

# Some innovative results for interpolative Kannan type and Reich-Rus-Ćirić type cyclic contractions

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## ABSTRACT

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*In this manuscript, we present innovative findings on the existence and uniqueness of fixed points for cyclic mappings. Our discussion covers results for interpolative Kannan-type cyclic contractions in two cases: when the sum of the interpolative exponents is less than one and when it is greater than one. Additionally, we provide results for interpolative Reich-Rus-Ćirić cyclic contractions specifically for cases where the sum of the interpolative exponents is greater than one. Furthermore, we verify our results with suitable examples. Our results are new and complement some results in the literature.*

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## 1. INTRODUCTION

After the formulation of the Banach fixed-point theorem, the field of fixed-point theory expanded significantly, encompassing various aspects such as the derivation of fixed-point results for different distance structures and types of contractions. In this context, Karapınar [5] was

the first to introduce the concept of interpolative Kannan-type contraction, identifying fixed points for such mappings. This advancement in fixed-point theory by Karapınar inspired many researchers to explore this newly established contraction, leading to significant new insights in the field. Additionally, in [6], Karapınar revisited the renowned Meir-Keeler contractions by incorporating interpolation theory in the context of complete metric spaces.

In 2003, Kirk et al. [10] introduced the concept of cyclic contractive mappings in their paper, where they generalized the notion of contraction mappings to cyclic contractive mappings and demonstrated that these mappings also possess a unique fixed point. Following this, many researchers continued this work and obtained numerous significant results related to cyclic mappings, as evidenced in [1, 4, 8, 3, 9, 11]. In [8], Krapınar and Erhan discussed the existence and uniqueness of fixed points of a class of cyclic operators defined on a closed subset of a Banach space. Additionally, in [7], Krapınar and Erhan obtained some proximity points using different types of cyclic contractions, introduced a generalized cyclic Meir-Keeler contraction, and stated a new fixed point theorem for this cyclic mapping.

**Definition 1.1.** Let  $(X, d)$  be a metric space,  $A$  and  $B$  be two nonempty subsets of  $X$ . A mapping  $T : A \cup B \rightarrow A \cup B$  is said to be a cyclic mapping provided that  $T(A) \subseteq B, T(B) \subseteq A$ . A point  $x \in A \cup B$  is called a best proximity point if  $d(x, Tx) = d(A, B)$  where  $d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$ .

In 2011, Karapınar et al. established the following result in [7].

**Definition 1.2** ([7]). Let  $A$  and  $B$  be nonempty subsets of a metric space  $(X, d)$ . A cyclic mapping  $T : A \cup B \rightarrow A \cup B$  is said to be a Kannan type cyclic contraction if there exist  $k \in (0, 1/2)$  such that

$$d(Tx, Ty) \leq k[d(Tx, x) + d(Ty, y)], \quad \forall x \in A, \forall y \in B.$$

**Theorem 1.3** ([7]). *Let  $A$  and  $B$  be nonempty closed subsets of a complete metric space  $(X, d)$  and let  $T : A \cup B \rightarrow A \cup B$  be a Kannan type cyclic contraction. Then  $T$  has a unique fixed point.*

Moreover, Edraoui et al. [2] extended the results of Karapınar et al. [7] to interpolative Kannan type cyclic contractions and interpolative Reich-Rus-Ćirić type cyclic contractions, and they established the following results:

**Definition 1.4** ([2]). Let  $(X, d)$  be a metric space,  $A$  and  $B$  be nonempty subsets of  $X$ . A cyclic mapping  $T : A \cup B \rightarrow A \cup B$  is said to be an interpolative Kannan type cyclic contraction, if there exist  $k \in [0, 1)$  and  $\alpha \in (0, 1)$  such that

$$d(Tx, Ty) \leq kd(x, Tx)^\alpha d(y, Ty)^{1-\alpha}, \quad \forall x \in A, \forall y \in B, \quad (1.1)$$

where  $x, y \notin \text{Fix}\{T\}$ .

**Theorem 1.5** ([2]). Let  $(X, d)$  be a complete metric space,  $A$  and  $B$  be nonempty subsets of  $X$  and let  $T : A \cup B \rightarrow A \cup B$  be an interpolative Kannan type cyclic contraction. Then  $T$  has a unique fixed point in  $A \cap B$ .

**Definition 1.6** ([2]). Let  $(X, d)$  be a metric space,  $A$  and  $B$  be nonempty subsets of  $X$ . A cyclic mapping  $T : A \cup B \rightarrow A \cup B$  is said to be an interpolative Reich-Rus-Ćirić type cyclic contraction, if there exists  $k \in (0, 1)$  and real numbers  $\alpha, \beta \in (0, 1)$  with  $\alpha + \beta < 1$  such that

$$d(Tx, Ty) \leq kd(x, y)^\beta d(Tx, x)^\alpha d(y, Ty)^{1-\alpha-\beta}, \quad \forall x \in A, \forall y \in B,$$

where  $x, y \notin \text{Fix}\{T\}$ .

**Theorem 1.7** ([2]). Let  $(X, d)$  be a complete metric space,  $A$  and  $B$  be nonempty subsets of  $X$  and let  $T : A \cup B \rightarrow A \cup B$  be a Reich-Rus-Ćirić type cyclic contraction. Then  $T$  has a unique fixed point in  $A \cap B$ .

In their paper [2], Edraoui et al. only addressed the specific case of interpolative Kannan type cyclic contraction when the sum of the interpolative exponents equals one. Similarly, for interpolative Reich-Rus-Ćirić type cyclic contraction, they solely discussed the scenario where the sum of interpolative exponents is less than one.

In this manuscript, our goal is to further explore the concept by addressing additional cases of interpolative exponents that were not covered by [2]. Specifically, for interpolative Kannan type cyclic contractions, we examine two scenarios turn by turn: when the sum of interpolative exponents is less than 1 and when the sum of interpolative exponents is greater than 1. Additionally, for interpolative Reich-Rus-Ćirić type cyclic contractions, we investigate case where the sum of interpolative exponents is greater than 1. Moreover, we will support our findings with relevant examples. Furthermore, we verify the validity of the results by discussing examples that are not covered by corresponding results in [2].

2. MAIN RESULT

2.1. Interpolative Kannan type cyclic contractions.

**Definition 2.1.** Let  $(X, d)$  be a metric space,  $A$  and  $B$  be nonempty subsets of  $X$ . A cyclic mapping  $T : A \cup B \rightarrow A \cup B$  is said to be a  $(k, \alpha + \beta < 1)$ -interpolative Kannan type cyclic contraction, if there exists  $k \in [0, 1)$  and  $\alpha, \beta \in (0, 1)$  with  $\alpha + \beta < 1$  such that

$$d(Tx, Ty) \leq kd(x, Tx)^\alpha d(y, Ty)^\beta \quad \forall x \in A, \forall y \in B, \tag{2.1}$$

where  $x, y \notin \text{Fix}\{T\}$ .

**Theorem 2.2.** Let  $(X, d)$  be a complete metric space,  $A$  and  $B$  be nonempty closed subsets of  $X$  and let  $T : A \cup B \rightarrow A \cup B$  be a  $(k, \alpha + \beta < 1)$ -interpolative Kannan type cyclic contraction. If for any  $x \in A \cup B$ ,  $d(x, Tx) \geq 1$ . Then  $T$  has a unique fixed point in  $A \cap B$ .

*Proof.* For any  $x \in A$ , by (2.1) it follows that

$$d(Tx, T^2x) = d(Tx, T(Tx)) \leq kd(x, Tx)^\alpha d(Tx, T^2x)^\beta,$$

this implies

$$d(Tx, T^2x)^{1-\beta} \leq kd(x, Tx)^\alpha.$$

Since  $\alpha + \beta < 1$  and  $d(x, Tx) \geq 1, \forall x \in A \cup B$ , so we have

$$d(Tx, T^2x)^{1-\beta} \leq kd(x, Tx)^{1-\beta},$$

or

$$d(Tx, T^2x) \leq k^{\frac{1}{1-\beta}} d(x, Tx) \leq kd(x, Tx).$$

Also

$$\begin{aligned} d(T^2x, T^3x) &= d(T(Tx), T(T^2x)) \leq kd(Tx, T^2x)^\alpha d(T^2x, T^3x)^\beta, \\ d(T^2x, T^3x)^{1-\beta} &\leq kd(Tx, T^2x)^\alpha, \end{aligned}$$

since  $\alpha + \beta < 1$  and  $d(x, Tx) \geq 1, \forall x \in A \cup B$ , so we have

$$d(T^2x, T^3x)^{1-\beta} \leq kd(Tx, T^2x)^{1-\beta},$$

$$d(T^2x, T^3x) \leq k^{\frac{1}{1-\beta}} d(Tx, T^2x) \leq kd(Tx, T^2x).$$

$$d(T^2x, T^3x) \leq k^2 d(x, Tx).$$

By mathematical induction, for any natural number  $n \in \mathbb{N}$ , we get

$$d(T^n x, T^{n+1} x) \leq k^n d(x, Tx).$$

Moreover, for any  $n, m \in \mathbb{N}$  with  $n \geq m$ , by using triangular inequality, we get

$$d(T^n x, T^m x) \leq (k^n + k^{n+1} + k^{n+2} + \dots + k^{m-1})d(x, Tx).$$

Thus by taking limits  $n, m \rightarrow \infty$ , we get  $\lim_{n, m \rightarrow \infty} d(T^n x, T^m x) = 0$ . Hence  $(T^n x)$  is a Cauchy sequence in  $A \cup B$ . Since  $A$  and  $B$  are closed subsets of  $X$ , and the finite union of closed subsets of a metric space is closed, moreover, every closed subspace of a complete metric space is complete. Hence,  $A \cup B$  is complete. Therefore, every Cauchy sequence in  $A \cup B$  converges to an element of  $A \cup B$ . Let  $(T^n x)$  converge to some  $z \in A \cup B$ , i.e.,  $T^n x \rightarrow z$ . Moreover, since  $x \in A$  by definition of the cyclic mapping, we have  $Tx \in B$ ,  $T^2x \in A$ ,  $T^3x \in B$ , and so on. Consequently, every odd term belongs to  $B$ , while every even term belongs to  $A$ . Therefore,  $(T^{2n}x)$  is a sequence in  $A$ , and  $(T^{2n+1}x)$  is a sequence in  $B$  and both sequences are subsequences of the convergent sequence  $(T^n x)$ , thus having the same limit  $z$ . As  $A$  and  $B$  are closed we conclude that  $z \in A \cap B$  that is  $A \cap B$  is nonempty. Now, we show that  $Tz = z$ , since

$$\begin{aligned} 0 \leq d(z, Tz) &= \lim_{n \rightarrow \infty} d(Tz, T^{2n}x) \\ &= \lim_{n \rightarrow \infty} d(Tz, T(T^{2n-1}x)) \\ &\leq \lim_{n \rightarrow \infty} kd(z, Tz)^\alpha d(T^{2n-1}x, T^{2n}x)^\beta \\ &\leq kd(z, Tz)^\alpha \lim_{n \rightarrow \infty} d(T^{2n-1}x, T^{2n}x)^\beta = 0. \end{aligned}$$

Hence  $Tz = z$ . Consequently  $z$  is the fixed point of  $T$ . Now we show that this fixed point is unique. For this assume that there exist another fixed point  $w \in A \cap B$  of  $T$ , such that  $z \neq w$  and  $Tw = w$ . So, there exist  $x, y \in A$  such that

$$\begin{aligned} d(z, w) &= d(Tz, Tw) \\ &= \lim_{n, m \rightarrow \infty} d(T^n x, T^m y) \\ &\leq \lim_{n, m \rightarrow \infty} kd(T^{n-1}x, T^n x)^\alpha d(T^{m-1}y, T^m y)^\beta \\ &= kd(z, z)^\alpha d(w, w)^\beta = 0, \end{aligned}$$

which yields  $d(z, w) = 0$ . Hence  $z = w$  and  $z$  is the unique fixed point of  $T$ . □

**Definition 2.3.** Let  $(X, d)$  be a metric space,  $A$  and  $B$  be nonempty subsets of  $X$ . A cyclic mapping  $T : A \cup B \rightarrow A \cup B$  is said to be a  $(k, \alpha + \beta > 1)$ -interpolative Kannan type cyclic contraction, if there exist  $k \in [0, 1)$  and  $\alpha, \beta \in (0, 1)$  with  $\alpha + \beta > 1$  such that

$$d(Tx, Ty) \leq kd(x, Tx)^\alpha d(y, Ty)^\beta, \quad \forall x \in A, \forall y \in B, \tag{2.2}$$

where  $x, y \notin \text{Fix}\{T\}$ .

**Theorem 2.4.** *Let  $(X, d)$  be a complete metric space,  $A$  and  $B$  be nonempty closed subsets of  $X$  and let  $T : A \cup B \rightarrow A \cup B$  is a  $(k, \alpha + \beta > 1)$ -interpolative Kannan type cyclic contraction, if there exist  $x \in A$  such that  $d(x, Tx) \leq 1$ . Then  $T$  has a unique fixed point in  $A \cap B$ .*

*Proof.* Fix  $x_0 \in A$  such that  $d(x_0, Tx_0) \leq 1$ , by (2.2), we have

$$d(Tx_0, T^2x_0) \leq kd(x_0, Tx_0)^\alpha d(Tx_0, T^2x_0)^\beta,$$

$$d(Tx_0, T^2x_0)^{1-\beta} \leq kd(x_0, Tx_0)^\alpha,$$

$$d(Tx_0, T^2x_0) \leq k^{\frac{1}{1-\beta}} d(x_0, Tx_0)^{\frac{\alpha}{1-\beta}},$$

since  $\frac{\alpha}{1-\beta} > 1$  and  $d(x_0, Tx_0) \leq 1$  so we have  $d(Tx_0, T^2x_0) \leq k$ . Also

$$d(T^2x_0, T^3x_0) \leq kd(Tx_0, T^2x_0)^\alpha d(T^2x_0, T^3x_0)^\beta,$$

$$d(T^2x_0, T^3x_0)^{1-\beta} \leq kd(Tx_0, T^2x_0)^\alpha,$$

$$d(T^2x_0, T^3x_0) \leq k^{\frac{1}{1-\beta}} d(Tx_0, T^2x_0)^{\frac{\alpha}{1-\beta}} \leq k^2.$$

Thus by mathematical induction, the relation  $d(T^n x_0, T^{n+1} x_0) \leq k^n$  holds for all natural numbers  $n \in \mathbb{N}$ . Moreover, for any  $n, m \in \mathbb{N}$  by using triangular inequality and then by applying limits, we get

$$\lim_{n, m \rightarrow \infty} d(T^n x_0, T^m x_0) = 0.$$

Hence  $(T^n x_0)$  is a Cauchy sequence in  $A \cup B$ , since  $A \cup B$  is closed as union of two closed subsets of  $X$ , moreover a closed subspace of a complete space is complete so every Cauchy sequence in  $A \cup B$  is convergent. Let  $T^n x_0 \rightarrow z$  for some  $z \in A \cup B$ . Since  $(T^{2n} x_0)$  is a sequence in  $A$  and  $(T^{2n+1} x_0)$  is a sequence in  $B$  having the same limit  $z$ , where  $A$  and  $B$  are closed, we conclude that  $z \in A \cap B$ . Thus  $A \cap B$  nonempty. Now we show that  $Tz = z$ , since

$$\begin{aligned} d(z, Tz) &= \lim_{n \rightarrow \infty} d(T^{2n} x_0, Tz) \\ &= \lim_{n \rightarrow \infty} d(T(T^{2n-1} x_0), Tz) \\ &\leq \lim_{n \rightarrow \infty} kd(T^{2n-1} x_0, T^{2n} x_0)^\alpha d(z, Tz)^\beta = 0. \end{aligned}$$

Thus  $0 \leq d(z, Tz) \leq 0$ , or  $d(z, Tz) = 0 \Rightarrow Tz = z$ . Hence  $z$  is the fixed point for  $T$ . Next we show that  $z$  is unique fixed point of  $T$ . For this let's assume there exist another fixed

point  $w \in A \cap B$  of  $T$  such that  $z \neq w$  and  $Tw = w$ . Thus, there exist  $x_0, y_0 \in A$  with  $d(x_0, Tx_0) \leq 1$  and  $d(y_0, Ty_0) \leq 1$  such that

$$\begin{aligned} d(z, w) &= d(Tz, Tw) \\ &= \lim_{n, m \rightarrow \infty} d(T^n x_0, T^m y_0) \\ &\leq \lim_{n, m \rightarrow \infty} kd(T^{n-1}x_0, T^n x_0)^\alpha d(T^{m-1}y_0, T^m y_0)^\beta \\ &= kd(z, z)^\alpha d(w, w)^\beta = 0, \end{aligned}$$

which yields,  $d(z, w) = 0$ . Hence  $z = w$  and  $z$  is the unique fixed point of  $T$ . □

**Example 2.5.** Let  $(X, d)$  be a metric space with  $X = \{-\frac{1}{4}, -\frac{1}{8}, 0, \frac{1}{8}, \frac{1}{4}\}$  and  $d(x, y) = |x - y|$ ,  $\forall x, y \in X$ . Consider  $A = \{-\frac{1}{4}, -\frac{1}{8}, 0\}$ ,  $B = \{0, \frac{1}{4}, \frac{1}{8}\}$  and  $T : A \cup B \rightarrow A \cup B$  defined as  $Tx = -x$ . Then obviously  $TA \subseteq B$  and  $TB \subseteq A$  and  $T$  is a cyclic mapping. Now for  $\alpha = \frac{1}{2}, \beta = \frac{1}{4}$  and  $k = \frac{99}{100}$ ,  $T$  is a  $(\frac{99}{100}, \alpha + \beta < 1)$ -interpolative Kannan type cyclic contraction. Thus by Theorem 2.2,  $0 \in A \cap B$  is a fixed point of  $T$ .

**Example 2.6.** Let  $(X, d)$  be a metric space with  $X = \{\frac{1}{4}, \frac{1}{2}, 1, 2, 4\}$  and  $d(x, y) = |x - y|$ ,  $\forall x, y \in X$ . Consider  $A = \{1, 2, 4\}$ ,  $B = \{\frac{1}{4}, \frac{1}{2}, 1\}$  and  $T : A \cup B \rightarrow A \cup B$  defined as  $Tx = \frac{1}{x}$ . Obviously  $TA \subseteq B$  and  $TB \subseteq A$ , thus  $T$  is a cyclic mapping. Now for  $\alpha = \frac{3}{4}, \beta = \frac{3}{4}$  and  $k = \frac{99}{100}$ , it is not hard to prove that  $T$  is a  $(\frac{99}{100}, \alpha + \beta > 1)$ -interpolativ Kannan type cyclic contraction. Thus by Theorem 2.4,  $1 \in A \cap B$  is a fixed point of  $T$ .

*Remark 2.7.* The examples mentioned above cannot be covered by Theorem 2.1 of [2], because Example 2.5 fails to satisfy the required condition for  $x = \frac{1}{4}$  and  $y = -\frac{1}{4}$ . Additionally, Example 2.6 fails to satisfy the conditions of that theorem when  $x = 2$  and  $y = \frac{1}{2}$ . Thus, this observation also ensures the strength of our results.

## 2.2. Interpolative Reich-Rus-Ćirić type cyclic contraction.

**Definition 2.8.** Let  $(X, d)$  be a metric space,  $A$  and  $B$  be nonempty subsets of  $X$ . A mapping  $T : A \cup B \rightarrow A \cup B$  is said to be a  $(k, \alpha + \beta > 1)$ -interpolative Reich-Rus-Ćirić type cyclic contraction, if there exist  $k \in [0, 1)$  and  $\alpha, \beta \in (0, 1)$  with  $\alpha + \beta > 1$  such that

$$d(Tx, Ty) \leq kd(x, y)^\beta d(x, Tx)^\alpha d(y, Ty)^{\alpha+\beta-1}, \quad \forall x \in A, \forall y \in B, \tag{2.3}$$

for all  $(x, y) \in A \times B$  with  $x, y \notin \text{Fix}\{T\}$ .

**Theorem 2.9.** *Let  $(X, d)$  be a complete metric space,  $A$  and  $B$  be nonempty closed subsets of  $X$  and  $T : A \cup B \rightarrow A \cup B$  is a  $(k, \alpha + \beta > 1)$ -interpolative Reich-Rus-Ćirić type cyclic contraction, if there exist  $x \in X$  such that  $d(x, Tx) \leq 1$ . Then  $T$  has a unique fixed point in  $A \cap B$ .*

*Proof.* Fix  $x_0 \in A$ , such that  $d(x_0, Tx_0) \leq 1$ , thus from (2.3) it follows that

$$d(Tx_0, T^2x_0) \leq kd(x_0, Tx_0)^\beta d(x_0, Tx_0)^\alpha d(Tx_0, T^2x_0)^{\alpha+\beta-1},$$

which yields

$$\begin{aligned} d(Tx_0, T^2x_0)^{2-(\alpha+\beta)} &\leq kd(x_0, Tx_0)^{\alpha+\beta}, \\ d(Tx_0, T^2x_0) &\leq k^{\frac{1}{2-(\alpha+\beta)}} d(x_0, Tx_0)^{\frac{\alpha+\beta}{2-(\alpha+\beta)}}, \end{aligned}$$

since  $\alpha, \beta \in (0, 1)$  and  $\alpha + \beta > 1$  so  $1 < \alpha + \beta < 2$ , thus  $\frac{\alpha+\beta}{2-(\alpha+\beta)} > 1$ . Moreover,  $d(x_0, Tx_0) \leq 1$ , thus

$$d(Tx_0, T^2x_0) \leq k^{\frac{1}{2-(\alpha+\beta)}} d(x_0, Tx_0) \leq k^{\frac{1}{2-(\alpha+\beta)}} \leq k.$$

Similarly, by (2.3)

$$\begin{aligned} d(T^2x_0, T^3x_0)^{2-(\alpha+\beta)} &\leq kd(Tx_0, T^2x_0)^{\alpha+\beta}, \\ d(T^2x_0, T^3x_0) &\leq k^{\frac{1}{2-(\alpha+\beta)}} d(Tx_0, T^2x_0)^{\frac{\alpha+\beta}{2-(\alpha+\beta)}} \leq k^2. \end{aligned}$$

Now suppose the relation holds for  $n - 1$ , that is

$$d(T^{n-1}x_0, T^n x_0) \leq k^{n-1}.$$

Thus by using our supposition and (2.3), we show that the relation holds for any  $n \in N$ , since

$$\begin{aligned} d(T^n x_0, T^{n+1} x_0) &\leq kd(T^{n-1} x_0, T^n x_0)^\beta d(T^{n-1} x_0, T^n x_0)^\alpha d(T^n x_0, T^{n+1} x_0)^{\alpha+\beta-1}, \\ d(T^n x_0, T^{n+1} x_0)^{2-(\alpha+\beta)} &\leq kd(T^{n-1} x_0, T^n x_0)^{\alpha+\beta} \\ d(T^n x_0, T^{n+1} x_0) &\leq k^{\frac{1}{2-(\alpha+\beta)}} d(T^{n-1} x_0, T^n x_0)^{\frac{\alpha+\beta}{2-(\alpha+\beta)}}, \\ d(T^n x_0, T^{n+1} x_0) &\leq k^n. \end{aligned}$$

Hence by mathematical induction, the above relation holds for all natural numbers  $n \in N$ . Thus for any  $n, m \in N$  by triangular inequality, we have

$$d(T^n x_0, T^m x_0) \leq k^n + k^{n+1} + k^{n+1} + \dots + k^{m-1}.$$

Thus by taking limits  $n, m \rightarrow \infty$ , we get  $\lim_{n, m \rightarrow \infty} d(T^n x_0, T^m x_0) = 0$ , we find that  $(T^n x_0)$  is a Cauchy sequence in  $A \cup B$ . Since  $A \cup B$  is closed so there exist  $z \in A \cup B$  such that  $T^n x_0 \rightarrow z$ . Obviously,  $(T^{2n} x_0)$  is a sequence in  $A$  and  $\{T^{2n+1} x_0\}$  is a sequence in  $B$  and both have the same

limit  $z$ . Moreover,  $A$  and  $B$  are closed, we conclude that  $z \in A \cap B$  that is  $A \cap B$  is nonempty. Now, we show that  $Tz = z$ .

$$\begin{aligned} d(z, Tz) &= \lim_{n \rightarrow \infty} d(T^{2n}x_0, Tz) \\ &= \lim_{n \rightarrow \infty} d(T(T^{2n-1}x_0), Tz) \\ &\leq \lim_{n \rightarrow \infty} kd(T^{2n-1}x_0, z)^\beta d(T^{2n-1}x_0, T^{2n}x_0)^\alpha d(z, Tz)^{\alpha+\beta-1} = 0. \end{aligned}$$

Thus we have  $d(z, Tz) = 0$  that is  $Tz = z$ . To prove the uniqueness of the fixed point  $z$ , assume that there exist  $w \in A \cap B$  such that  $z \neq w$  and  $Tw = w$ . Taking into account that  $T$  is cyclic and considering  $x_0, y_0 \in A$  with  $d(x_0, Tx_0) \leq 1$  and  $d(y_0, Ty_0) \leq 1$  such that  $T^n x_0 \rightarrow z$  and  $T^m y_0 \rightarrow w$ , we have

$$\begin{aligned} d(z, w) &= \lim_{n, m \rightarrow \infty} d(T^n x_0, T^m y_0) \\ &\leq \lim_{n, m \rightarrow \infty} kd(T^{n-1}x_0, T^{m-1}y_0)^\beta d(T^{n-1}x_0, T^n x_0)^\alpha d(T^{m-1}y_0, T^m y_0)^{\alpha+\beta-1} \\ &= kd(z, w)^\beta d(z, z)^\alpha d(w, w)^{\alpha+\beta-1} = 0. \end{aligned}$$

This yields that

$$0 \leq d(z, w) \leq 0 \Rightarrow d(z, w) = 0$$

and in turn,  $z = w$ . Hence  $z$  is the unique fixed point of  $T$ . □

**Example 2.10.** Let  $(X, d)$  be a metric space with  $X = \{-6, -4, -2, 0, 2, 4, 6\}$  and  $d(x, y) = |x - y|, \forall x, y \in X$ . Consider  $A = \{-6, -4, -2, 0\}, B = \{0, 2, 4, 6\}$  and  $T : A \cup B \rightarrow A \cup B$  is defined as  $Tx = -x$  then obviously  $TA \subseteq B$  and  $TB \subseteq A$  thus  $T$  is a cyclic mapping. Now for  $\alpha = \frac{3}{4}, \beta = \frac{1}{2}$  and  $k = \frac{99}{100}$ ,  $T$  is a  $(\frac{99}{100}, \alpha + \beta > 1)$ -interpolativ Reich-Ruc-Ćirić type cyclic contraction. Thus by Theorem 2.9,  $0 \in A \cap B$  is a fixed point of  $T$ .

*Remark 2.11.* Theorem 2.4 of [2] when  $x = 2$  and  $y = -2$  fails to hold through this example. This shows that our example independently covers our above result.

**Conclusion.** We introduced new results on the existence and uniqueness of fixed points for cyclic mappings, covering interpolative Kannan-type and Reich-Rus-Ćirić cyclic contractions. Our findings, validated with examples, extend existing literature. Future work could explore extending these results to different metric structures and mappings.

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