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# On the lattice of $e$-open sets and $od$-compact spaces

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ABSTRACT

*In this paper, we investigate $od$-compact spaces, a natural generalization of $e$-compact spaces (hence compact spaces), and study their interplay with the lattice-theoretic structure of $e$-open and open dense subsets, denoted by $E(X)$ and $OD(X)$, respectively. We first provide an algebraic characterization of $od$-compactness and $e$-compactness in terms of essential and Baer ideals of $C(X)$, respectively. We then observe the relationships between $od$-compactness, connectedness, hyperconnectedness, and compactness, and examine the behavior of these notions under continuous open maps and products. Furthermore, we analyze the lattice properties of $E(X)$ and $OD(X)$, showing that they are closely related to the lattice of Baer and essential ideals of $C(X)$, respectively. Several structural properties of these lattices, such as compactness, connectedness, and co-normality, are discussed, along with their preservation under homeomorphisms and ring isomorphisms. Finally, we establish isomorphisms between these lattices and their counterparts in the Stone–Čech compactification.*

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## 1. INTRODUCTION

Compactness-type properties play a fundamental role in general topology, and their algebraic characterizations via the ring $C(X)$ of continuous real-valued functions have been extensively studied. Classical notions such as compactness, pseudocompactness, and realcompactness have long been linked with the behavior of ideals in $C(X)$ ([11]). More recently, new concepts such as $e$-open sets, $e$-compact spaces,

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and related structures have been introduced and developed in [1, 2, 8]. These investigations reveal deep connections between topological covering properties, and the algebraic structure of continuous functions.

In this paper, we continue this line of research and study the notion of *od*-compact spaces, defined as spaces in which every open cover by dense subsets admits a finite subcover. Clearly, every *e*-compact (hence compact) space is *od*-compact, though the converse fails: for example, every infinite discrete space is *od*-compact but not *e*-compact. This shows that *od*-compactness forms a natural weakening of *e*-compactness, while retaining a close connection to ideals in $C(X)$.

After recalling the necessary background in Section 2, we begin Section 3 by developing the basic theory of *od*-compact spaces. In particular, we characterize *od*-compactness in algebraic terms: a completely regular space $X$ is *od*-compact precisely when the sum of every family of fixed essential ideals of $C(X)$ that is closed under finite sums is again a fixed ideal (Theorem 3.11). This is parallel to the characterization of *e*-compact spaces in terms of Baer ideals (Theorem 3.10). The relationship between Baer ideals and essential ideals of $C(X)$ is clarified further in Theorem 3.12, which shows that these two classes coincide in connected spaces. Several structural properties of *od*-compact spaces are also established. Proposition 3.7 relates *od*-compactness to the induced topology of open dense subsets, while Theorem 3.8 shows that *od*-compactness is preserved under continuous open maps. Examples are provided to illustrate that *od*-compactness and *e*-compactness differ essentially. We conclude this section with an algebraic characterization of ultranormal spaces (Theorem 3.20).

In Section 4, we turn to a lattice theoretic approach. We study the lattices $E(X)$ of *e*-open sets and $OD(X)$ of open dense subsets, and show that they are closely related to the lattices of Baer ideals and essential ideals of $C(X)$, respectively (Theorem 4.1). Structural properties of these latices are investigated in detail: for example, Theorem 4.4 establishes that $E(X)$ is a co-normal lattice and that compactness of $E(X)$ (resp., $OD(X)$) characterizes *e*-compactness (resp., *od*-compactness) of the underlying space. Several isomorphism results are also obtained. In particular, Theorems 4.5 and 4.7, together with Corollary 4.8, demonstrate that these lattices are preserved under homeomorphisms of spaces, isomorphisms of rings of continuous functions, and passage to Stone–Čech compactification.

## 2. Background and notation

2.1. **Rings of continuous functions and topological concepts.** In this paper, $C(X)$ (resp., $C^*(X)$) is denoted by the ring of all (resp., all bounded) real-valued continuous functions on a completely regular Hausdorff space $X$. For each $f \in C(X)$, the set $f^{-1}(\{0\})$ is called the *zero-set* of $f$, and is denoted by $Z(f)$, while $\mathrm{Coz}(f) = X \setminus Z(f)$ is the *cozero-set* of $f$. The space $\beta X$ is the *Stone–Čech compactification* of $X$, characterized as the compactification of $X$ in which $X$ is $C^*$-embedded as a dense subspace. The space

$\upsilon X$ is the *realcompactification* of $X$, in which $X$ is $C$-embedded as a dense subspace. For a completely regular Hausdorff space $X$, we have $X \subseteq \upsilon X \subseteq \beta X$, see [11].

An open subset $U$ of a topological space $X$ is called an *e-open* subset if its closure $\overline{U}$ is an open subset of $X$. A topological space $X$ is an *e-space* if every open subset of $X$ can be expressed as a union of a subfamily of $e$-open sets. Moreover, a topological space $X$ is called an *e-compact space* if every open cover by $e$-open sets admits a finite subcover. For further background on these concepts, we refer the reader to [1, 2, 8].

We need the following well-known lemma. The ideal generated by a subset $S$ of $C(X)$ denotes by $< S >= SC(X)$. For an ideal $I$ of $C(X)$, we have

$$\mathrm{Ann}(I) = \{f \in C(X) : \ fg = 0, \ \forall g \in I\}.$$

**Lemma 2.1.** *For ideals $I$ and $J$ of $C(X)$, $\mathrm{Ann}(I) = \mathrm{Ann}(J)$ if and only if* $\mathrm{int}_X(\bigcap Z[I]) = \mathrm{int}_X(\bigcap Z[J])$.

Recall from [16] that an ideal $I$ of a ring $C(X)$ is a *Baer ideal* if there exists an idempotent element $e$ of $C(X)$ such that $\mathrm{Ann}(I) = eC(X)$.

Also, following [4, 5, 13] that, an ideal $I$ of a ring $C(X)$ is an *essential ideal* if it intersects every non-zero ideal of $C(X)$. The following lemma which topologically characterizes essential ideals in $C(X)$ is proved in Theorem 3.1 of [5].

**Lemma 2.2.** *An ideal $I$ of $C(X)$ is an essential ideal if and only if* $\mathrm{int}_X \bigcap Z[I] = \varnothing$.

The socle of $C(X)$ denoted by $C_F(X)$ in [13]. It is well-known that $C_F(X)$ is the intersection of all essential ideals of $C(X)$.

2.2. **Basic lattice-theoretic facts and definitions.** Recall from [9], [7], [14] and [17] that a lattice $< L, \wedge, \vee, 0, 1 >$ is called a *co-normal* lattice whenever it is a distributive lattice and for all $a, b \in L$ with $a \wedge b = 0$ there exist $x, y \in L$ such that $x \vee y = 1$ and $x \wedge a = y \wedge b = 0$. Trivially, every Boolean algebra is a co-normal lattice.

Recall that a lattice $L$ with the top element 1 is *compact* if and only if whenever $1 = \bigvee A$ where $A \subseteq L$, then $1 = \bigvee A_0$ for some finite $A_0 \subseteq A$.

A bounded lattice $L$ with a top element 1 and a bottom element 0 is *connected* if whenever $1 = a \vee b$ and $a \wedge b = 0$, then either $a = 0$ or $b = 0$. For more details about lattice theory, the reader is referred to [12].

## 3. On $od$-compact spaces

In this section, we introduce the notion of $od$-compact spaces as a natural extension of $e$-compact spaces. We investigate some basic topological properties of this class of spaces, and we also provide an algebraic characterization of $e$-compact (resp., $od$-compact) spaces. As preparation, we first establish a connection between Baer ideals (resp., essential ideals) of $C(X)$ and $e$-open (resp., open dense) subsets of $X$, by showing the following lemma.

**Lemma 3.1.** *The following statements hold.*

(1) *An ideal $I$ of $C(X)$ is a Baer ideal if and only if the set $\bigcup \mathrm{Coz}[I]$ is an $e$-open set in $X$.*

(2) *An ideal $I$ of $C(X)$ is an essential ideal if and only if the set $\bigcup \mathrm{Coz}[I]$ is an open dense set in $X$.*

*Proof.* (1) Suppose $I$ is a Baer ideal of $C(X)$. Then there is an idempotent $e \in C(X)$ such that $\mathrm{Ann}(I) = \mathrm{Ann}(e)$. By Lemma 2.1, we obtain $\mathrm{int}_X(\bigcap Z[I]) = \mathrm{int}_X(Z(e)) = Z(e)$, which is a closed subset of $X$. Hence, $\bigcup \mathrm{Coz}[I]$ is an $e$-open set in $X$. Conversely, assume $\bigcup \mathrm{Coz}[I]$ is an $e$-open set. Then $\mathrm{cl}_X(\bigcup \mathrm{Coz}[I]))$ is open. Hence, there is an idempotent $e$ in $C(X)$ such that $\mathrm{cl}_X(\bigcup \mathrm{Coz}[I]) = X \setminus Z(e)$. This implies $\mathrm{int}_X(\bigcap Z[I]) = Z(e) = \mathrm{int}_X(Z(e))$. Therefore, $\mathrm{Ann}(I) = \mathrm{Ann}(e)$, by Lemma 2.1. Hence, $I$ is a Baer ideal.

(2) Let $I$ be an essential ideal of $C(X)$. Then $\mathrm{int} \bigcap Z[I] = \emptyset$, by Lemma 2.2. This shows that $\bigcup \mathrm{Coz}[I]$ is an open dense set. Conversely, suppose that $\bigcup \mathrm{Coz}[I]$ is an open dense set. Then $\mathrm{int} \bigcap Z[I] = \emptyset$. Hence, by Lemma 2.2, $I$ is an essential ideal. □

As an application of the above lemma, we conclude the following:

**Corollary 3.2.** *Let $X$ be a completely regular space.*

(1) *Every $e$-open subset of $X$ can be represented as $\bigcup \mathrm{Coz}[I]$, where $I$ is a Baer ideal of $C(X)$.*

(2) *Every open dense subset of $X$ is of the form $\bigcup \mathrm{Coz}[I]$, where $I$ is an essential ideal of $C(X)$.*

Now, we introduce a compactness-type notion based on dense open covers.

**Definition 3.3.** A topological space $X$ is called *od-compact* if every open cover of $X$ consisting of dense subsets has a finite subcover.

Clearly, every $e$-compact space (in particular, every compact space) is $od$-compact. The converse, however, does not hold in general, as the following example illustrates.

**Example 3.4.** (1) Let $X$ be an infinite discrete space. Then, it is easy to see that $X$ is an $od$-compact space. However, $X$ is not $e$-compact, since the cover by singletons has no finite $e$-subcover.

(2) Consider $\mathbb{R}$ with the usual topology. Put $A_n = \{n, n+1, n+2, ...\}$ for each $n \in \mathbb{N}$. Then, clearly, $\bigcap_{n\in\mathbb{N}} A_n = \emptyset$. On the other hand, for each $n \in \mathbb{N}$, $\mathbb{R} \setminus A_n$ is an open dense subset of $\mathbb{R}$ and we have $\mathbb{R} = \bigcup_{n\in\mathbb{N}}(\mathbb{R} \setminus A_n)$. If $\mathbb{R}$ were an $od$-compact space, then this open dense cover would admit a finite subcover; that is, there exists some $k \in \mathbb{N}$ such that $\mathbb{R} = \bigcup_{n=1}^{k}(\mathbb{R} \setminus A_n)$. This implies $\mathbb{R} = \mathbb{R} \setminus A_k$, a

contradiction. Thus $\mathbb{R}$ with the usual topology is not an $od$-compact space. Hence, it is not an $e$-compact space either.

The following well-known lemma will be needed in the sequel.

**Lemma 3.5.** *If $V$ is dense and $U$ is an open subset of $X$, then we have* $\mathrm{cl}_X(V \cap U) = \mathrm{cl}_X U$.

Now the proof of the following result is evident.

**Corollary 3.6.** *The intersection of any two open dense subsets in a space $X$ is an open dense subset.*

It is straightforward to see that any union of open dense subsets of $X$ is itself open and dense. Hence, $\tau_{od} = \{U \subseteq X : U \text{ is open and dense }\} \cup \{\varnothing\}$ defines a topology on $X$. Moreover, we clearly have $\tau_{od} \subseteq \tau_e \subseteq \tau$, where $\tau$ is the original topology on $X$ and $\tau_e$ is the topology generated by $e$-open subsets of $X$. For convenience, we denote the corresponding topological spaces by $X_{od} = (X, \tau_{od})$ and $X_e = (X, \tau_e)$. The following proposition is immediate.

**Proposition 3.7.**

(1) *A space $X$ is a connected space if and only if $X_e = X_{od}$ (i.e., every non-empty e-open set is dense).*
(2) *A space $X$ is a hyperconnected (irreducible) space if and only if $X = X_{od}$ (i.e., every open set is dense).*
(3) *If $X$ is an od-compact and hyperconnected (irreducible) space, then it is a compact space.*
(4) *If $X$ is e-compact and e-space, then $X$ is compact.*
(5) *A space $X$ is e-compact if and only if the space $X_e$ is compact.*
(6) *A space $X$ is od-compact if and only if the space $X_{od}$ is compact.*

**Theorem 3.8.**

(1) *The continuous open image of an od-compact space is od-compact.*
(2) *The continuous open image of an e-compact space is e-compact.*
(3) *If the product of some family of spaces is od-compact, then each member of the family is od-compact.*

*Proof.* (1) Let $f : X \to Y$ be continuous and open from an $od$-compact space $X$ onto a space $Y$. First, assume $U$ is an open and dense subset of $Y$. We claim that $f^{-1}(U)$ is open and dense in $X$. The continuity of $f$ ensures that $f^{-1}(U)$ is open in $X$. To see that it is dense, let $x \in X$ and $G$ be open in $X$ containing $x$. Then $f(G)$ is open in $Y$ and contains $f(x)$. Thus $U \cap f(G) \neq \varnothing$, by hypothesis. Hence, there exists $y \in U \cap f(G)$. So, $y = f(g)$, for some $g \in G$ and $f(g) \in U$ implies $g \in f^{-1}(U)$. Thus $G \cap f^{-1}(U) \neq \varnothing$, i.e., $\mathrm{cl}_X f^{-1}(U) = X$. Next, assume $Y = \bigcup_\alpha U_\alpha$, with each $U_\alpha$ is open and dense in $Y$. Then $X = \bigcup_{\alpha\in S} f^{-1}(U_\alpha)$, and for each $\alpha \in S$, $f^{-1}(U_\alpha)$ is open and dense in $X$. Since, $X$ is *od*-compact, there exist indices $\{\alpha_1, \alpha_2, ..., \alpha_n\} \subseteq S$ such that $X = \bigcup_{i=1}^n f^{-1}(U_{\alpha_i})$. It follows that $Y = \bigcup_{i=1}^n U_{\alpha_i}$. Thus, $Y$ is *od*-compact.

(2) Let $f : X \to Y$ be continuous and open from an $e$-compact $X$ onto $Y$. First, suppose $U$ is an $e$-open subset of $Y$. We claim that $\mathrm{cl}_X f^{-1}(U) = f^{-1}(\mathrm{cl}_Y U)$, i.e., $\mathrm{cl}_X f^{-1}(U)$ is open in $X$. Thus $f^{-1}(U)$ is $e$-open in $X$. To see it, consider $x \in f^{-1}(\mathrm{cl}_Y U)$ and suppose $G$ is an open set containing $x$ in $X$. Then $f(x) \in f(G) \cap \mathrm{cl}_Y U$. By hypothesis, $f(G)$ is open in $Y$, so $f(G) \cap U \neq \emptyset$. Thus, there exists $x_0 \in G$ such that $f(x_0) \in U$. Hence, $x_0 \in f^{-1}(U) \cap G$. That is $f^{-1}(\mathrm{cl}_Y U) \subseteq \mathrm{cl}_X f^{-1}(U)$. On the other hand, $f^{-1}(\mathrm{cl}_Y U)$ is closed in $X$ and contains $f^{-1}(U)$, so $\mathrm{cl}_X f^{-1}(U) \subseteq f^{-1}(\mathrm{cl}_Y U)$. So we are done. The rest of the proof now follows exactly as in Part (1).

(3) Suppose $X = \prod_{\alpha\in S} X_\alpha$ is an *od*-compact space. By Part (1), each $X_\alpha$ is *od*-compact. □

Theorem 3.8(2) implies Proposition 2.25 in [2].

*Remark* 3.9. The converse of Part 3 of the above theorem is not true. Consider the infinite discrete space $\mathbb{N}$. It is well-known that $\mathbb{N}$ is an *od*-compact space. However, the product space $\mathbb{N}^\mathbb{N}$ is not an *od*-compact space. For each $n \in \mathbb{N}$, put

$$A_n = \{(x_k)_{k\in\mathbb{N}} \in \mathbb{N}^\mathbb{N} : n \neq x_k, \forall k \in \mathbb{N}\}.$$

Then, $\bigcap_{n=1}^\infty A_n = \emptyset$. This implies $\mathbb{N}^\mathbb{N} = \bigcup_{n\in\mathbb{N}}(\mathbb{N}^\mathbb{N} \setminus A_n)$. On the other hand, it is easy to see that each $\mathbb{N}^\mathbb{N} \setminus A_n$ is open. We now show that each $\mathbb{N}^\mathbb{N} \setminus A_n$ is also dense. In fact, for each $n \in \mathbb{N}$, $A_n^\circ = \emptyset$. To the contrary, let $U$ be an open set in $\mathbb{N}^\mathbb{N}$ such that $U \subseteq A_n$ for some $n \in \mathbb{N}$. Then, there are open sets $U_1, U_2, ..., U_k$ in $\mathbb{N}$ such that $U_1 \times U_2 \times ... \times U_k \times \prod_{i\neq 1,2,...,k} X_i \subseteq A_n$, where for each $i \neq 1, 2, ..., k$, $X_i = \mathbb{N}$, which is a contradiction to the definition of $A_n$. Now, if $\mathbb{N}^\mathbb{N}$ were *od*-compact, then we would have $\mathbb{N}^\mathbb{N} = \bigcup_{k=1}^n(\mathbb{N}^\mathbb{N} \setminus A_k) = \mathbb{N}^\mathbb{N} \setminus \bigcap_{k=1}^n A_k$, a contradiction.

Now, we provide an algebraic characterization of $e$-compact (resp., *od*-compact) spaces. Denote by $\mathcal{I}(C(X))$, the set of all ideals of $C(X)$ . It is well-known that $< \mathcal{I}(C(X)), +, \cap, \subseteq >$ is a complete lattice.

**Theorem 3.10.** *The following statements are equivalent.*

(1) *A completely regular space $X$ is an $e$-compact space.*
(2) *The sum of every family of fixed Baer ideals of $C(X)$ that is closed under finite sum is a fixed ideal.*
(3) *Every subfamily of fixed Baer ideals of $C(X)$ that forms a sublattice of $\mathcal{I}(C(X))$ is itself a complete lattice.*

*Proof.* (1)⇒(2) Let $\mathcal{C} = \{I_\alpha : \alpha \in S\}$ be a family of fixed Baer ideals of $C(X)$ and set $I = \sum_{\alpha\in S} I_\alpha$. Suppose $I$ is a free ideal of $C(X)$. Then

$$\bigcap_{\alpha\in S}(\bigcap Z[I_\alpha]) = \bigcap Z[I] = \emptyset.$$

This implies

$$X = \bigcup_{\alpha\in S}(\bigcup_{f\in I_\alpha}(X \setminus Z(f))).$$

By Lemma 3.1, for each $\alpha \in S$ the set $\bigcup_{f\in I_\alpha}(X \setminus Z(f))$ is an $e$-open set. Since $X$ is $e$-compact, there exists a finite subfamily $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ of $S$ such that $X = \bigcup_{i=1}^{n}(\bigcup_{f\in I_{\alpha_i}}(X \setminus Z(f))$. Equivalently,

$$\bigcap Z[I_{\alpha_1} + ... + I_{\alpha_n}] = \bigcap_{i=1}^{n}(\bigcap Z[I_{\alpha_i}]) = \emptyset.$$

Thus, by Proposition 2.5 in [16], $I_{\alpha_1} + ... + I_{\alpha_n}$ is a Baer ideal which is a free ideal, contradicting the assumption that $\mathcal{C}$ is closed under finite sum.

(2)⇒(3) Let $\mathcal{C}$ be a subfamily of fixed Baer ideals which forms a sublattice of $\mathcal{I}(C(X))$. In particular, the sum of any two elements of $\mathcal{C}$ belongs to $\mathcal{C}$, so $\mathcal{C}$ is closed under arbitrary sums, hence, $\mathcal{C}$ is a complete lattice.

(3)⇒(1) Assume $X = \bigcup_{\alpha\in S} U_\alpha$, where each $U_\alpha$ is an $e$-open set in $X$. By Corolary 3.2, for each $\alpha \in S$ there is a Baer ideal $I_\alpha$ of $C(X)$ such that $U_\alpha = \bigcup \mathrm{Coz}[I_\alpha]$. Define $\mathcal{C} = \{I_\alpha : \alpha \in S\}$ and $I = \sum_{\alpha\in S} I_\alpha$. If some $I_\alpha$ is free, then $U_\alpha = X$ and the proof is complete. Otherwise, assume each $I_\alpha$ is fixed. We have,

$$\bigcap Z[I] = \bigcap_{\alpha\in S}(\bigcap Z[I_\alpha]) = \bigcap_{\alpha\in S}(X \setminus U_\alpha) = \emptyset.$$

Which shows that $I$ is a free ideal, hence, $I \notin \mathcal{C}$. By hypothesis, $\mathcal{C}$ can not be a sublattice of the lattice $\mathcal{I}(C(X))$, so there exist $\alpha_1, \alpha_2, ..., \alpha_n \in S$ such that $\sum_{i=1}^{n} I_{\alpha_i}$ is a free ideal. Consequently, $\bigcap_{i=1}^{n}(\bigcap Z[I_{\alpha_i}]) = \emptyset$, and therefore

$$X = \bigcup_{i=1}^{n}(\bigcup \mathrm{Coz}[I_{\alpha_i}]) = \bigcup_{i=1}^{n} U_{\alpha_i}.$$

Thus $X$ is $e$-compact. □

**Theorem 3.11.** *The following statements are equivalent.*

(1) *A completely regular space $X$ is an od-compact space.*

(2) *The sum of every family of fixed essential ideals of $C(X)$ that is closed under finite sum is a fixed ideal.*

(3) *Every subfamily of fixed essential ideals of $C(X)$ that forms a sublattice of $\mathcal{I}(C(X))$ is itself a complete lattice.*

*Proof.* The proof is similar to the proof of Theorem 3.10. □

It is evident that every essential ideal of $C(X)$ is a Baer ideal, since, if $I$ is an essential ideal of $C(X)$, then $\mathrm{Ann}(I) = 0$. However, a Baer ideal of $C(X)$ need not be an essential ideal. In the next result, we see that these two classes of ideals coincide precisely when $X$ is connected.

**Theorem 3.12.** *The following statements are equivalent.*

(1) *Every non-zero Baer ideal of $C(X)$ is an essential ideal.*

(2) *A completely regular space $X$ is a connected space.*

(3) *Every e-open set is an open dense set (i.e., $E(X) = OD(X)$).*

*Proof.* (1)⇒(2) let $e$ be a non-zero idempotent of $C(X)$. Then $I = eC(X)$ is a Baer ideal and hence essential. This implies $(1 - e)C(X) = \mathrm{Ann}(I) = 0$, i.e., $e = 1$. Therefore, the only idempotents of $C(X)$ are 0 and 1, which by [11, 1B.2], implies that $X$ is connected.

(2)⇒(3) Trivial.

(3)⇒(1) By Proposition 3.7(1), $X$ is a connected space, hence the only idempotents of $C(X)$ are $0, 1$, by [11, 1B.2]. This implies for every non-zero Baer ideal $I$ of $C(X)$, we must have $\mathrm{Ann}(I) = 0$. Therefore $I$ is an essential ideal. □

Consider $\mathbb{R}$ with the usual topology. Then $U = \mathbb{R} \setminus \mathbb{N}$ is an open and dense subset of it. But $U \cap \mathbb{N} = \emptyset$ is not dense in $\mathbb{N}$. This shows that the trace of an open dense subset on a subspace need not be an open dense in the subspace.

**Lemma 3.13.** *Let $A$ be a dense subset of $X$. Then*

$$OD(A) = \{U \cap A : U \in OD(X)\}.$$

*Proof.* Let $G \in OD(A)$. Then $G = U \cap A$, where $U$ is open in $X$. Thus,

$$A = \mathrm{cl}_A(G) = \mathrm{cl}_A(U \cap A) = \mathrm{cl}_X(U \cap A) \cap A = \mathrm{cl}_X U \cap A.$$

This implies $A \subseteq \mathrm{cl}_X U$. Hence $X = \mathrm{cl}_X A \subseteq \mathrm{cl}_X U$, i.e., $U$ is open and dnese in $X$. Now let $U$ be open and dense in $X$ then $U \cap A$ must be open in $A$ and we have,

$$\mathrm{cl}_A(U \cap A) = \mathrm{cl}_X(U \cap A) \cap A = \mathrm{cl}_X U \cap A = X \cap A = A.$$

This shows that $U \cap A \in OD(A)$. □

Let us call a subset $A$ of a topological space $X$ an *e-compact* (resp., *od-compact*) subset if $A \subseteq \bigcup_{\alpha\in S} U_\alpha$, where each $U_\alpha$ is $e$-open (resp., open and dense) in $X$, then there is a finite subset $\{\alpha_1, \alpha_2, ...., \alpha_n\}$ of $S$ such that $A \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$.

Note that an $e$-compact (resp., $od$-compact) subspace of a space need not be an $e$-compact (resp., $od$-compact) subset. For example, consider the natural number $\mathbb{N}$ as a subspace of $\mathbb{R}$ with the usual topology. Then, it is easy to see that $\mathbb{N}$ is not an $od$-compact subset. On the other hand, as a discrete space, it is an $od$-compact space.

In [2, Proposition 2.23], it was shown that $e$-closed subsets of an $e$-compact space are $e$-compact spaces. Here, we strengthen this result by proving that they are also $e$-compact subsets.

**Proposition 3.14.** *Let $A$ be an $e$-closed (resp., nowhere dense closed) subset of an $e$-compact (resp., $od$-compact) space $X$. Then $A$ is an $e$-compact (resp., $od$-compact) subset.*

*Proof.* Let $A \subseteq \bigcup_{\alpha \in S} U_\alpha$, where each $U_\alpha$ is $e$-open (resp., open and dense) in $X$. Then $X = \bigcup_{\alpha \in S} U_\alpha \cup (X \setminus A)$. By hypothesis, $X \setminus A$ is $e$-open (resp., open and dense). Hence, there is $\{\alpha_1, \alpha_2, ...., \alpha_n\} \subseteq S$ such that $X = \bigcup_{i=1}^{n} U_{\alpha_i} \cup (X \setminus A)$. Thus, $A \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$. □

**Proposition 3.15.** *Let $A$ be a dense subset of $X$. Then $A$ is an $od$-compact space if and only if it is an $od$-compact subset of $X$.*

*Proof.* First, assume $A$ is an $od$-compact space and $A \subseteq \bigcup_{\alpha \in S} U_\alpha$, where each $U_\alpha$ is an open dense subset of $X$. Then $A = \bigcup_{\alpha \in S}(U_\alpha \cap A)$. By Lemma 3.13, for each $\alpha \in S$, $U_\alpha \cap A$ is open and dense in $A$, hence $A = \bigcup_{i=1}^{n}(U_{\alpha_i} \cap A)$. Thus $A \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$, i.e., $A$ is an $od$-compact subset of $X$. Conversely, let $A = \bigcup_{\alpha \in S} G_\alpha$, where each $G_\alpha$ is open and dense in $A$. By Lemma 3.13, for each $\alpha \in S$, there is an open and dense subset $U_\alpha$ of $X$ such that $G_\alpha = U_\alpha \cap A$. Thus $A \subseteq \bigcup_{\alpha \in S} U_\alpha$. By hypothesis, $A \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$. This implies $A = \bigcup_{i=1}^{n} G_{\alpha_i}$. □

Consider $\mathbb{R}$ endowed with the topology in which every rational number is an isolated point while the neighborhood of each irrational number are the same as in the usual topology. In this topology, the set of rational numbers, $\mathbb{Q}$ is both dense and discrete. Hence, $\mathbb{Q}$ is an $od$-compact space, and hence $\mathbb{Q}$ is an $od$-compact subset of $\mathbb{R}$, by Proposition 3.15. On the other hand, it is not an $e$-compact space, since $\{\{r\} : r \in \mathbb{Q}\}$ is an $e$-open cover for $\mathbb{Q}$ which has no finite sub-cover.

Next, using Lemma 3.13, we have $OD(X) = \{U \cap X : U \in OD(\beta X)\}$. Therefore, a subset $A$ of $X$ is an $od$-compact subset of $X$ if and only if it is an $od$-compact subset of $\beta X$. This yields the following corollary.

**Corollary 3.16.** *The space $X$ is $od$-compact if and only if it is an $od$-compact subset of $\beta X$.*

**Lemma 3.17.** *Let $X$ be a completely regular Hausdorff space. Then*

$$E(X) = \{U \cap X : U \in E(\beta X)\}.$$

*Proof.* Let $U \subseteq X$ be an $e$-open set. Then $\mathrm{cl}_X(U)$ is open in $X$. By [11, Proposition 6.9 (c)], we have $\mathrm{cl}_{\beta X}(U) = \mathrm{cl}_{\beta X}(\mathrm{cl}_X U)$ which is open in $\beta X$. Hence, there is an open set $G$ in $\beta X$ such that $U = G \cap X$. Moreover, $\mathrm{cl}_{\beta X} U = \mathrm{cl}_{\beta X}(G \cap X) = \mathrm{cl}_{\beta X} G$. This implies $G$ is an $e$-open set in $\beta X$. On the other hand, by Proposition 2.4 in [1], the trace of every $e$-open set in a dense subset is an $e$-open set. Thus the equality follows. □

**Proposition 3.18.** *Let $A \subseteq X$. Then $A$ is an $e$-compact subset in $X$ if and only if it is an $e$-compact subset in $\beta X$.*

*Proof.* ($\Rightarrow$) First, assume $A$ is an $e$-compact subset in $X$ and $A \subseteq \bigcup_{\alpha \in S} U_\alpha$, where each $U_\alpha$ is $e$-open in $\beta X$. Then $A \subseteq \bigcup_{\alpha \in S}(U_\alpha \cap X)$. By Lemma 3.17, each $U_\alpha \cap X$ is $e$-open in $X$, hence there exists a finite subfamily $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ of $S$ such that $A \subseteq \bigcup_{i=1}^n (U_{\alpha_i} \cap X) \subseteq \bigcup_{i=1}^n U_{\alpha_i}$.

($\Leftarrow$) Let $A$ be an $e$-compact subset in $\beta X$ and $A \subseteq \bigcup_{\alpha \in S} U_\alpha$, where each $U_\alpha$ is $e$-open in $X$. Then, by Lemma 3.17, for each $\alpha \in S$, there exists an $e$-open $G_\alpha$ in $\beta X$ such that $U_\alpha = G_\alpha \cap X$. Thus, $A \subseteq \bigcup_{\alpha \in S} G_\alpha$. By hypothesis, there is a finite subset of $S$ say $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ such that $A \subseteq \bigcup_{i=1}^n G_{\alpha_i}$. Therefore, $A \subseteq \bigcup_{i=1}^n (G_{\alpha_i} \cap X) = \bigcup_{i=1}^n U_{\alpha_i}$. □

**Corollary 3.19.** *The space $X$ is $e$-compact if and only if it is an $e$-compact subset of $\beta X$.*

A Hausdorff space $X$ is called *ultranormal* if and only if whenever $A, B$ are disjoint closed sets in $X$, there is a clopen set $C$ with $A \subseteq C$ and $B \subseteq C^c = X \setminus C$. Clearly, the ultranormal spaces are precisely the spaces with large inductive dimension zero. A Hausdorff space is said to be zero-dimensional if it has a basis of clopen sets. Clearly every ultranormal space is normal. For instance, the space $\omega_1$ of all countable ordinals with the order topology is ultranormal. Here, we give an algebraic characterization of ultranormal spaces.

**Theorem 3.20.** *A space $X$ is ultranormal if and only if for each pair of ideals $I, J$ of $C(X)$ with $I + J$ free there are two orthogonal Baer ideals $I_1, J_1$ of $C(X)$ such that $I + I_1$ and $J + J_1$ are free.*

*Proof.* $\Rightarrow$ Let $I, J$ be two ideals of $C(X)$ with $I + J$ be free. Then

$$\bigcap Z[I] \cap Z[J] = \bigcap Z[I + J] = \emptyset.$$

By the ultranormality of $X$, there exists a clopen subset $C$ and hence an idempotent $e$ of $C(X)$ such that $\bigcap Z[I] \subseteq C = Z(e)$ and $\bigcap Z[J] \subseteq C^c = Z(1 - e)$. Set $I_1 =< 1 - e >$ and $J_1 =< e >$. Then, $I_1, J_1$ are two orthogonal Baer ideals of $C(X)$. Moreover, $\bigcap Z[I + I_1] = \bigcap Z[I] \cap Z[I_1] = \emptyset$, and $\bigcap Z[J + J_1] = \bigcap Z[J] \cap Z[J_1] = \emptyset$. Hence, $I + I_1$ and $J + J_1$ are free ideals, as required.

$\Leftarrow$ Let $A, B$ be two disjoint closed subsets of $X$. Then there are two ideals $I, J$ of $C(X)$ such $A = \bigcap Z[I]$ and $B = \bigcap Z[J]$. Since $A, B$ are disjoint, it follows that $I + J$ is free. By hypothesis, there are two Baer ideals $I_1, J_1$ such that $I + I_1$ and $J + J_1$ are free. By Lemma 3.1, the sets $\bigcup \mathrm{Coz}[I_1]$ and $\bigcup \mathrm{Coz}[J_1]$ are disjoint $e$-open sets. Since $I + I_1$ and $J + J_1$ are free, we have $\bigcap Z[I] \cap \bigcap Z[I_1] = \varnothing$ and $\bigcap Z[J] \cap \bigcap Z[J_1] = \varnothing$. Thus $\bigcap Z[I] \subseteq \mathrm{cl}_X(\bigcup \mathrm{Coz}[I_1])$ and $\bigcap Z[J] \subseteq \mathrm{cl}_X(\bigcup \mathrm{Coz}[J_1])$, where $\mathrm{cl}_X(\bigcup \mathrm{Coz}[I_1])$ and $\mathrm{cl}_X(\bigcup \mathrm{Coz}[J_1])$ are two disjoint clopen subsets of $X$. □

## 4. On the lattice of $E(X)$ (resp., $OD(X)$)

In this section, $X, Y$ denote any topological space unless stated otherwise. Put $E(X) = \{A \subseteq X : A$ is $e$-open$\}$. It is clear that the union of two elements of $E(X)$ belongs to $E(X)$, and by [1, Corollary 2.2], the intersection of any two elements of $E(X)$ also belongs to $E(X)$. Thus, $E(X)$ together with inclusion is a lattice with the following operations:

$$A \vee B = A \cup B \quad \text{and} \quad A \wedge B = A \cap B, \quad \text{where} \quad A, B \in E(X).$$

It follows that $E(X)$ is a complete lattice if and only if the set $E(X)$ is a topology on the space $X$. If $\tau$ is the original topology on $X$, then $E(X) \subseteq \tau$. Note that it may happen that $E(X)$ forms a topology on $X$ while $X$ is not an $e$-space. For example, consider the space $\mathbb{R}$ with the usual topology. Then we have $E(\mathbb{R}) = \{U : U$ is an open and dense subset of $\mathbb{R}\} \cup \{\varnothing, \mathbb{R}\}$, which is a topology on $\mathbb{R}$. However, this space is not an $e$-space.

Similarly, for a topological space $X$, put

$$OD(X) = \{U : U \text{ is open and dense in } X\}.$$

Then, it is straightforward that $(OD(X), \subseteq)$ is a complete lattice together with the following operations:

$$A \vee B = A \cup B \quad \text{and} \quad A \wedge B = A \cap B, \quad \text{where} \quad A, B \in OD(X).$$

On the algebraic side, let $EId(C(X))$ denotes the set of all essential ideals of $C(X)$. It is easy to check that the intersection of two essential ideals, as well as the sum of two essential ideals, is again an essential ideal. Hence, $(EId(C(X)), \subseteq, +, \cap)$ forms a complete lattice. However, this lattice does not necessarily have a least element (namely 0). In fact, in $C(X)$, every intersection of essential ideals need not be an essential ideal. In the next result, we provide an algebraic presentation of the lattice $E(X)$ (resp., $OD(X)$). Recall that the lattice of Baer ideals of a ring $R$ is denoted by $LB(R)$, see [10].

**Theorem 4.1.**

(1) *The lattice $E(X)$ is a quotient of the lattice $LB(C(X))$.*
(2) *The lattice $OD(X)$ is a quotient of the lattice $EId(C(X))$.*

*Proof.* (1) Define $\phi : LB(C(X)) \to E(X)$ by $\phi(I) = \bigcup \mathrm{Coz}[I]$. By Lemma 3.1 and Corollary 3.2(1), $\phi$ is well defined and onto. It remains to show that $\phi$ preserves $\wedge$ and $\vee$. Let $I, J \in LB(C(X))$. Then we have,

$$\phi(I \vee J) = \phi(I + J) = \bigcup \mathrm{Coz}[I + J] = \bigcup \mathrm{Coz}[I] \cup \bigcup \mathrm{Coz}[J] = \phi(I) \vee \phi(J),$$

and

$$\phi(I \wedge J) = \phi(I \cap J) = \bigcup \mathrm{Coz}[I \cap J] = \bigcup \mathrm{Coz}[I] \cap \bigcup \mathrm{Coz}[J] = \phi(I) \wedge \phi(J).$$

(2) Define $\psi : EId(C(X)) \to OD(X)$ by $\psi(I) = \bigcup \mathrm{Coz}[I]$. By Lemma 3.1 and Corollary 3.2(2), $\psi$ is well defined and onto. The remainder of the proof is similar to the part (1). □

Denote by $I(X)$ the set of isolated points of $X$. We need the following well-known lemma.

**Lemma 4.2.** *For a $T_1$-space $X$, $I(X) = \bigcap_{U \in OD(X)} U$.*

**Proposition 4.3.** *The following statements are equivalent.*

(1) *The lattice $EId(C(X))$ has a least element $0$.*
(2) *The set of isolated points of $X$ is dense in $X$.*
(3) *The socle of $C(X)$ (i.e., $C_F(X)$) is an essential ideal of $C(X)$.*
(4) *The lattice $OD(X)$ has a least element $0$.*

*Proof.* (1)⇒(3) Assume that $EId(C(X))$ has a least element $0$. Since $C_F(X)$ is the intersection of all essential ideals of $C(X)$, $C_F(X) = 0$ and the result follows.

(3)⇒(1) By hypothesis, $C_F(X)$ is the smallest essential ideal of $C(X)$, and hence it is the 0 element of the lattice $EId(C(X))$.

(2)⇔(3) This equivalence follows from Proposition 2.1 of [13].

(2)⇒(4) By hypothesis and Lemma 4.2, $I(X)$ is the least element of $OD(X)$.

(4)⇒(2) Let $OD(X)$ has the least element $0$. Then, by Lemma 4.2, $I(X) = 0$ and so an open dense subset. □

Recall from [3, 7, 9] and [14] that a lattice $< L, \wedge, \vee, 0, 1 >$ is called a co-normal lattice whenever it is a distributive lattice and for all $a, b \in L$ with $a \wedge b = 0$ there exist $x, y \in L$ such that $x \vee y = 1$ and $x \wedge a = y \wedge b = 0$.

Recall that a lattice $L$ with the top element 1 is compact if and only if whenever $1 = \bigvee A$ where $A \subseteq L$, then $1 = \bigvee A_0$ for some finite $A_0 \subseteq A$.

A bounded lattice $L$ with a top element 1 and a bottom element 0 is connected if whenever $1 = a \vee b$ and $a \wedge b = 0$, then either $a = 0$ or $b = 0$. We have:

**Theorem 4.4.**

(1) *The lattice $E(X)$ is a co-normal lattice.*

(2) *A space $X$ is e-compact if and only if $E(X)$ is a compact lattice.*

(3) *A space $X$ is od-compact if and only if $od(X)$ is a compact lattice.*

(4) *A space $X$ is connected if and only if $E(X)$ is connected.*

(5) *A space $X$ is connected if and only if $LB(C(X))$ is a connected lattice.*

(6) *For any $X$, $LB(C(X))$ is a compact lattice.*

*Proof.* (1) It is easy to see that $E(X)$ is a distributive lattice. Now, let $A, B \in E(X)$ and $A \wedge B = 0$. Then, $A \cap B = \emptyset$. Since, $A, B$ are $e$-open, $\overline{A} \cap \overline{B} = \emptyset$. Put $A_1 = X \setminus \overline{A}$ and $B_1 = X \setminus \overline{B}$. Then $A_1, B_1 \in E(X)$, $A \cap A_1 = \emptyset$, $B \cap B_1 = \emptyset$ and $A_1 \cup B_1 = X$. Thus, $A \wedge A_1 = B \wedge B_1 = 0$ and $A_1 \vee B_1 = 1$, so we are done.

(2) $\Rightarrow$. We know that the top element of $E(X)$ is $X$. Now, let $X = \bigvee_{\alpha \in S} U_\alpha = \bigcup_{\alpha \in S} U_\alpha$, where for each $\alpha \in S$, $U_\alpha \in E(X)$. Then, for each $\alpha \in S$, $U_\alpha$ is $e$-open, hence there is a finite subset $F$ of $S$ such that $X = \bigcup_{\alpha \in F} U_\alpha$. Thus, $E(X)$ is compact.

$\Leftarrow$ If $X = \bigcup_{\alpha \in S} U_\alpha$, where each $U_\alpha$ is an $e$-open subset of $X$, then by hypothesis, there is a finite subset $F$ of $S$ such that $X = \bigcup_{\alpha \in F} U_\alpha$. Thus, $X$ is $e$-compact.

(3) The proof is analogous to (2), replacing $E(X)$ with $OD(X)$.

(4)$\Rightarrow$ Let $U, V \in E(X)$ such that $U \vee V = 1$ and $U \wedge V = 0$. Then, $X = U \cup V$ and $U \cap V = \emptyset$. By connectedness of $X$, we must have $U = \emptyset$ or $V = \emptyset$. That is $U = 0$ or $V = 0$.

$\Leftarrow$ Suppose that $X = U \cup V$, where $U, V$ are two disjoint open subsets of $X$. Then, $U, V$ are two disjoint clopen subsets of $X$. Thus, in $E(X)$, $1 = U \vee V$ and $U \wedge V = 0$. By hypothesis, $U = 0$ or $V = 0$. Hence, $U = \emptyset$ or $V = \emptyset$. Therefore, $X$ is connected.

(5) $\Rightarrow$ Let $I$ and $J$ be two Baer ideals of $C(X)$ with $I \vee J = 1$ and $I \wedge J = 0$. Then, $I + J = C(X)$ and $I \cap J = 0$. This implies $I$ and $J$ are two direct summands of $R$, so there are two idempotents $e, f \in C(X)$ such that $I = eC(X)$ and $J = fC(X)$. By hypothesis and [11, 1B2], the only idempotents of $C(X)$ are 0 and 1. Thus, $I = 0$ or $J = 0$.

$\Leftarrow$ By [11, 1B2], it is enough to show that the only idempotents of $C(X)$ are 0 and 1. Let $e$ be an idempotent of $C(X)$. Then $I = eC(X)$ and $J = (1 - e)C(X)$ are two Baer ideals of $C(X)$ and we have $I \vee J = 1$ and $I \wedge J = 0$. Thus, by hypothesis, $eC(X) = I = 0$ or $(1 - e)C(X) = J = 0$. Hence, $e = 0$ or $e = 1$.

(6) We know that the top element of $LB(C(X))$ is $C(X)$. Now, let $C(X) = \bigvee_{\alpha \in S} I_\alpha$, where each $I_\alpha \in LB(C(X))$. Then, $1 \in \sum_{\alpha \in S} I_\alpha$. Hence, there exists a finite subset $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ of $S$ such that $1 \in \sum_{i=1}^{n} I_{\alpha_i}$. Thus, $C(X) = \sum_{i=1}^{n} I_{\alpha_i}$. That is $C(X) = \bigvee_{i=1}^{n} I_{\alpha_i}$. Therefore, $LB(C(X))$ is compact. □

**Theorem 4.5.**

(1) *If $X$ and $Y$ are homeomorphic spaces, then $E(X)$ and $E(Y)$ are isomorphic lattices.*
(2) *If $X$ and $Y$ are homeomorphic spaces, then $OD(X)$ and $OD(Y)$ are isomorphic lattices.*
(3) *If $C(X)$ and $C(Y)$ are isomorphic rings, then $EId(C(X))$ and $EId(C(Y))$ are isomorphic lattices.*
(4) *If $C(X)$ and $C(Y)$ are isomorphic rings, then $LB(C(X))$ and $LB(C(Y))$ are isomorphic lattices.*

*Proof.* (1) Let $f : X \to Y$ be a homeomorphism. Define $\phi : E(X) \to E(Y)$, by $\phi(U) = f(U)$. Since $f$ is a homeomorphism, for each $U \in E(X)$, $f(U)$ and $f(\overline{U})$ are open and $\overline{f(U)} = f(\overline{U})$. This shows that $\overline{f(U)}$ is open in $Y$ and hence $f(U)$ is an $e$-open set in $Y$. Thus, $\phi(U) \in E(Y)$. Clearly, $\phi$ is bijective, as $f$ is bijective. It is enough to show that $\phi$ preserves join and meet. Assume $U, V \in E(X)$. We have,

$$\phi(U \vee V) = \phi(U \cup V) = f(U \cup V) = f(U) \cup f(V) = \phi(U) \vee \phi(V),$$

and

$$\phi(U \wedge V) = \phi(U \cap V) = f(U \cap V) = f(U) \cap f(V) = \phi(U) \wedge \phi(V).$$

(2) The proof is identical to (1), replacing $E(X)$ with $OD(X)$.

(3) Let $\phi : C(X) \to C(Y)$ be a ring isomorphism. Define

$$\psi : EId(C(X)) \to EId(C(Y)), \text{ by } \psi(I) = \phi(I).$$

Since, $\phi$ is a ring isomorphism, for each $I \in EId(C(X))$, we have $\psi(I) = \phi(I) \in EId(C(Y))$. The map $\psi$ is bijective, as $\phi$ is bijective. Now, assume $I, J \in EId(C(X))$. Then,

$$\psi(I \vee J) = \psi(I + J) = \phi(I + J) = \phi(I) + \phi(J) = \phi(I) \vee \phi(J) = \psi(I) \vee \psi(J),$$

and

$$\psi(I \wedge J) = \psi(I \cap J) = \phi(I \cap J) = \phi(I) \cap \phi(J) = \phi(I) \wedge \phi(J) = \psi(I) \wedge \psi(J).$$

(4) The proof is analogous (3), replacing $EId(C(X))$ with $LB(C(X))$. □

An ideal $I$ of $C(X)$ is a *closed ideal in the m-topology* if

$$I = \overline{I} = M^{\theta(I)} = \bigcap_{I \subseteq M^p} M^p.$$

Moreover, we have $\theta(I) = \theta(\overline{I})$, see [11, 7Q]. Set

$$\mathbf{CBI}(X) = \{\overline{I} : I \text{ is a Baer ideal of } C(X)\}.$$

By Proposition 2.5 in [16], the sum of two Baer ideals in C(X) is a Baer ideal and by [10, Lemma 3.1], the intersection of two Baer ideals in $C(X)$ is a Baer ideal. Hence, $\mathbf{CBI}(X)$ partially ordered with inclusion and the following operations forms a lattice.

$$\overline{I} \vee \overline{J} = \overline{I + J} \text{ and } \overline{I} \wedge \overline{J} = \overline{I \cap J}.$$

Similarly, define

$$\mathbf{CEI}(X) = \{\overline{I} : \; I \text{ is an essential ideal of } C(X)\}.$$

Since, the sum and the intersection of two essential ideals are an essential ideals, it follows that $\mathbf{CEI}(X)$ partially ordered with inclusion and the following operations also forms a lattice.

$$\overline{I} \vee \overline{J} = \overline{I + J} \text{ and } \overline{I} \wedge \overline{J} = \overline{I \cap J}.$$

**Lemma 4.6.** *Let $X$ be a completely regular Hausdorff space.*

(1) $E(\beta X) = \{\beta X \setminus \theta(I) : I$ *is a Baer ideal of* $C(X)\}$.
(2) $OD(\beta X) = \{\beta X \setminus \theta(I) : I$ *is an essential ideal of* $C(X)\}$.

*Proof.* (1) Let $U \in E(\beta X)$. Since $U$ is open, there exists a collection $\{f_\alpha : \alpha \in S \;, f_\alpha \in C(X)\}$ such that $U = \bigcup_{\alpha \in S}(\beta X \setminus \mathrm{cl}_{\beta X} Z(f_\alpha))$. Put $I =< f_\alpha : \alpha \in S >$. Then,

$$U = \beta X \setminus \bigcap_{\alpha \in S} \mathrm{cl}_{\beta X} Z(f_\alpha) = \beta X \setminus \theta(I).$$

We have $\mathrm{cl}_{\beta X}(\beta X \setminus \theta(I))$ is open, hence there is an idempotent $e \in C^*(X)$ such that $\mathrm{cl}_{\beta X}(\beta X \setminus \theta(I)) = \beta X \setminus \mathrm{cl}_{\beta X} Z(e)$. Thus, $\mathrm{int}_{\beta X}\, \theta(I) = \mathrm{cl}_{\beta X} Z(e)$. This implies $\mathrm{int}_X \bigcap Z[I] = Z(e)$. By Lemma 2.1, $\mathrm{Ann}(I) = \mathrm{Ann}(e) = (1 - e)R$, i.e., $I$ is a Baer ideal of $C(X)$. Now, assume $I$ is a Baer ideal of $C(X)$. Then, there is an idempotent $e \in C^*(X)$ such that $\mathrm{Ann}(I) = \mathrm{Ann}(e)$. By Lemma 2.1, $\mathrm{int}_X \bigcap Z[I] = Z(e)$. This shows that $\mathrm{cl}_X(X \setminus \bigcap Z[I])$ is a clopen subset of $X$. By Lemma 3.5, we have

$$\mathrm{cl}_{\beta X}(\mathrm{cl}_X(X \setminus \bigcap Z[I])) = \mathrm{cl}_{\beta X}(X \setminus \bigcap Z[I]) = \mathrm{cl}_{\beta X}((\beta X \setminus \theta(I)) \cap X) = \mathrm{cl}_{\beta X}(\beta X \setminus \theta(I)).$$

By [11, 6.9 C], $\mathrm{cl}_{\beta X}(\beta X \setminus \theta(I))$ is a clopen subset of $\beta X$. Thus, $\beta X \setminus \theta(I) \in E(\beta X)$.

(2) The proof is entirely analogous to (1), replacing Baer ideals with essential ideals. □

**Theorem 4.7.** (1) *The lattices* $\mathbf{CBI}(X)$ *and* $E(\beta X)$ *are isomorphic.*
(2) *The lattices* $\mathbf{CEI}(X)$ *and* $OD(\beta X)$ *are isomorphic.*

*Proof.* (1) Define the map $\phi : \mathbf{CBI}(X) \to E(\beta X)$ by $\phi(\overline{I}) = \beta X \setminus \theta(I)$. Then, by Lemma 4.6, $\phi$ is well-defined. Let $\overline{I_1}, \overline{I_2} \in \mathbf{CBI}(X)$ and $\overline{I_1} = \overline{I_2}$. Then

$$\phi(\overline{I_1}) = \beta X \setminus \theta(I_1) = \beta X \setminus \theta(\overline{I_1}) = \beta X \setminus \theta(\overline{I_2}) = \beta X \setminus \theta(I_2) = \phi(\overline{I_2}).$$

Now, assume $\phi(\overline{I_1}) = \phi(\overline{I_2})$, where $\overline{I_1}, \overline{I_2} \in \mathbf{CBI}(X)$. Then, $\theta(I_1) = \theta(I_2)$. Thus,

$$\overline{I_1} = M^{\theta(I_1)} = M^{\theta(I_2)} = \overline{I_2}.$$

This shows that $\phi$ is injective. By Lemma 4.6, for every $e$-open subset $U$ of $\beta X$ there exists a Baer ideal $I$ of $C(X)$ such that $U = \beta X \setminus \theta(I)$. Thus, $\overline{I} \in \mathbf{CBI}(X)$ and we have $\phi(\overline{I}) = \beta X \setminus \theta(I) = U$. Thus, $\phi$ is surjective. It is enough to show the preserving joint and meet under $\phi$ as the following.

$$\phi(\overline{I_1} \vee \overline{I_2}) = \phi(\overline{I_1 + I_2}) = \beta X \setminus \theta(I_1 + I_2) = \beta X \setminus (\theta(I_1) \cap \theta(I_2)) =$$
$$(\beta X \setminus \theta(I_1)) \cup (\beta X \setminus \theta(I_2)) = \phi(\overline{I_1}) \vee \phi(\overline{I_2}),$$

and

$$\phi(\overline{I_1} \wedge \overline{I_2}) = \phi(\overline{I_1 \cap I_2}) = \beta X \setminus \theta(I_1 \cap I_2) = \beta X \setminus ((\theta(I_1) \cup \theta(I_2)) =$$
$$(\beta X \setminus \theta(I_1)) \cap (\beta X \setminus \theta(I_2)) = \phi(\overline{I_1}) \wedge \phi(\overline{I_2}).$$

(2) The proof is entirely analogous (1), replacing Baer ideals with essential ideals and $E(\beta X)$ with $OD(\beta X)$. □

**Corollary 4.8.**

(1) *For any completely regular space $X$, there exists a realcompact space $Y$ such that $EId(C(X))$ and $EId(C(Y))$ are isomorphic lattices.*

(2) *For any completely regular space $X$, there exists a realcompact space $Y$ such that $LB(C(X))$ and $LB(C(Y))$ are isomorphic lattices.*

(3) *If $C^*(X)$ and $C^*(Y)$ are isomorphic rings, then $\mathbf{CBI}(X)$ and $\mathbf{CBI}(Y)$ are isomorphic lattices.*

(4) *If $C^*(X)$ and $C^*(Y)$ are isomorphic rings, then $\mathbf{CEI}(X)$ and $\mathbf{CEI}(Y)$ are isomorphic lattices.*

*Proof.* (1) It is well-known that for any completely regular space $X$ there exists a realcompact space $\upsilon X$ such that $C(X)$ and $C(\upsilon X)$ are isomorphic rings. Put $Y = \upsilon X$. By Theorem 4.5(3), $EId(C(X))$ and $EId(C(Y))$ are isomorphic.

(2) Let $Y = \upsilon X$. By argument of Part (1) and Part (4) of Theorem 4.5, $LB(C(X))$ and $LB(C(Y))$ are isomorphic lattices.

(3) Since, $C^*(X)$ and $C^*(Y)$ are isomorphic to $C(\beta X)$ and $C(\beta Y)$, respectively. The hypothesis follows that $C(\beta X)$ and $C(\beta Y)$ are isomorphic rings. By [11, Theorem 4.9], $\beta X$ and $\beta Y$ are homeomorphic spaces. By Theorem 4.5(1), $E(\beta X)$ and $E(\beta Y)$ are isomorphic lattices. By Theorem 4.7(1), $\mathbf{CBI}(X)$ and $\mathbf{CBI}(Y)$ are isomorphic lattices.

(4) Following the same argument as in Part (3), $\beta X$ and $\beta Y$ are homeomorphic. By Theorem 4.5(2), $OD(\beta X)$ and $OD(\beta Y)$ are isomorphic lattices. By Theorem 4.7(2), $\mathbf{CEI}(X)$ and $\mathbf{CEI}(Y)$ are isomorphic lattices. □

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