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Space of quasicontinuous functions with the topology of uniform convergence on semi-compacta

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ABSTRACT

We introduce a new set-open topology on function spaces namely the semi-compact-open topology. The topology of uniform convergence on semi-compacta lies between the topology of pointwise convergence and the topology of uniform convergence. We show that for the space of quasicontinuous functions the topology of uniform convergence on semi-compacta coincides with the semi-compact-open topology. We also investigate results relating to induced functions, cardinal invariants and Ascoli like properties on the space of quasicontinuous functions when equipped with the topology of uniform convergence on semi-compacta.

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KEYWORDS: *quasicontinuous functions; function space; cardinality properties; induced functions; semi-compactness.*

1. INTRODUCTION

Quasicontinuous functions were defined by Kempisty [13] in 1932 though its origin can be traced back to the work of Baire [2]. Levine [15] defined semi-open sets in 1963 and studied quasicontinuous functions under the name of semi-continuous functions. Making use of semi-open sets, Dorsett [5] introduced and investigated semi-compactness. Holá and Holý studied quasicontinuous functions under pointwise convergence and uniform convergence investigating properties like locally boundedness,

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Baireness, compactness and metrizability ([10], [9], [12], [11]). Recently cardinal invariants, tightness and Fréchet-Urysohn property of quasicontinuous functions have been extensively studied ([7], [19], [20]). In this paper, we introduce a new function space topology that lies between the topology of pointwise convergence and uniform convergence, which coincides with a set-open topology namely the semi-compact-open topology for the space of quasicontinuous functions. We further prove results on induced functions, cardinal invariants and Ascoli like properties for the space of quasicontinuous functions when equipped with this new topology.

2. Definitions and Notations

Definition 2.1 ([15])**.** Any subset which is contained in the closure of its interior is called quasi-open (or semi-open).

Clearly, open sets are necessarily quasi-open.

Definition 2.2 ([5])**.** A topological space X is called semi-compact if for every semi-open cover of X has a finite subcover.

Semi-compact sets are necessarily compact. However, the converse is not always true. An example of a compact set which is not semi-compact is $\beta\mathbb{N}$.

Definition 2.3 ([14])**.** A space X is said to be locally semi-compact if each point of X has a neighborhood which is a semi-compact subset of X.

Definition 2.4 ([23])**.** A function $f : X \to \mathbb{R}$ is said to be cliquish at a point $x \in X$ if for each $\epsilon > 0$ and each neighborhood U of x there is a non-empty open set $G \subset U$ such that $|f(y) - f(z)| < \epsilon$ for each $y, z \in G$. The function f is said to be cliquish if it is cliquish at each point $x \in X$.

Definition 2.5 ([17])**.** A function $f : X \to Y$ is quasicontinuous at $x \in X$ if for every open set $V \subset Y$, $f(x) \in V$and open set $U \subset X$, $x \in U$ there is a non-empty open set $W \subset U$ such that $f(W) \subset V$. If f is quasicontinuous at every point of X, we say that f is quasicontinuous.

Continuous functions are necessarily quasicontinuous. A function $f : X \to Y$ is quasicontinuous if and only if $f^{-1}(V)$ is quasi-open for every open set $V \subset Y$.

We consider Y^X to be the set of all functions from X to Y, $Q(X, Y)$ to be the set of all quasicontinuous functions in Y^X and $C(X, Y)$ to be the space of all continuous functions in Y^X. $P(X)$ denotes the family of all non-empty finite subsets of X. The family of all non-empty compact subsets of X is denoted by $K(X)$

whereas $SK(X)$ denotes the family of all non-empty semi-compact subsets of X.

Let X and Y be topological spaces, $F(X, Y) \subset Y^X$ and λ be a non-empty family of subsets of X. Then the family of all sets of the form

$$[A, U] = \{f \in F(X, Y) : f(A) \subset U, A \in \lambda, U \text{ open in } Y\}$$

is a subbase for the λ-open topology $\mathscr{T}_\lambda$ on $F(X, Y)$. The space $F(X, Y)$ when equipped with $\mathscr{T}_\lambda$ is denoted by $F_\lambda(X, Y)$.

For $\lambda = P(X)$ and $\lambda = K(X)$, the set-open topologies $\mathscr{T}_p$ and $\mathscr{T}_k$ are called the point-open topology and the compact-open topology respectively. When $\lambda = SK(X)$, we call this new set-open topology $\mathscr{T}_{sk}$ the semi-compact-open topology. $F(X, Y)$ when equipped with the semi-compact-open topology $\mathscr{T}_{sk}$ is denoted by $F_{sk}(X, Y)$.

Similarly, when $\mathcal{U}$ is some uniformity on Y, $F(X, Y) \subset Y^X$ and λ is a non-empty family of subsets of X, the family of all sets of the form

$$W(A, M) = \{(f, g) \in F(X, Y) \times F(X, Y) : (f(x), g(x)) \in M, \forall x \in A\}$$

where $A \in \lambda$ and $M \in \mathcal{U}$ is a subbase for some uniformity on $F(X, Y)$. The topology $\mathscr{T}_{\lambda,u}$ generated by this uniformity is called the topology of uniform convergence on the elements of the family λ with respect to $\mathcal{U}$ on $F(X, Y)$. The space $F(X, Y)$ when equipped with $\mathscr{T}_{\lambda,u}$ is denoted by $F_{\lambda,u}(X, Y)$. For this topology, sets of the form

$$W(A, M)[f] = \{g \in F(X, Y) : (f(x), g(x)) \in M, \forall x \in A\}$$

where $M \in \mathcal{U}$ and $A \in \lambda$ denotes a basic neighborhood of $f \in F(X, Y)$. In particular, if (Y, d) be a metric space and $\mathcal{U}$ be the uniformity induced by the metric d, then $W(A, \epsilon)[f] = \{g \in F(X, Y) : d(f(x), g(x)) < \epsilon, \forall x \in A\}$ denotes a basic neighborhood of $f \in F(X, Y)$. Sometimes we write $W(A, \epsilon)[f]$ as $W(f, A, \epsilon)$.

For $\lambda = P(X)$ and $\lambda = K(X)$, the topologies $\mathscr{T}_{p,u}$ and $\mathscr{T}_{k,u}$ are called the topology of pointwise convergence and the topology of uniform convergence respectively. When $\lambda = SK(X)$, we call this new topology $\mathscr{T}_{sk,u}$ the topology of uniform convergence on semi-compacta with respect to $\mathcal{U}$ on $F(X, Y)$. The space $F(X, Y)$ when equipped with this new topology is denoted by $F_{sk,u}(X, Y)$.

From here on, we consider the topological space X to be Hausdorff if not mentioned otherwise. We denote the function space $F(X, Y)$ as $F(X)$ when $Y = \mathbb{R}$. For any subset A in X, the interior of A and closure of A are denoted by A^0 and $\overline{A}$ respectively.

3. Main Results

Theorem 3.1. *For any topological space X and some compatible uniformity $\mathcal{U}$ on Y, the semi-compact-open topology on the space of quasicontinuous functions $Q(X, Y)$ coincides with the topology of uniform convergence on semi-compacta with respect to $\mathcal{U}$, that is,*

$$Q_{sk}(X, Y) = Q_{sk,u}(X, Y).$$

Proof. Let $A \in SK(X)$ and U be open in Y such that $f \in [A, U]$ is open in $Q_{sk}(X, Y)$. For each $a \in A$, there exists $M_a \in \mathcal{U}$ such that the set $M_a[f(a)] = \{y \in Y : (f(a), y) \in M_a\} \subset U$. Now, let $N_a \in \mathcal{U}$ such that $N_a \circ N_a \subset M_a$. Then A being semi-compact, $f(A)$ is compact. There exists a finite subset A' of A such that $f(A) \subset \bigcup\{N_a[f(a)] : a \in A'\}$. Taking $N = \bigcap\{N_a : a \in A'\}$ we can show that $W(A, N)[f] \subset [A, U]$. Let $g \in W(A, N)[f]$ and $x \in A$. There exists some $a \in A'$ with $f(x) \in N_a[f(a)]$, so that $(f(a), f(x)) \in N_a$. Since $(f(x), g(x)) \in N \subset N_a$, $(f(a), g(x)) \in N_a \circ N_a \subset M_a$. Therefore, $g(x) \in M_a[f(a)] \subset U$.

Conversely, let $A \in SK(X)$, $M \in \mathcal{U}$ and $f \in W(A, M)$. Let N be a closed and symmetric element of $\mathcal{U}$ such that $N \circ N \circ N \subset M$. The set $f(A)$ being compact, there exists a finite subset A' of A so that $f(A) \subset \bigcup\{N[f(a)] : a \in A'\}$ where $N[f(a)] = \{y \in Y : (f(a), y) \in N\}$. For each $a \in A'$, let $A_a = A \cap f^{-1}(N[f(a)])$, which is again in $SK(X)$. Let V_a be the interior of $(N \circ N)[f(a)]$. Take $V = \bigcap\{[A_a, V_a] : a \in A'\}$, which is open in $Q_{sk}(X, Y)$. Since V_a contains $N[f(a)]$ for each $a \in A'$, we can show that $f \in V$. To see that $V \subset W(A, M)[f]$, let $g \in V$ and $x \in A$. There exists some $a \in A'$, with $f(x) \in N[f(a)]$, so that $(f(a), f(x)) \in N$. Also $g(x) \in V_a \subset (N \circ N)[f(a)]$, so $(f(a), g(x)) \in N \circ N$. Since N is symmetric, $(f(x), g(x)) \in N \circ N \circ N \subset M$, and thus $g \in W(A, M)[f]$. □

Similar results related to coincidence of the set-open and uniform topologies for the space of continuous functions $C(X, Y)$ can be found in the works of Nokhrin and Osipov ([18],[21]).

Induced functions are defined using composition of functions to create a bridge between two function spaces. For the space of continuous functions $C(X, Y)$ results relating to admissible topology, covering properties, joint continuity are established with the help of induced functions ([1], [16]). It may be noted that the composition of two quasicontinuous functions is not necessarily quasicontinuous.

Theorem 3.2. *Let X, Y be two given topological spaces and $\mathcal{U}$ be some compatible uniformity on Z. If $g : X \to Y$ be a quasicontinuous function, then the function $g^* : C_{k,u}(Y, Z) \to Q_{sk,u}(X, Z)$ where $g^*(h) = h \circ g$ is continuous.*

Proof. We know that $C_{k,u}(Y,Z) = C_k(Y,Z)$ (Theorem 3.3 of [22]). Using Theorem 3.1, we have $Q_{sk,u}(X,Z) = Q_{sk}(X,Z)$. Let $f_0 \in C(Y,Z)$ and $g^*(f_0) \in [A,V]$ where $A \in SK(X)$ and V is open in Z. Since quasicontinuous image of a semi-compact set is compact, therefore $g(A)$ is compact. Now consider $U = [g(A),V]$ which is open in $C(Y,Z)$. Clearly, $f_0 \in U$. For any $h \in U$, $g^*(h) \in [A,V]$ as $g^*(h)(A) = h \circ g(A) \subset V$. Therefore, $g^*(U) \subset [A,V]$. □

Theorem 3.3. *Let X, Y and Z be topological spaces and $g : Y \to Z$ be a continuous map. Then, the induced map $g_* : Q_{sk}(X,Y) \to Q_{sk}(X,Z)$ where $g_*(f) = gof$ is continuous. Moreover, if g is an embedding so is g_*.*

Proof. Let, $f \in Q(X,Y)$ and $g_*(f) \in [A,U] = \{h \in Q(X,Z) : h(A) \subset U\}$ where $A \in SK(X)$ and U is open in Z. Since $g : Y \to Z$ is continuous, $g^{-1}(U)$ is open in Y. Now,

$$\begin{aligned} g_*^{-1}[A,U] &= \{h \in Q(X,Y) : g_*(h) \in [A,U]\} \\ &= \{h \in Q(X,Y) : g \circ h \in [A,U]\} \\ &= \{h \in Q(X,Y) : g \circ h(A) \subset U\} \\ &= \{h \in Q(X,Y) : h(A) \subset g^{-1}(U)\} \\ &= [A, g^{-1}(U)]. \end{aligned}$$

Since $[A, g^{-1}(U)]$ is open in $Q_{sk}(X,Y)$, the map g_* is continuous.

Taking g to be an embedding, we can prove that g_* is also an embedding. For any $h_1, h_2 \in Q(X,Y)$ with $h_1 \neq h_2$, there must be some $x \in X$ such that $h_1(x) \neq h_2(x)$. If possible let $g_*(h_1) = g_*(h_2)$, that is, $g \circ h_1 = g \circ h_2$ and $g(h_1(x)) = g(h_2(x))$ which gives $h_1(x) = h_2(x)$, a contradiction. Therefore, g_* is one-one. Now for any open set $[A,U] \in Q_{sk}(X,Y)$, we prove that $g_*[A,U]$ is open in $Q_{sk}(X,Z)$. Since, g is an embedding from Y to Z, $g(U) = G \cap g(Y)$ where G is open in Z. Now,

$$\begin{aligned} g_*[A,U] &= \{g \circ f : f \in [A,U]\} \\ &= \{g \circ f : f \in [A, g^{-1}(G)]\} \\ &= \{g \circ f : g \circ f(A) \subset G\} \\ &= [A,G] \cap g_*(Q(X,Y)) \end{aligned}$$

which is open in $g_*(Q(X,Y))$. Hence proved. □

Proposition 3.4. *Let X and Y be topological spaces. Then the evaluation function at $x \in X$ defined as $e_x : Q_{sk}(X,Y) \to Y$ where $e_x(f) = f(x)$, is continuous.*

Proof. For $f \in Q(X,Y)$ take any open set V containing $e_x(f) = f(x)$. Then,

$$e_x^{-1}(V) = \{g \in Q(X,Y) : e_x(g) = g(x) \in V\} = [x, V]$$

which is a neighborhood of f in $Q_{sk}(X,Y)$. $\square$

Proposition 3.5. *Let X and Y be topological spaces. Then the evaluation function defined as $e : X \times Q(X,Y) \to Y$ where $e(x,f) = f(x)$, is quasicontinuous with respect to each variable when $Q(X,Y)$ is equipped with the semi-compact-open topology $\mathscr{T}_{sk}$.*

Proof. From Proposition 3.4, the function e is quasicontinuous at the first variable. For any fixed $f \in Q(X,Y)$, $e_f : X \to Y$ is defined by $e_f(x) = f(x)$, that is, $e_f = f$ which is quasicontinuous by definition. Therefore the function e is quasicontinuous with respect to each variable. $\square$

Remark 3.6. Sum of a continuous function with a quasicontinuous function gives a quasicontinuous function. Whereas sum of two real valued quasicontinuous functions gives a cliquish function [4].

Proposition 3.7. *For any topological space X and $\mathbb{R}$ with usual uniformity $\mathcal{U}$, the map $s : C_{k,u}(X) \times Q_{sk,u}(X) \to Q_{sk,u}(X)$ defined by $s(f,g) = f + g$ is continuous.*

Proof. Let $f_0 \in C(X)$, $g_0 \in Q(X)$ such that $s(f_0, g_0) \in W(A,\epsilon)[f_0 + g_0]$ where A is a semi-compact set in X and $\epsilon > 0$. $W(A, \frac{\epsilon}{2})[f_0]$ and $W(A, \frac{\epsilon}{2})[g_0]$ are neighborhoods of f_0 and g_0 in $C_{k,u}(X)$ and $Q_{sk,u}(X)$ respectively. To prove s to be continuous we show that $s(W(A, \frac{\epsilon}{2})[f_0] \times W(A, \frac{\epsilon}{2})[g_0]) \subset W(A,\epsilon)[f_0 + g_0]$. Take $(h,k) \in (W(A, \frac{\epsilon}{2})[f_0] \times W(A, \frac{\epsilon}{2})[g_0])$. Then $|h(a) - f_0(a)| < \frac{\epsilon}{2}$ and $|k(a) - g_0(a)| < \frac{\epsilon}{2}, \forall a \in A$. Consequently, $|(h+k)(a) - (f_0 + g_0)(a)| < \epsilon, \forall a \in A$. Therefore, $s(h,k) = h + k \in W(A,\epsilon)[f_0 + g_0]$. $\square$

Similarly we can prove the following proposition.

Proposition 3.8. *For any topological space X and $\mathbb{R}$ with usual uniformity $\mathcal{U}$, the map $s : Q_{sk,u}(X) \times Q_{sk,u}(X) \to CL_{sk,u}(X)$ defined by $s(f,g) = f + g$ is continuous where $CL(X)$ is the space of cliquish functions defined on X.*

From now on, we consider the cardinality of X to be infinite in addition to it being Hausdorff, the uniformity $\mathcal{U}$ on $\mathbb{R}$ is considered to be the usual one and the topology for the space $Q(X)$ is the topology of uniform convergence on semi-compacta with respect to $\mathcal{U}$ if not mentioned otherwise.

Cardinal invariants are used to prove certain properties of the function spaces using cardinals of the domain space. Topological properties like countability, separability, metrizability etc. can be proved using cardinal invariants ([9], [8], [7]).

The character of a point x in X is defined as,

$$\chi(x, X) = \aleph_0 + \min\{|O| : O \text{ is a base at } x\}.$$

The character of X is defined as,

$$\chi(X) = \sup\{\chi(x, X) : x \in X\}.$$

To define the π-character of X, we first need a notion of a local π-base. If $x \in X$, a collection $\mathcal{V}$ of non-empty open subsets of X is called a local π-base at x (see [16]) provided that for each open neighborhood U of x, there exists a $V \in \mathcal{V}$ which is contained in U.

The π-character of a point x in X is defined as,

$$\pi_\chi(x, X) = \aleph_0 + \min\{|\mathcal{V}| : \mathcal{V} \text{ is a local } \pi\text{-base at } x\}.$$

The π-character of X is defined as,

$$\pi_\chi(X) = \sup\{\pi_\chi(x, X) : x \in X\}.$$

We define SK-cofinality of a topological space X to be

$$SKcof(X) = \aleph_0 + \min\{|\mathcal{B}| : \mathcal{B} \text{ is a cofinal family in } SK(X)\}.$$

We call a topological space X hemi semi-compact if $SKcof(X) = \aleph_0$, that is, there is a countable family $\{K_n : K_n \in SK(X), n \in \mathbb{N}\}$ such that for every $K \in SK(X)$ there is $n \in \mathbb{N}$ with $K \subset K_n$.

Theorem 3.9. *Let X be a topological space and (Y, d) be a metric space which is not bounded, then*

$$\chi(Q_{sk,u}(X, Y)) = \pi_\chi(Q_{sk,u}(X, Y)) = SKcof(X).$$

Proof. First we show that $SKcof(X) \leq \pi_\chi(Q_{sk,u}(X, Y))$. Let $f_{y_0} : X \to Y$ be the constant function which takes all x of X to the single point y_0 in Y. Then f_{y_0} is quasicontinuous. Let $\{W(f_t, A_t, \epsilon_t) : f_t \in Q(X, Y), A_t \in SK(X), \epsilon_t > 0, t \in T\}$ be a local π-base of f_{y_0} in $Q(X, Y)$ with $|T| \leq \pi_\chi(Q_{sk,u}(X, Y))$ where

$$W(f_t, A_t, \epsilon_t) = \{g \in Q(X, Y) : d(f_t(x), g(x)) < \epsilon_t, \forall x \in A_t\}.$$

We show that the collection $\{A_t : t \in T\}$ is a cofinal family in $SK(X)$. Take any $A \in SK(X)$. Then there must be some $t \in T$ with $W(f_t, A_t, \epsilon_t) \subset W(f_{y_0}, A, 1)$. We show that $A \subset A_t$. If possible, let $a \in A \setminus A_t$. Now X being Hausdorff and A_t being compact, there is an open set U such that $a \in U$ and $\overline{U} \cap A_t = \phi$. Since (Y, d) is not bounded there is some $y_1 \in Y$ such that $d(y_0, y_1) > 1$.

Let $g : X \to Y$ be defined as follows,

$$g(x) = \begin{cases} y_1 & x \in \overline{U}, \\ f_t(x) & \text{otherwise.} \end{cases}$$

Then g is quasicontinuous and $g(s) = f_t(s)$ for $s \notin \overline{U}$ which gives $d(g(s), f_t(s)) = 0 < \epsilon_t, \forall s \in A_t$. Therefore, $g \in W(f_t, A_t, \epsilon_t)$, but $g \notin W(f_{y_0}, A, 1)$ since $d(g(a), f_{y_0}(a)) = d(y_1, y_0) > 1$. A contradiction. Thus,

$$SKcof(X) \leq \pi_\chi(Q_{sk,u}(X,Y)) \leq \chi(Q_{sk,u}(X,Y)).$$

To prove that $\chi(Q_{sk,u}(X,Y)) \leq SKcof(X)$, let $f \in Q(X,Y)$. Let $\mathcal{B}$ be a cofinal subfamily of $SK(X)$ with $|\mathcal{B}| = SKcof(X)$. Then the family $\{W(f, K, \frac{1}{n}) : K \in \mathcal{B}, n \in \mathbb{N}\}$ is a local base at f. □

Corollary 3.10. *Let X be a topological space. Then*

$$\chi(Q(X)) = \pi_\chi(Q(X)) = SKcof(X).$$

Corollary 3.11. *For a topological space X, the following are equivalent.*
1. X is hemi semi-compact.
2. $Q(X)$ is first countable.

Now we recall definitions of some other cardinal invariants which will help us to prove the metrizablility of the space $Q(X)$.

The weight of a topological space X is defined as

$$w(X) = \aleph_0 + \min\{|\beta| : \beta \text{ is a base in } X\}.$$

The density of a topological space X is defined as

$$d(X) = \aleph_0 + \min\{|\beta| : \beta \text{ is a dense subset in } X\}.$$

The cellularity of a topological space X is defined as

$$c(X) = \aleph_0 + \sup\{|\mathcal{U}| : \mathcal{U} \text{ is a pairwise disjoint family of non-empty open sets in } X\}.$$

The uniform weight for a Tychonoff space X is defined as

$$u(X) = \aleph_0 + \min\{m : \text{ there is a uniformity on } X \text{ of weight} \leq m\}.$$

From [6] we have, $w(X) = c(X) \times u(X), w(X) = e(X) \times u(X)$, where $e(X)$ is extent of X defined as

$$e(X) = \aleph_0 + \sup\{|E| : E \text{ is a closed discrete set in } X\}.$$

A collection $\mathcal{V}$ of non-empty open subsets of X is called a π-base provided that for each non-empty open set U in X, there exists a $V \in \mathcal{V}$ which is contained in U. The π-weight of X is defined by

$$\pi w(X) = \aleph_0 + min\{|\mathcal{B}| : \mathcal{B} \text{ is a } \pi\text{-base of } X\}.$$

Theorem 3.12. *For a topological space X, $u(Q(X)) = SKcof(X)$.*

Proof. We consider a cofinal family β in $SK(X)$ such that $SKcof(X) = |\beta|$. The family $\{W(K, \frac{1}{n}) : K \in \beta, n \in \mathbb{N}\}$ where $W(K, \frac{1}{n}) = \{(f, g) : \forall x \in K, |f(x) - g(x)| < \frac{1}{n}\}$ is a base of the uniformity on semi-compact convergence. Thus $u(Q(X)) \leq SKcof(X)$. For every Tychonoff space Z, we have $\chi(Z) \leq u(Z)$. By Theorem 3.9, $SKcof(X) = \chi(Q(X))$, therefore we must have $u(Q(X)) = SKcof(X)$. □

Corollary 3.13. *For a topological space X, the following are equivalent.*

1. X is hemi semi-compact.

2. $Q(X)$ is metrizable.

3. $Q(X)$ is first countable.

Corollary 3.14. *For a topological space X, $w(Q(X)) = \pi w(Q(X))$.*

Proof. We know (from [6]) that for a Tychonoff space Z, $w(Z) = c(Z) \times u(Z)$ and $w(Z) = e(Z) \times u(Z)$. Using Theorem 3.12, we have $w(Q(X)) = c(Q(X)) \times SKcof(X)$ and $w(Q(X)) = e(Q(X)) \times SKcof(X)$. By Theorem 3.9, $\pi w(Q(X)) \geq \pi_\chi(Q(X)) = \chi(Q(X)) = SKcof(X)$. Again, $\pi w(Q(X)) \geq d(Q(X))$ which gives the following inequality,

$$\pi w(Q(X)) \geq SKcof(X) \times d(Q(X)) \geq w(Q(X)).$$

□

Ascoli like theorems are used to characterize the compact subsets of function spaces where equicontinuity plays a vital role. For the space of quasicontinuous functions a property called densely equicontinuity is defined to characterize the compact subsets. Results related to the characterization of compact subsets in the space of quasicontinuous functions can be found in the works of Holý and Holá ([10], [12]).

Definition 3.15 ([12])**.** Let X be a topological space and (Y, d) be a metric space. A subset $\mathcal{S}$ of $F(X, Y)$ is said to be densely equiquasicontinuous at a point x in X provided that for every $\epsilon > 0$, there exists a finite family $\mathcal{B}$ of non-empty subsets of X which are either quasi-open or nowhere dense such that $\cup\mathcal{B}$ is a neighborhood of x and for every $f \in \mathcal{S}$, for every $B \in \mathcal{B}$,

$$d(f(p), f(q)) < \epsilon, \forall p, q \in B.$$

$\mathcal{S}$ is called densely equiquasicontinuous provided that it is densely equiquasicontinuous at every point of X.

Remark 3.16. Using open sets in place of quasi-open sets in the above gives an equivalent definition of densely equiquasicontinuity.

Proposition 3.17. *Let X be a locally semi-compact topological space and (Y, d) be a metric space. Then $Q(X, Y)$ is a closed set in $(Y^X, \mathscr{T}_{sk,u})$.*

Proof. Here, we use the fact that a uniform limit of a net of quasicontinuous functions is quasicontinuous. □

Theorem 3.18 ([12])**.** *Let X be a topological space and (Y, d) be a metric space. Let $\mathcal{S}$ be a densely equiquasicontinuous subset of Y^X. Then the topologies $\mathscr{T}_{p,u}$ and $\mathscr{T}_{k,u}$ on $\mathcal{S}$ coincide.*

Corollary 3.19. *Let X be a topological space and (Y, d) be a metric space. Let $\mathcal{S}$ be a densely equiquasicontinuous subset of Y^X. Then the topologies $\mathscr{T}_{p,u}$, $\mathscr{T}_{sk,u}$ and $\mathscr{T}_{k,u}$ on $\mathcal{S}$ coincide.*

We recall few definitions related to Ascoli like theorems.

A function f from a topological space X to a metric space (Y, d) is called locally bounded if for every $x \in X$, there is an open set $U \subset X$ containing x such that $f(U) = \{f(u) : u \in U\}$ is a bounded subset of Y. For any subset $F(X, Y) \subset Y^X$, the set of locally bounded functions from $F(X, Y)$ is denoted by $F^*(X, Y)$.

A metric space (Y, d) has the Heine–Borel property or (Y, d) is called boundedly compact [3] if closed and bounded subsets of Y are compact. Any subset $\mathcal{S} \subset Y^X$ is called pointwise bounded if the set $\mathcal{S}[x] = \{g(x) : g \in \mathcal{S}\}$ is a bounded subset of Y.

Theorem 3.20 ([12])**.** *Let X be a locally compact space and (Y, d) be a nontrivial metric space such that d has the Heine-Borel property. A subset $\mathcal{S}$ of $Q^*_{k,u}(X, Y)$ is compact if and only if it is closed, pointwise bounded and densely equiquasicontinuous.*

Theorem 3.21. *Let X be a locally semi-compact space and (Y, d) be a nontrivial metric space with Heine–Borel property. A subset $\mathcal{S} \subset Q_{sk,u}(X, Y)$ is compact if and only if it is closed, pointwise bounded and densely equiquasicontinuous.*

Proof. Since X is locally semi-compact, $Q(X, Y) = Q^*(X, Y)$. Let $\mathcal{S} \subset Q(X, Y)$ be closed, pointwise bounded and densely equiquasicontinuous. Now, $\mathcal{S}$ being closed in $Q_{sk,u}(X, Y)$, it is also closed in $Q_{k,u}(X, Y)$. Therefore $\mathcal{S}$ is compact in $Q_{k,u}(X, Y)$ which forces $\mathcal{S}$ to be compact in $Q_{sk,u}(X, Y)$.

Conversely, if $\mathcal{S}$ is compact in $Q_{sk,u}(X, Y)$, the space $Q_{sk,u}(X, Y)$ being Hausdorff, $\mathcal{S}$ is closed. Following the arguments of Theorem 2.8 of [12] we can prove that $\mathcal{S}$ is densely equiquasicontinuous. From Proposition 3.4, the evaluation mapping at x defined by $e_x(f) = f(x)$ for all $f \in Q(X, Y)$ is continuous with respect to $\mathscr{T}_{sk,u}$ topology on $Q(X, Y)$, therefore $e_x(\mathcal{S}) = \mathcal{S}[x]$ is compact and bounded. □

Theorem 3.22. *Let X be a locally semi-compact hemi semi-compact space. A subset $\mathcal{S} \subset Q_{sk,u}(X)$ is sequentially compact if and only if it is closed, pointwise bounded and densely equiquasicontinuous.*

Proof. Since the space X is hemi semi-compact, $Q_{sk,u}(X)$ is metrizable by Theorem 3.12 which makes the notions of compactness and sequential compactness equivalent for $\mathcal{S}$ in $Q(X)$. Now by Theorem 3.21 the result follows. $\square$

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