(G) \to \frac{\mathscr{F}_1(F)}{J_y(G)} \to \frac{C(F)}{J_{y,z}(G) \cdot \mathscr{O}_{n+k+r}} \to 0.$$

Furthermore, it is straightforward to verify that

Proposition 5.2 *The following formula holds:*

$$M_{\mathrm{rel}}(G) = \frac{(F^*)^{-1}(J_{y,z}(G) \cdot \mathscr{O}_{n+k+r})}{J_y(G)}.$$

In contrast with the case of $M(g)$, we have that $J_{y,z}(G) \cdot \mathscr{O}_{n+k+r} = J(G) \cdot \mathscr{O}_{n+k+r}$, beacuse of (3). Hence, they can be interchanged indistinctively in this formula.

As in the smooth case, the main property of $M_{\mathrm{rel}}(G)$ is that it specialises to $M(g)$ when $z = u = 0$:

Theorem 5.3 *If $n \geq 2$, then*

$$M_{\mathrm{rel}}(G) \otimes_{\mathscr{O}_{k+r}} \frac{\mathscr{O}_{k+r}}{\mathfrak{m}_{k+r}} \cong M(g).$$

Proof We follow the same steps as in the proof of the smooth case [4, Theorem 4.6]. On one hand, item 1 of [4, Proposition 4.4] gives

$$\frac{C(F)}{J_{y,z}(G)\cdot\mathscr{O}_{n+k+r}}\otimes_{\mathscr{O}_r}\frac{\mathscr{O}_r}{\mathfrak{m}_r}\cong\frac{C(\hat{f})}{J(\hat{g})\cdot\mathscr{O}_{n+k}}. \tag{6}$$

On the other hand, the same arguments to prove item 2 of [4, Proposition 4.4] imply in this case:

$$\frac{\mathscr{F}_1(F)}{J_y(G)}\otimes_{\mathscr{O}_r}\frac{\mathscr{O}_r}{\mathfrak{m}_r}\cong\frac{\mathscr{F}_1(\hat{f})}{J_y(\hat{g})}. \tag{7}$$

Now we consider the short exact sequence

$$0\to M_{\mathrm{rel}}(G)\to\frac{\mathscr{F}_1(F)}{J_y(G)}\to\frac{C(F)}{J_{y,z}(G)\cdot\mathscr{O}_{n+k+r}}\to 0 \tag{8}$$

By tensoring with $\mathscr{O}_r/\mathfrak{m}_r$, we obtain by (6) and (7) the associated long exact Tor-sequence:

$$\begin{aligned}\dots\to \operatorname{Tor}_1^{\mathscr{O}_r}&\left(\frac{C(F)}{J_{y,z}(G)\cdot\mathscr{O}_{n+k+r}},\frac{\mathscr{O}_r}{\mathfrak{m}_r}\right)\\ &\to M_{\mathrm{rel}}(G)\otimes_{\mathscr{O}_r}\frac{\mathscr{O}_r}{\mathfrak{m}_r}\to\frac{\mathscr{F}_1(\hat{f})}{J_y(\hat{g})}\to\frac{C(\hat{f})}{J(\hat{g})\cdot\mathscr{O}_{n+k}}\to 0.\end{aligned}$$

But we saw in (4) that

$$\frac{C(F)}{J_{y,z}(G)\cdot\mathscr{O}_{n+k+r}}=\frac{C(F)}{J(G)\cdot\mathscr{O}_{n+k+r}}\cong\frac{\mathscr{O}_{n+k+r}}{R(F)}$$

which is Cohen-Macaulay of dimension $n+k+r-2$. Again by (6) and by [9, Theorem 23.1], this implies that this module is $\mathscr{O}_r$-flat. Now we use [9, Theorem 7.8],

$$\operatorname{Tor}_1^{\mathscr{O}_r}\left(\frac{C(F)}{J_{y,z}(G)\cdot\mathscr{O}_{n+k+r}},\frac{\mathscr{O}_r}{\mathfrak{m}_r}\right)=0.$$

From the long exact Tor-sequence and the definition of $N(\hat{g})$, it follows that

$$M_{\mathrm{rel}}(G)\otimes_{\mathscr{O}_r}\frac{\mathscr{O}_r}{\mathfrak{m}_r}\cong N(\hat{g}),$$

Hence,

$$\begin{aligned}M_{\mathrm{rel}}(G)\otimes_{\mathscr{O}_{k+r}}\frac{\mathscr{O}_{k+r}}{\mathfrak{m}_{k+r}}&=\left(M_{\mathrm{rel}}(G)\otimes_{\mathscr{O}_r}\frac{\mathscr{O}_r}{\mathfrak{m}_r}\right)\otimes_{\mathscr{O}_k}\frac{\mathscr{O}_k}{\mathfrak{m}_k}\\ &=N(\hat{g})\otimes_{\mathscr{O}_k}\frac{\mathscr{O}_k}{\mathfrak{m}_k}=M(g),\end{aligned}$$

where the last equality holds by definition of $M(g)$. ◻

Lastly, an important result regarding the form of the module $M_{\mathrm{rel}}(G)$ when F is stable is the following:

Proposition 5.4 *Let F be a stable unfolding of $\hat{f}$ and G an equation of the image of F such that $G \in J(G)$. Then,*

$$M_{\mathrm{rel}}(G) = \frac{J(G)}{J_y(G)}.$$

Proof Since F is stable and $G \in J(G)$, we have $M(G) = 0$, by Proposition 3.3. In this case, the formula (1) yields that

$$M(G) = \frac{(F^*)^{-1}(J(G) \cdot \mathscr{O}_{n+k+r})}{J(G)},$$

and hence $(F^*)^{-1}(J(G) \cdot \mathscr{O}_{n+k+r}) = J(G)$. Furthermore, by (3) it follows that $J(G) \cdot \mathscr{O}_{n+k+r} = J_{y,z}(G) \cdot \mathscr{O}_{n+k+r}$. Hence, Proposition 5.2 yields

$$M_{\mathrm{rel}}(G) = \frac{(F^*)^{-1}(J_{y,z}(G) \cdot \mathscr{O}_{n+k+r})}{J_y(G)} = \frac{J(G)}{J_y(G)},$$

and the claim follows. ◻

Note that, since $\hat{f}$ is a finite mapping, the existence of a stable unfolding F of $\hat{f}$ is granted. Furthermore, as it is commented in [4], there is always a stable unfolding F which admits an equation G with the property that $G \in J(G)$. Such an equation is referred to as a *good defining equation*. Indeed, if $F(x, u) = (f_u(x), u)$ is a stable unfolding of $\hat{f}$, then let F' be the 1-parameter stable unfolding of F given by $F'(x, u, t) = (f_u(x), u, t)$. Let $G = 0$ be a reduced equation of the image of F and consider $G'(y, u, t) = e^t G(y, u)$. It therefore follows that $G' = 0$ is a reduced equation defining the image of F' with the property that $G' = \partial_t G' \in J(G')$.

6 Relation between $\dim_{\mathbb{C}} M(g)$ and $\operatorname{codim}_{\mathscr{A}_e}(X, f)$

In this section we extend the formula of Proposition 3.3 for mappings on ICIS :

Theorem 6.1 *Let $f : (X, S) \to (\mathbb{C}^{n+1}, 0)$ be $\mathscr{A}$-finite, where (X, S) is an ICIS of dimension $n \geq 2$. Then,*

$$\dim_{\mathbb{C}} M(g) = \dim_{\mathbb{C}} K(g) + \operatorname{codim}_{\mathscr{A}_e}(X, f),$$

where $K(g) = (J(g) + \langle g \rangle)/J(g)$.

Proof Let F be a stable unfolding of $\hat{f}$, and consider an equation G of the image of F, namely, $(Z, 0)$, that satisfies $G \in J(G)$. Then, the last result of the previous section showed that $M_{\text{rel}}(G) = J(G)/J_y(G)$. By [11, Theorems 1.3 and 1.4], we have

$$\operatorname{codim}_{\mathscr{A}_e}(X, f) = \dim_{\mathbb{C}} \frac{\theta(i)}{ti(\theta_{n+1}) + i^*(\operatorname{Derlog} Z)},$$

where $i : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{n+1+k+r}, 0)$ denotes the natural embedding. Recall that $\operatorname{Derlog} Z = \{\xi \in \theta_{n+1+k+r} : \xi(G) \in \langle G \rangle\}$, and $\operatorname{Derlog} G = \{\xi \in \theta_{n+1+k+r} : \xi(G) = 0\}$. Notice that $G \in J(G)$, so that $G = \sum_s a_s \partial_s G$, where $\partial_s G$ denotes the partial derivatives with respect to all the variables in $(y, z, u) \in \mathbb{C}^{n+1+k+r}$. Hence, the vector field $\epsilon = \sum_s a_s \partial_s$ satisfies that $\epsilon(G) = G$, where ∂_s denotes the coordinate vector field associated with the s-th coordinate, where $s \in \{1, \ldots, n + 1 + k + r\}$. Furthermore, $\operatorname{Derlog} Z = \operatorname{Derlog} G \oplus \langle \epsilon \rangle$. We therefore have that

$$\operatorname{codim}_{\mathscr{A}_e}(X, f) = \dim_{\mathbb{C}} \frac{\theta(i)}{ti(\theta_{n+1}) + i^*(\operatorname{Derlog} G) + i^*(\epsilon)}.$$

Notice that the evaluation mapping $\operatorname{ev} : \theta_{n+1+k+r} \to J(G)$ given by $\xi \mapsto \xi(G)$ is a surjective mapping with kernel $\operatorname{Derlog} G$. Hence, it induces an isomorphism

$$\frac{\theta_{n+1+k+r}}{\operatorname{Derlog} G} \cong J(G).$$

Thus,

$$\frac{\theta_{n+1+k+r}}{\left\langle \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_{n+1}} \right\rangle + \operatorname{Derlog} G} \cong \frac{J(G)}{J_y(G)} = M_{\text{rel}}(G).$$

Tensoring with $\mathscr{O}_{k+r}/\mathfrak{m}_{k+r}$ yields that

$$\frac{\theta(i)}{ti(\theta_{n+1}) + i^*\operatorname{Derlog} G} \cong M(g).$$

Now, notice that the evaluation map acting on ϵ gives $\epsilon(G) = G$, and hence $i^*(\operatorname{ev}(\epsilon)) = i^*(G) = g$. Therefore, if $K(g) = (J(g) + (g))/J(g)$, then

$$0 \to K(g) \to \frac{\theta(i)}{ti(\theta_{n+1}) + i^*\operatorname{Derlog} G} \to \frac{\theta(i)}{ti(\theta_{n+1}) + i^*\operatorname{Derlog} G + i^*(\epsilon)} \to 0$$

is a short exact sequence. Indeed, the evaluation map satisfies that $\operatorname{ev}(ti(\theta_{n+1})) = J(g)$ and that $\operatorname{ev}(i^*\operatorname{Derlog} G) = 0$. Hence, the evaluation map yields an isomorphism

$$\frac{ti(\theta_{n+1}) + i^*\operatorname{Derlog} G + i^*(\epsilon)}{ti(\theta_{n+1}) + i^*\operatorname{Derlog} G} \cong \frac{J(g) + (g)}{J(g)} = K(g).$$

After taking lengths in the exact sequence, and taking into account the previous assertions, we obtain the desired formula. □

The formula of Theorem 6.1 has the following direct consequence. We recall that two functions $g, g' \in \mathscr{O}_{n+1}$ are called $\mathscr{R}$-equivalent if there exists a biholomoprhism $\varphi\colon (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{n+1}, 0)$ such that $g' = g \circ \varphi$.

Corollary 6.2 *With the hypothesis of Theorem 6.1,* $\dim_{\mathbb{C}} M(g)$ *only depends on the $\mathscr{R}$-equivalence class of g, and neither depends on the mapping f nor on the chosen extension $\hat{f}$.*

When either $(n, n+1)$ are nice dimensions or (X, f) has corank one, all stable singularities are weighted homogeneous (see [12, Exercise 7.1.1 and Theorem 7.6]). This gives another consequence of Theorem 6.1:

Corollary 6.3 *Assume that either $(n, n+1)$ are nice dimensions or (X, f) has corank one. Then, $M(g) = 0$ if and only if (X, f) is stable.*

7 The Jacobian module and the Mond conjecture

This section is the centerpiece of this paper. We present one of the main results we aim to establish, namely, a formula for the image Milnor number expressed in terms of the Samuel multiplicity of the module $M_{\text{rel}}(G)$. Additionally, we prove that the generalised Mond conjecture holds provided the module $M_{\text{rel}}(G)$ is Cohen-Macaulay.

Theorem 7.1 *Assume that either $(n, n+1)$ are nice dimensions or (X, f) has corank one. Let F be a stable unfolding of $\hat{f}$. With the notations of Sect. 4,*

$$\mu_I(X, f) = e(\mathfrak{m}_{k+r}, M_{\text{rel}}(G)),$$

that is, $\mu_I(X, f)$ equals the Samuel multiplicity of the $\mathscr{O}_{k+r}$-module $M_{\text{rel}}(G)$ with respect to $\mathfrak{m}_{k+r}$.

Proof Take a representative of F and let $w \in \mathbb{C}^k \times \mathbb{C}^r$ be a generic value. The conservation of multiplicity implies that

$$e\Big(\mathfrak{m}_{k+r}, M_{\text{rel}}(G)\Big) = \sum_{p \in B_\epsilon} e\Big(\mathfrak{m}_{k+r,w}, M_{\text{rel}}(G)_{(p,w)}\Big),$$

where B_ϵ is a ball of radius $\epsilon > 0$ centered at the origin in $\mathbb{C}^{n+1}$. In order to compute the previous multiplicity, let us take first the points $p \in Y_w \cap B_\epsilon$, where Y_w is the image of $f_w : X_w \to \mathbb{C}^{n+1}$. Since the module $M_{\text{rel}}(G)$ specialises to $M(g)$, it follows that

$$M_{\text{rel}}(G)_{(p,w)} \otimes_{\mathscr{O}_{k+r,w}} \frac{\mathscr{O}_{k+r,w}}{\mathfrak{m}_{k+r,w}} \cong M(g_w)_p$$

as $\mathbb{C}$-vector spaces in virtue of Theorem 5.3. If F is a stable unfolding, it follows that, provided w is generic, f_w is a stable mapping, and hence $M(g_w)_p = 0$ for each $p \in Y_w$, by Corollary 6.3. Hence, the points $p \in Y_w \cap B_\epsilon$ do not contribute to the term $e(\mathfrak{m}_{k+r}, M_{\text{rel}}(G))$.

On the other hand, if $p \in B_\epsilon / Y_w$, then

$$M_{\text{rel}}(G)_{(p,w)} = \frac{\mathscr{O}_{\mathbb{C}^{k+r} \times B_\epsilon, (p,w)}}{J_y(G)}$$

is a module with dimension $\geq k+r$. Indeed, this follows from the exact sequence that defines $M_{\text{rel}}(G)$, since the localisation of $C(F)/(J_{y,z}(G) \cdot \mathscr{O}_{n+k+r})$ is zero if $p \notin Y_w$, and $\mathscr{F}_1(F)$ localises to $\mathscr{O}_{\mathbb{C}^{k+r} \times B_\epsilon, (p,w)}$. Moreover,

$$M_{\text{rel}}(G)_{(p,w)} \otimes_{\mathscr{O}_{k+r,w}} \frac{\mathscr{O}_{k+r,w}}{\mathfrak{m}_{k+r,w}} \cong \frac{\mathscr{O}_{B_\epsilon, p}}{J(g_w)}$$

is a module with dimension 0, since it has finite length due to the fact that g_u has isolated singularity. This implies that the dimension of $M_{\text{rel}}(G)$ is $\leq k+r$. Hence, $M_{\text{rel}}(G)_{(p,u)}$ is a complete intersection ring, and in particular a Cohen-Macaulay $\mathscr{O}_{k+r}$-module of dimension $k+r$. Hence,

$$\begin{aligned} e\Big(\mathfrak{m}_{k+r,w}, M_{\text{rel}}(G)_{(p,w)}\Big) &= \dim_{\mathbb{C}} \left(M_{\text{rel}}(G)_{(p,w)} \otimes \frac{\mathscr{O}_{k+r,w}}{\mathfrak{m}_{k+r,w}} \right) \\ &= \dim_{\mathbb{C}} \frac{\mathscr{O}_{B_\epsilon, p}}{J(g_w)} = \mu(g_w, p). \end{aligned}$$

By Siersma's Theorem 2.10, it follows that $\sum_{p \in B_\epsilon / Y_w} \mu(g_w, p) = \mu_I(X, f)$. □

Once this result has been proven, the following theorem is now an immediate consequence:

Theorem 7.2 *In the above conditions, the following statements are equivalent and imply the generalised Mond conjecture for* (X, f)*:*

(1) $\dim_{\mathbb{C}} M(g) = \mu_I(X, f)$*,*
(2) $M_{\text{rel}}(G)$ *is a Cohen-Macaulay.*

Furthermore, if g is weighted homogeneous and satisfies the generalised Mond conjecture, then the above assertions hold.

Proof Recent work by Nuño-Ballesteros and Giménez Conejero [5, 7] has shown that $M_{\text{rel}}(G)$ has dimension $k+r$. Recall that the Samuel multiplicity of an R-module M is generally smaller than the length of $M/(\mathfrak{m} \cdot M)$, with equality if and only if M is a Cohen-Macaulay module with the same dimension as R. Hence, $M_{\text{rel}}(G)$ is Cohen-Macaulay if and only if

$$\begin{aligned} \mu_I(X, f) = e\Big(\mathfrak{m}_{k+r}, M_{\text{rel}}(G)\Big) &= \dim_{\mathbb{C}} \frac{M_{\text{rel}}(G)}{\mathfrak{m}_{k+r} M_{\text{rel}}(G)} \\ &= \dim_{\mathbb{C}} M_{\text{rel}}(G) \otimes_{\mathscr{O}_{k+r}} \frac{\mathscr{O}_{k+r}}{\mathfrak{m}_{k+r}} = \dim_{\mathbb{C}} M(g). \end{aligned}$$

Therefore, if one of the items hold, it then follows that

$$\mu_I(X, f) = \dim_{\mathbb{C}} M(g) = \dim_{\mathbb{C}} K(g) + \operatorname{codim}_{\mathscr{A}_e}(X, f) \geq \operatorname{codim}_{\mathscr{A}_e}(X, f),$$

and hence, the generalised Mond conjecture holds for (X, f). Moreover, if g is weighted homogeneous, then $K(g) = 0$. Thus, if the generalised Mond conjecture holds for (X, f), then

$$\mu_I(X, f) = \operatorname{codim}_{\mathscr{A}_e}(X, f) = \dim_{\mathbb{C}} M(g) - \dim_{\mathbb{C}} K(g) = \dim_{\mathbb{C}} M(g).$$

Hence, the above assertions hold. □

This result shows that in order to establish the validity of the Mond conjecture for f, it suffices to check the Cohen-Macaulay property of the module $M_{\mathrm{rel}}(G)$. While it has been observed that this module is Cohen-Macaulay in every computed example, it remains an open question to provide a rigorous proof of this statement.

8 Proof of the generalised Mond conjecture for *n* = *2*

In this section, we achieve our main objective of the article, which is to show the validity of the generalised Mond conjecture for $n = 2$ by employing the results from the previous section.

Building on the main result of the previous section, to establish the generalised Mond conjecture for mappings $f : (X, 0) \to (\mathbb{C}^3, 0)$ where $(X, 0)$ is a 2-dimensional ICIS, it suffices to check that $M_{\mathrm{rel}}(G)$ is a Cohen-Macaulay module of dimension $k+r$. Notice first that the dimension of $M_{\mathrm{rel}}(G)$ is $\leq k + r$, due to the fact that

$$\dim_{\mathbb{C}} M_{\mathrm{rel}}(G) \otimes_{\mathscr{O}_{k+r}} \frac{\mathscr{O}_{k+r}}{\mathfrak{m}_{k+r}} = \dim_{\mathbb{C}} M(g) < +\infty.$$

Therefore, it is enough to show that depth $M_{\mathrm{rel}}(G) \geq k+r$. To achieve this, we apply the depth lemma [17, Tag 00LX] to the exact sequence

$$0 \to M_{\mathrm{rel}}(G) \to \frac{\mathscr{F}_1(F)}{J_y(G)} \to \frac{C(F)}{J_{y,z}(G) \cdot \mathscr{O}_{n+k+r}} \to 0$$

to obtain that

$$\operatorname{depth} M_{\mathrm{rel}}(G) \geq \min \left\{ \operatorname{depth} \frac{\mathscr{F}_1(F)}{J_y(G)}, \operatorname{depth} \frac{C(F)}{J_{y,z}(G) \cdot \mathscr{O}_{n+k+r}} + 1 \right\}.$$

Hence, both terms of the minimum have to be shown to be greater than or equal to $k + r$. For the module $C(F)/J_{y,z}(G) \cdot \mathscr{O}_{n+k+r}$, we already showed in (4) that it is Cohen-Macaulay of dimension $n+k+r-2$, provided $n \geq 2$. In particular, it follows that

$$\operatorname{depth} \frac{C(F)}{J_y(G) \cdot \mathscr{O}_{n+k+r}} + 1 = n + k + r - 1 \geq k + r.$$

Therefore, it is enough to verify that depth $\mathscr{F}_1(F)/J_y(G) \geq k + r$. Notice that, up to this point, the assumption $n = 2$ has not been used yet. In general, the module $\mathscr{F}_1(F)/J_y(G)$ has dimension

$$\dim \frac{\mathscr{F}_1(F)}{J_y(G)} = \max \left\{ \dim M_{\text{rel}}(G), \dim \frac{C(F)}{J_y(G) \cdot \mathscr{O}_{n+k+r,0}} \right\} = n + k + r - 2,$$

due to the fact that $\dim M_{\text{rel}}(G) \leq k + r \leq n + k + r - 2$ provided $n \geq 2$.

In fact, the same inequality gives depth $M_{\text{rel}}(G) \leq \dim M_{\text{rel}}(G) \leq k + r$ and again by the depth lemma, depth $\mathscr{F}_1(F)/J_y(G) \leq k + r$. This shows that $\mathscr{F}_1(F)/J_y(G)$ is not Cohen-Macaulay if $n > 2$. However, when $n = 2$ we make use of Pellikaan's Theorem (see [14, Proposition 3.4]):

Theorem 8.1 *If $J \subset F \subset R$ are ideals of R where J is generated by m elements,* $\operatorname{grade}(F/J) \geq m$ *and* $\operatorname{pd}(R/F) = 2$*, then F/J is a perfect module and* $\operatorname{grade}(F/J) = m$.

We recall that if R is a Noetherian ring, the grade of a finitely generated R-module M is defined in terms of the Ext functor as

$$\operatorname{grade} M = \inf\{i \mid \operatorname{Ext}^i_R(M, R) \neq 0\}.$$

The projective dimension, denoted by pd M, is the minimal length n of a projective resolution of M, that is, an exact sequence

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0,$$

where all P_i are projective. In general, $\operatorname{pd} M \geq \operatorname{grade} M$ (see [9, page 132]) and M is called perfect when $\operatorname{pd} M = \operatorname{grade} M$.

In the case that R is local and Cohen-Macaulay, we also have

$$\operatorname{grade} M = \operatorname{depth}(I, R) = \operatorname{ht} I = \dim R - \dim \frac{R}{I},$$

where $I = \operatorname{Ann} M$ (see [9, page 132 and Theorem 17.4]). Moreover, the Auslander-Buchsbaum formula (see [9, Theorem 19.1]) says that

$$\operatorname{pd} M = \operatorname{depth} R - \operatorname{depth} M,$$

which implies that M is perfect if and only if M is Cohen-Macaulay.

Theorem 8.2 *If $n = 2$, then $\mathscr{F}_1(F)/J_y(G)$ is a Cohen-Macaulay module.*

Proof We follow the notation of the previous result, taking $R = \mathscr{O}_{n+1+k+r}$, $F = \mathscr{F}_1(F)$ and $J = J_y(G)$. It follows from the proof of Theorem 3.4 of [13] that $R/F =$

$\mathcal{O}_{n+1+k+r}/\mathcal{F}_1(F)$ is a determinantal ring of dimension $n+k+r-1$, and hence Cohen-Macaulay. Hence, by the Auslander-Buchsbaum formula,

$$\begin{aligned}\operatorname{pd}(R/F) &= \operatorname{depth} R - \operatorname{depth} R/F = \dim R - \dim R/F \\ &= n+1+k+r-(n+k+r-1) = 2.\end{aligned}$$

Moreover, $J = J_y(G)$ is clearly generated by $n+1=3$ elements, namely the partial derivatives of G with respect to the variables $y_1, \dots, y_{n+1}$. Furthermore,

$$\begin{aligned}\operatorname{grade}(F/J) &= \operatorname{depth}(\operatorname{Ann}(F/J), \mathcal{O}_{n+1+r}) = \operatorname{ht}\operatorname{Ann}(F/J) = \\ &= \dim \mathcal{O}_{n+1+r} - \dim F/J = n+1+r-(n+r-2) = 3.\end{aligned}$$

Thus, Pellikaan's Theorem 8.1 states that F/J is a perfect $\mathcal{O}_{n+1+k+r}$-module. Furthermore, since $\mathcal{O}_{n+1+k+r}$ is a local Cohen-Macaulay ring, then F/J is Cohen-Macaulay. □

Now the main result of this article follows easily as an application of the results presented in the previous section.

Proof of Theorem 1.2 Theorem 8.2 implies that $M_{\mathrm{rel}}(G)$ is a Cohen-Macaulay module. Hence, Theorem 7.2 implies that the generalised Mond conjecture holds for (X, f). □

In this setting, the image Milnor number therefore satisfies that $\mu_I(X, f) = \dim_{\mathbb{C}} M(g)$. This gives an operative method to compute this number, as the following example shows:

Example 8.3 Let $(X, 0) \subset (\mathbb{C}^3, 0)$ be the hypersurface defined by $x^3+y^3-z^2=0$ and let $f : (X, 0) \to (\mathbb{C}^3, 0)$ be the $\mathscr{A}$-finite mapping $f(x, y, z) = (x, y, z^3+xz+y^2)$. In this case, $\hat{f}(x, y, z) = (x, y, z^3+xz+y^2, x^3+y^3-z^2)$ turns out to be a stable mapping. With SINGULAR [3], one easily obtains that $\operatorname{codim}_{\mathscr{A}_e}(X, f) = 6$ and $\mu_I(X, f) = \dim_{\mathbb{C}} M(g) = 6$.

In this example, the image $(Y, 0)$ of f has reduced equation

$$\begin{aligned}g(u, v, w) = &-u^9 - 2u^7 - 3u^6v^3 - u^5 - 4u^4v^3 - 3u^3v^6 \\ &- u^2v^3 - 2uv^6 - v^9 + v^4 - 2v^2w + w^2,\end{aligned}$$

which is not weighted homogeneous, although $g \in J(g)$. We do not know if there exists a coordinate change in $(\mathbb{C}^3, 0)$ such that the image has a weighted homogeneous equation.

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