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$\mathcal{I}$-Fréchet-Urysohn property in $C_\alpha(X)$

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Abstract

In this paper, we introduce $\mathcal{I}$-Fréchet-Urysohn, strongly $\mathcal{I}$-Fréchet-Urysohn and strictly $\mathcal{I}$-Fréchet-Urysohn spaces, discuses their properties of countable tightness and mappings that preserve these spaces. Meanwhile, we discuss the internal characterizations of these spaces in $C_\alpha(X)$. The following main theorem is obtained.

Theorem. *Let α be a network of X. The following are equivalent.*

(1) *$C_\alpha(X)$ is a strictly $\mathcal{I}$-Fréchet-Urysohn space.*
(2) *$C_\alpha(X)$ is a strongly $\mathcal{I}$-Fréchet-Urysohn space.*
(3) *$C_\alpha(X)$ is an $\mathcal{I}$-Fréchet-Urysohn space.*
(4) *Every open α-cover of X contains an $\mathcal{I}$-α-sequence.*
(5) *If $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ is a sequence of open α-cover of X, then there is an $\mathcal{I}$-α-sequence $\{U_n\}_{n\in\mathbb{N}}$ of X such that each $U_n \in \mathcal{U}_n$.*
(6) *$C^\omega_\alpha(X)$ is a strictly $\mathcal{I}$-Fréchet-Urysohn space.*

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Keywords: *$\mathcal{I}$-Fréchet-Urysohn spaces; strongly $\mathcal{I}$-Fréchet-Urysohn spaces; strictly $\mathcal{I}$-Fréchet-Urysohn spaces; countable tightness; set-open topology; function spaces .*

1. Introduction

The study of topological properties on spaces of continuous functions $C(X, L)$ has been a significant research topic in general topology in recent years. For spaces X and L, we denote by $C(X, L)$ the set of all continuous functions from X to L. It was endowed with the pointwise convergence topology denoted as

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$C_p(X, L)$, the compact-open topology denoted as $C_k(X, L)$, and the set-open topology denoted as $C_\alpha(X, L)$. The underlying problem is to find topological invariants (P, Q) for which X possesses property P if and only if $C(X, L)$ possesses property Q.

It is known that Fréchet-Urysohn spaces form a distinguished class lying strictly between first-countable spaces and sequential spaces, with profound applications in analysis and geometric topology. The Fréchet-Urysohn property is a desired property in a topological space since it allows us to handle the topology through their sequences. The study of Fréchet-Urysohn property encompasses rich content, key references are provided in works [6, 14, 18, 19]. In the study of ideal convergence, the closely related concept is $\mathcal{I}$-Fréchet-Urysohn spaces. Furthermore, we have conducted preliminary research on this class of spaces [16, 25, 26, 27].

In 1982 and 1993, D. Gerlits [7] and E.G. Pytkeev [21] characterized the Fréchet-Urysohn property of $C_p(X, \mathbb{R})$ using the topological properties of X, respectively. In 1986, A.V. Arhangel'skiĭ [3] characterized the countable fan-tightness of $C_p(X, \mathbb{R})$ using the topological properties of X. In 1994, S. Lin, C. Liu, and H. Teng [13] discussed both the Fréchet-Urysohn property and countable fan-tightness of $C_k(X, \mathbb{R})$.

On this base, we continue the discussion of the $\mathcal{I}$-Fréchet-Urysohn property in topological spaces and function spaces in this paper. We believe that the discussion on this topic has at least the following significance. First, these results generalize some theorems in the Fréchet-Urysohn property in topological spaces and function spaces. Second, we draw into $\mathcal{I}$-Fréchet-Urysohn, strongly $\mathcal{I}$-Fréchet-Urysohn and strictly $\mathcal{I}$-Fréchet-Urysohn spaces, which bring ideal convergence into function spaces, and greatly enrich the applications of ideal convergence. Third, these results prompt the study of the relationship between the spaces defined by ideal convergence and mappings, which further improves the mutual classification between spaces and mappings.

2. Preliminaries

In this section we recall the necessary concept, notation and terminology. The readers may refer to [5, 14] for concept, notation and terminology not explicitly given here.

The concept of $\mathcal{I}$-convergence in topological spaces is a generalization of statistical convergence and the usual convergence, which is based on the ideal of subsets of the set $\mathbb{N}$ of all positive integers. Let $\mathcal{A} = 2^{\mathbb{N}}$ be the family of all subsets of $\mathbb{N}$. An *ideal* $\mathcal{I} \subseteq \mathcal{A}$ is a hereditary family of subsets of $\mathbb{N}$ which is stable under finite unions [8, P. 670], i.e., the following are satisfied: if $B \subset A \in \mathcal{I}$, then $B \in \mathcal{I}$; if $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$. An ideal $\mathcal{I}$ is said to be *non-trivial*, if $\mathcal{I} \neq \emptyset$ and $\mathbb{N} \notin \mathcal{I}$. A non-trivial ideal $\mathcal{I} \subseteq \mathcal{A}$ is called *admissible* if $\mathcal{I} \supseteq \{\{n\} : n \in \mathbb{N}\}$. The family of all finite subsets of $\mathbb{N}$ is denoted by $\mathcal{I}_{fin}$. Then $\mathcal{I}_{fin}$ is the smallest non-trivial ideal contained in each admissible ideal.

Recall that a sequence $\{x_n\}_{n\in\mathbb{N}}$ in a topological space X is said to be *$\mathcal{I}$-convergent* to a point $x \in X$ provided for any neighborhood U of x, we have $\{n \in \mathbb{N} : x_n \notin U\} \in \mathcal{I}$, which is denoted by $x_n \xrightarrow{\mathcal{I}} x$ [8, Definition 3.1]. A subset $F \subseteq X$ is said to be *$\mathcal{I}$-closed* if for each sequence $\{x_n\}_{n\in\mathbb{N}}$ in F with $x_n \xrightarrow{\mathcal{I}} x \in X$, then $x \in F$. A subset $U \subseteq X$ is said to be *$\mathcal{I}$-open* if $X \setminus U$ is $\mathcal{I}$-closed.

For a topological space X, we denote by $\mathbb{F}(X)$ and $\mathbb{K}(X)$ the set of all nonempty finite and nonempty compact subsets of X, respectively. For spaces X and L, we denote by $C(X, L)$ the set of all continuous functions from X to L; if $L = \mathbb{R}$, we denote by $C(X) = C(X, \mathbb{R})$. Let $A \subseteq X$ and $B \subseteq L$. We denote by

$$[A, B] = \{f \in C(X, L) : f(A) \subseteq B\}.$$

It is easy to check that $[A, B_1 \cap B_2] = [A, B_1] \cap [A, B_2]$ and $[A_1 \cup A_2, B] = [A_1, B] \cap [A_2, B]$.

Let X be a topological space, a family α of subsets of X is a *network* of X if whenever $x \in U$ with U open in X, then there exists $A \in \alpha$ such that $x \in A \subseteq U$ [5, P. 127]. A network α of X is said to be a *closed network* (resp., *compact network*) if each element of α is closed (resp., compact) in X.

Definition 2.1 ([1, Definition 4.2])**.** A topology on $C(X, L)$ is said to be a *set-open topology* if there is a closed network α of X such that a subbase of this topology is formed by $\delta = \{[A, V] : A \in \alpha \text{ and } V \text{ is open in } L\}$. The continuous functions space with this topology denoted by $C_\alpha(X, L)$. Particularly when $L = \mathbb{R}$, it simply denoted by $C_\alpha(X)$.

If Y is a subspace of X, $C_\beta(Y, L)$ is still denoted by $C_\alpha(Y, L)$, where $\beta = \alpha|_Y$.

If $\alpha = \mathbb{F}(X)$ (or the family of all singletons), the set-open topology on $C(X, L)$ which generates by α is called *pointwise convergent topology*, and the continuous functions space with this topology denoted by $C_p(X, L)$. If $\alpha = \mathbb{K}(X)$, the set-open topology on $C(X, L)$ which generates by α is called *compact-open topology*, and the continuous functions space with this topology denoted by $C_k(X, L)$.

Definition 2.2. Let $\mathcal{I}$ be an ideal on $\mathbb{N}$ and α be a closed network of X.

(1) A cover $\mathcal{U}$ of X is called an *α-cover* provided each element from α is contained in a member of $\mathcal{U}$. When $\alpha = \mathbb{F}(X)$ (resp., $\alpha = \mathbb{K}(X)$), then α-cover is called *ω-cover* (resp., *k-cover*).

(2) A sequence $\{U_n\}_{n\in\mathbb{N}}$ of subsets of X is called an *$\mathcal{I}$-α-sequence* provided for each $A \in \alpha$, we have $\{n \in \mathbb{N} : A \nsubseteq U_n\} \in \mathcal{I}$.

An $\mathcal{I}_{fin}$-α-sequence is called *α-sequence*. It is obvious that each α-sequence of a space X is an $\mathcal{I}$-α-sequence of X. But the converse need not be true, we can refer to Example 3.2 in [4]. According to notations above it is clear what means the notation *$\mathcal{I}$-ω-sequence and $\mathcal{I}$-k-sequence*.

In this paper, if no otherwise specified, we consider $\mathcal{I}$ is always an admissible ideal on $\mathbb{N}$, all mappings are surjection, all topological spaces are supposed to be T_1 unless a stronger separation axiom is explicitly assumed.

3. $\mathcal{I}$-Fréchet-Urysohn spaces

In this section, we shall discuss some properties of ideal version of Fréchet-Urysohn spaces. First, we recall three properties of generalized sequences: Fréchet-Urysohn properties, strongly Fréchet-Urysohn properties and strictly Fréchet-Urysohn properties. A topological space X is said to be a *Fréchet-Urysohn* space if for each $A \subseteq X$ and each $x \in \overline{A}$ implies the existence of a sequence in A converging to x in X [6, P. 113]. A space X is said to be a *strongly Fréchet-Urysohn* space if $\{A_n\}_{n\in\mathbb{N}}$ is a decreasing sequence of subsets of X and $x \in \bigcap_{n\in\mathbb{N}} \overline{A_n}$ implies the existence of a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that each $x_n \in A_n$ with $x_n \to x$ [24, Definition 1.1]. A space X is said to be a *strictly Fréchet-Urysohn* space if $x \in \bigcap_{n\in\mathbb{N}} \overline{A_n}$ implies the existence of a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that each $x_n \in A_n$ with $x_n \to x$ [7, P. 151]. Similarly, we have the following definitions.

Definition 3.1. Let $\mathcal{I}$ be an ideal on $\mathbb{N}$ and X be a topological space.

(1) X is said to be an *$\mathcal{I}$-Fréchet-Urysohn* space if for each $A \subseteq X$ and each $x \in \overline{A}$ implies the existence of a sequence in A $\mathcal{I}$-converging to x in X [22, P. 90].

(2) X is said to be a *strongly $\mathcal{I}$-Fréchet-Urysohn* space if $\{A_n\}_{n\in\mathbb{N}}$ is a decreasing sequence of subsets of X and $x \in \bigcap_{n\in\mathbb{N}} \overline{A_n}$ implies the existence of a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that each $x_n \in A_n$ with $x_n \xrightarrow{\mathcal{I}} x \in X$.

(3) X is said to be a *strictly $\mathcal{I}$-Fréchet-Urysohn* space if $x \in \bigcap_{n\in\mathbb{N}} \overline{A_n}$ implies the existence of a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that each $x_n \in A_n$ with $x_n \xrightarrow{\mathcal{I}} x \in X$.

It is obvious that, first countable spaces $\Rightarrow$ strictly $\mathcal{I}$-Fréchet-Urysohn spaces $\Rightarrow$ strongly $\mathcal{I}$-Fréchet-Urysohn spaces $\Rightarrow$ $\mathcal{I}$-Fréchet-Urysohn spaces.

If $\mathcal{I} = \mathcal{I}_{fin}$, then $\mathcal{I}_{fin}$-Fréchet-Urysohn space, strongly $\mathcal{I}_{fin}$-Fréchet-Urysohn space and strictly $\mathcal{I}_{fin}$-Fréchet-Urysohn space coincide with Fréchet-Urysohn space, strongly Fréchet-Urysohn space and strictly Fréchet-Urysohn space, respectively.

Definition 3.2. Let X be a topological space.

(1) X is said to have *countable tightness* if whenever $A \subseteq X$ and $x \in \overline{A}$ in X, then $x \in \overline{C}$ for some countable subset $C \subseteq A$ [17, Definition 8.2].

(2) X is said to have *countable fan-tightness* if $x \in \bigcap_{n\in\mathbb{N}} \overline{A_n}, A_n \subseteq X$ implies that for each $n \in \mathbb{N}$ there exists a finite subset B_n of A_n such that $x \in \overline{\bigcup_{n\in\mathbb{N}} B_n}$ [3, P. 526].

(3) X is said to have *countable strong fan-tightness* if $x \in \bigcap_{n\in\mathbb{N}} \overline{A_n}, A_n \subseteq X$ implies that there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that each $x_n \in A_n$ and $x \in \overline{\{x_n : n \in \mathbb{N}\}}$ [23, P. 917].

Recall that X is called an *$\mathcal{I}$-sequential space* provided each $\mathcal{I}$-open subset of X is open [20, 26]. Obvious, X is an $\mathcal{I}$-sequential space if and only if each $\mathcal{I}$-closed subset of X is closed. An $\mathcal{I}_{fin}$-sequential space is called a *sequential space* [5, P. 53].

Proposition 3.3. *The following conclusions hold for a topological space X and an ideal $\mathcal{I}$ on $\mathbb{N}$.*

(1) *Each $\mathcal{I}$-Fréchet-Urysohn space has countable tightness.*

(2) *Each strongly $\mathcal{I}$-Fréchet-Urysohn space has countable fan-tightness.*

(3) *Each strictly $\mathcal{I}$-Fréchet-Urysohn space has countable strong fan-tightness.*

Proof. (1) Note that each $\mathcal{I}$-Fréchet-Urysohn space is an $\mathcal{I}$-sequential space [22, Theorem 2.5]; and each $\mathcal{I}$-sequential space is of countable tightness [26, Theorem 3.7].

(2) Let X be a strongly $\mathcal{I}$-Fréchet-Urysohn space, $\{A_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of subsets of X and $x \in \bigcap_{n\in\mathbb{N}} \overline{A_n}$. For each $n \in \mathbb{N}$, put $A'_n = \bigcup_{k\geq n} A_k$. It follows that $\{A'_n\}_{n\in\mathbb{N}}$ is a decreasing sequence of subsets of X and $x \in \bigcap_{n\in\mathbb{N}} \overline{A'_n}$. Since X is a strongly $\mathcal{I}$-Fréchet-Urysohn space, there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that each $x_n \in A'_n$ and $x_n \xrightarrow{\mathcal{I}} x \in X$. For each $n \in \mathbb{N}$, take $B_n = \{x_k : k \leq n\} \cap A_n$. Then B_n is a finite subset of A_n and $\bigcup_{n\in\mathbb{N}} B_n = \{x_n : n \in \mathbb{N}\}$. And because $x_n \xrightarrow{\mathcal{I}} x \in X$, it follows that $\{n \in \mathbb{N} : x_n \in U\} \in \mathbb{N} \setminus \mathcal{I}$ for any neighborhood U of x in X. Thus $x \in \overline{\bigcup_{n\in\mathbb{N}} B_n}$, and further X has countable fan-tightness.

(3) Let X be a strictly $\mathcal{I}$-Fréchet-Urysohn space, $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of subsets of X and $x \in \bigcap_{n\in\mathbb{N}} \overline{A_n}$. Then there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that each $x_n \in A_n$ with $x_n \xrightarrow{\mathcal{I}} x \in X$. Hence $\{n \in \mathbb{N} : x_n \in U\} \in \mathbb{N} \setminus \mathcal{I}$ for any neighborhood U of x in X, thus $x \in \overline{\{x_n : n \in \mathbb{N}\}}$, and further X has countable strong fan-tightness. □

Definition 3.4. Let X, Y be topological spaces and $f : X \to Y$ be a mapping.

(1) f is called *countably bi-quotient*, if for each $y \in Y$ and each countable family $\mathcal{U}$ of open subsets in X which covers $f^{-1}(y)$, then there is a finite subfamily $\mathcal{U}'$ of $\mathcal{U}$ such that $y \in [f(\bigcup \mathcal{U}')]^\circ$ [24, P. 149].

(2) f is called *strictly countably bi-quotient*, if for each $y \in Y$ and each countable family $\mathcal{U}$ of open subsets in X which covers $f^{-1}(y)$, then there is $m \in \mathbb{N}$ such that $y \in [f(U_m)]^\circ$ [15, Definition 2.2].

(3) f is said to be *pseudo-open*, if for each $y \in Y$ and an open subset U in X with $f^{-1}(y) \subseteq U$, then $f(U)$ is a neighborhood of y in Y[2, Definition 1].

(4) f is said to be *$\mathcal{I}$-quotient* provided $f^{-1}(U)$ is $\mathcal{I}$-open in X, then U is $\mathcal{I}$-open in Y [26, Definition 5.1].

(5) f is said to be *$\mathcal{I}$-covering*, if whenever $\{y_n\}_{n\in\mathbb{N}}$ is a sequence in Y $\mathcal{I}$-converging to y in Y, there exist a sequence $\{x_n\}_{n\in\mathbb{N}}$ of points $x_n \in f^{-1}(y_n)$ for each $n \in \mathbb{N}$ and $x \in f^{-1}(y)$ such that $x_n \xrightarrow{\mathcal{I}} x$ [26, Definition 5.1].

It is obvious that open mappings ⇒ strictly countably bi-quotient mappings ⇒ countably bi-quotient mappings ⇒ pseudo-open mappings.

It is known that each $\mathcal{I}$-convergence sequence is preserved by a continuous mapping [10, Theorem 3]; each $\mathcal{I}$-sequential space is preserved by a continuous and quotient mapping [26, Theorem 5.3]; each $\mathcal{I}$-covering mapping onto an $\mathcal{I}$-Fréchet-Urysohn is pseudo-open [26, Theorem 6.7].

Proposition 3.5. (1) *Each $\mathcal{I}$-Fréchet-Urysohn space is preserved by a continuous and pseudo-open mapping.*

(2) *Each strongly $\mathcal{I}$-Fréchet-Urysohn is preserved by a continuous and countably bi-quotient mapping.*

(3) *Each strictly $\mathcal{I}$-Fréchet-Urysohn is preserved by a continuous and strictly countably bi-quotient mapping.*

Proof. (1) It has been proved in [22, Proposition 2.3].

(2) Let X be a strongly $\mathcal{I}$-Fréchet-Urysohn space and $f : X \to Y$ be a continuous countably bi-quotient mapping. Suppose that $\{A_n\}_{n\in\mathbb{N}}$ is a decreasing sequence of subsets of Y and $y \in \bigcap_{n\in\mathbb{N}} \overline{A_n}$. Then there is $x \in f^{-1}(y)$ such that $x \in \bigcap_{n\in\mathbb{N}} \overline{f^{-1}(A_n)}$. Otherwise, $f^{-1}(y) \cap (\bigcap_{n\in\mathbb{N}} \overline{f^{-1}(A_n)}) = \varnothing$, i.e., $f^{-1}(y) \subseteq X \setminus (\bigcap_{n\in\mathbb{N}} \overline{f^{-1}(A_n)}) = \bigcup_{n\in\mathbb{N}}(X \setminus \overline{f^{-1}(A_n)})$. Since f is a countably bi-quotient mapping and $\{A_n\}_{n\in\mathbb{N}}$ is a decreasing sequence, there exists $n \in \mathbb{N}$ such that $y \in [f(X \setminus \overline{f^{-1}(A_n)})]^\circ$. Thus $\varnothing \neq A_n \cap f(X \setminus \overline{f^{-1}(A_n)}) \subseteq A_n \cap f(X \setminus f^{-1}(A_n)) = A_n \cap (Y \setminus A_n) = \varnothing$. This is a contradiction. Note that X is a strongly $\mathcal{I}$-Fréchet-Urysohn space, there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that each $x_n \in f^{-1}(A_n)$ and $x_n \xrightarrow{\mathcal{I}} x$. Thus $f(x_n) \in A_n$ and $f(x_n) \xrightarrow{\mathcal{I}} f(x) = y$ from f being continuous. This implies that Y is a strongly $\mathcal{I}$-Fréchet-Urysohn space.

(3) Similar to the proof in (2). □

Proposition 3.6. *Let Y be a strongly $\mathcal{I}$-Fréchet-Urysohn. Then each $\mathcal{I}$-covering mapping from a topological space X onto Y is countably bi-quotient.*

Proof. If $f : X \to Y$ is not a countably bi-quotient mapping, then there exist a point $y \in Y$ and a countable family $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ of open subsets in X which covers $f^{-1}(y)$ such that $y \in Y \setminus [f(\bigcup_{i\leq n} U_i)]^\circ = \overline{Y \setminus f(\bigcup_{i\leq n} U_i)}$ for each $n \in \mathbb{N}$. Since Y is a strongly $\mathcal{I}$-Fréchet-Urysohn space, there is $y_n \in Y \setminus f(\bigcup_{i\leq n} U_i)$ with $y_n \xrightarrow{\mathcal{I}} y \in Y$. And since f is an $\mathcal{I}$-covering mapping, there exist a sequence $\{x_n\}_{n\in\mathbb{N}}$ of points $x_n \in f^{-1}(y_n)$ for each $n \in \mathbb{N}$ and $x \in f^{-1}(y)$ such that $x_n \xrightarrow{\mathcal{I}} x \in X$. Note that $\{U_n : n \in \mathbb{N}\}$ is a family of open subsets in X and covers $f^{-1}(y)$, hence there is $m_0 \in \mathbb{N}$ such that $x \in U_{m_0}$, thus $\{n \in \mathbb{N} : x_n \in U_{m_0}\} = F \in \mathbb{N} \setminus \mathcal{I}$. Therefore there is $k \in F$ and $k > m_0$ such that $x_k \in U_{m_0}$, and further $y_k \in f(U_{m_0})$. This is a contradiction. Hence f is a countably bi-quotient mapping. □

Question 3.7. (1) *Let Y be a strongly $\mathcal{I}$-Fréchet-Urysohn and f be an $\mathcal{I}$-quotient mapping from a topological space X onto Y. Is f a countably bi-quotient mapping?*

(2) *Let Y be an a strictly $\mathcal{I}$-Fréchet-Urysohn. Is each $\mathcal{I}$-covering mapping from a topological space X onto Y strictly countably bi-quotient?*

(3) *Is X an $\mathcal{I}$-Fréchet-Urysohn space which has countable fan-tightness a strongly $\mathcal{I}$-Fréchet-Urysohn space?*

Example 3.8. There exists an $\mathcal{I}$-Fréchet-Urysohn space which is not a strongly $\mathcal{I}$-Fréchet-Urysohn space.

Proof. Let $\mathcal{I} = \mathcal{I}_{fin}$ and S_ω be the sequential fan. Then S_ω satisfies the requirements [11, Example 1.4.2]. □

At the end of this section, the following figure illustrates the relationship between these spaces and properties.

$$\begin{tikzcd}
\text{strictly FU-spaces} \arrow[r] \arrow[d] & \text{strongly FU-spaces} \arrow[r] \arrow[d] & \text{FU-spaces} \arrow[d] \\
\text{strictly } \mathcal{I}\text{-FU-spaces} \arrow[r] \arrow[d] & \text{strongly } \mathcal{I}\text{-FU-spaces} \arrow[r] \arrow[d] & \mathcal{I}\text{-FU-spaces} \arrow[d] \\
\text{countable strongly fan-tightness} \arrow[r] & \text{countable fan-tightness} \arrow[r] & \text{countable tightness}
\end{tikzcd}$$

4. $\mathcal{I}$-Fréchet-Urysohn property in $C_\alpha(X)$

It is known that some topological properties of $C_\alpha(X)$ can be well characterized by topological properties of X [12]. The purpose of this section is to characterize $\mathcal{I}$-Fréchet-Urysohn property of $C_\alpha(X)$ in terms of X.

In this section, we assume that all topological spaces are Tychonoff spaces and all closed networks α are hereditarily closed and compact. On one hand, the reason for this separation hypothesis is that it implies the existence of many continuous real-valued functions. On the other hand, if X is a Tychonoff space, A is compact in X and $B \subseteq X \setminus A$ is closed, then there is a $f \in C(X, [0, 1])$ such that $f(x) = 0, x \in A$; $f(x) = 1, x \in B$. In following, the symbol f_0 denotes the constantly zero function. Since $C_\alpha(X)$ is a topological vector space, we may confine ourselves to the point f_0 when studying local properties.

Theorem 4.1. *Let $\mathcal{I}$ be an ideal on $\mathbb{N}$ and α be a network of X. Then the following are equivalent.*

(1) *$C_\alpha(X)$ is a strictly $\mathcal{I}$-Fréchet-Urysohn space.*

(2) *$C_\alpha(X)$ is a strongly $\mathcal{I}$-Fréchet-Urysohn space.*

(3) *$C_\alpha(X)$ is an $\mathcal{I}$-Fréchet-Urysohn space.*

(4) *Every open α-cover of X contains an $\mathcal{I}$-α-sequence.*

(5) *If* $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ *is a sequence of open* α*-cover of* X*, then there is an* $\mathcal{I}$*-*α*-sequence* $\{U_n\}_{n\in\mathbb{N}}$ *of* X *such that each* $U_n \in \mathcal{U}_n$*.*

(6) $C^{\omega}_{\alpha}(X)$ *is a strictly* $\mathcal{I}$*-Fréchet-Urysohn space, where* $C^{\omega}_{\alpha}(X) = [C_{\alpha}(X)]^{\omega}$*.*

Proof. It is clear that (1) ⇒ (2) ⇒ (3) and (6) ⇒ (1). We only need to prove that (3) ⇒ (4) ⇒ (5) ⇒ (6).

(3) ⇒ (4) Let $C_\alpha(X)$ be an $\mathcal{I}$-Fréchet-Urysohn space and $\mathcal{U}$ be an open α-cover of X. For each $A \in \alpha$, there is $U_A \in \mathcal{U}$ such that $A \subseteq U_A$. Hence there exists $f_A \in C(X)$ such that $f_A(A) = \{0\}$ and $f_A(X \setminus U_A) \subseteq \{1\}$. It is easy to check that $f_0 \in \overline{\{f_A : A \in \alpha\}}$. It is follows from (3) that there is a sequence $\{A_n\}_{n\in\mathbb{N}}$ of subsets of α such that $\{f_{A_n}\}_{n\in\mathbb{N}}$ is a sequence in $C_\alpha(X)$ with $f_{A_n} \xrightarrow{\mathcal{I}} f_0$. Next, we shall show that $\{U_{A_n}\}_{n\in\mathbb{N}}$ is an $\mathcal{I}$-α-sequence of X. If $A \in \alpha$, then $f_0 \in [A, (-1, 1)]$, hence $\{n \in \mathbb{N} : f_{A_n} \notin [A, (-1, 1)]\} = I_A \in \mathcal{I}$. Thus if $x \in A$ and $n \in \mathbb{N} \setminus I_A$, then $f_{A_n}(x) < 1$, hence $x \notin X \setminus U_{A_n}$, i.e., $x \in U_{A_n}$. Therefore, $\{n \in \mathbb{N} : A \nsubseteq U_{A_n}\} \subseteq I_A \in \mathcal{I}$ for each $A \in \alpha$. This implies that $\{U_{A_n}\}_{n\in\mathbb{N}}$ is an $\mathcal{I}$-α-sequence of X.

(4) ⇒ (5) Let $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ be an open α-cover sequence of X. Without lose of generality, we may assume that each $\mathcal{U}_{n+1}$ refine $\mathcal{U}_n$. If $X \in \alpha$, then for each $n \in \mathbb{N}$, there is $U_n \in \mathcal{U}_n$ such that $X \subseteq U_n$. Thus $\{U_n\}_{n\in\mathbb{N}}$ is an $\mathcal{I}$-α-sequence of X. If $X \notin \alpha$, then $\{X \setminus \{x\}\}_{x\in X}$ is an open α-cover of X. It follows from (4) that there is a subset $\{x_n : n \in \mathbb{N}\}$ of X such that $\{X \setminus \{x_n\}\}_{n\in\mathbb{N}}$ is an $\mathcal{I}$-α-sequence of X, i.e., for each $A \in \alpha$, $\{n \in \mathbb{N} : A \nsubseteq X \setminus \{x_n\}\} = I'_A \in \mathcal{I}$. Put $\mathcal{B}_n = \{U \setminus \{x_n\} : U \in \mathcal{U}_n\}$ for each $n \in \mathbb{N}$. Then $\mathcal{B} = \bigcup_{n\in\mathbb{N}} \mathcal{B}_n$ is an α-cover of X. In fact, for each $A \in \alpha$, there is $n \in \mathbb{N} \setminus I'_A$ and $U \in \mathcal{U}_n$ such that $A \subseteq X \setminus \{x_n\}$ and $A \subseteq U$, thus $A \subseteq U \setminus \{x_n\}$. And by (4), $\mathcal{B}$ contains an $\mathcal{I}$-α-sequence $\{G_k\}_{k\in\mathbb{N}}$. Hence for each $A \in \alpha$, we have $\{k \in \mathbb{N} : A \nsubseteq G_k\} = I_A \in \mathcal{I}$. For each $k \in \mathbb{N}$, there is $n_k \in \mathbb{N}$ such that $G_k \in \mathcal{B}_{n_k}$, Thus there is $U_{n_k} \in \mathcal{U}_{n_k}$ such that $G_k = U_{n_k} \setminus \{x_{n_k}\}$. For each $n \in \mathbb{N}$, we have $\{x_1, x_2 \cdots , x_n\} \in \alpha$. Therefore, $\{x_i : i \leq n\} \subseteq G_k$ for $k \in \mathbb{N} \setminus I_A$. If $k \in \mathbb{N} \setminus I_A$ and $n_k \leq n$, then $x_{n_k} \in G_k = U_{n_k} \setminus \{x_{n_k}\}$, which is a contradiction. Thus $\{n_k : k \in \mathbb{N}\}$ is an infinite set. So there is an increasing subsequence $\{n_{k_i}\}_{i\in\mathbb{N}}$ of $\{n_k : k \in \mathbb{N}\}$. Since $\mathcal{U}_{n_{k_{i+1}}}$ refine $\mathcal{U}_n$ for $n_{k_i} < n < n_{k_{i+1}}$, there is $U_n \in \mathcal{U}_n$ such that $U_{n_{k_{i+1}}} \subseteq U_n$. Put

$$W_n = \begin{cases} U_{n_{k_i}}, & n = n_{k_i}, \\ U_n, & n \neq n_{k_i}, i \in \mathbb{N}. \end{cases}$$

Then $W_n \in \mathcal{U}_n$. For each $A \in \alpha$, there is $k_i \in \mathbb{N} \setminus I_A$ such that $A \subseteq G_{k_i} \subseteq U_{n_{k_i}}$, hence $A \subseteq W_n$. Note that $\{n \in \mathbb{N} : n < n_{k_i}\}$ is a finite set of $\mathbb{N}$, this implies that $\{n \in \mathbb{N} : A \nsubseteq W_n\} \subseteq I_A \cup \{n \in \mathbb{N} : n < n_{k_i}\} \in \mathcal{I}$. Consequently, $\{W_n\}_{n\in\mathbb{N}}$ is an $\mathcal{I}$-α-sequence of X.

(5) ⇒ (6) Since $C^{\omega}_{\alpha}(X)$ is homeomorphic to $C_\alpha(X, \mathbb{R}^\omega)$, it suffices to prove that $C_\alpha(X, \mathbb{R}^\omega)$ is a strictly $\mathcal{I}$-Fréchet-Urysohn space. Suppose that $\{A_n\}_{n\in\mathbb{N}}$ is a sequence of subsets of $C_\alpha(X, \mathbb{R}^\omega)$ and $f_0 \in \bigcap_{n\in\mathbb{N}} \overline{A_n}$. For each $n \in \mathbb{N}$, put

$$\mathcal{U}_n = \{f^{-1}(O_n) : f \in A_n\},$$

where $\{O_n\}_{n\in\mathbb{N}}$ is a decreasing open neighborhood base of $\mathbf{0} = (0,0,\cdots)$ in $\mathbb{R}^\omega$. Then each $\mathcal{U}_n$ is an α-cover sequence of X. In fact, for each $A \in \alpha$ and each $n \in \mathbb{N}$, $[A, O_n]$ is a base neighborhood of f_0 in $C_\alpha(X,\mathbb{R}^\omega)$. Since $f_0 \in \bigcap_{n\in\mathbb{N}} \overline{A_n}$, it follows that $[A, O_n] \cap A_n \neq \emptyset$. Thus there is $f \in A_n$ such that $f \in [A, O_n]$, hence $A \subseteq f^{-1}(O_n)$. It follows from (5) that there is an $\mathcal{I}$-α-sequence $\{U_n\}_{n\in\mathbb{N}}$ of X such that each $U_n \in \mathcal{U}_n$. Thus we have $\{n \in \mathbb{N} : A \nsubseteq U_n\} = I_A \in \mathcal{I}$ for each $A \in \alpha$. Take $f_n \in A_n$ such that $U_n = f_n^{-1}(O_n)$. Next, we shall show that $f_n \xrightarrow{\mathcal{I}} f_0$ in $C_\alpha(X,\mathbb{R}^\omega)$. Let $[A, V]$ be a basic neighborhood of f_0 in $C_\alpha(X,\mathbb{R}^\omega)$. Thus there is $m \in \mathbb{N}$ such that $O_m \subseteq V$. For each $n \in \mathbb{N} \setminus I_A$ and $n \geq m$, we have $A \subseteq U_n$, further $f_n(A) \subseteq f_n(U_n) \subseteq O_n \subseteq O_m \subseteq V$. Hence $\{n \in \mathbb{N} : f_n \notin [A, V]\} \subseteq I_A \cup \{n \in \mathbb{N} : n \leq m\} \in \mathcal{I}$. This implies that $f_n \xrightarrow{\mathcal{I}} f_0$. Hence $C_\alpha(X,\mathbb{R}^\omega)$ is a strictly $\mathcal{I}$-Fréchet-Urysohn space. □

Let Y be a subspace of X, we still denoted $C_\beta(X)$ by $C_\alpha(X)$, where $\beta = \alpha\mid_Y$.

Corollary 4.2. *Let $C_\alpha(X)$ be an $\mathcal{I}$-Fréchet-Urysohn space and Y be a closed subspace of X. Then $C_\alpha(Y)$ is an $\mathcal{I}$-Fréchet-Urysohn space.*

Proof. Let $C_\alpha(X)$ be an $\mathcal{I}$-Fréchet-Urysohn space and Y be a closed subspace of X. Then $\beta = \alpha\mid_Y$ is a hereditarily closed and compact network of Y. Suppose that $\mathcal{U}$ is an open β-cover of Y, then $\mathcal{V} = \{V : V = (X \setminus Y) \cup U, U \in \mathcal{U}\}$ is an open α-cover of X. In fact, for each $A \in \alpha$, there is $U \in \mathcal{U}$ and $V \in \mathcal{V}$ such that $A = (A \cap (X \setminus Y)) \cup (A \cap Y) \subseteq (X \setminus Y) \cup U = V$. Since $C_\alpha(X)$ is an $\mathcal{I}$-Fréchet-Urysohn space, it follows from Theorem 4.1 that there is an $\mathcal{I}$-α-sequence $\{V_n\}_{n\in\mathbb{N}}$ of $\mathcal{V}$. Hence for each $A \in \alpha$, we have $\{n \in \mathbb{N} : A \nsubseteq V_n\} \in \mathcal{I}$. Next, it suffices to show that $\{U_n\}_{n\in\mathbb{N}}$ is an $\mathcal{I}$-β-sequence of $\mathcal{U}$. For each $B \in \beta$, since Y is a closed subspace of X, it follows that $B \in \alpha$. Thus $\{n \in \mathbb{N} : B \nsubseteq V_n\} \in \mathcal{I}$. Note that $B \in \beta = \alpha\mid_Y$ and $V_n = (X \setminus Y) \cup U_n$, we have $\{n \in \mathbb{N} : B \nsubseteq U_n\} = \{n \in \mathbb{N} : B \nsubseteq V_n\} \in \mathcal{I}$. This implies that $\{U_n\}_{n\in\mathbb{N}}$ is an $\mathcal{I}$-β-sequence of $\mathcal{U}$. Therefore $C_\alpha(Y) = C_\beta(Y)$ is an $\mathcal{I}$-Fréchet-Urysohn space. □

Corollary 4.3. *Let $f : X \to Y$ be continuous and $C_p(X)$ be an $\mathcal{I}$-Fréchet-Urysohn space. Then $C_p(Y)$ is an $\mathcal{I}$-Fréchet-Urysohn space.*

Proof. Let $\mathcal{U}$ be an open ω-cover of Y. Since f is continuous and surjection, $f^{-1}(\mathcal{U}) = \{f^{-1}(U) : U \in \mathcal{U}\}$ is an open ω-cover of X. Note that $C_p(X)$ is an $\mathcal{I}$-Fréchet-Urysohn space, it follows from Theorem 4.1 that there is an $\mathcal{I}$-ω-sequence $\{f^{-1}(U_n) : U_n \in \mathcal{U}\}_{n\in\mathbb{N}}$ of $f^{-1}(\mathcal{U})$. For each $F \in \mathbb{F}(Y)$, there is $A \in \mathbb{F}(X)$ such that $f(A) = F$. Hence $\{n \in \mathbb{N} : F \nsubseteq U_n\} \subseteq \{n \in \mathbb{N} : A \nsubseteq f^{-1}(U_n)\} \in \mathcal{I}$. This means that $\{U_n\}_{n\in\mathbb{N}}$ is an $\mathcal{I}$-ω-sequence of $\mathcal{U}$. Hence $C_p(Y)$ is an $\mathcal{I}$-Fréchet-Urysohn space. □

Recall that a space X is called *zero-dimensional* if X has a base consisting of clopen subsets of X, and denoted it by ind$X$=0. A space X is called *strongly zero-dimensional* if for each closed subset A of X and an open set V with $A \subseteq V$, there is an open subset U of X such that $A \subseteq U \subseteq V$, and denoted it by

Ind$X$=0. A space X has *property* C'', if for any sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of open covers of X there is a sequence $\{U_n\}_{n\in\mathbb{N}}$ covers X such that each $U_n \in \mathcal{U}_n$ [9]. It is know that the unit closed interval $[0,1]$ does not have property C'' [23, P. 918].

Lemma 4.4 ([12, Corallary 2.1.11])**.** *Let X be a Lindelöf space. If ind$X$=0, then Ind$X$=0.*

Lemma 4.5 ([23, Theorem 1])**.** *The following are equivalent for a space X.*

(1) *Each finite product of X has property C''.*
(2) *$C_p(X)$ has countable strongly fan-tightness.*

Theorem 4.6. *Let $C_p(X)$ be an $\mathcal{I}$-Fréchet-Urysohn space. Then Ind$X$=0.*

Proof. Since $C_p(X)$ is an $\mathcal{I}$-Fréchet-Urysohn space, it follows from Proposition 3.3 that $C_p(X)$ has countable tightness. Hence X is a Lindelöf space [12, P. 179]. By Lemma 4.4, it suffices to show that ind$X$=0. For each $x \in X$ and each open neighborhood U of x in X, there is $f \in C_p(X,[0,1])$ such that $f(x) = 1$ and $f(X \setminus U) \subseteq \{0\}$. Note that $C_p(X)$ is an $\mathcal{I}$-Fréchet-Urysohn space, it follows from Corollary 4.3 that $C_p(f(X))$ is an $\mathcal{I}$-Fréchet-Urysohn space. By Theorem 4.1 and Proposition 3.3, $C_p(f(X))$ is a strictly $\mathcal{I}$-Fréchet-Urysohn space, and $C_p(f(X))$ has countable strong fan-tightness. Since unit closed interval $[0,1]$ does not have property C'', it follows from Lemma 4.5 that $f(X) \neq [0,1]$. Thus there is $y \in [0,1) \setminus f(X)$. Hence $x \in f^{-1}((y,1]) \subseteq U$ and $f^{-1}((y,1]) = f^{-1}([y,1])$ is clopen in X. Hence ind$X$=0, and further Ind$X$=0. □

Question 4.7. *Does there exist a function space $C_\alpha(X)$ being an $\mathcal{I}$-Fréchet-Urysohn space which is not Fréchet-Urysohn space for some ideal $\mathcal{I}$ on $\mathbb{N}$.*

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