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Strong convergence of an inertial iterative method for generalized nonexpansive mappings in Banach spaces

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ABSTRACT

We introduce an iterative technique with an inertial term that converges strongly to a fixed point of mappings satisfying Condition (E). Our results extend existing work by providing a robust numerical method for solving fixed point problems in Banach spaces. To demonstrate the effectiveness of our approach, we present numerical examples of a mapping that is not nonexpansive but satisfies Condition (E). Furthermore, we illustrate the convergence behavior of our algorithm for different choices of initial guesses and coefficients, using MATLAB to validate the theoretical results. This work contributes to the broader framework of fixed point theory and offers practical insights for solving nonlinear problems in applied mathematics.

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KEYWORDS: *condition (E); uniformly convex Banach space; strong convergence.*

1. INTRODUCTION

Fixed point theory is a fundamental area of nonlinear analysis, offering powerful tools to address a wide range of problems in mathematics and its applications. It plays a pivotal role in optimization, variational inequalities, differential equations, and many other fields. In this study, we explore the fixed point

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problem within the framework of Banach spaces, a setting that provides a rich structure for investigating operator properties and convergence behaviors. Let $\mathcal{E}$ be a nonempty, closed, and convex subset of a real Banach space Σ, equipped with its dual space Σ^*. Central to our analysis is the normalized duality mapping $J : \Sigma \to 2^{\Sigma^*}$, defined such that for any $v \in \Sigma$, it satisfies

$$\langle v, \chi \rangle = \|v\|^2 = \|\chi\|^2,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing [8]. This mapping becomes single-valued when Σ^* is strictly convex, a property we leverage throughout this work.

Banach spaces exhibit diverse geometric characteristics, such as uniform convexity and smoothness, which significantly influence the behavior of iterative methods. A Banach space Σ is uniformly convex if sequences on the unit sphere satisfying a specific limit condition collapse to the same point [1, 24]. Similarly, uniform smoothness is characterized via the modulus of smoothness $\rho_\Sigma(\tau)$, with Σ being uniformly smooth if $\rho_\Sigma(\tau)/\tau \to 0$ as $\tau \to 0$. A refined notion, q-uniform smoothness, imposes a bound $\rho_\Sigma(\tau) \leq K_q \tau^q$ for some $q > 1$ and constant $K_q > 0$, implying uniform smoothness. These properties underpin our investigation of mappings $\Phi : \mathcal{E} \to \mathcal{E}$, particularly those that are L-Lipschitzian, where

$$\|\Phi(v) - \Phi(\vartheta)\| \leq L\|v - \vartheta\| \quad \text{for all } v, \vartheta \in \mathcal{E}.$$

When $L = 1$, the mapping is nonexpansive, and various methods have been explored to numerically solve the fixed point equation $v = \Phi(v)$, assuming that the fixed point set $\mathcal{F}(\Phi) = \{v \in \mathcal{E} \mid v = \Phi(v)\}$ is nonempty.

The simplest approach to finding fixed points, known as the Banach-Picard iteration, involves the recursive formula $v_{m+1} = \Phi(v_m)$. For contraction mappings ($L < 1$), this method guarantees strong convergence to a unique fixed point, as established by the Banach-Picard theorem. However, when Φ is merely nonexpansive, convergence is not assured, as demonstrated by the counterexample $\Phi = -\mathrm{Id}$ with a nonzero starting point. To address this limitation, Krasnoselskii introduced an averaged iteration, applying the Banach-Picard scheme to the operator $(1/2)\mathrm{Id} + (1/2)\Phi$ [14]. This evolved into the Krasnoselskii-Mann iteration, given by

$$v_{m+1} = (1 - \eta_m)v_m + \eta_m \Phi(v_m),$$

where $\{\eta_m\} \subset (0, 1)$ is a control sequence. Under mild conditions on $\{\eta_m\}$ and the existence of fixed points, this iteration yields weak convergence to a point in $\mathcal{F}(\Phi)$ [13, 7]. Despite its success, weak convergence remains a limitation, especially in Hilbert spaces, prompting further refinements [12].

A key step in establishing the convergence of the iterates in (1.3) involves demonstrating that $v_m - \Phi(v_m) \to 0$ as $m \to +\infty$, as shown by Browder and Petryshyn in [18] for the constant case where

$\eta_m \equiv \eta \in (0,1)$. Subsequently, the weak convergence of these iterates was investigated across various contexts in [18, 17].

It is worth noting that, even in real Hilbert spaces, all prior adaptations of the Krasnoselskii-Mann method for nonexpansive mappings yield only weak convergence; additional details can be found in [19].

Recently, Bot et al. [5] introduced a novel formulation of Mann's method to tackle these challenges. Given an arbitrary v_0 in a real Hilbert space $\mathcal{H}$, for all $m \geq 0$, the iteration is defined as:

$$v_{m+1} = \eta_m v_m + \zeta_m \left(\Phi(\eta_m v_m) - \eta_m v_m\right).$$

They demonstrated that the sequence of iterates in (1.4) exhibits strong convergence.

Recent advancements, such as those by [5] and Baewnoi et al. [4], have sought to achieve strong convergence. Baewnoi et al. demonstrated that an inertial-enhanced algorithm in uniformly convex Banach spaces with uniformly Gâteaux differentiable norms ensures strong convergence to fixed points of nonexpansive mappings. Building on this, Suzuki [23] introduced a broader class of mappings satisfying condition (C), later generalized by Garcia-Falset as condition (E), offering new avenues for analysis.

A mapping $\Phi : \mathcal{E} \to \mathcal{E}$ is said to satisfy condition (C) (also known as Suzuki-generalized nonexpansiveness) if for all $v, \vartheta \in \mathcal{E}$,

$$\frac{1}{2}\|v - \Phi(v)\| \leq \|v - \vartheta\| \implies \|\Phi(v) - \Phi(\vartheta)\| \leq \|v - \vartheta\|.$$

This condition generalizes the concept of nonexpansiveness and has been widely studied due to its applicability in fixed point theory [23]. García-Falset [11] considered the class of mapping satisfying condition (E). A mapping $\Phi : \mathcal{E} \to \mathcal{E}$ satisfies condition (E) if there exists $\mu \geq 1$ such that for all $v, \vartheta \in \mathcal{E}$,

$$\|v - \Phi(\vartheta)\| \leq \mu\|v - \Phi(v)\| + \|v - \vartheta\|.$$

This condition extends the class of mappings that can be analyzed using fixed point methods, providing a broader framework for solving nonlinear problems in Banach spaces. For further insights, we refer the reader to some interesting results derived from condition (E) and the related literature cited therein [22, 21]. Recently, Paimsang and Thianwan [15] introduced modified Noor iterative scheme and proved strong and weak convergence results for the mappings satisfying condition (E).

In this paper, we extend the results of Baewnoi et al. [4] to the more general class of Suzuki-generalized nonexpansive mappings and those satisfying condition (E). We propose a numerical method for solving the fixed point problem and establish a strong convergence theorem under suitable conditions.

Additionally, we provide an illustrative example and use MATLAB to analyze the convergence behavior, contributing to the ongoing development of robust iterative techniques in Banach spaces.

2. Preliminaries

Throughout this paper, the space Σ is a real uniformly convex Banach space (RUCBS, in short), for the mapping $\Phi : \mathcal{E} \to \mathcal{E}$ we denote the set of fixed points of the mapping Φ by $F(\Phi)$.

Definition 2.1. A mapping $\Phi : \Sigma \to \Sigma$ is said to be

(a) monotone, if
$$\langle \Phi(\zeta) - \Phi(\eta), \zeta - \eta \rangle \geq 0, \forall \zeta, \eta \in \Sigma;$$

(b) pseudomonotone, if
$$\langle \Phi(\zeta), \eta - \zeta \rangle \geq 0 \implies \langle \Phi(\eta), \eta - \zeta \rangle \geq 0, \forall \zeta, \eta \in \Sigma;$$

(c) contraction, if $\exists$ a constant $k \in (0, 1)$ such that
$$\|\Phi(\zeta) - \Phi(\eta)\| \leq k\|\zeta - \eta\|, \forall \zeta, \eta \in \Sigma;$$

(d) L-Lipschitz continuous, if
$$\|\Phi(\zeta) - \Phi(\eta)\| \leq L\|\zeta - \eta\|, \forall \zeta, \eta \in \Sigma;$$

(e) nonexpansive mapping if
$$\|\Phi(\zeta) - \Phi(\eta)\| \leq \|\zeta - \eta\|, \forall \zeta, \eta \in \Sigma;$$

(f) mapping satisfying Condition (C) if
$$\frac{1}{2}\|\zeta - \Phi(\zeta)\| \leq \|\zeta - \eta\| \implies \|\Phi(\zeta) - \Phi(\eta)\| \leq \|\zeta - \eta\|, \forall \zeta, \eta \in \Sigma;$$

Lemma 2.2 ([25])**.** *Suppose that Σ is a RUCBS. For any $\nu > 0$, $\Xi_\nu(0) = \{\mu \in \Sigma : \|\mu\| \leq \nu\}$ and $\beta \in [0, 1]$ then there exists a continuous, convex and strictly increasing function $r : [0, 2\nu] \to \mathbb{R}, r(0) = 0$ such that*
$$\|\beta\mu + (1 - \beta)\vartheta\|^2 \leq \beta\|\mu\|^2 + (1 - \beta)\|\vartheta\|^2 - \beta(1 - \beta)r(\|\mu - \vartheta\|).$$

Lemma 2.3 ([9])**.** *Suppose Σ is a real normed linear space, then for any $\mu, \vartheta \in \Sigma$, $j(\mu + \vartheta) \in J(\mu + \vartheta)$, then the following holds*
$$\|\mu + \vartheta\|^2 \leq \|\mu\|^2 + 2\langle \vartheta, j(\mu + \vartheta) \rangle.$$

Definition 2.4. Suppose that J is a normalized duality mapping on a real Banach space $\mathcal{X}$, $J : \mathcal{X} \to 2^{\mathcal{X}^*}$ defined as $J(x) = \{h \in \mathcal{X}^* : \langle x, h\rangle = \|x\|\|h\|, \|h\| = \|x\|\}$, for all $x \in \mathcal{X}$, where $\mathcal{X}^*$ is dual space of $\mathcal{X}$ and the pair $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing. If the dual space $\mathcal{X}^*$ is strictly convex, then the normalized duality J is single valued, and the single valued duality mapping is denoted by j.

Definition 2.5 ([2]). Let Σ be a Banach space and $S_\Sigma = \{\zeta \in \Sigma : \|\zeta\| = 1\}$ the unit sphere of Σ. Then

(1) the norm of Σ is said to be Gâteaux differentiable at point $\zeta \in S_\Sigma$ if for $\eta \in S_\Sigma$

$$\lim_{t\to 0} \frac{\|\zeta + t\eta\| - \|\zeta\|}{t}, \tag{2.1}$$

exists. The norm of Σ is said to be Gâteaux differentiable if it is Gâteaux differentiable at each point of S_Σ.

(2) the Banach space Σ is said to be smooth if the limit (2.1) exists for all $\zeta, \eta \in S_\Sigma$.

(3) the norm of the Banach space Σ is said to be Frećhet differentiable if for each $\zeta, \eta \in S_\Sigma$, the limit (2.1) exists uniformly.

Lemma 2.6 ([6]). *Suppose Σ is a uniformly convex Banach space and $\mathcal{E} \neq \emptyset$ is a closed, and convex subset of Σ. Suppose that $\Phi : \mathcal{E} \to \mathcal{E}$ is a mapping with $F(\Phi) \neq \emptyset$ and $\{\zeta_n\}$ is a sequence in $\mathcal{E}$ such that $\{\zeta_n\}$ converges weakly to $\zeta^\dagger$ and $\lim_{n\to\infty} \|\zeta_n - \Phi(\zeta_n)\| = d$ then $\lim_{n\to\infty} \|\zeta^\dagger - \Phi(\zeta^\dagger)\| = d$.*

Lemma 2.7 ([26]). *Suppose $(\mu_0, \mu_1, \mu_2, \dots) \in \ell_\infty$, is so that $\Lambda_n\mu_n \leq 0$ for all Banach limits Λ. If $\limsup_{n\to\infty}(\mu_{n+1} - \mu_n) \leq 0$, then $\limsup_{n\to\infty} \mu_n \leq 0$.*

Lemma 2.8 ([20]). *Suppose $\{\tau_n\}$ is a sequence of nonnegative real numbers such that*

$$\tau_{n+1} \leq (1 - \rho_n)\tau_n + \rho_n\sigma_n + \delta_n,\ n \geq 1.$$

If

(1) $\{\rho_n\} \subset [0, 1], \sum \rho_n = \infty, \limsup \sigma_n \leq 0,$

(2) *for any* $n \geq 0, \delta_n \geq 0, \sum \delta_n < \infty,$

then $\lim_{n\to\infty} \tau_n = 0$.

Lemma 2.9. *Let Φ be a mapping on a subset $\mathcal{E}$ of a Banach space Σ. Assume that Φ satisfies the condition (C). Then*

$$\|\chi - \Phi(\vartheta)\| \leqslant 3\|\Phi(\chi) - \chi\| + \|\chi - \vartheta\|$$

holds for all $\chi, \vartheta \in \mathcal{E}$.

Definition 2.10 ([16]). Let $\mathcal{E}$ be a nonempty subset of a Banach space Σ. A mapping $\Phi : \mathcal{E} \to \mathcal{E}$ is called a generalized α-Reich-Suzuki nonexpansive mapping if, for some $\alpha \in [0, 1)$, the following condition holds:

$$\frac{1}{2}\|\chi - \Phi(\chi)\| \leq \|\chi - \vartheta\| \text{ implies } \|\Phi(\chi) - T\Phi(\vartheta)\| \leq \max\{P(\chi, \vartheta), Q(\chi, \vartheta)\}$$

for all $\chi, \vartheta \in \mathcal{E}$, where

$$P(\chi, \vartheta) = \alpha\|\Phi(\chi) - \chi\| + \alpha\|\Phi(\vartheta) - \vartheta\| + (1 - 2\alpha)\|\chi - \vartheta\|$$
$$Q(\chi, \vartheta) = \alpha\|\Phi(\chi) - \vartheta\| + \alpha\|\Phi(\vartheta) - \chi\| + (1 - 2\alpha)\|\chi - \vartheta\|.$$

Definition 2.11 ([3]). Let $\mathcal{E}$ be a nonempty subset of a Banach space Σ. A mapping $T : \mathcal{E} \to \mathcal{E}$ is called a generalized contraction of Suzuki type if there exist $\lambda \in (0, 1)$ and $\alpha_1, \alpha_2, \alpha_3 \in [0, 1]$ where $\alpha_1 + 2\alpha_2 + 2\alpha_3 = 1$ such that for all $x, y \in X$

$$\begin{aligned}\frac{1}{2}\|\chi - \Phi(\chi)\| \leq \|\chi - \vartheta\| \Longrightarrow \|\Phi(\chi) - \Phi(\vartheta)\| \leq& \alpha_1\|\chi - \vartheta\| + \alpha_2[\|\Phi(\chi) - \chi\| + \|\Phi(\vartheta) - \vartheta\|] \\ &+ \alpha_3[\|\Phi(\vartheta) - \chi\| + \|\Phi(\chi) - \vartheta\|].\end{aligned}$$

Proposition 2.12 ([11]). *Let $\mathcal{E}$ be a nonempty subset of a Banach space Σ. If $\Phi : \mathcal{E} \to \mathcal{E}$ is a mapping satisfying condition (E) with $F(\Phi) \neq \emptyset$ then Φ is quasi-nonexpansive.*

3. Main Results

Theorem 3.1. *Suppose $\mathcal{E} \neq \emptyset$ is a convex and closed subset of a RUCBS Σ having uniformly Gâteaux differentiable norm, and $\Phi : \mathcal{E} \to \mathcal{E}$ is a mapping satisfying condition (E) with $F(\Phi) \neq \emptyset$. Assume the following conditions satisfied:*

(a) $\lim_{n\to\infty} \omega_n = 0$, $\lim_{n\to\infty} \rho_n = 0$, $\sum_{n=1}^{\infty} \rho_n = \infty$,
(b) $\upsilon_n \geq 0, \forall n \in \mathbb{N}$ *and* $\sum_{n=1}^{\infty} \upsilon_n < \infty$,
(c) $\omega_n, \rho_n \in (0, 1), \gamma_n \in [a, b] \subset (0, 1)$.

Suppose $\chi_0, \chi_1 \in \mathcal{E}$ then for all $n \geq 1$ the sequence $\{\chi_n\}$ generated by

$$\begin{cases}\vartheta_n = \chi_n + \upsilon_n(\chi_n - \chi_{n-1}),\\ \xi_n = (1 - \omega_n)(1 - \rho_n)\vartheta_n,\\ \chi_{n+1} = (1 - \gamma_n)\xi_n + \gamma_n\Phi(\xi_n),\end{cases} \tag{3.1}$$

converges strongly to a point $\chi^\dagger \in F(\Phi)$.

Proof. Suppose $\chi^\dagger \in F(\Phi)$, using (3.1), we have

$$\begin{aligned}
\|\chi_{n+1} - \chi^\dagger\| &= \left\|(1-\gamma_n)\xi_n + \gamma_n\Phi(\xi_n) - \chi^\dagger\right\| \\
&= \|\gamma_n(\Phi(\xi_n) - \chi^\dagger) + (1-\gamma_n)(\xi_n - \chi^\dagger)\| \\
&\leq \gamma_n\|\Phi(\xi_n) - \chi^\dagger\| + (1-\gamma_n)\|\xi_n - \chi^\dagger\|.
\end{aligned}$$

Using Proposition 2.12, we get

$$\begin{aligned}
\|\chi_{n+1} - \chi^\dagger\| &\leq \gamma_n\|\xi_n - \chi^\dagger\| + (1-\gamma_n)\|\xi_n - \chi^\dagger\| \\
&= \|\xi_n - \chi^\dagger\|. \qquad (3.2)
\end{aligned}$$

Let $(1-\rho_n)\vartheta_n = \varpi_n$, we have

$$\begin{aligned}
\|\xi_n - \chi^\dagger\| &= \|(1-\omega_n)(1-\rho_n)\vartheta_n - \chi^\dagger\| = \|(1-\omega_n)\varpi_n - \chi^\dagger\| \\
&= \|(1-\omega_n)(\varpi_n - \chi^\dagger) - \omega_n\chi^\dagger\| \\
&\leq (1-\omega_n)\|\varpi_n - \chi^\dagger\| + \omega_n\|\chi^\dagger\| \\
&\leq (1-\omega_n)\|(1-\rho_n)\vartheta_n - \chi^\dagger\| + \omega_n\|\chi^\dagger\| \\
&= (1-\omega_n)\|(1-\rho_n)(\vartheta_n - \chi^\dagger) - \rho_n\chi^\dagger\| + \omega_n\|\chi^\dagger\| \\
&\leq (1-\omega_n)[(1-\rho_n)\|\vartheta_n - \chi^\dagger\| + \rho_n\|\chi^\dagger\|] + \omega_n\|\chi^\dagger\| \\
&= (1-\omega_n)(1-\rho_n)\|\vartheta_n - \chi^\dagger\| + (1-\omega_n)\rho_n\|\chi^\dagger\| + \omega_n\|\chi^\dagger\| \\
&\leq (1-\omega_n)(1-\rho_n)\|\vartheta_n - \chi^\dagger\| + (1-\omega_n)\|\chi^\dagger\| + \omega_n\|\chi^\dagger\| \\
&\leq (1-\rho_n)\|\vartheta_n - \chi^\dagger\| + \|\chi^\dagger\| \\
&= (1-\rho_n)\|\chi_n + \upsilon_n(\chi_n - \chi_{n-1}) - \chi^\dagger\| + \|\chi^\dagger\| \\
&= (1-\rho_n)\|(\chi_n - \chi^\dagger) + \upsilon_n(\chi_n - \chi_{n-1})\| + \|\chi^\dagger\| \\
&\leq (1-\rho_n)\|\chi_n - \chi^\dagger\| + (1-\rho_n)\upsilon_n\|\chi_n - \chi_{n-1}\| + \|\chi^\dagger\| \\
&\leq \max\{\|\chi_n - \chi^\dagger\|, \|\chi_n - \chi_{n-1}\|, \|\chi^\dagger\|\}.
\end{aligned}$$

From here we can say that the sequence $\{\chi_n\}$ is bounded and hence $\{\vartheta_n\}$, $\{\xi_n\}$ are also bounded.

Since $(1-\rho_n)\vartheta_n = \varpi_n$ then $\xi_n = (1-\omega_n)\varpi_n$, using Lemma 2.2 and Lemma 2.3, we get

$$\begin{aligned}
\|\chi_{n+1}-\chi^\dagger\|^2 &= \|(1-\gamma_n)\xi_n + \gamma_n\Phi(\xi_n) - \chi^\dagger\|^2 \\
&\le (1-\gamma_n)\|\xi_n-\chi^\dagger\|^2 + \gamma_n\|\Phi(\xi_n)-\chi^\dagger\|^2 - \gamma_n(1-\gamma_n)r(\|\Phi(\xi_n)-\xi_n\|) \\
&\le (1-\gamma_n)\|\xi_n-\chi^\dagger\|^2 + \gamma_n\|\xi_n-\chi^\dagger\|^2 - \gamma_n(1-\gamma_n)r(\|\Phi(\xi_n)-\xi_n\|) \\
&= \|\xi_n-\chi^\dagger\|^2 - \gamma_n(1-\gamma_n)r(\|\Phi(\xi_n)-\xi_n\|) \\
&= \|(1-\omega_n)\varpi_n-\chi^\dagger\|^2 - \gamma_n(1-\gamma_n)r(\|\Phi(\xi_n)-\xi_n\|) \\
&\le \|\varpi_n-\chi^\dagger\|^2 + 2\omega_n\langle\varpi_n-\chi^\dagger, j(\xi_n-\chi^\dagger)\rangle - \gamma_n(1-\gamma_n)r(\|\Phi(\xi_n)-\xi_n\|) \\
&\le \|\vartheta_n-\chi^\dagger\|^2 + 2\rho_n\langle\vartheta_n-\chi^\dagger, j(\varpi_n-\chi^\dagger)\rangle + 2\omega_n\langle\varpi_n-\chi^\dagger, j(\xi_n-\chi^\dagger)\rangle \\
&\quad - \gamma_n(1-\gamma_n)r(\|\Phi(\xi_n)-\xi_n\|) \\
&\le \|\chi_n-\chi^\dagger\|^2 + 2\upsilon_n\langle\chi_n-\chi^\dagger, j(\vartheta_n-\chi^\dagger)\rangle + 2\rho_n\langle\vartheta_n-\chi^\dagger, j(\varpi_n-\chi^\dagger)\rangle \\
&\quad + 2\omega_n\langle\varpi_n-\chi^\dagger, j(\xi_n-\chi^\dagger)\rangle - \gamma_n(1-\gamma_n)r(\|\Phi(\xi_n)-\xi_n\|).
\end{aligned}$$

Simplifying the above equation we get

$$\begin{aligned}
\gamma_n(1-\gamma_n)r(\|\Phi(\xi_n)-\xi_n\|) &\le \|\chi_n-\chi^\dagger\|^2 - \|\chi_{n+1}-\chi^\dagger\|^2 + 2\upsilon_n\langle\chi_n-\chi^\dagger, j(\vartheta_n-\chi^\dagger)\rangle \\
&\quad + 2\rho_n\langle\vartheta_n-\chi^\dagger, j(\varpi_n-\chi^\dagger)\rangle + 2\omega_n\langle\varpi_n-\chi^\dagger, j(\xi_n-\chi^\dagger)\rangle. \qquad (3.3)
\end{aligned}$$

Since the sequences $\{\chi_n\}$, $\{\vartheta_n\}$, $\{\varpi_n\}$ and $\{\xi_n\}$ are bounded and if we take $\Gamma_1, \Gamma_2, \Gamma_3 > 0$ then for all $n \ge 1$

$$\begin{cases}
\langle\chi_n-\chi^\dagger, j(\vartheta_n-\chi^\dagger)\rangle \le \Gamma_1, \\
\langle\vartheta_n-\chi^\dagger, j(\varpi_n-\chi^\dagger)\rangle \le \Gamma_2, \\
\langle\varpi_n-\chi^\dagger, j(\xi_n-\chi^\dagger)\rangle \le \Gamma_3.
\end{cases} \qquad (3.4)$$

Applying (3.4) in (3.3), we get

$$\gamma_n(1-\gamma_n)r(\|\Phi(\xi_n)-\xi_n\|) \le \|\chi_n-\chi^\dagger\|^2 - \|\chi_{n+1}-\chi^\dagger\|^2 + 2\upsilon_n\Gamma_1 + 2\rho_n\Gamma_2 + 2\omega_n\Gamma_3. \qquad (3.5)$$

It gives that the sequence $\{\chi_n\}$ converges to $\chi^\dagger$. To prove strong convergence we consider the following cases.

Case(1): If the sequence $\{\|\chi_n-\chi^\dagger\|\}$ is monotonically decreasing. We have

$$\|\chi_{n+1}-\chi^\dagger\|^2 - \|\chi_n-\chi^\dagger\|^2 \to 0, \text{ as } n\to\infty.$$

From (3.5), we get

$$\lim_{n\to\infty} \gamma_n(1-\gamma_n)r(\|\Phi(\xi_n)-\xi_n\|) = 0.$$

Since $\gamma_n \in [a, b] \subset (0, 1)$, using the property of r as given in Lemma 2.2, we get

$$\lim_{n\to\infty} \|\Phi(\xi_n) - \xi_n\| = 0. \tag{3.6}$$

Now we get

$$\lim_{n\to\infty} \|\chi_{n+1} - \xi_n\| = \lim_{n\to\infty} \|\gamma_n(\Phi(\xi_n) - \xi_n)\| = 0, \tag{3.7}$$

and using (a), we get

$$\lim_{n\to\infty} \|\xi_n - \varpi_n\| = \lim_{n\to\infty} \omega_n\|\varpi_n\| = 0, \tag{3.8}$$

and

$$\lim_{n\to\infty} \|\varpi_n - \vartheta_n\| = \lim_{n\to\infty} \rho_n\|\vartheta_n\| = 0. \tag{3.9}$$

Now, using (3.8) and (3.9), we have

$$\lim_{n\to\infty} \|\xi_n - \vartheta_n\| \leq \lim_{n\to\infty} \|\xi_n - \varpi_n\| + \lim_{n\to\infty} \|\varpi_n - \vartheta_n\| = 0. \tag{3.10}$$

Since $\sum_{n=1}^{\infty} \upsilon_n\|\chi_n - \chi_{n-1}\| < \infty$, we have

$$\lim_{n\to\infty} \|\vartheta_n - \chi_n\| = \lim_{n\to\infty} \upsilon_n\|\chi_n - \chi_{n-1}\| = 0. \tag{3.11}$$

We can also get

$$\lim_{n\to\infty} \|\xi_n - \chi_n\| \leq \lim_{n\to\infty} \|\xi_n - \vartheta_n\| + \lim_{n\to\infty} \|\vartheta_n - \chi_n\| = 0, \tag{3.12}$$

and

$$\lim_{n\to\infty} \|\chi_{n+1} - \chi_n\| \leq \lim_{n\to\infty} \|\chi_{n+1} - \xi_n\| + \lim_{n\to\infty} \|\xi_n - \chi_n\| = 0,$$

and

$$\begin{aligned} \|\Phi(\varpi_n) - \varpi_n\| &\leq \|\Phi(\varpi_n) - \xi_n\| + \|\xi_n - \varpi_n\| \\ &\leq \mu\|\xi_n - \Phi(\xi_n)\| + \|\xi_n - \varpi_n\| + \|\xi_n - \varpi_n\|. \end{aligned}$$

Applying $\lim_{n\to\infty}$ and using (3.6), (3.8), we get

$$\lim_{n\to\infty} \|\Phi(\varpi_n) - \varpi_n\| = 0. \tag{3.13}$$

Since the sequence $\{\chi_n\}$ is bounded so there is a subsequence $\{\chi_{n_k}\}$ of $\{\chi_n\}$ such that it converges weakly to $\chi^\dagger \in \Sigma$. Using Lemma 2.6, we get $\chi^\dagger \in F(\Phi)$.
We also have

$$\begin{aligned}\xi_n &= (1-\omega_n)\varpi_n \\ &= (1-\omega_n)(1-\rho_n)\vartheta_n \\ &\leq (1-\rho_n)\vartheta_n.\end{aligned}$$

Thus

$$\begin{aligned}\|\xi_n - \chi^\dagger\|^2 &\leq \|(1-\rho_n)\vartheta_n - \chi^\dagger\|^2 \\ &\leq \|(1-\rho_n)(\vartheta_n - \chi^\dagger) - \rho_n\chi^\dagger\|^2.\end{aligned} \tag{3.14}$$

By applying inequality (3.14), Lemma 2.3, and $\sum\limits_{n=1}^{\infty} \upsilon_n\|\chi_n - \chi_{n-1}\| < \infty$, we get

$$\begin{aligned}\|\chi_{n+1} - \chi^\dagger\|^2 &= \|(1-\gamma_n)(\xi_n - \chi^\dagger) + \gamma_n(\Phi(\xi_n) - \zeta^\dagger)\|^2 \\ &\leq (1-\gamma_n)\|\xi_n - \chi^\dagger\|^2 + \gamma_n\|\Phi(\xi_n) - \chi^\dagger\|^2 \\ &\leq \|\xi_n - \chi^\dagger\|^2 \\ &\leq \|(1-\rho_n)(\vartheta_n - \chi^\dagger) - \rho_n\chi^\dagger\|^2 \\ &= (1-\rho_n)\|\vartheta_n - \chi^\dagger\|^2 + 2\rho_n\langle -\chi^\dagger, j(\varpi_n - \chi^\dagger)\rangle \\ &\leq (1-\rho_n)\|(\chi_n - \chi^\dagger) + \upsilon_n(\chi_n - \chi_{n-1})\|^2 + 2\rho_n\langle -\chi^\dagger, j(\varpi_n - \chi^\dagger)\rangle \\ &= (1-\rho_n)\|\chi_n - \chi^\dagger\|^2 + 2\rho_n\langle -\chi^\dagger, j(\varpi_n - \chi^\dagger)\rangle.\end{aligned} \tag{3.15}$$

Now, we prove that $\langle -\chi^\dagger, j(\varpi_n - \chi^\dagger)\rangle \leq 0$.

Define a continuous convex function f from $\mathcal{E}$ into $[0,\infty)$ by

$$f(\chi) = \limsup_{n\to\infty} \|\varpi_n - \chi\| \text{ for all } \chi \in \mathcal{E}.$$

Since $\mathcal{E}$ is closed and bounded, Σ is uniformly convex Banach space, and f is weakly lower semicontinuous, there exists $z \in \mathcal{E}$ such that

$$f(z) = \min\{f(\chi) : \chi \in \mathcal{E}\}.$$

Define

$$\hat{\mathfrak{R}} = \left\{v \in \mathcal{E} : f(v) = \min_{a\in\mathcal{E}} f(a)\right\} \neq \emptyset.$$

Let $z \in \hat{\mathfrak{R}}$, we have

$$\|\varpi_n - \Phi(z)\| \leqslant \mu \|\Phi(\varpi_n) - \varpi_n\| + \|\varpi_n - z\|.$$

From (3.13), we have $f(\Phi z) \leqslant f(z)$. Since $f(z)$ is the minimum, $f(\Phi z) = f(z)$ holds. If $\Phi z \neq z$, then since f is strictly quasiconvex (see [23, Lemma 2]), we have

$$f(z) \leqslant f\left(\frac{z + \Phi z}{2}\right) < \max\{f(z), f(\Phi z)\} = f(z).$$

This is a contradiction. Hence $\Phi(z) = z$. Hence, Φ has a fixed point in $\hat{\mathfrak{R}}$, so $\hat{\mathfrak{R}} \cap \mathcal{F}(\Phi) \neq \emptyset$. Without losing the general case, as a particular instance, suppose that $z = \chi^\dagger \in \hat{\mathfrak{R}} \cap \mathcal{F}(\Phi)$. Let $\delta \in (0,1)$ then we can easily get $f(\chi^\dagger) \leq f(\chi^\dagger - \delta\chi^\dagger)$. Since $f(\chi) \in [0,\infty)$, we can have $[f(\chi^\dagger)]^2 \leq [f(\chi^\dagger - \delta\chi^\dagger)]^2$. Applying Lemma 2.3 to the expression $\|\varpi_n - \chi^\dagger + \delta\chi^\dagger\|^2$, we get

$$\|\varpi_n - \chi^\dagger + \delta\chi^\dagger\|^2 \leq \|\varpi_n - \chi^\dagger\|^2 + 2\delta\langle \chi^\dagger, j(\varpi_n - \chi^\dagger + \delta\chi^\dagger)\rangle.$$

Applying $\limsup$ both the sides, we have

$$\begin{aligned}\limsup_{n\to\infty} \|\varpi_n - \chi^\dagger + \delta\chi^\dagger\|^2 \leq \limsup_{n\to\infty} \|\varpi_n - \chi^\dagger\|^2 \\ + 2\limsup_{n\to\infty} \delta\langle \chi^\dagger, j(\varpi_n - \chi^\dagger + \delta\chi^\dagger)\rangle.\end{aligned}$$

Using the property of f that $f(\chi) = \limsup_{n\to\infty} \|\varpi_n - \chi\|$ we get

$$[f(\chi^\dagger - \delta\chi^\dagger)]^2 \leq [f(\chi^\dagger)]^2 + 2\limsup_{n\to\infty} \delta\langle \chi^\dagger, j(\varpi_n - \chi^\dagger + \delta\chi^\dagger)\rangle.$$

And

$$2\limsup_{n\to\infty} \delta\langle -\chi^\dagger, j(\varpi_n - \chi^\dagger + \delta\chi^\dagger)\rangle \leq [f(\chi^\dagger)]^2 - [f(\chi^\dagger - \delta\chi^\dagger)]^2 \leq 0.$$

It gives us

$$\langle -\chi^\dagger, j(\varpi_n - \chi^\dagger + \delta\chi^\dagger)\rangle \leq 0.$$

We can also have

$$\begin{aligned}\langle -\chi^\dagger, j(\varpi_n - \chi^\dagger)\rangle &\leq \langle -\chi^\dagger, j(\varpi_n - \chi^\dagger) - j(\varpi_n - \chi^\dagger + \delta\chi^\dagger)\rangle \\ &\quad + \langle -\chi^\dagger, j(\varpi_n - \chi^\dagger + \delta\chi^\dagger)\rangle \\ &\leq \langle -\chi^\dagger, j(\varpi_n - \chi^\dagger) - j(\varpi_n - \chi^\dagger + \delta\chi^\dagger)\rangle. \qquad (3.16)\end{aligned}$$

Since the normalized duality mapping is norm to weak* uniformly continuous on bounded subsets of Σ. We also have, for fixed n $\delta \to 0$, thus

$$\begin{aligned}&\langle -\chi^{\dagger}, j(\varpi_n - \chi^{\dagger}) - j(\varpi_n - \chi^{\dagger} + \delta\chi^{\dagger})\rangle \\ &\leq \langle -\chi^{\dagger}, j(\varpi_n - \chi^{\dagger})\rangle - \langle -\chi^{\dagger}, j(\varpi_n - \chi^{\dagger} + \delta\chi^{\dagger})\rangle \to 0.\end{aligned}$$

Thus for each $\varepsilon > 0$ there exists $\varsigma_{\varepsilon} > 0$ such that for all $\delta \in (0, \varsigma_{\varepsilon})$

$$\langle -\chi^{\dagger}, j(\varpi_n - \chi^{\dagger})\rangle - \langle -\chi^{\dagger}, j(\varpi_n - \chi^{\dagger} + \delta\chi^{\dagger})\rangle < \varepsilon.$$

Since ε is arbitrary, using (3.9) we get

$$\langle -\chi^{\dagger}, j(\varpi_n - \chi^{\dagger})\rangle \leq 0.$$

Now

$$\|\varpi_{n+1} - \varpi_n\| \leq \|\varpi_{n+1} - \vartheta_{n+1}\| + \|\vartheta_{n+1} - \chi_{n+1}\| + \|\chi_{n+1} - \xi_n\| + \|\xi_n - \varpi_n\|.$$

Applying above conditions and $n \to \infty$, we get

$$\lim_{n\to\infty} \|\varpi_{n+1} - \varpi_n\| = 0.$$

Again since the normalized duality mapping is norm to weak* uniformly continuous on bounded subsets of Σ, we get

$$\lim_{n\to\infty} (\langle -\chi^{\dagger}, j(\varpi_n - \chi^{\dagger})\rangle - \langle -\chi^{\dagger}, j(\varpi_{n+1} - \chi^{\dagger})\rangle) = 0.$$

Applying Lemma 2.7, we get

$$\limsup_{n\to\infty} (\langle -\chi^{\dagger}, j(\varpi_n - \chi^{\dagger})\rangle \leq 0.$$

Thus applying Lemma 2.8 at (3.15), we get the sequence $\{\chi_n\}$ converges to $\chi^{\dagger}$.

Case(2): If the sequence $\{\|\chi_n - \chi^{\dagger}\|\}$ is not monotonically decreasing, let $\aleph_n = \|\chi_n - \chi^{\dagger}\|^2$. Let $\prod : \mathbb{N} \to \mathbb{N}$ defined as

$$\prod(n) \max\{\vartheta \in \mathbb{N} : \vartheta \leq n : \aleph_{\vartheta} \leq \aleph_{\vartheta+1}\}.$$

Here, $\prod$ is a nonincreasing sequence so $\lim_{n\to\infty} \prod(n) = \infty$ and $\aleph_{\prod(n)} \leq \aleph_{\prod(n)+1}$ $\forall$ $n \geq n_0$ for some sufficient large n_0. Now using (3.5), we get

$$\begin{aligned}\gamma_{\prod_{(n)}}\left(1-\gamma_{\prod_{(n)}}\right) r\left(\left\|\Phi\left(\xi_{\prod_{(n)}}\right)-\xi_{\prod_{(n)}}\right\|\right) &\leq \left\|\chi_{\prod_{(n)}}-\chi^{\dagger}\right\|^2 - \left\|\chi_{\prod_{(n)}+1}-\chi^{\dagger}\right\|^2 \\ &\quad + 2\upsilon_{\prod_{(n)}}\Gamma_1 + 2\rho_{\prod_{(n)}}\Gamma_2 + 2\omega_{\prod_{(n)}}\Gamma_3 \\ &= \aleph_{\prod_{(n)}} - \aleph_{\prod_{(n)}+1} \\ &\quad + 2\upsilon_{\prod_{(n)}}\Gamma_1 + 2\rho_{\prod_{(n)}}\Gamma_2 + 2\omega_{\prod_{(n)}}\Gamma_3 \\ &\leq 2\upsilon_{\prod_{(n)}}\Gamma_1 + 2\rho_{\prod_{(n)}}\Gamma_2 + 2\omega_{\prod_{(n)}}\Gamma_3.\end{aligned}$$

Since

$$\lim_{n\to\infty} 2\upsilon_{\prod_{(n)}}\Gamma_1 + \lim_{n\to\infty} 2\rho_{\prod_{(n)}}\Gamma_2 + \lim_{n\to\infty} 2\omega_{\prod_{(n)}}\Gamma_3 = 0.$$

We can also get

$$\lim_{n\to\infty}\left\|\Phi\left(\xi_{\prod_{(n)}}\right)-\xi_{\prod_{(n)}}\right\| = 0.$$

Following the same proof as in Case(a), we can get $\chi_{\prod_{(n)}} \rightharpoonup \chi^{\dagger}$ as $\prod_{(n)} \to \infty$ and $\limsup_{\prod_{(n)}\to\infty}\langle -\chi^{\dagger}, j(\varpi_{\prod_{(n)}} - \chi^{\dagger})\rangle \leq 0$. For all $n \geq n_0$, from (3.15), we get

$$0 \leq \left\|\chi_{\prod_{(n)+1}}-\chi^{\dagger}\right\|^2 - \left\|\chi_{\prod_{(n)}}-\chi^{\dagger}\right\|^2 \leq \rho_{\prod_{(n)}}\left[2\langle -\chi^{\dagger}, j(\varpi_{\prod_{(n)}}-\chi^{\dagger})\rangle - \left\|\chi_{\prod_{(n)}}-\chi^{\dagger}\right\|^2\right].$$

It gives us

$$\left\|\chi_{\prod_{(n)}}-\chi^{\dagger}\right\|^2 \leq 2\left\langle -\chi^{\dagger}, j\left(\varpi_{\prod_{(n)}}-\chi^{\dagger}\right)\right\rangle.$$

Since $\limsup_{\prod_{(n)}\to\infty}\left\langle -\chi^{\dagger}, j\left(\varpi_{\prod_{(n)}}-\chi^{\dagger}\right)\right\rangle \leq 0$, we get

$$\lim_{n\to\infty}\left\|\chi_{\prod_{(n)}}-\chi^{\dagger}\right\|^2 = 0.$$

Thus

$$\lim_{n\to\infty}\aleph_{\prod_{(n)}} = \lim_{n\to\infty}\aleph_{\prod_{(n)+1}} = 0.$$

Moreover, for each $n \geq n_0$, it can be easily seen that $\aleph_n \leq \aleph_{\prod_{(n)+1}}$ if $n \neq \prod_{(n)}$ i.e. $\prod_{(n)} < n$. Since $\aleph_i > \aleph_{i+1}$ for $\prod_{(n)+1} \leq i \leq n$. For all $n \geq n_0$, we get

$$0 \leq \aleph_n \leq \max\left\{\aleph_{\prod_{(n)}}, \aleph_{\prod_{(n)+1}}\right\} = \aleph_{\prod_{(n)+1}}.$$

Hence, $\lim_{n\to\infty}\aleph_n = 0$, it gives us that the sequence $\{\chi_n\}$ converges strongly to $\chi^{\dagger}$.

□

Corollary 3.2. *Suppose $\mathcal{E} \neq \emptyset$ is a convex and closed subset of a RUCBS Σ having uniformly Gâteaux differentiable norm, and $\Phi : \mathcal{E} \to \mathcal{E}$ is a mapping satisfying condition (C) with $F(\Phi) \neq \emptyset$. Assume the following conditions satisfied:*

(a) $\lim\limits_{n\to\infty} \omega_n = 0, \lim\limits_{n\to\infty} \rho_n = 0, \sum\limits_{n=1}^{\infty} \rho_n = \infty,$

(b) $\upsilon_n \geq 0, \forall n \in \mathbb{N}$ *and* $\sum\limits_{n=1}^{\infty} \upsilon_n < \infty,$

(c) $\omega_n, \rho_n \in (0,1), \gamma_n \in [a,b] \subset (0,1).$

Suppose $\chi_0, \chi_1 \in \mathcal{E}$ then $\forall n \geq 1$ the sequence $\{\chi_n\}$ generated by

$$\begin{cases} \vartheta_n = \chi_n + \upsilon_n(\chi_n - \chi_{n-1}), \\ \xi_n = (1-\omega_n)(1-\rho_n)\vartheta_n, \\ \chi_{n+1} = (1-\gamma_n)\xi_n + \gamma_n\Phi(\xi_n), \end{cases} \tag{3.17}$$

converges strongly to a point $\chi^\dagger \in F(\Phi)$.

Corollary 3.3. *Suppose $\mathcal{E} \neq \emptyset$ is a convex and closed subset of a RUCBS Σ having uniformly Gâteaux differentiable norm, and $\Phi : \mathcal{E} \to \mathcal{E}$ is a generalized contraction of Suzuki type mapping with $F(\Phi) \neq \emptyset$. Assume the following conditions satisfied:*

(a) $\lim\limits_{n\to\infty} \omega_n = 0, \lim\limits_{n\to\infty} \rho_n = 0, \sum\limits_{n=1}^{\infty} \rho_n = \infty,$

(b) $\upsilon_n \geq 0, \forall n \in \mathbb{N}$ *and* $\sum\limits_{n=1}^{\infty} \upsilon_n < \infty,$

(c) $\omega_n, \rho_n \in (0,1), \gamma_n \in [a,b] \subset (0,1).$

Suppose $\chi_0, \chi_1 \in \mathcal{E}$ then $\forall n \geq 1$ the sequence $\{\chi_n\}$ generated by

$$\begin{cases} \vartheta_n = \chi_n + \upsilon_n(\chi_n - \chi_{n-1}), \\ \xi_n = (1-\omega_n)(1-\rho_n)\vartheta_n, \\ \chi_{n+1} = (1-\gamma_n)\xi_n + \gamma_n\Phi(\xi_n), \end{cases} \tag{3.18}$$

converges strongly to a point $\chi^\dagger \in F(\Phi)$.

Corollary 3.4. *Suppose $\mathcal{E} \neq \emptyset$ is a convex and closed subset of a RUCBS Σ having uniformly Gâteaux differentiable norm, and $\Phi : \mathcal{E} \to \mathcal{E}$ is a generalized α-Reich-Suzuki nonexpansive mapping with $F(\Phi) \neq \emptyset$. Assume the following conditions satisfied:*

(a) $\lim\limits_{n\to\infty} \omega_n = 0, \lim\limits_{n\to\infty} \rho_n = 0, \sum\limits_{n=1}^{\infty} \rho_n = \infty,$

(b) $\upsilon_n \geq 0, \forall n \in \mathbb{N}$ *and* $\sum\limits_{n=1}^{\infty} \upsilon_n < \infty,$

(c) $\omega_n, \rho_n \in (0,1), \gamma_n \in [a,b] \subset (0,1).$

Suppose $\chi_0, \chi_1 \in \mathcal{E}$ *then for all* $n \geq 1$ *the sequence* $\{\chi_n\}$ *generated by*

$$\begin{cases} \vartheta_n = \chi_n + \upsilon_n(\chi_n - \chi_{n-1}), \\ \xi_n = (1-\omega_n)(1-\rho_n)\vartheta_n, \\ \chi_{n+1} = (1-\gamma_n)\xi_n + \gamma_n \Phi(\xi_n), \end{cases} \tag{3.19}$$

converges strongly to a point $\chi^\dagger \in F(\Phi)$.

4. Examples

Example 4.1. Define the mapping $\Phi : [0,1] \to [0,1]$ by

$$\Phi(\zeta) = \begin{cases} 1-\zeta, \ \text{if } \zeta \in \left[0, \frac{1}{6}\right), \\ \frac{\zeta+5}{6}, \ \text{if } \zeta \in \left[\frac{1}{6}, 1\right]. \end{cases}$$

First, we verify that the mapping Φ satisfies condition (E). We follow these cases

Case (1): Let $\zeta \in \left[0, \frac{1}{6}\right)$, then

$$\frac{1}{2}\|\zeta - \Phi(\zeta)\| = \frac{1-2\zeta}{2} \in \left(\frac{2}{6}, \frac{1}{2}\right],$$

for $\frac{1}{2}\|\zeta - \Phi(\zeta)\| \leq \|\zeta - \eta\|$, we must have

$$\frac{1-2\zeta}{2} \leq \eta - \zeta, \ \text{i.e.} \ \frac{1}{2} \leq \eta, \ \text{hence} \ \eta \in \left[\frac{1}{2}, 1\right].$$

We have

$$\|\Phi(\zeta) - \Phi(\eta)\| = \left\|(1-\zeta) - \left(\frac{\eta+5}{6}\right)\right\| = \left|\frac{\eta + 6\zeta - 1}{6}\right| < \frac{1}{6},$$

and

$$\|\zeta - \eta\| = |\zeta - \eta| > \frac{1}{3}.$$

Hence

$$\frac{1}{2}\|\zeta - \Phi(\zeta)\| \leq \|\zeta - \eta\| \implies \|\Phi(\zeta) - \Phi(\eta)\| \leq \|\zeta - \eta\|.$$

Case (2): Let $\zeta \in \left[\frac{1}{6}, 1\right]$, then

$$\frac{1}{2}\|\zeta - \Phi(\zeta)\| = \frac{1}{2}\left\|\frac{\zeta+5}{6} - \zeta\right\| = \frac{1-\zeta}{2} \in \left[0, \frac{5}{12}\right],$$

for $\frac{1}{2}\|\zeta - \Phi(\zeta)\| \leq \|\zeta - \eta\|$ we must have $\frac{1-\zeta}{2} \leq \|\eta - \zeta\|$ which gives two possibilities:

Case (a): $\zeta < \eta$, then $\frac{1-\zeta}{2} \le \eta - \zeta \implies \eta \ge \frac{1+\zeta}{2} \implies \eta \in \left[\frac{7}{12}, 1\right] \subset \left[\frac{1}{6}, 1\right]$. So

$$\|\Phi(\zeta) - \Phi(\eta)\| = \left\|\frac{\zeta+5}{6} - \frac{\eta+5}{6}\right\| = \frac{1}{6}\|\zeta - \eta\| \le \|\zeta - \eta\|.$$

Hence

$$\frac{1}{2}\|\zeta - \Phi(\zeta)\| \le \|\zeta - \eta\| \implies \|\Phi(\zeta) - \Phi(\eta)\| \le \|\zeta - \eta\|.$$

Case (b): Let $\zeta > \eta$, then

$$\frac{1-\zeta}{2} \le \zeta - \eta \implies \eta \le \frac{3\zeta - 1}{2} \implies \eta \in \left[\frac{-3}{24}, 1\right].$$

Since $\eta \in [0, 1]$, so

$$\eta \le \frac{3\zeta - 1}{2} \implies \zeta \ge \frac{2\eta + 1}{3} \implies \zeta \in \left[\frac{1}{3}, 1\right].$$

So the case is $\zeta \in \left[\frac{1}{3}, 1\right]$ and $\eta \in [0, 1]$. Since $\zeta \in \left[\frac{1}{3}, 1\right]$ and $\eta \in \left[\frac{1}{6}, 1\right]$ is already included in the Case (a). So let $\zeta \in \left[\frac{1}{3}, 1\right]$ and $\eta \in \left[0, \frac{1}{6}\right)$, then

$$\|\Phi(\zeta) - \Phi(\eta)\| = \left\|\frac{\zeta+5}{6} - (1-\eta)\right\| = \left|\frac{\zeta + 6\eta - 1}{6}\right|.$$

First, we consider $\zeta \in \left[\frac{1}{3}, \frac{1}{2}\right]$ and $\eta \in \left[0, \frac{1}{6}\right]$, then

$$\|\Phi(\zeta) - \Phi(\eta) = \left|\frac{\zeta + 6\eta - 1}{6}\right| \le \frac{1}{12},$$

and

$$\|\zeta - \eta\| \ge \left|\frac{1}{3} - \frac{1}{6}\right| = \frac{1}{6}.$$

Hence

$$\|\Phi(\zeta) - \Phi(\eta)\| \le \|\zeta - \eta\|.$$

Next consider $\zeta \in \left[\frac{1}{2}, 1\right]$, $\eta \in \left[0, \frac{1}{6}\right)$. Then

$$\|\Phi(\zeta) - \Phi(\eta)\| \le \frac{1}{6} \text{ and } \|\zeta - \eta\| \ge \frac{1}{3} \implies \|\Phi(\zeta) - \Phi(\eta)\| \le \|\zeta - \eta\|.$$

Hence

$$\frac{1}{2}\|\zeta - \Phi(\zeta)\| \le \|\zeta - \eta\| \implies \|\Phi(\zeta) - \Phi(\eta)\| \le \|\zeta - \eta\|.$$

Hence the mapping Φ is satisfying Condition (C).

Using Lemma 2.9, we can say that the mapping Φ is satisfying condition (E) for $\mu = 3$.
On the other hand if we take $\zeta = \frac{3}{19}, \eta = \frac{1}{6}$

$$\|\Phi(\zeta) - \Phi(\eta)\| = \left\|\left(1 - \frac{3}{19}\right) - \left(\frac{\frac{1}{6} + 5}{6}\right)\right\| = \frac{13}{684} > \frac{1}{114} = \|\zeta - \eta\|.$$

Therefore Φ is not a nonexpansive mapping.

Example 4.2. Let $\Sigma = \mathbb{R}^2$ and $\mathcal{E} = \{\zeta = (\zeta_1, \zeta_2) \in [0,1] \times [0,1]\}$ be a subset of Σ with norm $\|\zeta\| = \|(\zeta_1, \zeta_2)\| = (|\zeta_1|^2 + |\zeta_2|^2)^{1/2}$. The mapping $\Phi : \mathcal{E} \to \mathcal{E}$ is defined by

$$\Phi(\zeta_1, \zeta_2) = \begin{cases} (1 - \zeta_1, 1 - \zeta_2), & (\zeta_1, \zeta_2) \in \left[0, \frac{1}{2}\right] \times [0,1], \\ \frac{1}{3}(1 + \zeta_1, 1 + \zeta_2), & (\zeta_1, \zeta_2) \in \left(\frac{1}{2}, 1\right] \times [0,1]. \end{cases}$$

For this, we consider the following cases:

Case(a): $\zeta, \eta \in \left[0, \frac{1}{2}\right] \times [0,1]$. Then

$$\begin{aligned} \|\zeta - \Phi(\eta)\| &\le \|\zeta - \Phi(\zeta)\| + \|\Phi(\zeta) - \Phi(\eta)\| \\ &= \|\zeta - \Phi(\zeta)\| + \left(|\zeta_1 - \eta_1|^2 + |\zeta_2 - \eta_2|^2\right)^{1/2} \\ &= \|\zeta - \Phi(\zeta)\| + \|\zeta - \eta\|. \end{aligned}$$

Case(b): $\zeta, \eta \in \left(\frac{1}{2}, 1\right] \times [0,1]$. Then

$$\begin{aligned} \|\zeta - \Phi(\eta)\| &\le \|\zeta - \Phi(\zeta)\| + \|\Phi(\zeta) - \Phi(\eta)\| \\ &= \|\zeta - \Phi(\zeta)\| + \frac{1}{3}\left(|\zeta_1 - \eta_1|^2 + |\zeta_2 - \eta_2|^2\right)^{1/2} \\ &\le \|\zeta - \Phi(\zeta)\| + \left(|\zeta_1 - \eta_1|^2 + |\zeta_2 - \eta_2|^2\right)^{1/2} \\ &= \|\zeta - \Phi(\zeta)\| + \|\zeta - \eta\|. \end{aligned}$$

Case(c): $\zeta \in \left[0, \frac{1}{2}\right] \times [0,1]$, $\eta \in \left(\frac{1}{2}, 1\right] \times [0,1]$. Then

$$\begin{aligned} \|\zeta - \Phi(\eta)\| &= \left\|(\zeta_1, \zeta_2) - \left(\frac{1 + \eta_1}{3}, \frac{1 + \eta_2}{3}\right)\right\| \\ &= \left\|\left(\zeta_1 - \left(\frac{1 + \eta_1}{3}\right)\right), \left(\zeta_2 - \left(\frac{1 + \eta_2}{3}\right)\right)\right\| \\ &= \left\|\left(\frac{3\zeta_1 - \eta_1 - 1}{3}, \frac{3\zeta_2 - \eta_2 - 1}{3}\right)\right\| \\ &= \left(\left|\frac{3\zeta_1 - \eta_1 - 1}{3}\right|^2 + \left|\frac{3\zeta_2 - \eta_2 - 1}{3}\right|^2\right)^{1/2}, \end{aligned}$$

and

$$\|\zeta - \Phi(\zeta)\| = \left(|2\zeta_1 - 1|^2 + |2\zeta_2 - 1|^2\right)^{1/2}.$$

Since

$$\begin{aligned}\left|\frac{3\zeta_1 - \eta_1 - 1}{3}\right| &\leq |2\zeta_1 - 1 + \zeta_1 - \eta_1| \leq |2\zeta_1 - 1| + |\zeta_1 - \eta_1|,\\ \left|\frac{3\zeta_2 - \eta_2 - 1}{3}\right| &< |2\zeta_2 - 1 + \zeta_2 - \eta_2| \leq |2\zeta_2 - 1| + |\zeta_2 - \eta_2|,\end{aligned}$$

we have,

$$\|\zeta - \Phi(\eta)\| \leq \|\zeta - \Phi(\zeta)\| + \|\zeta - \eta\|.$$

Therefore in all the cases Φ satisfies condition (E).

On the other hand, for $\zeta = (0, 0)$ and $\eta = \left(\frac{51}{100}, \frac{25}{100}\right)$, we have

$$\|\Phi(\zeta) - \Phi(\eta)\| = 0.766 > 0.567 = \|\zeta - \eta\|.$$

Therefore Φ is not a nonexpansive mapping.

Now, we present the convergence behaviour of the Algorithm (3.1) for the Example (4.1). We present the convergence behaviour for the two cases, first we present for the different choices of initial guesses and then for the different choices of $\upsilon_n, \omega_n, \rho_n, \gamma_n$.

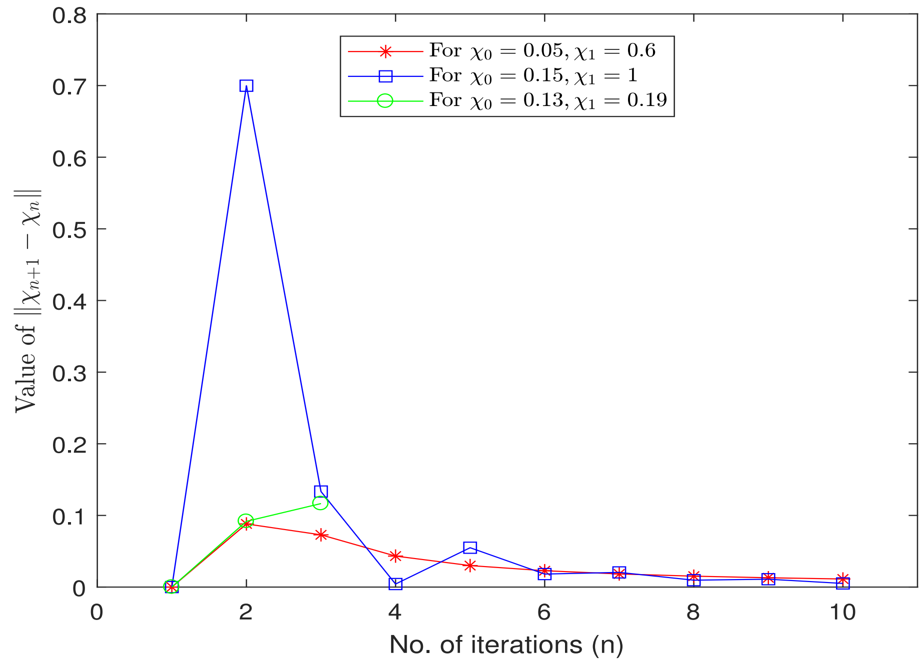

Figure 1: Convergence behaviour for the different choices of initial guesses.

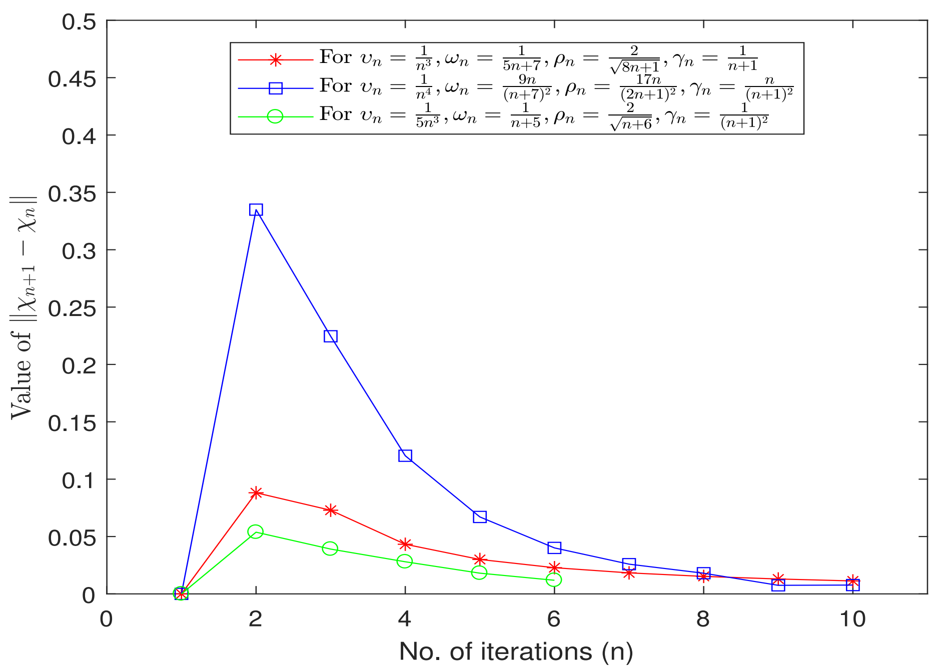

Figure 2: Convergence behaviour for different choices of υ_n, ω_n, ρ_n, γ_n.

Now, we present the comparison of Algorithm (3.1) with the algorithms presented in Baewnoi et al. [4] and Dong et al. [10]. We have taken Example (4.1) for Algorithm (3.1) and mapping $\Phi(\zeta) = \frac{\zeta}{3}$ for the algorithms presented in Baewnoi et al. [4] and Dong et al. [10].

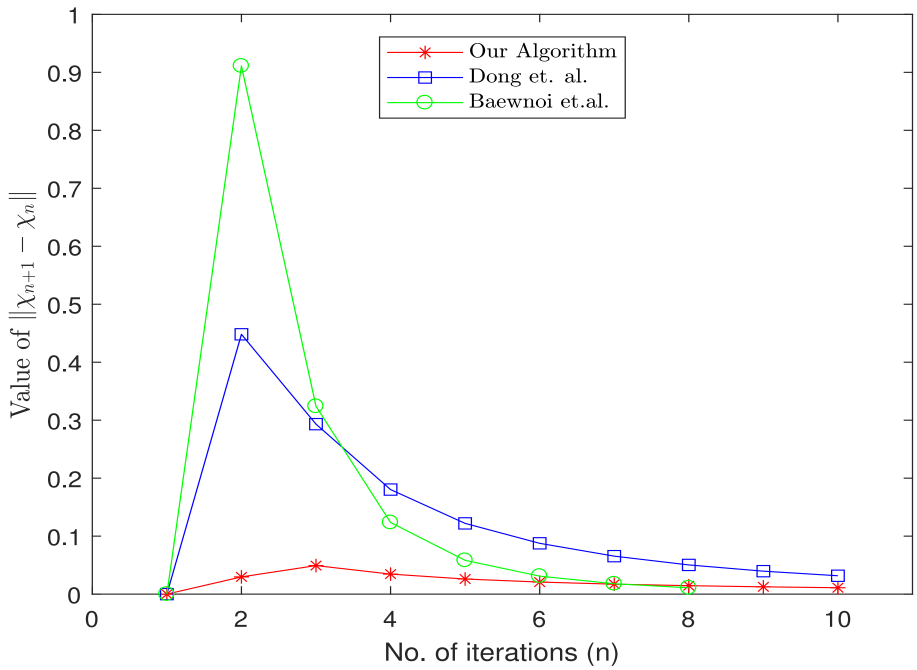

Figure 3: Comparison.

Open Problem

We conclude this paper by proposing the following open question:

Question 4.3. *Can Algorithm* (3.1) *be extended to approximate fixed points of quasi-nonexpansive mappings in the setting of Banach spaces?*

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