APPLIED GENERAL TOPOLOGY

Appl. Gen. Topol. 26, no. 2 (2025), 799-813
doi:10.4995/agt.2025.23158

On n-ellipse contractive mappings over some extended metric spaces

Kushal Roy [a], Zoran D. Mitrović [b] and Vahid Parvaneh [c]

[a] Department of Mathematics, Dr. B. C. Roy Engineering College, Jemua Road, Fuljhore, Durgapur-713206, West Bengal, India (kushal.roy93@gmail.com)

[b] University of Banja Luka, Faculty of Electrical Engineering, patre 5, 78000 Banja Luka, Bosnia and Herzegovina (zoran.mitrovic@etf.unibl.org)

[c] Department of Mathematics, Gilan-E-Gharb Branch, Islamic Azad University, Gilan-E-Gharb, Iran (zam.dalahoo@gmail.com)

Communicated by R. Pant

Abstract

In this paper, the concept of a mapping contracting axes of an n-ellipse is introduced and its fixed point set has been determined. Investigation of fixed point of mappings over different metric-type spaces is always an interesting study in fixed point theory. We consider a mapping with special characteristic, which is contracting axes of an n-ellipse over two different metric-type structures, Δ-metric space and $\mathcal{F}$-metric space. An enriched version of a mapping contracting axes of an n-ellipse is also introduced, and this class of mappings includes several non-expansive mappings. Moreover, the paper is furnished by some suitable examples that support our results and show that our mapping is distinct from the usual contractive type mappings.

2020 MSC: *47H10; 54H25.*

Keywords: *fixed point; axes contractive mapping of n-ellipse; Δ-metric space; $\mathcal{F}$-metric space; Banach space.*

Received 3 January 2025 – Accepted 28 March 2025

1. Introduction and Preliminaries

It is well known that an element ξ is called a fixed point of a self mapping $\mathcal{T}$ defined on a non-empty set $\mathcal{X}$ if $\mathcal{T}\xi = \xi$. The set $Fix(\mathcal{T})$ denotes the set of all fixed points of the mapping $\mathcal{T}$. If a mapping $\mathcal{T}$ has more than one fixed point, that is, $Fix(\mathcal{T})$ is not a singleton then it can contain some geometrical shapes like a circle, disc or different types of conics. A geometrical pattern $\mathcal{G}$ is called a fixed figure of a mapping $\mathcal{T}$ if $\mathcal{T}\xi = \xi$ for all $\xi \in \mathcal{G}$.

Özgür and Taş [10, 11] first introduced the concept of fixed circle theory in the year 2017. They showed that under some relevant conditions, a mapping can fix a circle on arbitrary metric spaces. On such a topic, several researchers have contributed nowadays (see [18, 19]). Next in the year 2021, Joshi et al. [8] introduced the notion of a fixed ellipse in metric spaces.

The famous Banach contraction principle [2] can reduce the radius of a circle with center as the unique fixed point of a contraction mapping, that is, the perimeter and the area of the circle also. E. Petrov has introduced a mapping contracting perimeters of triangles [15] in the year 2023. From the title of the mapping, it is clearly understood that this mapping can shrink the lengths of sides of a given triangle. Recently, the concept of a mapping that can decrease the lengths of axes of an ellipse has been given by K. Roy (see [17]).

Definition 1.1 (Mapping contracting perimeters of triangles, [15]). A self mapping $\mathcal{T}$ defined on a metric space $(\mathcal{X}, D)$ with $|\mathcal{X}| \geq 3$ is called a mapping contracting perimeters of triangles if there exists $k_1 \in [0, 1)$ such that the inequality

$$D(\mathcal{T}\xi, \mathcal{T}\eta) + D(\mathcal{T}\eta, \mathcal{T}\sigma) + D(\mathcal{T}\sigma, \mathcal{T}\xi) \leq k_1[D(\xi, \eta) + D(\eta, \sigma) + D(\sigma, \xi)] \tag{1.1}$$

holds for all three pairwise distinct points $\xi, \eta, \sigma \in \mathcal{X}$.

Definition 1.2 ([17]). Let $(\mathcal{X}, D)$ be a metric space and $\theta, \vartheta \in \mathcal{X}$ be two distinct elements. A mapping $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ is said to be a mapping contracting axes of ellipse if for all $\xi \in \mathcal{X} \setminus \{\theta, \vartheta\}$ with $\xi \neq \mathcal{T}\xi$,

$$D(\mathcal{T}\xi, \theta) + D(\mathcal{T}\xi, \vartheta) \leq k_2(D(\xi, \theta) + D(\xi, \vartheta)), \tag{1.2}$$

for some $k_2 \in [0, 1)$.

In the year 2021, N. Taş et al. [21] introduced new fixed-figure results using the notion of k-ellipse on a metric space. Succeeding one is the definition of k-ellipse.

Definition 1.3 ([21]). Let $(\mathcal{X}, D)$ be a metric space. A k-ellipse is defined by

$$E[\xi_1, \ldots, \xi_k; r] = \{\xi \in \mathcal{X} : D(\xi, \xi_1) + \cdots + D(\xi, \xi_k) = r\} \tag{1.3}$$

Clearly, for $k = 2$ we get an ellipse and if $k = 1$ then we get a circle in metric space $(\mathcal{X}, D)$.

In this manuscript, we introduce a mapping $\mathcal{T}$ which can contract the axes of n-ellipse, a generalized version of ellipse.

2. Mapping contracting axes of an n-ellipse in a Δ-metric space

First, we recall the definition and related properties of a Δ-metric space.
Let the set $\mathcal{S}_\Delta$ be defined by

$$\mathcal{S}_\Delta = \{\Delta : \mathcal{X} \times \mathcal{X} \to [0, +\infty) : \Delta(\xi, \xi) = 0 \text{ for all } \xi \in \mathcal{X}\},$$

where $\mathcal{X}$ is a nonempty set.

Definition 2.1 ([20])**.** Let $\mathcal{X}$ be a nonempty set and $\rho_\Delta : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ be a mapping. Then ρ_Δ is said to be a Δ-metric if there exists some $\Delta \in \mathcal{S}_\Delta$ such that for all $\xi, \eta, \zeta \in \mathcal{X}$ it satisfies the following conditions:
($\rho_\Delta 1$) $0 \le \Delta(\xi, \eta) \le \rho_\Delta(\xi, \eta)$;
($\rho_\Delta 2$) $\rho_\Delta(\xi, \eta) = \Delta(\xi, \eta)$ if and only if $\xi = \eta$;
($\rho_\Delta 3$) $\rho_\Delta(\xi, \eta) = \rho_\Delta(\eta, \xi)$;
($\rho_\Delta 4$) $\rho_\Delta(\xi, \zeta) \le \rho_\Delta(\xi, \eta) + \rho_\Delta(\eta, \zeta) + \Delta(\xi, \zeta)$.
The pair (X, ρ_Δ) is called a Δ-metric space.

Remark 2.2 ([20]). (i) If in Definition 2.1. we put

$$\Delta(\xi, \eta) = 0, \text{ for all } \xi, \eta \in \mathcal{X},$$

then ρ_Δ will give an usual metric.
(ii) If in Definition 2.1. we put

$$\Delta(\xi, \eta) = \frac{k-1}{k}\rho_\Delta(\xi, \eta), \text{ for all } \xi, \eta \in \mathcal{X},$$

and for some $k > 1$ then ρ_Δ will give a b-metric (see [1, 6]) with coefficient $k > 1$.
(iii) If we take

$$\Delta(\xi, \eta) = \frac{\theta(\xi, \eta) - 1}{\theta(\xi, \eta)}\rho_\Delta(\xi, \eta), \text{ for all } \xi, \eta \in \mathcal{X},$$

and for some function $\theta : \mathcal{X}^2 \to [1, +\infty)$ in Definition 2.1 then ρ_Δ will give an extended b-metric [9].
(iv) Let $\Omega : [0, \infty) \to [0, +\infty)$ be a strictly increasing continuous function with

$$\Omega^{-1}(t) \le t \le \Omega(t) \text{ and } \Omega^{-1}(0) = 0 = \Omega(0).$$

Then if we consider

$$\Delta(\xi, \eta) = \rho_\Delta(\xi, \eta) - \Omega^{-1}(\rho_\Delta(\xi, \eta)), \text{ for all } \xi, \eta \in \mathcal{X},$$

then ρ_Δ will give a $p-$metric, see [14].

Definition 2.3 ([20]). Let $(\mathcal{X}, \rho_\Delta)$ be a Δ-metric space and $\{\xi_n\}$ be a sequence in $\mathcal{X}$. Then:

(i) the sequence $\{\xi_n\}$ is ρ_Δ-convergent and converges to some $\xi \in \mathcal{X}$, if $\rho_\Delta(\xi_n, \xi) \to 0$ as $n \to +\infty$,

(ii) the sequence $\{\xi_n\}$ is ρ_Δ-Cauchy if $\rho_\Delta(\xi_n, \xi_m) \to 0$ as $n, m \to +\infty$,

(iii) $(\mathcal{X}, \rho_\Delta)$ is complete if every ρ_Δ-Cauchy sequence in $\mathcal{X}$ is ρ_Δ-convergent.

We give an example of complete Δ-metric space.

Example 2.4 ([20]). Let $\mathcal{X} = \mathbb{N}$ and $\rho_\Delta : \mathcal{X} \times \mathcal{X} \to [0, +\infty)$ satisfies $\rho_\Delta(\xi, \eta) = \rho_\Delta(\eta, \xi)$ for all $\xi, \eta \in \mathcal{X}$ and be given by

$$\rho_\Delta(\xi, \eta) = \begin{cases} 0, & \text{if } \xi = \eta; \\ \frac{1}{\eta}, & \text{if } \xi = 1 \text{ and } \eta > 1; \\ 2, & \text{if } \xi, \eta \in \mathbb{N} \setminus \{1\} \text{ and } \xi \neq \eta. \end{cases} \tag{2.1}$$

Then ρ_Δ is a Δ-metric space, which is complete, and $\Delta \in \mathcal{S}_\Delta$ given by

$$\Delta(\xi, \eta) = \begin{cases} 2 - \frac{\xi+\eta}{\xi\eta}, & \text{if } \xi, \eta \in \mathbb{N} \setminus \{1\} \text{ and } \xi \neq \eta; \\ 0, & \text{otherwise.} \end{cases} \tag{2.2}$$

Definition 2.5 ([20]). Let $(\mathcal{X}, \rho_\Delta)$ be a Δ-metric space and $\{\xi_n\}$ be a sequence in $\mathcal{X}$. Then

(i) the sequence $\{\xi_n\}$ is said to be Δ-convergent to an element $\xi \in \mathcal{X}$ if $\Delta(\xi_n, \xi) \to 0$ as $n \to +\infty$.

(ii) the sequence $\{\xi_n\}$ is called Δ-Cauchy if $\Delta(\xi_n, \xi_m) \to 0$ as $n, m \to +\infty$.

Remark 2.6 ([20]). (i) The concepts of ρ_Δ-convergence and Δ-convergence is same in the case of b-metric spaces and p-metric spaces. A ρ_Δ-Cauchy sequence is also Δ-Cauchy in b-metric spaces and p-metric spaces, and vice versa;

(ii) A Δ-metric is not always continuous;

(iii) In a Δ-metric space, any ρ_Δ-convergent sequence is also Δ-convergent, but the converse is not true in general;

(iv) In a Δ-metric space, a ρ_Δ-convergent sequence may not be always a ρ_Δ-Cauchy sequence;

(v) A Δ-metric space is always Hausdorff, that is, if in a Δ-metric space $\mathcal{X}$ a sequence $\{\xi_n\}$ is ρ_Δ-convergent to two elements ξ and η, then $\xi = \eta$;

(vi) If in a Δ-metric space $\mathcal{X}$ a sequence $\{\xi_n\}$ is ρ_Δ-convergent to some $\xi \in X$ then $\lim_{n,m \to +\infty} [\rho_\Delta(\xi_n, \xi_m) - \Delta(\xi_n, \xi_m)] = 0$. Moreover if $\{\xi_n\}$ is a Δ-Cauchy sequence then $\{\xi_n\}$ will be a ρ_Δ-Cauchy sequence in $\mathcal{X}$;

(vii) If in a Δ-metric space $\mathcal{X}$ two sequences $\{\xi_n\}$ and $\{\eta_n\}$ both are ρ_Δ-convergent to an element $\xi \in \mathcal{X}$ then $\lim_{n \to +\infty} [\rho_\Delta(\xi_n, \eta_n) - \Delta(\xi_n, \eta_n)] = 0$. Moreover if $\lim_{n \to +\infty} \Delta(\xi_n, \eta_n) = 0$ then $\lim_{n \to +\infty} \rho_\Delta(\xi_n, \eta_n) = 0$.

Before going to our main theorem, we define orbital continuity and p-continuity, where $p \geq 1$.

Definition 2.7 (Orbitally continuous mapping)**.** For a self mapping $\mathcal{T}$ over a Δ-metric space $(\mathcal{X}, \rho_\Delta)$ the set

$$O(\xi, \mathcal{T}) := \{\mathcal{T}^n \xi : n = 0, 1, 2, \ldots\}, \xi \in \mathcal{X},$$

is called an orbit of the mapping $\mathcal{T}$ and $\mathcal{T}$ is said to be orbitally continuous at a point $u \in \mathcal{X}$ if for any sequence $\{\xi_n\} \subset O(\xi, \mathcal{T})$ (for some $\xi \in \mathcal{X}$) $\xi_n \to u$ (ρ_Δ-convergence) implies $\mathcal{T}\xi_n \to \mathcal{T}u$ as $n \to +\infty$. Moreover, if $\mathcal{T}$ is orbitally continuous at each point of $\mathcal{X}$ then $\mathcal{T}$ is said to be orbitally continuous on $\mathcal{X}$.

Definition 2.8 (p-continuous mapping)**.** A mapping $\mathcal{T} : (\mathcal{X}, \rho_\Delta) \to (\mathcal{X}, \rho_\Delta)$ in a Δ-metric space is called p-continuous for some $p \in \mathbb{N}$ if for all sequence $\{\chi_n\} \in \mathcal{X}$, $\mathcal{T}^{p-1}(\chi_n) \to l$ (ρ_Δ-convergence) as $n \to +\infty$ implies $\mathcal{T}^p(\chi_n) \to \mathcal{T}(l)$.

Now we are in a position to prove some fixed point theorems for mappings contracting axes of an n-ellipse in a Δ-metric space.

Definition 2.9. Let the mapping $\mathcal{T}$ on a Δ-metric space $(\mathcal{X}, \rho_\Delta)$ be such that

$$\rho_\Delta(\mathcal{T}\xi, \lambda_1) + \cdots + \rho_\Delta(\mathcal{T}\xi, \lambda_n) \leq k[\rho_\Delta(\xi, \lambda_1) + \cdots + \rho_\Delta(\xi, \lambda_n)] \tag{2.3}$$

for all $\xi \in \mathcal{X} \setminus \{\lambda_1, \ldots, \lambda_n\}, \xi \neq \mathcal{T}\xi$, where $k \in (0, 1)$ and $\lambda_1, \ldots, \lambda_n$ are all pairwise distinct elements. Then the mapping $\mathcal{T}$ is a mapping contracting axes of an n-ellipse with Lipschitz constant $k \in (0, 1)$.

Theorem 2.10. *Let $(\mathcal{X}, \rho_\Delta)$ be a complete Δ-metric space and $\mathcal{T}$ be a mapping contracting axes of n-ellipse with Lipschitz constant $k \in (0, 1)$. Also, let $\mathcal{T}$ satisfies the following three conditions:*

(a) For some $\xi_0 \in \mathcal{X}$, the Picard iterating sequence $\{\xi_m\}$, $\xi_m = \mathcal{T}^m \xi_0$ we have that $\xi_m \in \mathcal{X} \setminus \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ for every $m = 1, 2, \ldots$;

(b) The Picard iterating sequence $\{\xi_m\}_{m \geq 1}$ is Δ-Cauchy;

(c) $\mathcal{T}$ is orbitally continuous or, p-continuous for some $p \geq 1$ in $\mathcal{X}$.

Then $\mathcal{T}$ has at least one fixed point in $\mathcal{X}$.

Proof. If $\xi_m = \xi_{m+1}$ for some $m \in \mathbb{N} \cup \{0\}$, then we are done. Without loss of generality, therefore we assume that $\xi_m \neq \xi_{m+1}$ for $m = 0, 1, \ldots$. Then for any $m \geq 1$,

$$\begin{aligned}
&\rho_\Delta(\xi_m, \lambda_1) + \rho_\Delta(\xi_m, \lambda_2) + \cdots + \rho_\Delta(\xi_m, \lambda_n) \\
&\leq k[\rho_\Delta(\xi_{m-1}, \lambda_1) + \rho_\Delta(\xi_{m-1}, \lambda_2) + \cdots + \rho_\Delta(\xi_{m-1}, \lambda_n)] \\
&\leq k^2[\rho_\Delta(\xi_{m-2}, \lambda_1) + \rho_\Delta(\xi_{m-2}, \lambda_2) + \cdots + \rho_\Delta(\xi_{m-2}, \lambda_n)] \\
&\vdots \\
&\leq k^m[\rho_\Delta(\xi_0, \lambda_1) + \rho_\Delta(\xi_0, \lambda_2) + \cdots + \rho_\Delta(\xi_0, \lambda_n)].
\end{aligned} \tag{2.4}$$

Now, for every $m \geq 1$

$$\begin{aligned}
\rho_\Delta(\xi_m, \xi_{m+q}) &\leq \rho_\Delta(\xi_m, \lambda_1) + \rho_\Delta(\xi_{m+q}, \lambda_1) + \Delta(\xi_m, \xi_{m+q}) \\
&\leq (k^m + k^{m+q})[\rho_\Delta(\xi_0, \lambda_1) + \cdots + \rho_\Delta(\xi_0, \lambda_n)] + \Delta(\xi_m, \xi_{m+q}) \\
&= k^m(1 + k^q)[\rho_\Delta(\xi_0, \lambda_1) + \cdots + \rho_\Delta(\xi_0, \lambda_n)] + \Delta(\xi_m, \xi_{m+q}) \\
&\text{for } q = 1, 2, \ldots \qquad (2.5)
\end{aligned}$$

Applying condition (b) and since $0 < k < 1$, we have $\{\xi_m\}$ is a ρ_Δ-Cauchy sequence. Due to the completeness of $\mathcal{X}$ we get $\zeta \in \mathcal{X}$ such that $\rho_\Delta(\xi_m, \zeta) \to 0$ as $m \to +\infty$. Using condition (c) from the Definition 2.7. and Definition 2.8. we obtain that $\rho_\Delta(\xi_{m+1}, \mathcal{T}\zeta) \to 0$ as $m \to +\infty$. Triangle inequality of Δ-metric space implies that

$$\rho_\Delta(\mathcal{T}\zeta, \zeta) \leq \rho_\Delta(\xi_m, \mathcal{T}\zeta) + \rho_\Delta(\xi_m, \zeta) + \Delta(\mathcal{T}\zeta, \zeta) \text{ for all } m \geq 1. \qquad (2.6)$$

Thus

$$\Delta(\mathcal{T}\zeta, \zeta) \leq \rho_\Delta(\mathcal{T}\zeta, \zeta) \leq \Delta(\mathcal{T}\zeta, \zeta),$$

so, $\rho_\Delta(\mathcal{T}\zeta, \zeta) = \Delta(\mathcal{T}\zeta, \zeta)$. Now, we conclude that $\mathcal{T}\zeta = \zeta$, that is $\mathcal{T}$ has a fixed point in $\mathcal{X}$. □

Example 2.11. Let us consider the complete Δ-metric space defined in Example 2.4. Define $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ by

$$\mathcal{T}(\xi) = \begin{cases} 1, & \text{if } \xi \in 2\mathbb{N}; \\ \xi, & \text{otherwise.} \end{cases} \qquad (2.7)$$

Then for all $\xi \in 2\mathbb{N} \setminus \{2\}$

$$\begin{aligned}
&\rho_\Delta(\mathcal{T}\xi, 2) + \rho_\Delta(\mathcal{T}\xi, 3) + \rho_\Delta(\mathcal{T}\xi, 5) + \rho_\Delta(\mathcal{T}\xi, 7) + \rho_\Delta(\mathcal{T}\xi, 11) \\
=&\rho_\Delta(1, 2) + \rho_\Delta(1, 3) + \rho_\Delta(1, 5) + \rho_\Delta(1, 7) + \rho_\Delta(1, 11) \\
\approx&1.267
\end{aligned}$$

and

$$\rho_\Delta(\xi, 2) + \rho_\Delta(\xi, 3) + \rho_\Delta(\xi, 5) + \rho_\Delta(\xi, 7) + \rho_\Delta(\xi, 11) = 10.$$

Therefore $\mathcal{T}$ is a mapping contracting axes of 5-ellipse for the Lipschitz constant $k = \frac{3}{20}$, with the node points (foci) $2, 3, 5, 7$ and 11. Moreover, $\mathcal{T}$ satisfies all the conditions of Theorem 2.10 and we see that $\mathcal{T}$ has infinitely many fixed points in $\mathcal{X}$.

Corollary 2.12. *Let $\mathcal{X}$ be a complete metric space (or, a b-metric space or, a p-metric space) and $\mathcal{T}$ be a mapping contracting axes of n-ellipse for some Lipschitz constant $k \in (0,1)$. Also let $\mathcal{T}$ satisfies the following two conditions:*

(a) For some $\xi_0 \in \mathcal{X}$ the Picard iterating sequence $\{\xi_m\}$ so that $\xi_m = \mathcal{T}^m\xi_0$, is such that $\xi_m \in \mathcal{X} \setminus \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ for every $m \in \mathbb{N}$;

(b) $\mathcal{T}$ is orbitally continuous or, p-continuous for some $p \geq 1$ in $\mathcal{X}$.

Then $\mathcal{T}$ has at least one fixed point in $\mathcal{X}$.

Let us take the collection Φ of all functions

$$F : \underbrace{[0,+\infty) \times \cdots \times [0,+\infty)}_{n \text{ times}} \to [0,+\infty)$$

satisfying the following two conditions:

(i) F is continuous;

(ii) $\max\{t_1, t_2, \ldots, t_n\} \leq F(t_1, t_2, \cdots, t_n)$ for all $t_1, t_2, \ldots, t_n \in [0,+\infty)$.

Example 2.13. There are several examples of above defined mappings. For all $(t_1, t_2, \cdots, t_n) \in \underbrace{[0,+\infty) \times \cdots \times [0,+\infty)}_{n \text{ times}}$ one may consider the following mappings:

(i) $F_1(t_1, t_2, \cdots, t_n) = t_1 + t_2 + \cdots + t_n$;

(ii) $F_2(t_1, t_2, \cdots, t_n) = \sqrt[r]{t_1^r + t_2^r + \cdots + t_n^r}$ for any positive integer r;

(iii) $F(t_1, t_2, \cdots, t_n) = e^{t_1} + e^{t_2} + \cdots + e^{t_n}$.

Definition 2.14. A mapping $\mathcal{T}$ on a Δ-metric space $(\mathcal{X}, \rho_\Delta)$ is said to be a generalized mapping contracting axes of n-ellipse ($n \geq 2$) if it satisfies

$$F(\rho_\Delta(\mathcal{T}\xi, \lambda_1), \ldots, \rho_\Delta(\mathcal{T}\xi, \lambda_n)) \leq kF(\rho_\Delta(\xi, \lambda_1), \ldots, \rho_\Delta(\xi, \lambda_n)) \tag{2.8}$$

for all $\xi \notin Fix(\mathcal{T}) \cup \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$; for some $k \in (0,1)$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are all pairwise distinct elements.

Theorem 2.15. *Let $(\mathcal{X}, \rho_\Delta)$ be a complete Δ-metric space and $\mathcal{T}$ be a generalized mapping contracting axes of n-ellipse for some Lipschitz constant $k \in (0,1)$. Also let $\mathcal{T}$ satisfies the following conditions:*

(a) For some $\xi_0 \in \mathcal{X}$ the Picard iterating sequence $\{\xi_m\}$ so that $\xi_m = \mathcal{T}^m\xi_0$, is such that $\xi_m \in \mathcal{X} \setminus \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ for every $m = 1, 2, \ldots$;

(b) The Picard iterating sequence $\{\xi_m\}_{m\geq 1}$ is Δ-Cauchy;

(c) $\mathcal{T}$ is orbitally continuous or, p-continuous for some $p \geq 1$ in $\mathcal{X}$.

Then $\mathcal{T}$ has at least one fixed point in $\mathcal{X}$.

Proof. Let us assume that the sequence $\{\xi_m\}$ does not contain any fixed point of $\mathcal{T}$. Then we see that

$$\begin{aligned}
&F(\rho_\Delta(\xi_m,\lambda_1),\rho_\Delta(\xi_m,\lambda_2),\cdots,\rho_\Delta(\xi_m,\lambda_n))\\
&\le kF(\rho_\Delta(\xi_{m-1},\lambda_1),\rho_\Delta(\xi_{m-1},\lambda_2),\ldots,\rho_\Delta(\xi_{m-1},\lambda_n))\\
&\le k^2F(\rho_\Delta(\xi_{m-2},\lambda_1),\rho_\Delta(\xi_{m-2},\lambda_2),\ldots,\rho_\Delta(\xi_{m-2},\lambda_n))\\
&\vdots\\
&\le k^mF(\rho_\Delta(\xi_0,\lambda_1),\rho_\Delta(\xi_0,\lambda_2),\ldots,\rho_\Delta(\xi_0,\lambda_n)).
\end{aligned} \tag{2.9}$$

The remaining proof is similar to the proof of Theorem 2.10. □

Corollary 2.16. *Let $(\mathcal{X},\rho_\Delta)$ be a complete Δ-metric space and $\mathcal{T}$ satisfies*

$$\max\{\rho_\Delta(\mathcal{T}\xi,\lambda_1),\ldots,\rho_\Delta(\mathcal{T}\xi,\lambda_n)\}\le k\max\{\rho_\Delta(\xi,\lambda_1),\ldots,\rho_\Delta(\xi,\lambda_n)\} \tag{2.10}$$

for all $\xi\notin Fix(\mathcal{T})\cup\{\lambda_1,\ldots,\lambda_n\}$, for some $k\in(0,1)$, where $\lambda_1,\ldots,\lambda_n$ are all pairwise distinct elements. Also let $\mathcal{T}$ fulfils the following two conditions:

(a) For some $\xi_0\in\mathcal{X}$ for Picard sequence $\xi_m=\mathcal{T}^m\xi_0$ holds

$$\xi_m\in\mathcal{X}\setminus\{\lambda_1,\ldots,\lambda_n\},$$

for every $m=1,2,\ldots$;

(b) $\mathcal{T}$ is orbitally continuous or, p-continuous for some $p\ge 1$ in $\mathcal{X}$.

Then $\mathcal{T}$ has at least one fixed point in $\mathcal{X}$.

3. Mapping contracting axes of an n-ellipse on $\mathcal{F}$-metric space

In this section we prove some fixed point theorems for mappings contracting axes of n-ellipse in an $\mathcal{F}$-metric space, which was introduced by Jleli and Samet [7] in the year 2018.

Definition 3.1 ($\mathcal{F}$-metric space, [7]). Let $\mathcal{X}$ be a non-empty set. A mapping $\sigma_F:\mathcal{X}\times\mathcal{X}\to[0,+\infty)$ is said to be an $\mathcal{F}$-metric on $\mathcal{X}$ if for all $\xi,\eta\in\mathcal{X}$, σ_F satisfies:

1. $\sigma_F(\xi,\eta)=0$ if and only if $\xi=\eta$;
2. $\sigma_F(\xi,\eta)=\sigma_F(\eta,\xi)$;
3. For every $m\in\mathbb{N}$ with $m\ge 2$, and for every $u_1,\ldots,u_m\in\mathcal{X}$ with $(u_1,u_m)=(\xi,\eta)$, we have

$$\Omega(\sigma_F(\xi,\eta))\le\Omega\left(\sum_{i=1}^{m-1}\sigma_F(u_i,u_{i+1})\right)+c, \tag{3.1}$$

for $\xi \neq \eta$, where $\Omega : (0, +\infty) \to (-\infty, +\infty)$ is an increasing function, $\Omega(t_n) \to -\infty$ for all sequence $\{t_n\}$ with $t_n \to 0$ and $c \in [0, +\infty)$.
The pair $(\mathcal{X}, \sigma_F)$ is called an $\mathcal{F}$-metric space.

Definition 3.2 ([7])**.** Let $(\mathcal{X}, \sigma_F)$ be an $\mathcal{F}$-metric space. Also, let $\{\xi_n\}$ be a sequence in $\mathcal{X}$ and $\zeta \in \mathcal{X}$.
(i) The sequence $\{\xi_n\}$ is $\mathcal{F}$-convergent and converges to ζ if $\lim_{n\to+\infty} \sigma_F(\xi_n, \zeta) = 0$.
(ii) The sequence $\{\xi_n\}$ is $\mathcal{F}$-Cauchy if $\lim_{n,m\to+\infty} \sigma_F(\xi_n, \xi_m) = 0$.
(iii) The $\mathcal{F}$-metric space $(\mathcal{X}, \sigma_F)$ is $\mathcal{F}$-complete if any $\mathcal{F}$-Cauchy sequence in $\mathcal{X}$ is $\mathcal{F}$-convergent.

Example 3.3 ([7])**.** Let $\mathcal{X} = \mathbb{N}$ and $\sigma_F : \mathcal{X}^2 \to \mathbb{R}^+ \cup \{0\}$ be defined by

$$\sigma_F(\xi, \eta) = \begin{cases} e^{|\xi-\eta|}, & \text{if } \xi \neq \eta; \\ 0, & \text{if } \xi = \eta. \end{cases} \tag{3.2}$$

Then $(\mathcal{X}, \sigma_F)$ is an $\mathcal{F}$-complete $\mathcal{F}$-metric space.

Remark 3.4 ([7]). (i) Though any metric space is an $\mathcal{F}$-metric space but there are several examples of b-metric which are not $\mathcal{F}$-metric. The concepts of b-metrics and $\mathcal{F}$-metric are independent of each others.

(ii) An $\mathcal{F}$-metric space is always Hausdorff, that is, if in a $\mathcal{F}$-metric space $\mathcal{X}$ a sequence $\{\xi_n\}$ satisfies $\lim_{n\to+\infty} \sigma_F(\xi_n, \xi) = 0$ and $\lim_{n\to+\infty} \sigma_F(\xi_n, \eta) = 0$, then $\xi = \eta$;

(iii) In an $\mathcal{F}$-metric space an $\mathcal{F}$-convergent sequence is also $\mathcal{F}$-Cauchy.

Remark 3.5. The definition of orbital continuity and p-continuity, $p \in \mathbb{N}$, for a mapping $\mathcal{T}$ in an $\mathcal{F}$-metric space is the same as for a Δ-metric space, (see Definition 2.8 and Definition 2.9).

Definition 3.6. A mapping $\mathcal{T}$ on an $\mathcal{F}$-metric space $(\mathcal{X}, \sigma_F)$ which satisfies

$$\sigma_F(\mathcal{T}\xi, \lambda_1) + \cdots + \sigma_F(\mathcal{T}\xi, \lambda_n) \leq k[\sigma_F(\xi, \lambda_1) + \cdots + \sigma_F(\xi, \lambda_n)] \tag{3.3}$$

for all $\xi \notin Fix(\mathcal{T}) \cup \mathcal{X} \setminus \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, for some $k \in (0, 1)$, where λ_1,λ_2,. . .,λ_n are all pairwise distinct elements, is called a mapping contracting axes of n-ellipse ($n \geq 2$).

Theorem 3.7. *Let $\mathcal{X}$ be an $\mathcal{F}$-complete $\mathcal{F}$-metric space and $\mathcal{T}$ be a mapping contracting axes of n-ellipse for some Lipschitz constant $k \in (0, 1)$. Also, let $\mathcal{T}$ satisfies the following two conditions:*

(*a*) *For some $\xi_0 \in \mathcal{X}$ the Picard iterating sequence $\{\xi_m\}$, $\xi_m = \mathcal{T}^m \xi_0$ is such that $\xi_m \notin \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ for every $m \in \mathbb{N}$;*

(*b*) *$\mathcal{T}$ is orbitally continuous or, p-continuous for some $p \geq 1$ in $\mathcal{X}$.*

Then $\mathcal{T}$ has at least one fixed point in $\mathcal{X}$.

Proof. Without loss of generality, we assume that $\{\xi_m\}$ does not contain any fixed point of $\mathcal{T}$. Then for any $m \geq 1$ we have

$$\begin{aligned}
&\sigma_F(\xi_m, \lambda_1) + \sigma_F(\xi_m, \lambda_2) + \cdots + \sigma_F(\xi_m, \lambda_n) \\
&\leq k[\sigma_F(\xi_{m-1}, \lambda_1) + \sigma_F(\xi_{m-1}, \lambda_2) + \cdots + \sigma_F(\xi_{m-1}, \lambda_n)] \\
&\leq k^2[\sigma_F(\xi_{m-2}, \lambda_1) + \sigma_F(\xi_{m-2}, \lambda_2) + \cdots + \sigma_F(\xi_{m-2}, \lambda_n)] \\
&\vdots \\
&\leq k^m[\sigma_F(\xi_0, \lambda_1) + \sigma_F(\xi_0, \lambda_2) + \cdots + \sigma_F(\xi_0, \lambda_n)].
\end{aligned}$$

For all $m, t \in \mathbb{N}$ we have

$$\begin{aligned}
\Omega(\sigma_F(\xi_m, x_{m+t})) &\leq \Omega(\sigma_F(\xi_m, \lambda_n) + \sigma_F(\xi_{m+t}, \lambda_n)) + c \\
&\leq \Omega(\{\sigma_F(\xi_m, \lambda_1) + \sigma_F(\xi_m, \lambda_2) + \cdots + \sigma_F(\xi_m, \lambda_n)\} + \\
&\{\sigma_F(\xi_{m+t}, \lambda_1) + \sigma_F(\xi_{m+t}, \lambda_2) + \cdots + \sigma_F(\xi_{m+t}, \lambda_n)\}) + c \\
&\leq \Omega((k^m + k^{m+t})\{\sigma_F(\xi_0, \lambda_1) + \sigma_F(\xi_0, \lambda_2) + \cdots + \\
&\sigma_F(\xi_0, \lambda_n)\}) + c \\
&\leq \Omega(k^m(1 + k^t)\{\sigma_F(\xi_0, \lambda_1) + \sigma_F(\xi_0, \lambda_2) + \cdots + \\
&\sigma_F(\xi_0, \lambda_n)\}) + c.
\end{aligned}$$

Since $0 < k < 1$, for any $t = 1, 2, \ldots,$

$$\lim_{m \to +\infty} \Omega(\sigma_F(\xi_m, x_{m+t})) = -\infty \Rightarrow \lim_{m \to +\infty} \sigma_F(\xi_m, x_{m+t}) = 0.$$

Which shows that $\{\xi_m\}$ is an $\mathcal{F}$-Cauchy sequence. Here $\mathcal{X}$ is $\mathcal{F}$-complete and thus we get some $\eta \in \mathcal{X}$ such that $\sigma_F(\xi_m, \eta) \to 0$ as $m \to +\infty$. Now, using condition (b) it is clear from the Definition 2.8. and Definition 2.9. that $\sigma_F(\xi_{m+1}, \mathcal{T}\eta) \to 0$ as $m \to +\infty$. As $(\mathcal{X}, \sigma_F)$ is Hausdorff, it implies that $\mathcal{T}\eta = \eta$ and η is a fixed point of $\mathcal{T}$. □

Example 3.8. Consider the $\mathcal{F}$-complete $\mathcal{F}$-metric space given in Example 3.3. Let $\mathcal{P}$ be the set of all primes and $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be defined by

$$\mathcal{T}(\xi) = \begin{cases} 9, & \text{if } \xi \in \mathcal{P}; \\ \xi, & \text{otherwise.} \end{cases}$$

Then for any $\xi \in \mathcal{P}$,

$$\sigma_F(\mathcal{T}\xi, 8) + \sigma_F(\mathcal{T}\xi, 10) \leq \frac{2}{e^2 + 1}(\sigma_F(\xi, 8) + \sigma_F(\xi, 10)).$$

Therefore $\mathcal{T}$ is a mapping contracting axes of 2-ellipse, with the node points (foci) $8, 10$. Moreover $\mathcal{T}$ satisfies all the conditions of Theorem 3.7 and we see that $\mathcal{T}$ has infinitely many fixed points in $\mathcal{X}$.

Corollary 3.9. *Let $\mathcal{X}$ be a complete metric space and $\mathcal{T}$ be a mapping contracting axes of n-ellipse for some Lipschitz constant $k \in (0,1)$. Also, let $\mathcal{T}$ satisfies the following two conditions:*
(a) For some $\xi_0 \in \mathcal{X}$ the Picard iterating sequence $\{\xi_m\}$, $\xi_m = \mathcal{T}^m\xi_0$ is such that $\xi_m \in \mathcal{X} \setminus \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ for every $m \in \mathbb{N}$;
(b) $\mathcal{T}$ is orbitally continuous or, p-continuous for some $p \geq 1$ in $\mathcal{X}$.
Then $\mathcal{T}$ has at least one fixed point in $\mathcal{X}$.

Definition 3.10. A mapping $\mathcal{T}$ on an $\mathcal{F}$-metric space $(\mathcal{X}, \sigma_F)$ is said to be a generalized mapping contracting axes of n-ellipse ($n \geq 2$) if it satisfies

$$F(\sigma_F(\mathcal{T}\xi, \lambda_1), \ldots, \sigma_F(\mathcal{T}\xi, \lambda_n)) \leq kF(\sigma_F(\xi, \lambda_1), \ldots, \sigma_F(\xi, \lambda_n)) \tag{3.4}$$

for all $\xi \notin Fix(\mathcal{T}) \cup \mathcal{X} \setminus \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, for some $k \in (0,1)$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are all pairwise distinct elements and $F \in \Phi$.

Theorem 3.11. *Let $(\mathcal{X}, \sigma_F)$ be an $\mathcal{F}$-complete $\mathcal{F}$-metric space and $\mathcal{T}$ be a generalized mapping contracting axes of n-ellipse for some Lipschitz constant $k \in (0,1)$. Also let $\mathcal{T}$ satisfies the following two conditions:*

(a) For some $\xi_0 \in \mathcal{X}$ the Picard iterating sequence $\{\xi_m\}$, $\xi_m = \mathcal{T}^m\xi_0$ is such that $\xi_m \notin \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ for every $m \in \mathbb{N}$;

(b) $\mathcal{T}$ is orbitally continuous or p-continuous for some $p \geq 1$ in $\mathcal{X}$.
Then $\mathcal{T}$ has at least one fixed point in $\mathcal{X}$.

Proof. Assume that $\xi_m \neq \xi_{m+1}$ for any $m \geq 1$. Then it is seen that

$$\begin{aligned}
&F(\sigma_F(\xi_m, \lambda_1), \sigma_F(\xi_m, \lambda_2), \ldots, \sigma_F(\xi_m, \lambda_n)) \\
&\leq kF(\sigma_F(\xi_{m-1}, \lambda_1), \sigma_F(\xi_{m-1}, \lambda_2), \ldots, \sigma_F(\xi_{m-1}, \lambda_n)) \\
&\leq k^2F(\sigma_F(\xi_{m-2}, \lambda_1), \sigma_F(\xi_{m-2}, \lambda_2), \ldots, \sigma_F(\xi_{m-2}, \lambda_n)) \\
&\vdots \\
&\leq k^mF(\sigma_F(\xi_0, \lambda_1), \sigma_F(\xi_0, \lambda_2), \ldots, \sigma_F(\xi_0, \lambda_n)).
\end{aligned} \tag{3.5}$$

The remaining part of the proof is same as Theorem 3.7. □

4. Enriched mapping contracting axes of n-ellipse

Recently, in 2020, V. Berinde and M. Păcurar [3] introduced the notion of enriched contraction mappings over a normed linear space, which is a larger class than the several known contraction and contractive-type mappings; and using the Krasnoselskij iterative algorithm, it has been shown that enriched contraction mapping possesses a unique fixed point in a Banach space.

Enriched contraction is essentially defined as follows:

Definition 4.1 ([3])**.** In a normed linear space $\mathcal{X}$, a self mapping $\mathcal{T}$ is said to be an enriched contraction if there exist $b \geq 0$ and $\alpha \in [0, b+1)$ such that for all $\xi, \eta \in \mathcal{X}$,

$$\|b(\xi - \eta) + \mathcal{T}\xi - \mathcal{T}\eta\| \leq \alpha\|\xi - \eta\|. \tag{4.1}$$

After that, several researchers have been generalized and extended various well known fixed-point results by considering enriched contraction mapping; for this we may refer [4, 5, 12].

Motivating form the above discussed literatures, and the way of research, in this article we introduce enriched mapping contracting axes of n-ellipse.

Definition 4.2. For n pairwise distinct points $\lambda_1, \lambda_2, \cdots, \lambda_n$, a self mapping $\mathcal{T}$ in a normed linear space $\mathcal{X}$ is said to be an enriched mapping contracting axes of n-ellipse ($n \geq 2$) if there exist $b \geq 0$ and $\alpha \in [0, b+1)$ such that for all $\xi \in \mathcal{X} \setminus (Fix(\mathcal{T}) \cup \{\lambda_1, \lambda_2, \ldots, \lambda_n\})$,

$$\begin{aligned}
&\|b(\xi - \lambda_1) + (\mathcal{T}\xi - \lambda_1)\| + \cdots + \|b(\xi - \lambda_n) + (\mathcal{T}\xi - \lambda_n)\| \\
&\leq \alpha(\|\xi - \lambda_1\| + \cdots + \|\xi - \lambda_n\|).
\end{aligned} \tag{4.2}$$

Theorem 4.3. *Let $(\mathcal{X}, \|\|)$ be a Banach space and $\mathcal{T}$ be an enriched mapping contracting axes of n-ellipse. Also let $\mathcal{T}$ satisfies the following two conditions:*
(a) For some $\xi_0 \in \mathcal{X}$ the iterating sequence $\{\xi_m\}$, $\xi_m = \frac{b\xi_{m-1} + \mathcal{T}\xi_{m-1}}{b+1}$ is such that $\xi_m \in \mathcal{X} \setminus \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ for every $m = 1, 2, \ldots$;
(b) The mapping $\frac{bI + \mathcal{T}}{b+1}(\xi)$, $\xi \in \mathcal{X}$, where I is the identity mapping on $\mathcal{X}$, is orbitally continuous or p-continuous for some $p \geq 1$ in $\mathcal{X}$.
Then $\mathcal{T}$ has at least one fixed point in $\mathcal{X}$.

Proof. Consider the averaged mapping

$$\mathcal{T}^*\xi = (1 - \mu)\xi + \mu\mathcal{T}\xi,$$

for all $\xi \in \mathcal{X}$, where $\mu = \frac{1}{b+1}$. Then it is clear that $Fix(\mathcal{T}) = Fix(\mathcal{T}^*)$. Now $\mathcal{T}$ satisfies the contractive condition (4.2), therefore for all

$$\xi \in \mathcal{X} \setminus (Fix(\mathcal{T}^*) \cup \{\lambda_1, \lambda_2, \ldots, \lambda_n\}),$$

we get

$$\begin{aligned}&\|b(\xi-\lambda_1)+(\mathcal{T}\xi-\lambda_1)\|+\cdots+\|b(\xi-\lambda_n)+(\mathcal{T}\xi-\lambda_n)\|\\&\leq\alpha(\|\xi-\lambda_1\|+\cdots+\|\xi-\lambda_n\|),\end{aligned}$$

this implies

$$\begin{aligned}&\|(b\xi+\mathcal{T}\xi)-(b+1)\lambda_1\|+\cdots+\|(b\xi+\mathcal{T}\xi)-(b+1)\lambda_n\|\\&\leq\alpha(\|\xi-\lambda_1\|+\cdots+\|\xi-\lambda_n\|),\end{aligned}$$

so,

$$\left\|\frac{b\xi+\mathcal{T}\xi}{b+1}-\lambda_1\right\|+\cdots+\left\|\frac{b\xi+\mathcal{T}\xi}{b+1}-\lambda_n\right\|\leq\alpha(\|\xi-\lambda_1\|+\cdots+\|\xi-\lambda_n\|),$$

now, we have

$$\|\mathcal{T}^*\xi-\lambda_1\|+\cdots+\|\mathcal{T}^*\xi-\lambda_n\|\leq\alpha(\|\xi-\lambda_1\|+\cdots+\|\xi-\lambda_n\|) \tag{4.3}$$

Thus $\mathcal{T}^*$ is a mapping contracting axes of n-ellipse on the Banach space $(\mathcal{X},\|\|)$ and the Krasnoselskij iterating sequence $\{\xi_m\}$, defined by

$$\xi_m=(1-\mu)\xi_{m-1}+\mu\mathcal{T}\xi_{m-1}, m\in\mathbb{N},$$

turns out to be the Picard iterating sequence of $\mathcal{T}^*$. Moreover, conditions (a) and (b) imply
$(a)'$ For some $\xi_0\in\mathcal{X}$ the iterative sequence $\{\xi_m\}$, $\xi_m=\mathcal{T}^*\xi_{m-1}$ is such that $\xi_m\in\mathcal{X}\setminus\{\lambda_1,\lambda_2,\cdots,\lambda_n\}$ for every $m=1,2,\ldots$;
$(b)'$ The mapping $\mathcal{T}^*$ is orbitally continuous or, p-continuous for some $p\geq 1$ in $\mathcal{X}$.
Just by approaching in the same way as that in Theorem 2.10 and Theorem 3.7 (see Corollary 2.12 and Corollary 3.9) we can establish that $Fix(\mathcal{T}^*)\neq\emptyset$. Hence it can be concluded that $Fix(\mathcal{T})\neq\emptyset$. □

Example 4.4. Let us consider the Banach space $\mathcal{X}=\mathbb{R}$ with the usual norm. Choose the mapping $\mathcal{T}:\mathcal{X}\to\mathcal{X}$ which is defined by

$$\mathcal{T}(\xi)=\begin{cases}1-\xi, & \text{if } \xi\in[\frac{17}{18},1);\\ \xi, & \text{otherwise.}\end{cases} \tag{4.4}$$

Then $\mathcal{T}$ satisfies the contractive condition (4.2) for the node points (foci) $\lambda_1=0$, $\lambda_2=\frac{1}{2}$ and $\lambda_3=1$, for $\alpha=\frac{3}{2}$ and $b=1$, that is, $\mathcal{T}$ is an enriched mapping contracting axes of 3-ellipse and also all the conditions of Theorem 4.3 are satisfied. We see that $\mathcal{T}$ has infinitely many fixed points in $\mathcal{X}$.

The continuity of contractive definitions at fixed points in metric spaces was investigated by Rhoades [16] first, who showed that the usual contractive type mappings are generally continuous at their fixed points. Rhoades [16] also posed an open question regarding the existence of a contractive condition that permits discontinuity of a mapping at its fixed point. Pant [13] was the first researcher to provide an affirmative solution to this interesting problem within the framework of metric spaces in the year 1999.

Remark 4.5. Our proposed mapping gives a new solution to Rhoades open problem. The mapping $\mathcal{T}$ in Example 4.4 is discontinuous at $\xi = 1$, which is a fixed point of $\mathcal{T}$.

5. Conclusion and motivation for future work

This paper deals with a new contractive type mapping known as mapping contracting axes of n-ellipse. This mapping is totally different from usual contractive mappings. This particular mapping may possesses more than one fixed point, it may also be discontinuous at its fixed points. An n-ellipse is an extended version of usual ellipse, whose number of foci is n. A recent work of K. Roy [17] motivates us to introduce this mapping, which generalizes mapping contracting axes of ellipse. Though the area or perimeter of an n-ellipse in two dimensional Euclidean space is not known clearly except for $n = 2$, it is our believe that this particular mapping definitely can shrink the perimeter or area of an n-ellipse. So many generalizations have been surely possible for this mapping and also this type of mappings can be applied to different areas of science and engineering. Since for $n = 2$, a mapping contracting axes of n-ellipse is actually a mapping contracting axes of ellipse and over $\mathbb{R}^2$ it can shrink area of a given ellipse and over $\mathbb{R}^3$ it can reduce volume of a given ellipsoid, therefore it is our believe that these type of mappings can certainly be used in image processing. This was our actual motivation.

Acknowledgements: The authors are thankful to the eminent reviewers for their important suggestions for the improvement of this paper.

Funding: This research has not received external funding.

Author Contributions: Conceptualization, methodology and software K.R., Z.D.M. and V.P.; investigation K.R., Z.D.M. and V.P.; writing—original draft preparation K.R., Z.D.M. and V.P.; writing—review and editing K.R., Z.D.M. and V.P.; supervision K.R. All authors have read and agreed to the published version of the manuscript.

References

[1] I. A. Bakhtin, Contracting mapping principle in an almost metric space, (Russian), Funkts. Anal. 30 (1989), 26–37.

[2] S. Banach, Sur les operations dans les ensembles abstraits et leur application aux equations untegrales, Fund. Math. 3 (1922), 133–181.

[3] V. Berinde and M. Păcurar, Approximating fixed points of enriched contractions in Banach spaces, J. Fixed Point Theory Appl. 22 (2020), 38.

[4] V. Berinde and M. Păcurar, Kannan's fixed point approximation for solving split feasibility and variational inequality problems, J. Comput. Appl. Math. 386 (2021), 113217.

[5] V. Berinde and M. Păcurar, Approximating fixed points of enriched Chatterjea contractions by Krasnoselskij iterative algorithm in Banach spaces, J. Fixed Point Theory Appl. 23 (2021), 66.

[6] S. Czerwik, Contraction mappings in $b-$metric spaces, Acta Math. Inform. Univ. Ostrav. 1 (1993), 5–11.

[7] M. Jleli and B. Samet, On a new generalization of metric spaces, J. Fixed Point Theory Appl. 20, no. 3 (2018), 128.

[8] M. Joshi, A. Tomar, and S. K. Padaliya, Fixed point to fixed ellipse in metric spaces and discontinuous activation function, Appl. Math. E-Notes 21 (2021), 225–237.

[9] T. Kamran, M. Samreen, and Q. U. Ain, A generalization of $b-$metric space and some fixed point theorems, Mathematics 5 (2017), 19.

[10] N. Y. Özgür and N. Taş, Some fixed-circle theorems on metric spaces, Bull. Malays. Math. Sci. Soc. 42 (2019), 1433–1449.

[11] N. Y. Özgür and N. Taş, Some fixed-circle theorems and discontinuity at fixed circle, AIP Conf. Proc. 1926 (2018), 020048.

[12] S. Panja, K. Roy, and M. Saha, Wardowski type enriched contractive mappings with their fixed points, J. Anal. 32 (2024), 169–281.

[13] R. P. Pant, Discontinuity and fixed points, J. Math. Anal. Appl. 240 (1999), 284–289.

[14] V. Parvaneh and S. J. H. Ghoncheh, Fixed points of $(\psi, \phi)_{\Omega}$-contractive mappings in ordered p-metric spaces, Glob. Anal. Discret. Math. 4, no. 1 (2020), 15–29.

[15] E. Petrov, Fixed point theorem for mappings contracting perimeters of triangles, arXiv:2308.01003v1.

[16] B. E. Rhoades, Contractive definitions and continuity, Contemp. Math. 72 (1988), 233–245.

[17] K. Roy, Mappings contracting axes of ellipse, J. Anal. 32 (2024), 3557–3563.

[18] K. Roy, D. Dey, and M. Saha, Certain fixed point results on $\mathfrak{U}$-metric space using Banach orbital contraction and asymptotic regularity, Math. Slovaca 73, no. 2 (2023), 485–500.

[19] K. Roy and M. Saha, Fixed point theorems for a class of extended JS-contraction mappings over a generalized metric space with an application to fixed circle problem, Proyecciones 41, no. 6 (2022), 1551–1572.

[20] K. Roy, M. Saha, V. Parvaneh, and M. Khorshidi, A new variant of symmetric distance spaces and an extension of the Banach fixed point theorem, J. Math. 2022 (2022), 282762.

[21] N. Taş, H. Aytïmur, and Ş. Güvenç, New fixed figure results with the notion of k-ellipse, arXiv:2112.10204v1.