APPLIED GENERAL TOPOLOGY

Appl. Gen. Topol. 26, no. 1 (2025), 241-253
doi:10.4995/agt.2025.21612

On the topology of topological fundamental groupoids

ALI PAKDAMAN

Department of Mathematics, Faculty of Sciences, Golestan University, P.O.Box 155, Gorgan, Iran. (a.pakdaman@gu.ac.ir)

Communicated by F. Lin

ABSTRACT

This paper is devoted to the Lasso and the Whisker topology of the fundamental groupoid. We prove that the topological fundamental groupoid of a given space X is locally path connected in general and is path connected if X is simply connected. We show that for locally path connected space X, the unit map $1 : X \longrightarrow \pi X$ is an embedding if and only if X is a semilocally simply connected space. Also, we give conditions that guarantee Hausdorffness of the topological fundamental groupoid.

2020 MSC: *22A22; 55Q05; 57M05.*

KEYWORDS: *topological fundamental groupoid; Lasso topology; Whisker topology.*

1. INTRODUCTION

Fundamental groupoid is equipped with several topologies: Lifted topology [3], Lasso topology [18], Whisker topology [19], compact-open (CO) topology and universal covering (UC) topology [11].

- πX by the lifted topology is a topological groupoid for locally path connected and semilocally simply connected spaces.
- πX by the Lasso topology is a topological groupoid for locally path connected spaces.

Received 24 April 2024 – Accepted 30 November 2024

- πX by the Whisker topology is a topological groupoid for locally path connected and semilocally small loop transfer spaces.
- πX by the CO topology and UC topology is a topological groupoid for locally path connected and semilocally simply connected spaces.

When the given space X is locally path connected and semilocally simply connected, all topologies defined on its fundamental group are equivalent and this also happens for its universal path space [24]. This rule is not an exception for the topological fundamental groupoids and hence the topologies that have been introduced so far are equivalent under these nice local conditions (locally path connectedness and semilocally simply connectedness) [11, 16, 20].

- The lifted topology and the Lasso topology on the topological fundamental groupoid of locally path connected and semilocally simply connected spaces are equivalent [20, Proposition 3.10 and Theorem 3.17].
- The Lasso topology and the Whisker topology on the topological fundamental groupoid of locally path connected and semilocally simply connected spaces are equivalent [16].
- CO topology and UC topology on the topological fundamental groupoid of locally path connected and semilocally simply connected spaces are equivalent [11].

Except the Lasso and the Whisker topologies, the other topologies are only defined for locally path connected and semilocally simply connected spaces (of course, they can probably be generalized for more general spaces). Therefore, we work with the Lasso and the Whisker topologies to study the topological fundamental groupoid of spaces that do not have these good local properties.

This article examines the topological properties of fundamental groupoids, such as path connectivity, local path connectivity and separation axioms.

In fact, we prove that the topological fundamental groupoid is always locally path connected. This result is important because it simply follows that the topological fundamental groupoid cannot be *étale* [16]. If the topological fundamental groupoid is *étale*, then the target map $T : \pi X \longrightarrow X$ will be a local homeomorphism and the space X must also be locally path connected, but it may not be so. Moreover, we prove that if the space X is simply connected, its topological fundamental groupoid will be path connected.

Also, we will clear the ambiguity of the existing definitions of the topological groupoid and we will specify some sensitivities. In particular, we prove that the unit map in the topological fundamental topological groupoid of locally path connected and semilocally simply connected spaces is an imbedding and vice versa. We will also specify the necessary and sufficient condition for $1(X)$ to be closed in πX.

Finally, we will examine the conditions that make the fundamental groupoid Hausdorff. In [11], it is proved that for locally path connected and semilocally simply connected space X, πX is Hausdorff if and only if X is Hausdorff. We will prove a same result here (for more general spaces) with a weaker condition.

2. Preliminaries

A groupoid is a small category in which every arrow is invertible. Thus a groupoid can be denoted as a (G, G_0) where G is the small category with G_0 as the set of objects. For an arrow $a \in G$, $T(a)$ and $S(a)$ and $i(a)$ are, respectively, the target of a, the source of a and a^{-1} such that $i \circ i$ is the identity map on G, and $T = S \circ i$. For a groupoid G, $m : G_2 \longrightarrow G$ is called the composition map (which is abbreviated $m(a, b)$ as ab), where $G_2 \subseteq G \times G$ denotes the set of composable arrows and $1 : G_0 \longrightarrow G; x \mapsto 1_x$, is called the unit map.

The groupoid G is called topological if G and G_0 also carry two topologies such that these 5 structure maps are continuous (for the continuity of the multiplication map, G_2 is equipped with subspace topology of $G \times G$).

The set of arrows from $x \in G_0$ to $y \in G_0$ is denoted by $G(x, y)$ and, in particular, $G(x) := G(x, x)$ is called the object group (or vertex group) at x. Also, we denote $S^{-1}(x)$ by G_x and $T^{-1}(x)$ by G^x. A groupoid G is called 1-connected if for every $x, y \in G_0$ we have $|G(x, y)| = 1$.

The fundamental groupoid πX, for a topological space X, is a groupoid that has homotopy classes of paths in X as the set of morphisms and has the set X as its set of objects. Composition of morphisms $[\alpha], [\beta]$ is $[\alpha * \beta]$ and the identity in $\pi X(x, x)$ is the $e_x = [c_x]$, where c_x is the constant path at $x \in X$. The object group at x, $\pi X(x)$ considered to be the well-known fundamental group $\pi_1(X, x)$ [2].

Lasso topology: Here, we recall the Lasso topology on the fundamental groupoid from [18].

If $\mathcal{U}$ is an open cover of X, the *Spanier group with respect to* $\mathcal{U}$, denoted by $\pi(\mathcal{U}, x)$ [8, 22] is the subgroup of $\pi_1(X, x)$ consisting of the homotopy classes of loops that can be represented by a product of the following type:

$$\prod_{j=1}^{n} \alpha_j * (\beta_j * \alpha_j^{-1}),$$

where the α_j's are arbitrary paths starting at the base point x and each β_j is a loop inside one of the neighborhoods $U_i \in \mathcal{U}$. The Spanier group of X, denoted by $\pi_1^{sp}(X, x)$ is the intersection of all the Spanier groups with respect to open covers of X. Spanier groups are important in covering theory [1, 15].

Let $\mathcal{U}$ be an open cover of a given space X and for $x, y \in X$, let $[\alpha] \in \pi X(x, y)$. If V, $W \in \mathcal{U}$ are open neighborhoods of x, y, respectively. Consider

$$N([\alpha], \mathcal{U}, V, W) =$$

$$\{[\beta] \in \pi X \mid \ \beta \simeq \gamma * \mu * \alpha * \mu' * \lambda,\ \gamma : I \to V, \lambda : I \to W, [\mu] \in \pi_1(\mathcal{U}, x), [\mu'] \in \pi_1(\mathcal{U}, y)\}.$$

The family

$$\big\{N([\alpha], \mathcal{U}, V, W) \big|\ \mathcal{U} \text{ is an open cover of } X; V, W \in \mathcal{U},\ [\alpha] \in \pi X(x, y),\ x \in V,\ y \in W\big\}$$

form a basis for a topology on πX, named **Lasso topology** and denoted by $\pi^l X$, in which makes it a topological groupoid, when X is locally path connected [18]. Hence the operation π^l is a functor from the category of locally path connected topological spaces and continuous map to the category of topological groupoids [18].

Whisker topology: For a given topological space X, let $[\alpha] \in \pi X(x, y)$, where $x, y \in X$. If V, W are open neighborhoods of x, y, respectively, one can define [19]

$$N([\alpha], V, W) := \{[\beta] \in \pi X | \beta \simeq \gamma * \alpha * \lambda,\ \gamma(I) \subseteq V,\ \lambda(I) \subseteq W\},$$

where $\gamma(1) = x$ and $\alpha(1) = \lambda(0) = y$. Then the family

$$\{N([\alpha], V, W);\ [\alpha] \in \pi X(x, y),\ x \in V,\ y \in W\},$$

forms a basis for a topology on fundamental groupoid, named **Whisker topology** and denoted by $\pi^w X$.

The fundamental groupoid with the Whisker topology is a topological groupoid if X is a small loop transfer space [19]. A topological space X is called a *small loop transfer* (SLT) if for every $x \in X$ and every open neighborhood $U \subseteq X$ containing x there is an open cover $\mathcal{U}$ of X at x such that $\pi(\mathcal{U}, x) \leq i_* \pi_1(U, x)$ [6].

In this article, all the homotopies are path homotopy.

3. Open and Closed Sets

For simply connected spaces, we know that the fundamental groupoid is 1-connected, which is not a topological property of the fundamental groupoid. We are interested in investigating under what conditions the topological fundamental groupoid will become path connected or locally path connected? The following propositions will partially answer this question.

Proposition 3.1. *Let X be a simply connected space. Then $\pi^w X$ ($\pi^l X$) is path connected.*

Proof. Let $[\alpha] \in \pi^w X(x,y)$ and $[\beta] \in \pi^w X(z,p)$. Since X is path connected, there are two paths: λ from x to z and μ from y to p such that $\lambda^{-1} * \alpha * \mu \simeq \beta$ because X is simply connected. For every $s \in I$, define paths $\lambda_s,\ \mu_s$ in X by $\lambda_s(t) = \lambda(st)$ and $\mu_s(t) = \mu(st)$. It can be easily checked that

$$\lambda_s(0) = x, \quad \lambda_s(1) = \lambda(s), \quad \lambda_0 = c_x, \quad \lambda_1 = \lambda,$$

$$\mu_s(0) = y, \quad \mu_s(1) = \mu(s), \quad \mu_0 = c_y, \quad \mu_1 = \mu.$$

For every $s \in I$, consider the path $\varphi_s := \lambda_s^{-1} * (\alpha * \mu_s)$ from $\lambda(s)$ to $\mu(s)$ and define $\varphi : I \longrightarrow \pi^w X$ by

$$\varphi(s) = [\varphi_s].$$

Observe that $\varphi(0) = [\alpha]$ and that $\varphi(1) = [\lambda^{-1} * \alpha * \mu] = [\beta]$. It remains to prove that φ is continuous.

Let $s_0 \in I$ and let $N(\varphi(s_0), U, V)$ be a basic open neighborhood of $\varphi(s_0)$. We have $\varphi_{s_0}(0) = \lambda(s_0) \in U$ and $\varphi_{s_0}(1) = \mu(s_0) \in V$. Since λ and μ are continuous, there are open intervals W and W' of s_0 with $\lambda(W) \subseteq U$ and $\mu(W') \subseteq V$. Let $O = W \cap W'$; we will have $\varphi(O) \subseteq N(\varphi(s_0), U, V)$ because for every $s \in O$, φ_s is a concatenation of φ_{s_0} from both sides in U and V. In fact, $\varphi_s = f_s * \varphi_{s_0} * g_s$, where f_s is the path homotopy class of the path $(\lambda|_{[s_0,s]})^{-1}$ when $s > s_0$ and $(\lambda|_{[s,s_0]})^{-1}$ when $s < s_0$ and also, g_s is the path homotopy class of the path $\mu|_{[s_0,s]}$ when $s > s_0$ and $\mu|_{[s,s_0]}$ when $s < s_0$. □

If a topological fundamental groupoid G is *étale*, then we can conclude that there is a similar local behavior between G and G_0. But topological fundamental groupoid is not *étale* [16] and we can not conclude that locally path connectedness of X is equivalent to locally path connectedness of $\pi^w X$. The next proposition shows that without the locally path connectedness of X, its fundamental groupoid becomes locally path connected and it is another proof for this assertion that topological fundamental groupoids are not *étale*.

Proposition 3.2. *For a given space X, its topological fundamental groupoid $\pi^w X$ ($\pi^l X$) is locally path connected.*

Proof. Let $[\alpha] \in \pi^w X(x,y)$ and O be an open neighborhood of $[\alpha]$. There exists a basic open neighborhoods $N([\alpha], U, V)$ contained in O. We claim that $N([\alpha], U, V)$ is path connected neighborhood of $[\alpha]$.

For every $[\beta] \in N([\alpha], U, V)$, there are paths λ in U and μ in V such that $\beta \simeq \lambda * \alpha * \mu$. Define

$\varphi : I \longrightarrow N([\alpha], U, V)$ by $\varphi(s) = [\lambda_s^{-1} * \alpha * \mu_s]$, where λ_s and μ_s are the paths introduced in the Proposition 3.1.
Observe that $\varphi(0) = [\alpha]$ and that $\varphi(1) = [\lambda^{-1} * \alpha * \mu] = [\beta]$. Continuity of φ is also obtained by a method similar to Proposition 3.1. This completes the proof. $\square$

Remark 3.3. A point worth pondering is that in some references [13, 2, 3], in the definition of topological groupoid G over G_0, there are two topologies, one on the G and the other on the G_0, so that 5 structure maps: $m,\ i,\ 1,\ S,\ T$ are continuous. But in some other references [21] (especially those who go into the topics of algebraic operators, etc.) there is only one topology on the G such that maps $m,\ i$ are continuous. Here, G_0 has the induced topology from G, by considering G_0 as a subset of G by identifying it with $1(G_0)$. In the first case, continuity of S, T comes from the continuity of m but this does not happen for unit map 1. In the second case, continuity of S, T also comes from the continuity of m and the unit map will be automatically continuous and in addition, will be an imbedding. When we work with fundamental groupoids, we have a topology on the space X and we cannot consider the induced topology on X from the $\pi^l X$ or $\pi^w X$. Now, if in the definition of the topological groupoid, only the continuity of m and i is included, we must check the continuity of the unit map 1.
At first note that $1 : X \longrightarrow \pi^l X$ by $1(x) = e_x = [c_x]$. For every basic (Lasso) open neighborhood $N(e_x, \mathcal{U}, U, V)$ of e_x we have $1^{-1}(N(e_x, \mathcal{U}, U, V))$ is the path component of $U \cap V$ containing x. So 1 will be continuous if X is locally path connected, which makes its path components open. This happens for the whisker topology as well.

We can solve the continuity problem by assuming locally path connectedness of X, but it requires a stronger assumption that the unit map be an imbedding.

Theorem 3.4. *For locally path connected space X, the unit map $1 : X \longrightarrow \pi^w X$ ($1 : X \longrightarrow \pi^l X$) is an embedding if and only if X is a semilocally simply connected space.*

Proof. Let X be a locally path connected and semilocally simply connected space. By the above discussion, $1 : X \longrightarrow \pi^w X$ is continuous. It suffices to show that $1(U)$ is open in $1(X)$ when $U \subseteq X$ is open. For $e_x \in 1(U)$ assume that V is a path connected open neighborhood of x such that $i_*(\pi_1(V, y)) = \{e_y\}$, for every $y \in V$, where $i : V \hookrightarrow X$ is the inclusion map. Then $1(V) = N(e_x, V, V) \cap 1(X) \subseteq 1(U)$ which implies that $1(U)$ is open in $1(X)$.
Conversely, assume that the unit map $1 : X \longrightarrow \pi^w X$ is an embedding. For $x \in X$, Let U be the path connected open neighborhood of x. Then $1(U)$ is open in $1(X)$. Hence there exists basic open neighborhood $N(e_x, V, W)$ of e_x in $\pi^w X$ such that $N(e_x, V, W) \subseteq 1(U)$. Let P be the path component of $V \cap W$ containing x. We claim that $i_*(\pi_1(P, x)) = \{e_x\}$, where

$i : P \hookrightarrow X$ is the inclusion map. For if, $[\alpha] \in i_*(\pi_1(P,x))$, then α is a loop in P based at x and $[\alpha * c_x] \in N(e_x, V, W) \subseteq 1(U)$. Thus $[\alpha] = 1(x) = e_x$, as desired. □

Target map and source map are continuous and so for T_1 space X and $x \in X$, πX_x and πX^x are closed subspaces of $\pi^l X$ ($\pi^w X$) because they are inverse image of the closed set $\{x\}$. Also, $\pi_1(X,x)$ is closed in πX because it is $T^{-1}(x) \cap S^{-1}(x)$.

According to Remark 3.3 and the previous theorem, we cannot consider general space X topologically as a subset of πX by identifying it with the set of unit arrows on the objects. But we are interested to know whether $1(X)$ in πX is closed or open? As we said in Remark 3.3, it is important what definition we work with. By definition of algebraic operatorists, when G is a Hausdorff topological groupoid, G_0 is closed in G [21]. If we do not work with this definition, other results will be obtained.

If the space X is Hausdorff, the single points are closed, and in a topological group such as H, the set $\{e_H\}$ is closed if and only if H is a Hausdorff topological group. In the following, we provide a similar rule (a groupoid version) for $1(X)$ to be closed in the topological fundamental groupoid πX.

We recall that a space X is called *homotopically Hausdorff* if given any point x in X and any nontrivial homotopy class $[\alpha] \in \pi_1(X,x)$, then there is a neighborhood U of x which contains no representative for $[\alpha]$, [9].

Theorem 3.5. *For a Hausdorff space X, $1(X)$ is closed in πX if and only if X is homotopically Hausdorff.*

Proof. Our proof is for the Whisker topology, and for the Lasso topology the argument is similar.

Let X be homotopically Hausdorff and $[\alpha] \notin 1(X)$.

(i) If $[\alpha] \in \pi X(x,y)$, for $x \neq y \in X$, then there are open neighborhoods U and V containing x and y, respectively such that $U \cap V = \varnothing$. Hence $N([\alpha], U, V) \cap 1(X) = \varnothing$ which implies that $1(X)$ is closed in $\pi^w X$ and so in $\pi^l X$.

(ii) If $e_x \neq [\alpha] \in \pi X(x,x)$, for $x \in X$, then there exists open neighborhood U of x such that contains no representative for $[\alpha]$, Since X is homotopically Hausdorff. We claim that $N([\alpha], U, U) \cap 1(X) = \varnothing$. By contrary, assume that there exists $y \in X$ such that $e_y = 1(y) \in N([\alpha], U, U) \cap 1(X)$. There are paths λ and μ in U from x to y such that $\lambda^{-1} * \alpha * \mu \simeq c_y$. Hence $\alpha \simeq \lambda * \mu^{-1}$ which is contradiction because $\lambda * \mu^{-1}$ is a loop in U which contains no representative for $[\alpha]$.

Conversely, assume that $1(X)$ is closed in πX. By contrary assume that X is not homotopically Hausdorff. Then there exists $x \in X$ and $e_x \neq [\alpha] \in \pi(X,x)$ such that for every open neighborhood U of x, there is a loop $\alpha_U : I \longrightarrow U$ based at x such that $[\alpha] = [\alpha_U]$. Now, $[\alpha] \notin 1(X)$ but every basic open neighborhood $N(e_x, U, V)$ of $e_x = 1(x)$ contains $[\alpha] = [\alpha_U] = [\alpha_V]$. This means $[\alpha] \in \overline{1(X)} = 1(X)$ which is contradiction. □

4. Separation Properties

In this section we study separation properties of fundamental groupoid, equipped by the Lasso topology and the Whisker topology.

Note that a topological group G is Hausdorff if and only if $\{e\}$ is closed in G, where e is the identity element [13]. In fact, continuity of $f : G \times G \to G$ defined by $f(g,h) = gh^{-1}$ makes the diagonal subset $f^{-1}(e)$ of $G \times G$ closed, when G is Hausdorff.

We recall that the Lasso topology on the fundamental group [5], has a basis containing the sets in the following form

$$N([\alpha],\mathcal{U}) = \{[\beta] \in \pi_1(X,x) | \ \beta \simeq \alpha * \mu, \ for\ some\ [\mu] \in \pi_1(\mathcal{U},x)\},$$

where $[\alpha] \in \pi_1(X,x)$ and $\mathcal{U}$ is an open cover of X and is denoted by $\pi_1^l(X,x)$. The Lasso topology on fundamental groupoid is a generalization of the Lasso topology on $\pi_1^l(X,x)$ [18].

Lemma 4.1. *Let X be a topological space and $x \in X$. If the fundamental group is endowed with the Lasso topology, then $\pi_1^{sp}(X,x) = \overline{\{e_x\}}$.*

Proof. Let $[\alpha] \in \pi_1^{sp}(X,x)$, $\mathcal{U}$ be an open cover of the topological space X and $O := N([\alpha],\mathcal{U})$ be a basic open neighborhood of $[\alpha]$. Since $[\alpha] \in \pi_1^{sp}(X,x)$, $[\alpha]$, $[\alpha]^{-1} \in \pi(\mathcal{U},x)$. Thus $e_x = [c_x] = [\alpha * \alpha^{-1}] = [\alpha] * [\alpha]^{-1} \in O$. Therefore $O \cap \{e_x\} \neq \varnothing$. Since the intersection of every basic open neighborhood of $[\alpha]$ with $\{e_x\}$ is nonempty, we have $[\alpha] \in \overline{\{e_x\}}$.

Conversely, if $[\gamma] \in \overline{\{e_x\}}$, then for every basic open neighborhood $N([\gamma],\mathcal{U})$ of $[\gamma]$, $N([\gamma],\mathcal{U}) \cap \{e_x\} \neq \varnothing$. Thus $e_x \in N([\gamma],\mathcal{U})$ which implies that $c_x \simeq \gamma * \beta$, for some $[\beta] \in \pi(\mathcal{U},x)$. Hence $e_x = [\gamma] * [\beta]$. Obviously γ is a loop based at x and so for every open cover $\mathcal{U}$ of X, $[\gamma] = e_x * [\beta^{-1}] \in \pi(\mathcal{U},x)$ which implies $[\gamma] \in \pi_1^{sp}(X,x)$. □

Corollary 4.2. *For every space X and every $x \in X$, $\pi_1^l(X,x)$ is Hausdorff if and only if $\pi_1^{sp}(X,x) = \{e_x\}$.*

It seems that by the condition $\pi_1^{sp}(X,x) = \{e_x\}$ on the Hausdorff space X, it can be proved that the fundamental groupoid is also Hausdorff by the Lasso topology. We prove this claim in the next theorem.

Theorem 4.3. *Let X be a Hausdorff space and $x \in X$. Then $\pi^l X$ is Hausdorff if $\pi_1^{sp}(X,x) = \{e_x\}$.*

Proof. Assume that the Spanier group $\pi_1^{sp}(X,x)$ is trivial and let $[\alpha] \neq [\beta] \in \pi^l X$.

- If $\alpha(0) \neq \beta(0)$:
  There are open neighborhoods U and V of $\alpha(0)$ and $\beta(0)$ respectively, such that $U \cap V = \varnothing$. Let $\mathcal{U}'$ be an open cover of the topological space X and $\mathcal{U} := \mathcal{U}' \cup \{U, V\}$. If W and W' are elements of $\mathcal{U}$ containing $\alpha(1)$ and $\beta(1)$ respectively, then $O := N([\alpha], \mathcal{U}, U, W)$ and $O' := N([\beta], \mathcal{U}, V, W')$ are basic open neighborhoods of $[\alpha]$ and $[\beta]$ respectively. We show that $O \cap O' = \varnothing$.

  Let $[\lambda]$ and $[\gamma]$ be arbitrary elements of O and O', respectively. Then $\lambda(0) \in U$ and $\gamma(0) \in V$. Since $U \cap V = \varnothing$, we have $[\lambda] \neq [\gamma]$ and hence $O \cap O' = \varnothing$. If $\alpha(1) \neq \beta(1)$, by a similar proof we have $O \cap O' = \varnothing$.
- If $\alpha(0) = \beta(0) = x$, $\alpha(1) = \beta(1) = y$, $x \neq y$:
  Let $\mathcal{U}'$ be an arbitrary open cover of the topological space X, U and V be open neighborhoods in X containing x and y, respectively such that $U \cap V = \varnothing$ and U' and V' be elements of $\mathcal{U}'$ contaning x and y respectively. Let $U'' = U' \cap U$ and $V'' = V' \cap V$. Now $\mathcal{U} := \mathcal{U}' \cup \{U'', V''\}$ is an open cover of X. Suppose that $O := N([\alpha], \mathcal{U}, U'', V'')$ and $O' := N([\beta], \mathcal{U}, U'', V'')$ are basic neighborhoods of $[\alpha]$ and $[\beta]$. We show that $O \cap O' = \varnothing$. By contradiction, suppose that $[\eta] \in O \cap O'$, then by [18, Lemma 2.3]
  $$N([\alpha], \mathcal{U}, U'', V'') = N([\eta], \mathcal{U}, U'', V'') = N([\beta], \mathcal{U}, U'', V''),$$
  which implies $O = O'$ and hence $[\beta] \in O' = O$. Therefore $\beta \simeq \lambda * \mu * \alpha * \mu' * \lambda'$ where λ and λ' are paths in open neighborhoods U'' and V'', respectively, $[\mu] \in \pi_1(\mathcal{U}, x)$ and $[\mu'] \in \pi_1(\mathcal{U}, y)$. By the hypothesis $\lambda(0) = \beta(0) = \alpha(0) = x = \lambda(1)$ and $\lambda'(1) = \beta(1) = \alpha(1) = y = \lambda'(0)$. Thus λ and λ' are loops in U'' and V'', respectively which implies that $[\lambda * \mu] \in \pi(\mathcal{U}, x)$ and $[\mu' * \lambda'] \in \pi_1(\mathcal{U}, y)$. Obviously
  $$\beta * \alpha^{-1} \simeq \lambda * \mu * \alpha * \mu' * \lambda' * \alpha^{-1}.$$
  Since $[\alpha * (\mu' * \lambda') * \alpha^{-1}] \in \pi(\mathcal{U}, x)$, we have $[\beta * \alpha^{-1}] \in \pi(\mathcal{U}, x) \leq \pi_1(\mathcal{U}', x)$. But $\mathcal{U}'$ is an arbitrary open cover of X which implies that $[\beta * \alpha^{-1}] \in \pi_1^{sp}(X, x)$. Therefore $[\beta * \alpha^{-1}] = e_x$ that means $\beta \simeq \alpha$, which is a contradiction. Thus $O \cap O' = \varnothing$.
- If $\alpha(0) = \beta(0) = \alpha(1) = \beta(1) = x$:
  In this case, $[\alpha] \neq [\beta] \in \pi^l X(x,x) = \pi_1^l(X,x)$ and the proof is obvious by Corollary 4.2.

□

Theorem 4.4. *Let X be a Hausdorff topological space. The following assertions are equivalent.*

(i) $\pi^l X$ *is* T_0.

(ii) $\pi^l X$ *is* T_1.

(iii) $\pi^l X$ *is* T_2.

(iv) *For every* $x \in X$, $\pi_1^{sp}(X,x) = \{e_x\}$.

Proof. We proved in Theorem 4.3 that if $\pi_1^{sp}(X,x) = \{e_x\}$ then $\pi^l X$ is Hausdorff (T_2). Since every T_2 space is T_1 and every T_1 space is T_0, $(iv) \Rightarrow (iii)$, $(iii) \Rightarrow (ii)$ and $(ii) \Rightarrow (i)$ are obvious. It remains to prove $(i) \Rightarrow (iv)$.

Let $e_x \neq [\alpha] \in \pi^l(X,x)$. Since $\pi^l(X,x)$ is T_0, either there exists an open neighborhood containing e_x but not $[\alpha]$, or there exists an open neighborhood containing $[\alpha]$ but not e_x.

Let $N([c_x],\mathcal{U})$ be an open neighborhood of e_x such that $[\alpha] \notin N([c_x],\mathcal{U})$. Since $N([c_x],\mathcal{U}) = \pi(\mathcal{U},x)$, there is an open cover $\mathcal{U}$ of X such that $[\alpha] \notin \pi(\mathcal{U},x)$ i.e. $[\alpha] \notin \bigcap_{\mathcal{U}} \pi(\mathcal{U},x) = \pi_1^{sp}(X,x)$. Thus $\pi_1^{sp}(X,x) = \{e_x\}$.

Let $N([\alpha],\mathcal{U})$ be an open neighborhood of $[\alpha]$ such that $e_x \notin N([\alpha],\mathcal{U})$, then $[\alpha] \notin \overline{\{e_x\}}$. By Lemma 4.1, $[\alpha] \notin \pi_1^{sp}(X,x)$. Thus $\pi_1^{sp}(X,x) = \{e_x\}$. $\square$

For locally path connected and semilocally simply connected spaces, the Spanier group is trivial [8] and using Theorem 3.4 and Theorem 4.4 we have the following corollary.

Corollary 4.5. *Let X be a locally path connected and semilocally simply connected space. Then the following assertions are equivalent:*

(i) X *is Hausdorff.*

(ii) $\pi^l X$ *is* T_0.

(iii) $\pi^l X$ *is* T_1.

(iv) $\pi^l X$ *is* T_2.

Now, we want to examine Theorem 4.3 and Theorem 4.4 for the Whisker topology. The structure of the proofs is not very different, but the details are a little sensitive. At first, we prove a copy of Lemma 4.1.

Lemma 4.6. *Let X be a SLT topological space and $x \in X$. If the fundamental group is endowed with the Whisker topology, then* $\pi_1^{sp}(X,x) = \overline{\{e_x\}}$.

Proof. Let $[\alpha] \in \pi_1^{sp}(X,x)$ and $O = N([\alpha],U)$ be a basic open neighborhood of $[\alpha]$. By definition of SLT space, there exists an open cover $\mathcal{U}$ of X such that $\pi(\mathcal{U},x) \leq i_*\pi(U,x)$. Since $[\alpha] \in \pi_1^{sp}(X,x)$, we have $[\alpha] \in \pi(\mathcal{U},x)$ and hence $[\alpha] \in i_*\pi_1(U,x)$. Thus there exists a loop β in U

such that $\alpha \simeq \beta$. Then $c_x \simeq \alpha * \beta^{-1}$ which implies $[c_x] \in N([\alpha], U)$. Hence for every open neighborhood U of x we have $N([\alpha], U) \cap \{e_x\} \neq \varnothing$, i.e. $[\alpha] \in \overline{\{e_x\}}$.

Conversely let $[\alpha] \in \overline{\{e_x\}}$ and $\mathcal{U}$ be an arbitrary open cover of X. Let U be an element of $\mathcal{U}$ containing x. Since $[\alpha] \in \overline{\{e_x\}}$, we have $N([\alpha], U) \cap \{e_x\} \neq \varnothing$ that means $e_x \in N([\alpha], U)$. Thus there exists $[\beta] \in \pi_1(U, x)$ such that $e_x \simeq \alpha * \beta$ and hence $\alpha \simeq e_x * \beta^{-1}$, where $[\beta^{-1}] \in \pi_1(U, x)$. Therefore $[\alpha] \in \pi(\mathcal{U}, x)$ for every open cover $\mathcal{U}$ of X. Hence $[\alpha] \in \bigcap_{\mathcal{U}} \pi(\mathcal{U}, x) = \pi_1^{sp}(X, x)$. □

Corollary 4.7. *For every SLT space X and every $x \in X$, $\pi_1^w(X, x)$ is Hausdorff if and only $\pi_1^{sp}(X, x) = \{e_x\}$.*

The next theorem is the Whisker version of Theorem 4.3 and its proof is just a rewrite with the definition of the Whisker topology and its left to the reader.

Theorem 4.8. *Let X be an SLT Hausdorff space and $x \in X$. If $\pi_1^{sp}(X, x) = \{e_x\}$, then $\pi^{wh} X$ is Hausdorff.*

Theorem 4.9. *Let X be SLT topological space. The following propositions are equivalent.*
(i) $\pi^w X$ is T_0.
(ii) $\pi^w X$ is T_1.
(iii) $\pi^w X$ is T_2.
(iv) For every $x \in X$, $\pi_1^{sp}(X, x) = \{e_x\}$.

Proof. We just prove $(i) \Rightarrow (iv)$. Let $e_x \neq [\alpha] \in \pi^w(X, x)$. Since $\pi^w(X, x)$ is T_0, either there exists an open neighborhood containing e_x but not $[\alpha]$, or there exists an open neighborhood containing $[\alpha]$ but not e_x.

Let $N([c_x], U)$ be an open neighborhood of e_x such that $[\alpha] \notin N([c_x], U)$. Since X is SLT, by definition there is an open cover $\mathcal{U}$ of X such that $\pi(\mathcal{U}, x) \leq i_* \pi_1(U, x)$. Hence $[\alpha] \notin \pi(\mathcal{U}, x)$ which implies $[\alpha] \notin \pi_1^{sp}(X, x)$. This means $\pi_1^{sp}(X, x) = \{e_x\}$.

Let $N([\alpha], U)$ be an open neighborhood of $[\alpha]$ such that $e_x \notin N([\alpha], U)$. Then $[\alpha] \notin \overline{\{e_x\}}$. By Lemma 4.6, $[\alpha] \notin \pi_1^{sp}(X, x)$. Thus $\pi_1^{sp}(X, x) = \{e_x\}$. □

Note that Corollary 4.5 also holds for $\pi^w X$ because the given space is locally path connected and semilocally simply connected that makes the Lasso topology and the Whisker topology equivalent.

Acknowledgements. *The authors would like to thank the referee for his/her careful reading and useful suggestions.*

References

[1] J. Brazas, Thick Spanier groups and the First Shape Group, The Rocky Mountain Journal of Mathematics, 44 (2014), 1415–1444.

[2] R. Brown, Topology and Groupoids, Deganwy, United Kingdom: BookSurge LLC, 2006.

[3] R. Brown and G. Danesh-Naruie, Topological fundamental groupoid as topological groupoid, Proc. Edinburgh Math. Soc. 19 (1975), 237–244.

[4] R. Brown and J. PL. Hardy, Topological groupoid: I. Universal constructions, Math. Nachr. 71 (1976) 273-286.

[5] N. Brodskiy, J. Dydac, B. Labuz, and A. Mitra, Covering maps for locally path-connected spaces, Fundamenta Mathematicae 218 (2012), 13–46.

[6] N. Brodskiy, J. Dydac, B. Labuz, and A. Mitra, Topological and uniform structures on universal covering spaces, arXiv:1206.0071.

[7] C. Ehresmann, Catégories topologiques et catégories différentiables, Colloque Géom. Diff. Globale (Bruxelles, 1958), Centre Belge Rech. Math., Louvain, 1959, pp. 137-150 (French).

[8] H. Fischer, D. Repovš, Ž. Virk, and A. Zastrow, On semilocally simply connected spaces, Topology and its Applications 158 (2011), 397–408.

[9] H. Fischer, and A. Zastrow, Generalized universal covering spaces and the shape group, Fundam. Math. 197 (2007), 167–196.

[10] P. J. Higgins, Notes on Categories and Groupoids, Van Nostrand 1971.

[11] R. D. Holkar and MD. A. Hossain, Topological fundamental groupoid. I., arXiv:2302.01583.

[12] O. Mucuk, T. Sahan, N. Alemdar, Normality and quotients in crossed modules and group-groupoids, Applied Categorical Structures 23 (2015), 415–428.

[13] K. MacKenzie, Lie Groupoids and Lie Algebroids in Differential Geometry, London Mathematical Society Lecture Note Series, (124) 1987.

[14] S. Mac Lane, Categories for the working mathematician, Springer-Verlag, 1971.

[15] B. Mashayakhy, A. Pakdaman, and H. Torabi, Spanier spaces and covering theory of non-homotopically path Hausdorff spaces, Georgian Mathematical Journal 20 (2013), 303–317.

[16] A. Pakdaman, Local triviallity and comparison of topological fundamental groupoids, submitted.

[17] A. Pakdaman, Small loops in topological fundamental groupoid, submitted.

[18] A. Pakdaman, and F. Shahini, The fundamental groupoid as a topological groupoid: Lasso topology, Topology and its applications 302 (2021), 107837.

[19] A. Pakdaman, and F. Shahini, Fundamental groupoid and whisker topology, Mathematical Researches 9 (2023) 55–71.

[20] A. Pakdaman, and F. Shahini, Topological fundamental groupoids: Brown's topology, Hacettepe Journal of mathematics and statistics 52 (2023), 896–906.

[21] J. Renault, A Groupoid Approach to C*-Algebras, Lecture Notes in Mathematics, Springer, 1980.

[22] H. Spanier, Algebraic Topology, McGraw-Hill Book Co, New York-Toronto, Ont.-London, 1966.

[23] Z. Virk, Small loop spaces, Topology and its Applications 157 (2010), 451–455.

[24] Z. Virk, and A. Zastrow, The comparison of topologies related to various concepts of generalized covering spaces, Topology and its Applications 170 (2014), 52–62.

[25] Z. Virk, and A. Zastrow, A new topology on the universal path space, Topology and its Applications 231 (2017), 186–196.