

# Remarks on fixed point assertions in digital topology, 11

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## ABSTRACT

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*The topic of fixed points in digital metric spaces has drawn yet more publications with assertions that are incorrect, incorrectly proven, trivial, or incoherently stated. We discuss publications with bad assertions concerning fixed points of self-functions on digital images, as in some of our previous papers.*

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## 1. INTRODUCTION

**Fixed points in digital topology** - this is a topic that has inspired some beautiful results. It has also been at the center of many assertions that are incorrect, incorrectly proven, trivial, or not presented coherently. In the current paper, we continue the work of [13, 3, 4, 5, 6, 7, 8, 9, 10, 11] as we discuss several other papers with assertions that merit at least one of the descriptions in the previous sentence. Papers drawing our disapproval in the current work have all come to our attention since acceptance for publication of [11].

## 2. PRELIMINARIES

**2.1. Adjacencies, continuity, fixed point.** Much of the material in this section is quoted or paraphrased from [11].

A *digital image* is a pair  $(X, \kappa)$ , where  $X$  is a nonempty subset of  $\mathbb{Z}^n$  for some integer  $n > 0$ , and  $\kappa$  is an adjacency relation on pairs of members of  $X$ . Thus, a digital image may be considered to be a graph. In the current paper, we do not consider a point to be adjacent to itself.

In a digital image  $(X, \kappa)$ , if  $x, y \in X$ , we use the notation  $x \leftrightarrow_{\kappa} y$  to mean  $x$  and  $y$  are  $\kappa$ -adjacent; we may write  $x \leftrightarrow y$  when  $\kappa$  can be understood. We write  $x \rightleftharpoons_{\kappa} y$ , or  $x \rightleftharpoons y$  when  $\kappa$  can be understood, to mean  $x \leftrightarrow_{\kappa} y$  or  $x = y$ .

The most commonly used adjacencies in the study of digital images are the  $c_u$  adjacencies. These are defined as follows.

**Definition 2.1.** Let  $X \subset \mathbb{Z}^n$ . Let  $u \in \mathbb{Z}$ ,  $1 \leq u \leq n$ . Let  $x = (x_1, \dots, x_n)$ ,  $y = (y_1, \dots, y_n) \in X$ . Then  $x \leftrightarrow_{c_u} y$  if

- $x \neq y$ ,
- for at most  $u$  distinct indices  $i$ ,  $|x_i - y_i| = 1$ , and
- for all indices  $j$  such that  $|x_j - y_j| \neq 1$  we have  $x_j = y_j$ .

**Definition 2.2** (See [21]). Let  $(X, \kappa)$  be a digital image. Let  $x, y \in X$ . Suppose there is a set  $P = \{x_i\}_{i=0}^n \subset X$  such that  $x = x_0$ ,  $x_i \leftrightarrow_{\kappa} x_{i+1}$  for  $0 \leq i < n$ , and  $x_n = y$ . Then  $P$  is a  $\kappa$ -*path* (or just a *path* when  $\kappa$  is understood) in  $X$  from  $x$  to  $y$ , and  $n$  is the *length* of this path.

**Definition 2.3** ([24]). A digital image  $(X, \kappa)$  is  $\kappa$ -*connected*, or just *connected* when  $\kappa$  is understood, if given  $x, y \in X$  there is a  $\kappa$ -path in  $X$  from  $x$  to  $y$ . The  $\kappa$ -*component* of  $x$  in  $X$  is the maximal  $\kappa$ -connected subset of  $X$  containing  $x$ .

**Definition 2.4** ([24, 1]). Let  $(X, \kappa)$  and  $(Y, \lambda)$  be digital images. A function  $f : X \rightarrow Y$  is  $(\kappa, \lambda)$ -*continuous*, or  $\kappa$ -*continuous* if  $(X, \kappa) = (Y, \lambda)$ , or *digitally continuous* when  $\kappa$  and  $\lambda$  are understood, if for every  $\kappa$ -connected subset  $X'$  of  $X$ ,  $f(X')$  is a  $\lambda$ -connected subset of  $Y$ .

**Theorem 2.5** ([1]). A function  $f : X \rightarrow Y$  between digital images  $(X, \kappa)$  and  $(Y, \lambda)$  is  $(\kappa, \lambda)$ -continuous if and only if for every  $x, y \in X$ , if  $x \leftrightarrow_{\kappa} y$  then  $f(x) \rightleftharpoons_{\lambda} f(y)$ .

A *fixed point* of a function  $f : X \rightarrow X$  is a point  $x \in X$  such that  $f(x) = x$ .

As a convenience, if  $x$  is a point in the domain of a function  $f$ , we will often abbreviate " $f(x)$ " as " $fx$ ".

**2.2. Digital metric spaces.** A *digital metric space* [18] is a triple  $(X, d, \kappa)$ , where  $(X, \kappa)$  is a digital image and  $d$  is a metric on  $X$ . The metric is usually taken to be the Euclidean metric or some other  $\ell_p$  metric; alternately,  $d$  might be taken to be the shortest path metric. These are defined as follows.

- Given  $x = (x_1, \dots, x_n) \in \mathbb{Z}^n$ ,  $y = (y_1, \dots, y_n) \in \mathbb{Z}^n$ ,  $p > 0$ ,  $d$  is the  $\ell_p$  metric if

$$d(x, y) = \left( \sum_{i=1}^n |x_i - y_i|^p \right)^{1/p}.$$

Note the special cases: if  $p = 1$  we have the *Manhattan metric*; if  $p = 2$  we have the *Euclidean metric*.

- [14] If  $(X, \kappa)$  is a connected digital image,  $d$  is the *shortest path metric* if for  $x, y \in X$ ,  $d(x, y)$  is the length of a shortest  $\kappa$ -path in  $X$  from  $x$  to  $y$ .

We say a metric space  $(X, d)$  is *uniformly discrete* if there exists  $\varepsilon > 0$  such that  $x, y \in X$  and  $d(x, y) < \varepsilon$  implies  $x = y$ .

*Remark 2.6.* If

- $X$  is finite, or
- [4]  $d$  is an  $\ell_p$  metric, or
- $(X, \kappa)$  is connected and  $d$  is the shortest path metric,

then  $(X, d)$  is uniformly discrete.

For an example of a digital metric space that is not uniformly discrete, see Example 2.10 of [5].

We say a sequence  $\{x_n\}_{n=0}^{\infty}$  is *eventually constant* if for some  $m > 0$ ,  $n > m$  implies  $x_n = x_m$ . The notions of convergent sequence and complete digital metric space are often trivial, e.g., if the digital image is uniformly discrete, as noted in the following, a minor generalization of results of [20, 13].

**Proposition 2.7** ([5]). *If  $(X, d)$  is a uniformly discrete metric space, then any Cauchy sequence in  $X$  is eventually constant, and  $(X, d)$  is a complete metric space.*

**2.3. On fixed points.** If  $X$  is a topological space or if  $(X, \kappa)$  is a digital image,  $X$  or, respectively,  $(X, \kappa)$  has the *fixed point property* (FPP) if for every continuous (respectively,  $\kappa$ -continuous)  $f : X \rightarrow X$  has a fixed point. But the FPP turns out to be trivial in digital topology, as shown by the following.

**Theorem 2.8** ([12]). *A digital image  $(X, \kappa)$  has the FPP if and only if  $\#X = 1$ .*

### 3. [16]'s "NOVEL APPROACH"

One of the flaws of [16] is its extensive use of unnecessarily complex notation. Where possible, we simplify the notation of [16].

**Definition 2.6:** Let the digital image  $(P, \kappa) \subset Z^q$ . Let  $P$  represent a nonempty (NE) set with  $\rho_D: P \times P \times (0, +\infty) \rightarrow [0, +\infty)$

Let  $(P, \kappa) \subset Z^q$  be a digital image. Let  $P$  be a NE set and a function  $\rho_D: P \times P \times (0, +\infty) \rightarrow [0, +\infty)$  is called a PMM on  $P$  if it fulfills the following circumstances for all  $t > 0$  and all  $q_*, q_2, q_\Delta \in P$

- (i)  $\rho_D(q_*, q_2, t) = 0 \Leftrightarrow q_* = q_2$
- (ii)  $\rho_D(q_*, q_2, t) = \rho_D(q_2, q_*, t)$
- (iii)  $\rho_D(q_*, q_2, t) \leq \rho_D(q_*, q_\Delta, t) + \rho_D(q_\Delta, q_2, t)$

$(P, \rho_D, \kappa)$  is called digital PMS.

The digital image and the function  $\rho_D$  together form a DMS with  $\kappa$ -adjacency.

FIGURE 1. Definition 2.6 of [16]

3.1. **The introduction.** We mention many errors that appear in the Introduction of [16]. Among these are attributions of definitions and theorems to papers that used these but are not the papers in which they first appeared.

Definition 2.1 of [16] -  $Z^q$  - is both incorrectly attributed and incorrectly stated. This definition is attributed in [16] to [17] rather than assuming the reader will know it is classical. The definition stated is

$$Z^q = \{(a_1, a_2, a_3, \dots, a_n) \mid a_i \in \mathbb{Z}, 1 \leq i \leq q\}$$

rather than

$$Z^q = \{(a_1, a_2, a_3, \dots, a_q) \mid a_i \in \mathbb{Z}, 1 \leq i \leq q\}.$$

Definition 2.2 of [16] - the  $c_u$  (also called  $k_u$  and  $\ell_u$  in the literature) adjacency - is both incorrectly attributed and incorrectly stated. The given attribution is to [18, 17] but should be to [2]. The definition stated in [16] claims that points  $x = (x_1, \dots, x_q)$  and  $y = (y_1, \dots, y_q)$  are adjacent in this adjacency if at most  $u$  indices  $i$  satisfy  $|x_i - y_i| = 1$  and for all other indices  $j$ ,  $|x_i - y_i| \neq 1$ . The latter inequality should be  $|x_j - y_j| = 0$ .

Other incorrect attributions:

<u>Entry</u>	<u>[16] attribution</u>	<u>Better attribution</u>
Def. 2.3 (dig. img.)	[18, 17]	[23, 24]
Def. 2.4 (fixed pt.)	[18, 17]	classic
Def. 2.4 (FPP)	[18, 17]	[24]
Thm. 2.5 (dig. Banach princ.)	[18]	[18]; corrected proof [11]

**Theorem 3.1:** Consider the triple  $(P, \rho_D, \kappa)$  be a Complete DPMS and  $D$  is a continuous mapping satisfying the following condition:

$$\rho_D(Dq_*, Dq_2, t) \leq \delta_1 \max \left[ \begin{array}{l} \rho_D(q_*, q_2, t), \rho_D(q_*, D(q_*), t), \\ \rho_D(q_2, D(q_2), t), \rho_D(q_*, D(q_2), t), \rho_D(D(q_*), q_2, t) \end{array} \right]$$

For all  $q_*, q_2 \in P, q_* \neq q_2$  and  $\forall t > 0$  with  $\delta_1 \in [0,1)$ . The mapping  $D$  possesses a unique FP in  $P$ .

FIGURE 2. "Theorem" 3.1 of [16]

Definition 2.6 of [16] (see Figure 1) defines a function that has the properties of a metric, but also has a third parameter  $t > 0$  that is not used for any purpose in the paper. Thus by taking  $t$  to be a positive constant, we see that we can omit this parameter and replace this function by an actual metric.

Definition 2.8 of [16] fails to clarify in its statement that the continuity it uses is of the classical  $\varepsilon - \delta$  type. Example 4.1 of [13] shows that the classical  $\varepsilon - \delta$  continuity does not imply digital continuity.

Lemma 2.9 of [16] is unattributed. It should be attributed to Theorem 3.7 of [18] and to Theorem 3.1 of [11]; the former does not have a correct proof, and the latter does.

Lemma 2.10 of [16] essentially duplicates Lemma 2.9 of [16] and is also unattributed.

3.2. [16]’s “**Theorem**” 3.1. ”Theorem” 3.1 of [16] (see Figure 2) is restated in this section in a much simpler and correct form, and is given a proof that is clearer and correct.

The argument offered as proof in [16] of this assertion depends on a constant  $\delta_1 \in [0, 1)$ . From this, another constant, here denoted  $\mathcal{L}$  as we don’t see how to duplicate in L<sup>A</sup>T<sub>E</sub>X the symbol used in [16], is derived via

$$\mathcal{L} = \frac{\delta_1}{1 - \delta_1}.$$

The authors claim  $\mathcal{L}$  is less than 1. But since  $\delta_1$  can be a member of  $[1/2, 1)$ , the claim is not true. The consequences of this error propagate through the “proof” argument, so ”Theorem” 3.1 of [16] is unproven.

A self-mapping on a digital image that satisfies the inequality of Figure 2 is a *quasi-contraction* [15]. Fixed point results for quasi-contractions on digital metric spaces, using the shortest-path metric, have been obtained in [19, 8].

We modify the statement of ”Theorem” 3.1 of [16] as follows, and give a proof. We make the following changes from ”Theorem” 3.1 of [16]:

- We omit the continuity and completeness assumptions.
- We add the assumption of uniform discreteness.

- We take  $1/2$ , not  $1$ , as the upper bound for the constant on the right side of (3.1) below.
- We simplify notation.

**Theorem 3.1.** *Let  $(X, d, \kappa)$  be a digital metric space, where  $d$  is uniformly discrete. Let  $f : X \rightarrow X$  be a function and let  $c$  be a constant,  $0 \leq c < 1/2$ , such that for all  $x, y \in X$ ,*

$$d(fx, fy) \leq c \cdot \max\{d(x, y), d(x, fx), d(y, fy), d(x, fy), d(y, fx)\}. \quad (3.1)$$

*Then  $f$  has a unique fixed point in  $X$ .*

*Proof.* We use ideas of [16]. Let  $x_0 \in X$  and  $x_{i+1} = fx_i$ ,  $i \geq 0$ . For all  $i$ ,

$$\begin{aligned} d(x_{i+1}, x_{i+2}) &= d(fx_i, fx_{i+1}) \\ &\leq c \cdot \max\{d(x_i, x_{i+1}), d(x_i, fx_i), d(x_{i+1}, fx_{i+1}), d(x_i, fx_{i+1}), d(x_{i+1}, fx_i)\} \\ &= c \cdot \max\{d(x_i, x_{i+1}), d(x_i, x_{i+1}), d(x_{i+1}, x_{i+2}), d(x_i, x_{i+2}), d(x_{i+1}, x_{i+1})\} \\ &\leq c \cdot \max\{d(x_i, x_{i+1}), d(x_{i+1}, x_{i+2}), d(x_i, x_{i+1}) + d(x_{i+1}, x_{i+2}), 0\} \\ &= c \cdot [d(x_i, x_{i+1}) + d(x_{i+1}, x_{i+2})]. \end{aligned}$$

Therefore,  $(1 - c) \cdot d(x_{i+1}, x_{i+2}) \leq c \cdot d(x_i, x_{i+1})$ , or

$$d(x_{i+1}, x_{i+2}) \leq \mathcal{L} \cdot d(x_i, x_{i+1}),$$

where  $\mathcal{L} = \frac{c}{1-c} < 1$ . An easy induction yields that  $d(x_i, x_{i+m}) \leq \mathcal{L}^{i+m} d(x_0, x_1)$ . It follows from Proposition 2.7 that there exists  $N$  such that  $i > N$  implies  $x_i = x_N$ . Thus,  $x_N$  is a fixed point of  $f$ .

To show the uniqueness of this fixed point, suppose  $x'$  is a fixed point of  $f$ . Then

$$\begin{aligned} d(x_N, x') &= d(fx_N, fx') \leq \\ &c \cdot \max\{d(x_N, x'), d(x_N, fx_N), d(x', fx'), d(x_N, fx'), d(fx_N, x')\} = \\ &c \cdot \max\{d(x_N, x'), 0, 0, d(x_N, x'), d(x_N, x')\} = c \cdot d(x_N, x'). \end{aligned}$$

Since  $0 \leq c < 1/2$ , we have  $d(x_N, x') = 0$ , i.e.,  $x_N = x'$ . □

**Theorem 3.2:** Let  $D$  be continuous mapping that fulfils the succeeding condition, and let  $(P, \rho_D, \kappa)$  be a CDPMS.

$$\begin{aligned} & \rho_D(Dq_*, Dq_2, t) \\ & \leq \delta_1[\rho_D(q_*, D(q_*), t), \rho_D(q_2, D(q_2), t)] \quad (3.2) \\ & \quad + \delta_2[\rho_D(q_*, D(q_2), t), \rho_D(D(q_*), q_2, t)] \end{aligned}$$

for all  $q_*, q_2 \in P$

And  $\delta_1 + \delta_2 < \frac{1}{2}$ ,  $\delta_1, \delta_2 \in [0, \frac{1}{2})$ . Then  $T$  has a FP in  $P$ .

FIGURE 3. [16]’s “Theorem” 3.2

3.3. [16]’s **Theorem 3.2.** Figure 3 shows [16]’s Theorem 3.2. We feel our rewriting is desirable because the flaws and difficult notation of [16]’s version make both the assertion and its proof difficult to understand. The assertion is flawed as follows:

- It is not clear if the continuity hypothesized is of the  $\varepsilon - \delta$  variety, or the digital variety; however, we will show that neither is required.
- The right side of the inequality marked as “(3.2)” is undefined. It appears that there are operators missing - apparently, the “+” in each term according to the argument given as proof.
- In the conclusion, “ $T$ ” is apparently meant to be “ $D$ ”.

We correct the statement of [16]’s “Theorem” 3.2. Changes made in the statement of the assertion:

- We add the assumption that the metric, which we denote as  $d$ , is uniformly discrete.
- We omit the unnecessary assumptions of continuity and completeness (the “C” in “CDPMS”).
- We insert the missing “+” operators.
- We use only one symbol for the function with which we are concerned.
- We simplify notation.

With these changes, Theorem 3.2 of [16] can be rewritten from the version shown in Figure 3 to the following.

**Theorem 3.2.** *Let  $(X, d, \kappa)$  be a digital image. Let  $f : X \rightarrow X$  satisfy, for all  $x, y \in X$  and some nonnegative constants  $a, b$  such that  $a + b < 1/2$ ,*

$$d(fx, fy) \leq a \cdot [d(x, fx) + d(y, fy)] + b \cdot [d(x, fy) + d(y, fx)]. \quad (3.2)$$

*Then  $f$  has a fixed point in  $X$ .*

*Proof.* Our argument is largely that of [16]. Using our change of notation, let  $x_0 \in X$  and let a sequence  $\{x_i\}_{i=1}^{\infty} \subset X$ , where  $x_{i+1} = fx_i$ , be established. From the inequality of the assertion, we obtain

$$d(x_{i+1}, x_{i+2}) = d(fx_i, fx_{i+1}) \leq$$

$$a \cdot [d(x_i, fx_i) + d(x_{i+1}, fx_{i+1})] + b \cdot [d(x_i, fx_{i+1}) + d(x_{i+1}, fx_i)] =$$

$$a \cdot [d(x_i, x_{i+1}) + d(x_{i+1}, x_{i+2})] + b \cdot [d(x_i, x_{i+2}) + d(x_{i+1}, x_{i+1})] \leq$$

(using the Triangle Inequality)

$$a \cdot [d(x_i, x_{i+1}) + d(x_{i+1}, x_{i+2})] + b \cdot [d(x_i, x_{i+1}) + d(x_{i+1}, x_{i+2}) + 0].$$

Thus

$$(1 - (a + b))d(x_{i+1}, x_{i+2}) \leq (a + b)d(x_i, x_{i+1}).$$

Since  $a + b < 1/2$ , we have

$$0 \leq \frac{a + b}{1 - (a + b)} < 1.$$

Let  $c = \frac{a+b}{1-(a+b)}$ . Then  $0 \leq c < 1$  and

$$d(x_{i+1}, x_{i+2}) \leq c \cdot d(x_i, x_{i+1}).$$

An easy induction yields that for all positive integers  $n, m$ ,

$$d(x_n, x_{n+m}) \leq c^{m+n} \cdot d(x_0, x_1) \rightarrow_{m \rightarrow \infty} 0.$$

Thus,  $\{x_n\}_{n=0}^{\infty}$  is a Cauchy sequence. By Proposition 2.7, there exists  $x_N \in X$  such that for almost all  $i$ ,  $x_i = x_N$ . Thus,  $x_N$  is a fixed point of  $f$ .

To show the uniqueness of  $x_N$  as a fixed point, suppose  $x'$  is a fixed point of  $f$ . Then

$$d(x_N, x') = d(fx_N, fx') \leq$$

$$a \cdot [d(x_N, fx_N) + d(x', fx')] + b[d(x_N, fx') + d(fx_N, x')] =$$

$$a \cdot (0 + 0) + b \cdot [d(x_N, x') + d(x_N, x')] = 2b \cdot d(x_N, x').$$

Since  $2b < 1$ , we have  $d(x_N, x') = 0$ , i.e.,  $x_N = x'$ . □

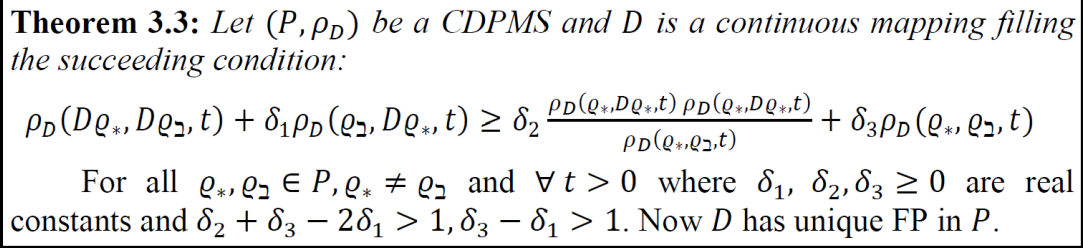


FIGURE 4. "Theorem" 3.3 of [16]

3.4. [16]’s “**Theorem**” 3.3. “Theorem” 3.3 of [16] is shown in Figure 4. As written, the assertion is false, as shown by the following, stated in simpler notation.

**Example 3.3.** Let  $X = \{2^n \mid n \in \mathbb{N}\}$ . Let  $d(x, y) = |x - y|$ . Let  $f : X \rightarrow X$  be defined by  $fx = 2x$ . Then for any adjacency  $\kappa$  on  $X$ ,  $(X, d, \kappa)$  is a uniformly discrete digital metric space that satisfies the hypotheses of the assertion shown in the equivalent notation of Figure 4. However,  $f$  has no fixed point.

*Proof.* It is elementary that  $(X, d, \kappa)$  is a uniformly discrete digital metric space, hence is complete, and that  $f$  has no fixed point. Let

$$\delta_1 = \delta_2 = 0, \quad \delta_3 = 1.5.$$

These values satisfy the restrictions on  $\delta_1, \delta_2, \delta_3$  shown in Figure 4 and reduce the distance relation to

$$d(fx, fy) \geq 1.5 \cdot d(x, y) \tag{3.3}$$

Then we have

$$d(f(2^i), f(2^j)) = |2^{i+1} - 2^{j+1}| = 2 \cdot d(2^i, 2^j) \geq 1.5 \cdot d(2^i, 2^j),$$

so  $f$  satisfies (3.3). □

To obtain a true assertion from Figure 4, we make the following changes:

- We assume uniform discreteness.
- In the 3rd line of Figure 4, we use “ $\leq$ ” in place of “ $\geq$ ”.
- We use “for some constants” rather than “for all constants”.
- We use “ $b + c < 1$  instead of “ $b + c - 2a > 1$ ”.
- We use simpler notation.
- We further show that the fixed point is unique.

The assertion can then be restated as follows.

**Proposition 3.4.** Let  $(X, d, \kappa)$  be a uniformly discrete digital metric space. Let  $f : X \rightarrow X$  be such that for some nonnegative constants  $a, b, c$  such that  $b + c < 1$ , and for all  $x, y \in X$  such that  $x \neq y$  we have

$$d(fx, fy) + a \cdot d(y, fx) \leq b \cdot \frac{d(x, fx) \cdot d(x, fx)}{d(x, y)} + c \cdot d(x, y).$$

Then  $f$  has a unique fixed point.

*Proof.* Let  $x_0 \in X$  and let  $x_{i+1} = fx_i$  for  $i \geq 0$ . If  $x_{i+1} = x_i$  then  $x_i$  is a fixed point. Otherwise,

$$d(fx_i, fx_{i+1}) + a \cdot d(x_{i+1}, fx_i) \leq b \cdot \frac{d(x_i, fx_i) \cdot d(x_i, fx_i)}{d(x_i, x_{i+1})} + c \cdot d(x_i, x_{i+1}).$$

This simplifies as

$$d(x_{i+1}, x_{i+2}) + a \cdot 0 \leq b \cdot \frac{[d(x_i, x_{i+1})]^2}{d(x_i, x_{i+1})} + c \cdot d(x_i, x_{i+1})$$

or

$$d(x_{i+1}, x_{i+2}) \leq (b + c) \cdot d(x_i, x_{i+1}) = k \cdot d(x_i, x_{i+1}), \quad (3.4)$$

where  $0 < k = b + c < 1$ . Thus

$$d(x_i, x_{i+m}) \leq k^m d(x_0, x_1) \rightarrow_{i,m \rightarrow \infty} 0.$$

Therefore,  $\{x_n\}_{n=0}^{\infty}$  is a Cauchy sequence. By Proposition 2.7, for some  $N$ ,  $x_N$  is a fixed point of  $f$ .

To show the uniqueness of this fixed point, let  $x'$  be a fixed point of  $f$  and suppose  $x' \neq x_N$ . Then

$$d(fx_N, fx') + a \cdot d(x', fx_N) \leq b \cdot \frac{[d(x_N, fx_N)]^2}{d(x_N, x')} + c \cdot d(x_N, x')$$

or

$$d(x_N, x') + a \cdot d(x', x_N) \leq 0 + c \cdot d(x_N, x')$$

$$(1 + a) \cdot d(x_N, x') \leq c \cdot d(x_N, x')$$

$$(1 + a - c) \cdot d(x_N, x') \leq 0.$$

Since  $c < 1$ , the previous inequality contradicts the assumption that  $x' \neq x_N$ . □

#### 4. [22]'S ASSERTIONS

Two assertions are presented as new theorems in [22]. We show that one of these reduces to triviality, and we show the other is not correctly proven.

4.1. [22]’s “Theorem” 3.1. We use the following.

**Definition 4.1** ([18]). Let  $(X, d, \kappa)$  be a digital metric space and  $f : X \rightarrow X$ . If for some  $\alpha \in [0, 1)$  and all  $x, y \in X$  we have

$$d(fx, fy) < \alpha d(x, y)$$

then  $f$  is a *digital contraction map*.

The digital Banach contraction principle is the following.

**Theorem 4.2** ([18]; corrected proof [11]). *Let  $(X, d, \kappa)$  be a uniformly discrete digital metric space and let  $f : X \rightarrow X$  be a digital contraction map. Then  $f$  has a unique fixed point.*

The paper [22] states the following as its “Theorem” 3.1.

*Assertion 4.3.* Let  $(X, d, u)$  be a complete digital metric space. Let  $G : X \rightarrow X$  be a mapping such that, for all  $x, y \in X$  and for all non-negative  $a, b, c$  with  $a + b + c < 1$ ,

$$(G_1) \quad d(Gx, Gy) \leq a \cdot d(x, y)$$

$$(G_2) \quad d(Gx, Gy) \leq b \cdot [d(x, Gx) + d(y, Gy)]$$

$$(G_3) \quad d(Gx, Gy) \leq c \cdot [d(x, Gy) + d(y, Gx)]$$

and

$$d(Gx, Gy) < \beta \cdot \max \left\{ d(x, y), \frac{d(x, Gx) + d(y, Gy)}{2}, \frac{d(x, Gy) + d(y, Gx)}{2} \right\}.$$

Let  $P : X \rightarrow X$  be a mapping such that, for all  $x, y \in X$ ,

$$(P_1) \quad d(Px, Py) \leq a \cdot d(x, y)$$

$$(P_2) \quad d(Px, Py) \leq b \cdot [d(x, Px) + d(y, Py)]$$

$$(P_3) \quad d(Px, Py) \leq c \cdot [d(x, Py) + d(y, Px)]$$

and

$$d(Px, Py) < \beta \cdot \max \left\{ d(x, y), \frac{d(x, Px) + d(y, Py)}{2}, \frac{d(x, Py) + d(y, Px)}{2} \right\}.$$

Then, if  $G$  and  $P$  commute, they have a unique and common fixed point.

Notice:

- The digital Banach Fixed Point Principle ([18]; corrected proof [11]) requires only a particular  $a \in [0, 1)$ , not all  $a, b, c \in [0, 1)$ , and does not require  $(G_2)$  or  $(G_3)$  to show the existence of a fixed point for a function  $G$  that satisfies  $(G_1)$ . Similarly for the function  $P$ .
- No restriction on  $\beta$  is given in the statement of Assertion 4.3.

- Perhaps more importantly, in spite of the multipage argument offered as its proof in [22], this proposition reduces to triviality, as we show below, despite omitting several of the hypotheses of Assertion 4.3.
- Also, the argument offered as “proof” has flaws including:
  - an undefined notion, used in the second line of the argument, that a point is “contained” in another point; and
  - an undefined notion, used in the fifth line of the argument, that a distance is “contained” in another distance.

We modify Assertion 4.3 to obtain the following Proposition 4.4, which shows how Assertion 4.3 reduces to triviality for a finite digital metric space.

By the *diameter* of a finite metric space  $(X, d)$  we mean

$$\text{diam}(X) = \max\{d(x, y) \mid x, y \in X\}.$$

**Proposition 4.4.** *Let  $(X, d, u)$  be a finite digital metric space. Let  $G$  and  $P$  be self-maps on  $X$  satisfying  $(G_1)$  and  $(P_1)$  of Assertion 4.3, respectively, for some coefficient  $a$  such that*

$$0 \leq a < \frac{\min\{d(u, v) \mid u, v \in X, u \neq v\}}{\text{diam}(X)}.$$

*Then each of  $G, P$  has a unique fixed point,  $u_G, u_P$ , respectively. If  $G$  and  $P$  commute, then  $G$  and  $P$  are equal constant functions.*

*Proof.* By  $(G_1)$  and  $(P_1)$ , each of  $G, P$  is a digital contraction map. By Theorem 4.2, each has a unique fixed point,  $u_G, u_P$ , respectively.

Then  $(G_1)$  implies for all distinct  $x, y \in X$ ,

$$d(Gx, Gy) \leq a \cdot d(x, y) < \frac{\min\{d(u, v) \mid u, v \in X, u \neq v\}}{\text{diam}(X)} \cdot d(x, y) \leq$$

$$\min\{d(u, v) \mid u, v \in X, u \neq v\}$$

so  $d(Gx, Gy) = 0$ , and similarly,  $d(Px, Py) = 0$ . Thus  $G$  and  $P$  are constant functions taking values  $u_G, u_P$ , respectively.

If  $G$  and  $P$  commute, we have

$$u_G = G(P(u_G)) = P(G(u_G)) = u_P.$$

Thus  $G = P$ . □

4.2. [22]’s “**Theorem**” 3.2. Assuming [22]’s “ $(G(Gx), G(Gy))$ ” is intended to be “ $d(G(Gx), G(Gy))$ ” and “ $(P(Px), P(Py))$ ” is intended to be “ $d(P(Px), P(Py))$ ”, the following is a corrected statement of [22]’s “**Theorem**” 3.2.

*Assertion 4.5.* Let  $(X, d, u)$  be a digital metric space. Let  $G, P : X \rightarrow X$ . Suppose for all  $x, y \in X$  and non-negative constants  $e, f, g, h, i$  such that

$$e + f + g + h + i < 1,$$

we have

$$d(G(Gx), G(Gy)) \leq e \cdot d(Gx, Gy) + f \cdot d(Gx, G(Gx)) + g \cdot d(Gy, G(Gy)) + h \cdot d(Gx, G(Gy)) + i \cdot d(Gy, G(Gx))$$

and

$$d(P(Px), P(Py)) \leq e \cdot d(Px, Py) + f \cdot d(Px, P(Px)) + g \cdot d(Py, P(Py)) + h \cdot d(Px, P(Py)) + i \cdot d(Py, P(Px)).$$

Then, if  $G$  and  $P$  commute, they have a unique and common fixed point.

That Assertion 4.5 is not correctly proven is shown as follows. Errors in the argument offered as proof in [22] include:

- In the first two lines of the “proof” we find “If  $G(x) \subset P(x)$ ”, an undefined notion.
- On the 3rd line of the “proof” we have the claim that

$$(G(Gx), G(Gy)) \leq (P(Px), P(Py))$$

which is unsupported even if rewritten as

$$d(G(Gx), G(Gy)) \leq d(P(Px), P(Py)).$$

- On the 2nd line of the “proof’s” second page, we find “Since  $G(x) \subset P(x)$ ” [as above, undefined] “it follows that  $G(x) \leq P(x)$ ” [also undefined since  $X$  is not assumed a subset of  $\mathbb{Z}$ ].
- Later in the “proof”,  $r$  is defined by

$$r = \frac{e + f + h}{1 - g - h}$$

and it is claimed that  $r < 1$ , which is false for some choices of  $e, f, g, h, i$ .

We must conclude that Assertion 4.5 is unproven.

## 5. [25]'S ASSERTION

The paper [25] claims the following as its "Theorem" 3.1.

*Assertion 5.1.* Let  $(X, d)$  be a complete digital metric space. Let  $T : X \rightarrow X$  be a mapping. Let  $k_1, k_2, k_3$  be nonnegative constants such that

$$k_1^2 + k_2^2 + k_3^2 < 1 \quad (5.1)$$

and for all  $a, b \in X$ ,

$$d(Ta, Tb) \leq$$

$$k_1^2 \cdot d(a, b) + k_2^2 \cdot [d(a, Ta) + d(b, Tb)] + k_3^2 \cdot \sqrt{d(a, b) \cdot \min\{d(a, Ta), d(b, Tb)\}}. \quad (5.2)$$

Then  $T$  has a unique fixed point in  $X$ .

That Assertion 5.1 is false is shown by the following example.

**Example 5.2.** Let  $X = [0, 1]_{\mathbb{Z}}$  and let  $T : X \rightarrow X$  be given by  $T(x) = 1 - x$ . Notice  $T$  has no fixed point. However, we can choose  $k_1, k_2, k_3$  to satisfy the hypotheses of Assertion 5.1.

*Proof.* Take  $d$  to be the usual metric for  $\mathbb{Z}$ ,  $d(x, y) = |x - y|$ . Let  $k_1 = 0$ ,  $k_2 = \sqrt{0.9}$ ,  $k_3 = 0.3$ . Then

$$k_1^2 + k_2^2 + k_3^2 = 0 + 0.9 + 0.09 = 0.99 < 1$$

and (5.2) reduces to

$$d(Ta, Tb) \leq 0.9 \cdot [d(a, Ta) + d(b, Tb)] + 0.09 \cdot \sqrt{d(a, b) \cdot \min\{d(a, Ta), d(b, Tb)\}}. \quad (5.3)$$

To show that (5.3) holds, we consider the following cases.

- If  $a = b$ , the left side of (5.3) is 0, so (5.3) holds in this case.
- If  $a \neq b$ , without loss of generality,  $a = 0$  and  $b = 1$ . Then

$$d(Ta, Tb) = d(1, 0) = 1 < 1.89 = 0.9(2) + 0.09 \sqrt{1 \cdot \min\{1, 1\}} =$$

$$0.9 \cdot [d(a, Ta) + d(b, Tb)] + 0.09 \sqrt{d(a, b) \cdot \min\{d(a, Ta), d(b, Tb)\}},$$

satisfying (5.3).

Thus for all  $a, b \in X$ , (5.3) holds. □

## 6. FURTHER REMARKS

We paraphrase [9]:

We have discussed several papers that seek to advance fixed point assertions for digital metric spaces. Many of these assertions are incorrect, incorrectly proven, or reduce to triviality; all of these reflect badly on the authors, and also on the referees and editors who approved their publication, and, perhaps, predatory journals that accept payment for publication regardless of quality.

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