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Various notions of topological transitivity in non-autonomous discrete dynamical systems

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Abstract

We consider various strengthenings of the notion of topological transitivity in non-autonomous discrete dynamical systems. We give many equivalent conditions for each of these notions and present the implications among them. We also consider rearrangements of non-autonomous dynamical systems.

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1. Introduction

In this paper, we discuss various notions of transitivity for non-autonomous discrete dynamical systems (NDDS). Our study is along the lines of that which is made for autonomous discrete dynamical systems (ADDS) in [1] and for topological dynamical systems (TDS) in [13]. Throughout this paper, we assume that the underlying space is a compact metric space.

A *non-autonomous discrete dynamical system (NDDS)* is a pair $(X, f_{1,\infty})$ where X is a compact metric space and $f_{1,\infty} = (f_n)_{n\in\mathbb{N}}$ is a sequence of continuous self maps on X. For $n \in \mathbb{N}$, the composition

$$f_1^n := f_n \circ f_{n-1} \circ \cdots \circ f_2 \circ f_1$$

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is said to be n^{th}- iterate of $f_{1,\infty}$. We denote $(f_1^n)^{-1}$ by f_1^{-n}. i.e., $f_1^{-n} = f_1^{-1} \circ f_2^{-1} \circ \cdots \circ f_n^{-1}$. The case when $f_{1,\infty}$ is a constant sequence (f), the pair $(X, f_{1,\infty})$ reduces to the classical discrete dynamical system (X, f) which is also known as autonomous discrete dynamical system (ADDS). Transitivity for NDDS generated by uniformly convergent sequences was well studied in [16]. Dynamics of non-autonomous systems generated by finite collection of maps was studied in [5] and [15]. The authors of [2], [3], [4], [6], [17], and [18] studied ω-limit sets of non-autonomous discrete dynamical systems.

In [1] and [9], the authors provided some equivalent conditions of some dynamical properties for autonomous discrete dynamical systems and such results for topological dynamical systems are in [13]. In the present paper, we investigate whether such results hold for non-autonomous discrete dynamical systems. For instance, it is proved in [1] that an ADDS is topologically transitive if and only if there exists an element whose orbit is dense. In the present paper, we prove that a similar result holds for NDDS also. In other words, we prove that an NDDS is topologically transitive if and only if there exists an element with dense orbit.

In the case of autonomous systems, a *topological conjugacy* from (X, f) to (Y, g) is a homeomorphism $\pi : X \to Y$ such that $\pi \circ f = g \circ \pi$. It follows that $\pi \circ f^n = g^n \circ \pi$ for every $n \in \mathbb{N}$. However, in the case of non-autonomous system $(X, f_{1,\infty})$ and $(Y, g_{1,\infty})$, we need to ensure this as a part of the definition. Notions like toplogical transitivity, strong transitivity and locally eventually onto are preseved under conjugacy.

Consider an NDDS $(X, f_{1,\infty})$ such that $f_m \circ f_n = f_n \circ f_m$ for all $m, n \in \mathbb{N}$. In Propositions 3.24 and 3.25, we prove that if $(X, f_{1,\infty})$ is either topologically transitive or mixing, then $(X, g_{1,\infty})$ also exhibits identical notion for every finite rearrangement $g_{1,\infty}$ of $f_{1,\infty}$. See Proposition 3.26 for a similar result about locally eventually onto systems.

The following is the list of various notions of transitivity that we study in this paper:

(i) topological transitivity (TT)
(ii) strong transitivity (ST)
(iii) very strong transitivity (VST)
(iv) exact transitivity (ET)
(v) strong exact transitivity (SET)
(vi) topological mixing (TM)
(vii) locally eventually onto (LEO)

For literature on topological transitivity and mixing for non-autonomous discrete dynamical systems see [7], [8], [10], [11], [12], [14], and [17].

2. Preliminaries

We begin with some elementary results.

For a map $f : X \to Y$ with $A \subset X, B \subset Y$

$$f(A) \cap B \neq \emptyset \Leftrightarrow A \cap f^{-1}(B) \neq \emptyset \tag{2.1}$$

$$f(A) \subset B \Leftrightarrow A \subset f^{-1}(B) \Leftrightarrow f^{-1}(B^c) \subset A^c \tag{2.2}$$

$$\textit{If } f \textit{ is surjective, then } f^{-1}(B) \subset A \Rightarrow B \subset f(A). \tag{2.3}$$

Now we give some basic definitions and then provide some results related to these definitions.

Definition 2.1. Let $(X, f_{1,\infty})$ be an NDDS.

(i) The *orbit* of $x \in X$ is $O(x, f_{1,\infty}) = \{f_1^n(x) : n \in \mathbb{N}\}$. For $N \in \mathbb{N}$, $O_N(x, f_{1,\infty}) = \{f_1^n(x) : 1 \leq n \leq N\}$ is said to be a *partial orbit* of x.

(ii) The *negative orbit* of $x \in X$ is
$O^-(x, f_{1,\infty}) = \{y \in X : f_1^n(y) = x \text{ for some } n \in \mathbb{N}\} = \bigcup_{n \in \mathbb{N}} f_1^{-n}(x)$.
For $N \in \mathbb{N}$, $O_N^-(x, f_{1,\infty}) = \bigcup_{1 \leq n \leq N} f_1^{-n}(x)$ is said to be a *partial negative orbit* of x.

(iii) The set of limit points of an orbit $O(x, f_{1,\infty})$ is the *ω-limit set* of x, which is denoted as $\omega(x, f_{1,\infty})$, i.e., $\omega(x, f_{1,\infty}) = \bigcap_{N \in \mathbb{N}} \overline{\{f_1^n(x) : n \geq N\}}$, and $O(x, f_{1,\infty}) \cup \omega(x, f_{1,\infty}) = \overline{O(x, f_{1,\infty})}$ is the orbit closure of x.

(iv) A point $x \in X$ is called *recurrent* when $x \in \omega(x, f_{1,\infty})$.

Definition 2.2. Let $(X, f_{1,\infty})$ be an NDDS and A be a subset of X. Then A is said to be

(i) a $+$ *invariant set* if $f_1^n(A) \subset A$ for all $n \in \mathbb{N}$.

(ii) a $-$ *invariant set* if $f_1^{-n}(A) \subset A$ for all $n \in \mathbb{N}$.

(iii) a *weakly $-$ invariant set* if $A \subset f_1^n(A)$ for all $n \in \mathbb{N}$.

(iv) an *invariant set* if $f_1^n(A) = A$ for all $n \in \mathbb{N}$.

It is easy to see that A is invariant if and only if $f_n(A) = A$ for all $n \in \mathbb{N}$.

We now prove that the $-$ invariant sets are precisely the complements of $+$ invariant sets.

Lemma 2.3. *For an NDDS $(X, f_{1,\infty})$, a subset $A \subset X$ is $+$ invariant if and only if A^c is $-$ invariant.*

Proof. Assume A is $+$ invariant, i.e., $f_1^n(A) \subset A$ and so $f_1^{-n}(A^c) \subset A^c$ for all $n \in \mathbb{N}$ by equation (2.2). Hence A^c is $-$ invariant. Now suppose that A^c is $-$ invariant. Thus, $f_1^{-n}(A^c) \subset A^c$ and so $f_1^n(A) \subset A$ for all $n \in \mathbb{N}$ by equation (2.2). Therefore A is $+$ invariant. □

The next lemma gives an equivalent condition for a set to be weakly $-$ invariant.

Lemma 2.4. *For an NDDS $(X, f_{1,\infty})$, a subset $A \subset X$ is weakly $-$ invariant if and only if $f_1^n(A \cap f_1^{-n}(A)) = A$, for all $n \in \mathbb{N}$.*

Proof. We prove the necessity part and the sufficiency part is trivial. To prove the necessity part, assume A is weakly $-$ invariant and fix $n \in \mathbb{N}$. Then $A \subset f_1^n(A)$ and so $f_1^n(A \cap f_1^{-n}(A)) \subset f_1^n(A) \cap A = A$. Fix $a \in A$. Since $A \subset f_1^n(A)$, there exists $b \in A$ such that $f_1^n(b) = a$. Then $b \in f_1^{-n}(A)$ which implies that $b \in A \cap f_1^{-n}(A)$. As $a = f_1^n(b) \in f_1^n(A \cap f_1^{-n}(A))$, we get $A \subset f_1^n(A \cap f_1^{-n}(A))$. Hence $A = f_1^n(A \cap f_1^{-n}(A))$, for all $n \in \mathbb{N}$. □

Remark 2.5. In an NDDS $(X, f_{1,\infty})$,

(i) the orbit of an element need not be $+$ invariant.
(ii) the negative orbit of an element need not be $-$ invariant.
(iii) a subset $A \subset X$ is $+$ invariant if and only if $O(x, f_{1,\infty}) \subset A$ for every $x \in A$.
(iv) a subset $A \subset X$ is $-$ invariant if and only if $O^{-}(x, f_{1,\infty}) \subset A$ for every $x \in A$.
(v) the ω-limit set of an element need not be $+$ invariant. (For examples see [17]).

In an autonomous discrete dynamical system (X, f), if f is surjective and A is either invariant or weakly $-$ invariant subset of X, then $\overline{A}$ satisfies the corresponding property. If f is open surjective map then interior of any $+$ invariant subset of X is also $+$ invariant and closure of any $-$ invariant subset of X is $-$ invariant. For the proof of these results see [1]. It is natural to ask whether similar results hold in the case of NDDS. The following two lemmas give an affirmative answer.

Lemma 2.6. *If f_n is surjective for every $n \in \mathbb{N}$ and A is a subset of X, then the following statements hold:*

(i) *If A is $-$ invariant, then A is weakly $-$ invariant.*
(ii) *The set A is invariant if and only if A is both $+$ invariant and weakly $-$ invariant. Also, $A = f_1^{-n}(A)$ for all $n \in \mathbb{N}$ if and only if A is both $+$ invariant and $-$ invariant, in that case, it is invariant.*
(iii) *If A is either $+$ invariant, weakly $-$ invariant or invariant, then $\overline{A}$ satisfies the corresponding property.*
(iv) *If f_n is open map for every $n \in \mathbb{N}$ and if A is $+$ invariant then A° is $+$ invariant. Also, if A is $-$ invariant then $\overline{A}$ is $-$ invariant.*

Proof. (i) follows from equation (2.3).
(ii) follows from equations (2.2) and (2.3).
Proof of (iii): If A is $+$ invariant, then $f_1^n(A) \subset A$ and so $f_1^n(\overline{A}) \subset \overline{f_1^n(A)} \subset \overline{A}$, for every $n \in \mathbb{N}$. Therefore $\overline{A}$ is $+$ invariant.
Suppose A is weakly $-$ invariant, and let $n \in \mathbb{N}$. Then $A \subset f_1^n(A)$ or $\overline{A} \subset \overline{f_1^n(A)}$. Continuity of f_1^n gives

us $\overline{f_1^n(A)} \subset f_1^n(\overline{A})$. The reverse inclusion is always true. Hence we have $\overline{f_1^n(A)} = f_1^n(\overline{A})$. Thus, we have $\overline{A} \subset f_1^n(\overline{A})$ or $\overline{A}$ is weakly $-$ invariant. Hence $\overline{A}$ is invariant if A is invariant.
Proof of (iv): Assume f_n is an open map for every $n \in \mathbb{N}$. Since A° is open, $f_n(A^\circ)$ is open for all $n \in \mathbb{N}$. Since A is $+$ invariant and f_1^n is continuous, we have $f_1^n(A^\circ) \subset (f_1^n(A))^\circ \subset A^\circ$ for all $n \in \mathbb{N}$. Thus, A° is $+$ invariant. If A is $-$ invariant, then A^c is $+$ invariant. By above, $(A^c)^\circ$ is $+$ invariant, hence $\overline{A}= ((A^c)^\circ)^c$ is $-$ invariant. □

Definition 2.7. Let $\epsilon > 0$. A subset $U \subset X$ is called ϵ *dense* if U intersects every open ball of radius ϵ in X.

Some notions of topological transitivity can be written in terms of ϵ dense sets. To establish such results we recall the following result from [1].

Lemma 2.8. *Let (U_n) be a sequence of subsets of X. Then*

(i) $\bigcup_{n=1}^{\infty} U_n$ *is dense if and only if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that* $\bigcup_{n=1}^{N} U_n$ *is ϵ dense.*

(ii) *If each U_n is open and* $\bigcup_{n=1}^{\infty} U_n = X$, *then there exists $N \in \mathbb{N}$ such that* $\bigcup_{n=1}^{N} U_n = X$.

3. Main Results

3.1. **Topological transitivity.** For any two nonempty subsets $U, V \subset X$, we define the *set of hitting times* $N(U, V)$ as: $N(U, V) = \{n \in \mathbb{N} : f_1^n(U) \cap V \neq \emptyset\} = \{n \in \mathbb{N} : U \cap f_1^{-n}(V) \neq \emptyset\}$. For an element $x \in X$, we write $N(U, \{x\})$ as $N(U, x)$.

Definition 3.1 (Topological transitivity)**.** A system $(X, f_{1,\infty})$ is called *topologically transitive* if for every opene(=nonempty open) $V \subset X$, $\bigcup_{n=1}^{\infty} f_1^n(V)$ is dense in X. Equivalently, if for every opene pair $U, V \subset X$, the set of hitting times $N(U, V)$ is nonempty. i.e., for every opene pair $U, V \subset X$, there exists $n \in \mathbb{N}$ such that $f_1^n(U) \cap V \neq \emptyset$.

A point $x \in X$ is called a *transitive point* if for every opene $V \subset X$, the set of hitting times $N(x, V)$ is nonempty. Equivalently, if the orbit $O(x, f_{1,\infty}) = \{f_1^n(x) : n \in \mathbb{N}\}$ is dense in X. The set of all transitive points is denoted by $T(f_{1,\infty})$.

In Theorem 3.3 we will prove that an NDDS $(X, f_{1,\infty})$ is topologically transitive if and only if the set of transitive points is a G_δ subset of X. To prove these implications we need the following proposition.

Proposition 3.2. *Let $(X, f_{1,\infty})$ be an NDDS. Then the set $T(f_{1,\infty})$ of transitive points equals $\{x : \omega(x, f_{1,\infty}) = X\}$.*

Proof. Let $x \in X$ be such that $\omega(x, f_{1,\infty}) = X$. Then $\overline{O(x, f_{1,\infty})} = O(x, f_{1,\infty}) \cup \omega(x, f_{1,\infty}) = X$, which implies $x \in T(f_{1,\infty})$. Conversely, assume $x \in T(f_{1,\infty})$. Let $y \in X$ and U_y be any neighbourhood of y. Since $O(x, f_{1,\infty})$ is dense in X, there exists a sequence in the orbit of x converging to y. This means that y is an ω-limit point of x, and $\omega(x, f_{1,\infty}) = X$. □

The following theorem gives some equivalent conditions for topological transitivity.

Theorem 3.3. *Let $(X, f_{1,\infty})$ be an NDDS, where X is a perfect space(i.e., has no isolated points). Then the following are equivalent:*

(i) *The system is topologically transitive.*
(ii) *For every pair of opene sets U and V in X, there exists $n \in \mathbb{N}$ such that $f_1^{-n}(U) \cap V \neq \emptyset$.*
(iii) *For every pair of opene sets U, V in X the set $N(U, V)$ is nonempty.*
(iv) *For every pair of opene sets U, V in X the set $N(U, V)$ is infinite.*
(v) *There exists $x \in X$ such that the orbit $O(x, f_{1,\infty})$ is dense in X.*
(vi) *The set $T(f_{1,\infty})$ of transitive points is a dense, G_δ subset of X.*
(vii) *For every opene set $U \subset X$, $\bigcup_{n=1}^{\infty} f_1^n(U)$ is dense in X.*
(viii) *For every opene set $U \subset X$, and $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\bigcup_{n=1}^{N} f_1^n(U)$ is ϵ dense in X.*
(ix) *For every opene set $U \subset X$, $\bigcup_{n=1}^{\infty} f_1^{-n}(U)$ is dense in X.*
(x) *For every opene set $U \subset X$, and $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\bigcup_{n=1}^{N} f_1^{-n}(U)$ is ϵ dense in X.*

Proof. Notice that condition (vii) is the definition of topological transitivity and so, of course, (i)⇔(vii). (ii)⇔(iii), (iii)⇔(vii) and (iii)⇔(ix) follow from the definition of $N(U, V)$. (vii)⇔(viii), and (ix)⇔(x) follow from Lemma 2.8. So far, we have established equivalence of (i), (ii), (iii), (vii), (viii), (ix), (x). To complete the proof we prove that (ix)⇒(vi)⇒(v)⇒(iv)⇒(iii).

To prove (ix)⇒(vi): Let $x \in T(f_{1,\infty})$, then for all opene $U \subset X$, there exists $n \in \mathbb{N}$ such that $f_1^n(x) \in U$. So $x \in \bigcup_{n=1}^{\infty} f_1^{-n}(U)$ for all opene U. Since every compact metric space is second countable, we have a countable base $\mathcal{B}$ and so $T(f_{1,\infty}) \subset \bigcap_{U \in \mathcal{B}} \bigcup_{n=1}^{\infty} f_1^{-n}(U)$.

Now let $U \in \mathcal{B}$ and $x \in \bigcup_{n=1}^{\infty} f_1^{-n}(U)$. This implies that $x \in \bigcup_{n=1}^{\infty} f_1^{-n}(U)$, and hence $x \in f_1^{-n}(U)$ for some $n \in \mathbb{N}$. So $f_1^n(x) \in U$ for some $n \in \mathbb{N}$, which implies that $N(U, x) \neq \emptyset$, and hence $x \in T(f_{1,\infty})$. Therefore $\bigcap_{U \in \mathcal{B}} (\bigcup_{n=1}^{\infty} f_1^{-n}(U)) \subset T(f_{1,\infty})$. Hence $\bigcap_{U \in \mathcal{B}} (\bigcup_{n=1}^{\infty} f_1^{-n}(U)) = T(f_{1,\infty})$. Now by (ix), $\bigcup_{n=1}^{\infty} f_1^{-n}(U)$ is dense in X for all opene $U \subset X$. Thus, $T(f_{1,\infty})$ is countable intersection of opene dense subsets in X. Since X is a compact metric space, $T(f_{1,\infty})$ is a dense, G_δ subset of X. (vi)⇒(v) follows by Proposition 3.2.

To prove (v)⇒(iv): If X is perfect and $x \in T(f_{1,\infty})$, then the orbit $O(x, f_{1,\infty})$ of x intersects every opene set in an infinite set. Thus, $N(U, V)$ is infinite for every opene $U, V \subset X$. To conclude the proof, observe that (iv)⇒(iii) is trivial. □

Corollary 3.4. *Let $(X, f_{1,\infty})$ be a topologically transitive system. Then:*

(i) *any opene, $-$ invariant subset $U \subset X$ is dense in X.*
(ii) *if $A \subset X$ is closed, $+$ invariant, then either $A = X$ or A is nowhere dense in X.*

Proof. We first prove (i). Let the system $(X, f_{1,\infty})$ be topologically transitive, then by condition (ix) of Theorem 3.3, we have for every opene $U \subset X$, $\bigcup_{n=1}^{\infty} f_1^{-n}(U)$ is dense in X. Suppose $U \subset X$ is opene and $-$ invariant, then $f_1^{-n}(U) \subset U$ for all $n \in \mathbb{N}$. Therefore $\bigcup_{n=1}^{\infty} f_1^{-n}(U) \subset U$ and hence U is dense in X. From Lemma 2.3, both (i) and (ii) are equivalent. □

3.2. **Strong transitivity and very strong transitivity.** In this section, we define some notions which are strengthen to topological transitivity.

Definition 3.5 (Strongly transitive)**.** A system $(X, f_{1,\infty})$ is called *strongly transitive* if for every opene subset $V \subset X$, $\bigcup_{n=1}^{\infty} f_1^{n}(V) = X$.

Definition 3.6 (Very strongly transitive)**.** A system $(X, f_{1,\infty})$ is called *very strongly transitive* if for every opene subset $V \subset X$, there exists $k \in \mathbb{N}$ such that $\bigcup_{n=1}^{k} f_1^{n}(V) = X$.

From the definitions it is easy to observe that

$$\text{very strong transitivity} \Rightarrow \text{strong transitivity} \Rightarrow \text{topological transitivity.}$$

We now provide some equivalent conditions for strong transitivity in Theorem 3.7 and for very strong transitivity in Theorem 3.9.

Theorem 3.7. *For an NDDS $(X, f_{1,\infty})$ the following are equivalent:*

(i) *The system $(X, f_{1,\infty})$ is strongly transitive.*
(ii) *For every opene set $U \subset X$ and every point $x \in X$, there exists $n \in \mathbb{N}$ such that $x \in f_1^{n}(U)$.*
(iii) *For every opene set $U \subset X$ and every point $x \in X$, the set $N(U, x)$ is nonempty.*
(iv) *For every $x \in X$, the negative orbit $O^{-}(x, f_{1,\infty})$ is dense in X.*
(v) *For every $x \in X$, and $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $O_N^{-}(x, f_{1,\infty})$ is ϵ dense in X.*

Proof. It is straight forward to observe (i)⇔(ii) from the definition of strong transitivity.
(ii)⇔(iii) follows from the definition of $N(U, x)$.

(ii)⇔(iv): Assume (ii) and suppose $O^{-}(x, f_{1,\infty})$ is not dense in X. Then there exists opene subset $V \subset X$ such that $\bigcup_{n\in\mathbb{N}} f_1^{-n}(x) \cap V = \emptyset$ which implies that $f_1^{-n}(x) \cap V = \emptyset$, for all $n \in \mathbb{N}$. i.e., $x \notin f_1^{n}(V)$ for any $n \in \mathbb{N}$. This contradicts condition (ii). Hence, $O^{-}(x, f_{1,\infty})$ is dense in X. Converse can be proved along the same. (iv)⇔(v) follows by Lemma 2.8. □

Corollary 3.8. *Let $(X, f_{1,\infty})$ be an NDDS. If the system is strongly transitive, then every nonempty, $-$ invariant subset U of X is dense in X.*

Proof. Let $x \in U$. Since U is $-$ invariant, $O^{-}(x, f_{1,\infty}) \subset U$. By the condition (iv) of Theorem 3.7, $O^{-}(x, f_{1,\infty})$ is dense in X. Hence, U is dense in X. □

Theorem 3.9. *For an NDDS $(X, f_{1,\infty})$ the following are equivalent:*

(i) *The system $(X, f_{1,\infty})$ is very strongly transitive.*
(ii) *For all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $O_N^{-}(x, f_{1,\infty})$ is ϵ dense in X for every $x \in X$.*

Proof. First we will prove (i)⇒(ii). Let $\epsilon > 0$ and consider a cover $\{V_i\}_{i\in\mathbb{I}}$ of X by $\epsilon/2$ balls. Since X is compact, there exists a finite subcover say $\{V_1, V_2, \ldots, V_m\}$. Since $(X, f_{1,\infty})$ is very strong transitive for every $i \in \{1, 2, \ldots, m\}$, there exists $N_i \in \mathbb{N}$ such that $\bigcup_{n=1}^{N_i} f_1^{n}(V_i) = X$, for $i = 1, 2, \ldots, m$. Take $N = max\{N_1, N_2, \ldots, N_m\}$. Then $\bigcup_{n=1}^{N} f_1^{n}(V_i) = X$ for $i = 1, 2, \cdots, m$. Therefore, for every $x \in X$ there exists $i \in \{1, 2, \ldots, m\}$ and $m \in \{1, 2, \cdots, N\}$ such that $x \in f_1^{n}(V_i)$. i.e., $f_1^{-n}(x) \cap V_i \neq \emptyset$. Hence, $O_N^{-}(x, f_{1,\infty})$ intersects V_i for every $i \in \{1, 2, \ldots, m\}$. Therefore, $O_N^{-}(x, f_{1,\infty})$ is dense in X.
For (ii)⇒(i): Let $\epsilon > 0$ and let $N \in \mathbb{N}$ be such that $O_N^{-}(x, f_{1,\infty})$ is ϵ dense in X for every $x \in X$. Let V be an ϵ ball in X. Then for every $x \in X$, $f_1^{-n}(x) \cap V \neq \emptyset$ for some n with $1 \leq n \leq N$. Therefore $\bigcup_{n=1}^{N} f_1^{n}(V) = X$. Also, this works for all $\epsilon > 0$ and hence is true for every opene $V \subset X$. This implies that the system is very strongly transitive. □

For surjective open maps we see in the next theorem that very strong transitivity is equivalent to strong transitivity.

Theorem 3.10. *Let $(X, f_{1,\infty})$ be an NDDS and f_n is surjective open map for every $n \in \mathbb{N}$. Then the following are equivalent:*

(i) *The system $(X, f_{1,\infty})$ is very strongly transitive.*
(ii) *The system $(X, f_{1,\infty})$ is strongly transitive.*

Proof. (i)⇔(ii): We can see that (i)⇒(ii) is true by the definitions.
To prove (ii)⇒(i): Assume that for every opene subset $U \subset X$, $\bigcup_{n=1}^{\infty} f_1^{n}(U) = X$. Since f_n is an open map

for all $n \in \mathbb{N}$, then $f_1^n(U)$ is open for all $n \in \mathbb{N}$. Thus, $\bigcup\limits_{n=1}^{\infty} f_1^n(U)$ is an open cover of X and compactness of X implies that there exists $N \in \mathbb{N}$ such that $\bigcup\limits_{n=1}^{N} f_1^n(U) = X$. Hence, the system is very strongly transitive. □

3.3. **Exact transitivity and strong exact transitivity.**

Definition 3.11 (Exact and fully exact)**.** Let $(X, f_{1,\infty})$ be an NDDS. If for every pair of opene subsets $U, V \subset X$, there exists $N \in \mathbb{N}$ such that $f_1^N(U) \cap f_1^N(V) \neq \emptyset$, then the system is called *exact*. The system is called *fully exact* if for all opene $U, V \subset X$, there exists $N \in \mathbb{N}$ such that $(f_1^N(U) \cap f_1^N(V))^\circ \neq \emptyset$.

For open maps, given any opene set $U \subset X$, $f_1^n(U)$ and $(f_1^n(U))^\circ$ are the same. Thus, in case of open maps, a system is exact if and only if it is fully exact.
Next, we give a result for exact systems and an equivalent condition for a system to be fully exact.

Theorem 3.12. (i) *If a system $(X, f_{1,\infty})$ is exact and f_n is injective for every $n \in \mathbb{N}$, then the system is trivial, i.e., X is singleton.*

(ii) *The system $(X, f_{1,\infty})$ is fully exact if and only if for every opene subsets $U, V \subset X$ we have $(\bigcup\limits_{n=1}^{\infty} f_1^n(U) \cap f_1^n(V))^\circ \neq \emptyset$.*

Proof. (i) If X is not singleton, then there exists disjoint opene sets $U, V \subset X$. Since f_n is injective for every $n \in \mathbb{N}$, we have $f_1^n(U) \cap f_1^n(V) = \emptyset$ for all $n \in \mathbb{N}$. This contradicts our assumption that $(X, f_{1,\infty})$ is exact. Hence, X must be singleton.
(ii) The necessity part is trivial. To prove the sufficiency part, assume that for any two opene subsets $U, V \subset X$ we have $(\bigcup\limits_{n=1}^{\infty} f_1^n(U) \cap f_1^n(V))^\circ \neq \emptyset$. Fix two opene subsets $U, V \subset X$. Let $P \subset U$, $Q \subset V$ be closed sets with nonempty interior. By our assumption the open set $W = (\bigcup\limits_{n=1}^{\infty} f_1^n(P) \cap f_1^n(Q))^\circ$ is nonempty, and so it is a Baire space with a countable, relatively closed cover $\{W \cap f_1^n(P) \cap f_1^n(Q) : n \in \mathbb{N}\}$. So by the Baire Category Theorem, some $f_1^n(P) \cap f_1^n(Q)$ has nonempty interior in W and so in X. Hence $\bigcup\limits_{n=1}^{\infty} (f_1^n(P) \cap f_1^n(Q))^\circ \neq \emptyset$ which gives $(\bigcup\limits_{n=1}^{\infty} f_1^n(U) \cap f_1^n(V))^\circ \neq \emptyset$. Therefore $(X, f_{1,\infty})$ is fully exact. □

Definition 3.13 (Exact transitivity and strong exact transitivity)**.** A system $(X, f_{1,\infty})$ is called *exact transitive* if for every opene pair $U, V \subset X$, $\bigcup\limits_{n=1}^{\infty} (f_1^n(U) \cap f_1^n(V))$ is dense in X, and if $\bigcup\limits_{n=1}^{\infty} (f_1^n(U) \cap f_1^n(V)) = X$, then the system is called *strongly exact transitive*.

Remark 3.14. It is easy to see that

(i) strong exact transitivity implies exact transitivity.

(ii) if $(X, f_{1,\infty})$ is exact transitive, then it is both exact and topologically transitive,

(iii) if $(X, f_{1,\infty})$ is strongly exact transitive, then it is fully exact.

Following result gives some characterizations of strongly exact transitive systems.

Theorem 3.15. *Let $(X, f_{1,\infty})$ be an NDDS. Then the following are equivalent:*

(i) *The system $(X, f_{1,\infty})$ is strongly exact transitive.*

(ii) *For every pair of opene sets $U, V \subset X$, $\bigcup_{n\in\mathbb{N}} (f_1^n \times f_1^n)(U \times V)$ contains the diagonal Id_X.*

(iii) *For every $x \in X$, the negative orbit $O^-_{f_{1,\infty}\times f_{1,\infty}}(x, x)$ is dense in $X \times X$.*

(iv) *For every $x \in X$ and opene subsets $U, V \subset X$, there exists $N \in \mathbb{N}$ such that $x \in f_1^N(U) \cap f_1^N(V)$.*

Proof. To complete the proof we establish that (i), (ii) and (iii) are equivalent to (iv). (i)⇔(iv): The system $(X, f_{1,\infty})$ is strongly exact transitive.

⇔ for every opene pair $U, V \subset X$, $\bigcup_{n=1}^{\infty} (f_1^n(U) \cap f_1^n(V)) = X$.

⇔ for every $x \in X$ and opene $U, V \subset X$, there exists $N \in \mathbb{N}$ such that $x \in f_1^N(U) \cap f_1^N(V)$.

(ii)⇔ (iv): For every pair of opene sets $U, V \subset X$, $\bigcup_{n\in\mathbb{N}} (f_1^n \times f_1^n)(U \times V)$ contains the diagonal Id_X.

⇔ for every $x \in X$ and opene $U, V \subset X$, there exists $N \in \mathbb{N}$ such that $(x, x) \in f_1^N(U) \times f_1^N(V)$.

⇔ for every $x \in X$ and opene $U, V \subset X$, there exists $N \in \mathbb{N}$ such that $x \in f_1^N(U) \cap f_1^N(V)$.

(iii)⇔ (iv): For every $x \in X$, the negative orbit $O^-_{f_{1,\infty}\times f_{1,\infty}}(x, x)$ is dense in $X \times X$.

⇔ for every $x \in X$ and opene $U, V \subset X$ there exists $N \in \mathbb{N}$ such that $(x, x) \in (f_1^N \times f_1^N)(U, V) = (x, x) \in f_1^N(U) \times f_1^N(V)$.

⇔ for every $x \in X$ and opene $U, V \subset X$, there exists $N \in \mathbb{N}$ such that $x \in f_1^N(U) \cap f_1^N(V)$. □

3.4. **Topologically mixing and locally eventually onto.** In this section, we introduce the notion of topologically mixing and locally eventually onto system and then we give some equivalent conditions for these dynamical properties.

Definition 3.16 (Topologically mixing)**.** An NDDS $(X, f_{1,\infty})$ is called *topologically mixing* if for every pair of opene subsets $U, V \subset X$, there exists $k \in \mathbb{N}$ such that $f_1^n(U) \cap V \neq \emptyset$ for all $n \geq k$.

Definition 3.17 (Locally eventually onto)**.** If for every opene subset $V \subset X$, there exists $k \in \mathbb{N}$ such that $f_1^k(V) = X$, then the system is called *locally eventually onto*.

In autonomous discrete dynamical system, locally eventually onto implies topologically mixing but in non-autonomous discrete dynamical system, if $(X, f_{1,\infty})$ is locally eventually onto and each f_n is surjective, then the system is topologically mixing, strongly exact transitive and exact transitive.

Theorem 3.18. *For an NDDS $(X, f_{1,\infty})$ the following are equivalent:*

(i) *The system is topologically mixing.*
(ii) *For every pair of opene subsets $U, V \subset X$, $N(U, V)$ is co-finite.*
(iii) *For every opene subset $U \subset X$, and $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $f_1^{-n}(U)$ is ϵ dense in X for all $n \geq N$.*
(iv) *For every opene subset $U \subset X$, and $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $f_1^{n}(U)$ is ϵ dense in X for all $n \geq N$.*

Proof. (i)⇔(ii) follows from the definition of topologically mixing.
For (iii)⇒(ii): Let $U, V \subset X$ be opene sets. We choose $\epsilon > 0$ such that U contains an ϵ ball. By condition (iii), there exists $n \in \mathbb{N}$ such that $f_1^{-n}(V)$ is ϵ dense in X for all $n \geq N$. Then it follows that $U \cap f_1^{-n}(V) \neq \emptyset$ for all $n \geq N$. Hence, $N(U, V)$ is co-finite. (iv)⇒(ii) is similar as (iii)⇒(ii).
Now to prove (ii)⇒(iv): Let $\{U_1, U_2, \dots, U_m\}$ be a finite subcover of X by $\epsilon/2$ balls. Let V be any opene set in X, then by condition (ii), $N(V, U_1), N(V, U_2), \dots N(V, U_m)$ are co-finite sets. Clearly, finite intersection of these co-finite sets is again co-finite. So, there exists $N \in \mathbb{N}$ such that for every $i \in \{1, 2, \dots, m\}$, $f_1^n(V) \cap U_i \neq \emptyset$ for all $n \geq N$. Since $U_i' s$ are $\epsilon/2$ balls covering X, $f_1^n(V)$ intersects any ϵ ball in X for all $n \geq N$. Hence, $f_1^n(V)$ is ϵ dense in X for all $n \geq N$. (ii)⇒(iii) follows similarly from the fact that $N(U_1, V), N(U_2, V), \dots, N(U_m, V)$ are co-finite sets. □

Theorem 3.19. *For an NDDS $(X, f_{1,\infty})$ the following are equivalent:*

(i) *The system is locally eventually onto.*
(ii) *For all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $f_1^{-N}(x)$ is ϵ dense in X for every $x \in X$.*
(iii) *For all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $f_1^{-n}(x)$ is ϵ dense in X for every $x \in X$ and every $n \geq N$.*

Proof. To complete the proof we prove that (i)⇒(iii)⇒(ii)⇒(i).
(i)⇒(iii): Consider a finite subcover $\{V_1, V_2, \dots, V_m\}$ of X by $\epsilon/2$ balls using compactness of X. By condition (i), for every $i \in \{1, 2, \dots, m\}$, there exists $N_i \in \mathbb{N}$ such that $f_1^n(V_i) = X$ for all $n \geq N_i$. Now take $N = max\{N_1, N_2, \dots, N_m\}$, then for all $i \in \{1, 2, \dots, m\}$ and $n \geq N$, $f_1^n(V_i) = X$. This implies that for every V_i, $f_1^{-n}(x) \cap V_i \neq \emptyset$ for every $x \in X$ and every $n \geq N$. Since $V_i' s$ are $\epsilon/2$ balls covering X, condition (iii) follows. (iii)⇒(ii) is trivial.
To prove (ii)⇒(i): Let V be any opene set in X. We choose $\epsilon > 0$ such that V contains an ϵ ball. Then by condition (ii), there exists $N \in \mathbb{N}$ such that $f_1^{-N}(x)$ is ϵ dense in X. Then $f_1^{-N}(x) \cap V \neq \emptyset$ for every $x \in X$, i.e., $\{x\} \cap f_1^N(V) \neq \emptyset$ for every $x \in X$. Thus $f_1^N(V) = X$. □

3.5. **Semiconjugacy.** Let $(X, f_{1,\infty})$ and $(Y, g_{1,\infty})$ be two non-autonomous discrete dynamical systems. If $\pi : X \to Y$ is a continuous surjection such that $\pi \circ f_1^n = g_1^n \circ \pi$ for each $n \in \mathbb{N}$, then π is called a *semiconjugacy*.

Theorem 3.20. *Let $\pi : (X, f_{1,\infty}) \to (Y, g_{1,\infty})$ be a semiconjugacy. If $(X, f_{1,\infty})$ is topologically transitive, strongly transitive, very strongly transitive, exact transitive, strongly exact transitive, exact or locally eventually onto, then $(Y, g_{1,\infty})$ satisfies the corresponding property.*

Proof. Suppose $(X, f_{1,\infty})$ is strongly transitive. Then for any opene subset U in Y, $\pi^{-1}(U)$ is opene in X and so $X = \bigcup_{n=1}^{\infty} f_1^n(\pi^{-1}(U))$. It follows that $Y = \pi(X) = \bigcup_{n=1}^{\infty} \pi f_1^n(\pi^{-1}(U)) = \bigcup_{n=1}^{\infty} g_1^n \pi(\pi^{-1}(U)) = \bigcup_{n=1}^{\infty} g_1^n(U)$. Thus $(Y, g_{1,\infty})$ is strongly transitive.
If $(X, f_{1,\infty})$ is very strongly transitive then we use the same proof with $\bigcup_{n=1}^{\infty} f_1^n(\pi^{-1}(U))$ is replaced by $\bigcup_{n=1}^{N} f_1^n(\pi^{-1}(U))$ for sufficiently large N depending on $\pi^{-1}(U)$ and obtain the result with $\bigcup_{n=1}^{\infty} g_1^n(U)$ replaced by $\bigcup_{n=1}^{N} g_1^n(U)$. If $(X, f_{1,\infty})$ is locally eventually onto, we replace $\bigcup_{n=1}^{\infty} f_1^n(\pi^{-1}(U))$ by $f_1^N(\pi^{-1}(U))$ and obtain the result with $\bigcup_{n=1}^{\infty} g_1^n(U)$ replaced by $g_1^N(U)$. The remaining properties can be proved similarly by using the inclusion $\pi[f_1^n(\pi^{-1}(U)) \cap f_1^n(\pi^{-1}(V))] \subset g_1^n(U) \cap g_1^n(V)$. □

Given two sequences $f_{1,\infty} = (f_n)_{n\in\mathbb{N}}$ and $g_{1,\infty} = (g_n)_{n\in\mathbb{N}}$ of self maps on X and Y respectively, we denote the sequence $(f_n \times g_n)_{n\in\mathbb{N}}$ of self maps on $X \times Y$ by $f_{1,\infty} \times g_{1,\infty}$.

Using the fact that every projection is a semiconjugacy, we can easily prove the following theorem.

Theorem 3.21. *If $(X \times Y, f_{1,\infty} \times g_{1,\infty})$ is strongly transitive, very strongly transitive, exact transitive, strongly exact transitive, exact or locally eventually onto, then both $(X, f_{1,\infty})$ and $(Y, g_{1,\infty})$ satisfy the corresponding property.*

Theorem 3.22. *Assume $(Y, g_{1,\infty})$ is topologically mixing. If $(X, f_{1,\infty})$ is topologically transitive or topologically mixing, then $(X \times Y, f_{1,\infty} \times g_{1,\infty})$ satisfies the corresponding property.*

Proof. Let $U_1, V_1 \subset X$ and $U, V \subset Y$ be opene subsets. Since $(Y, g_{1,\infty})$ is topologically mixing, $N(U, V)$ is co-finite. Hence $N(U_1, V_1) \cap N(U, V)$ is infinite whenever $N(U_1, V_1)$ is infinite, therefore $(X \times Y, f_{1,\infty} \times g_{1,\infty})$ is topologically transitive whenever $(X, f_{1,\infty})$ is topologically transitive. Similarly, $N(U_1, V_1) \cap N(U, V)$ is co-finite whenever $N(U_1, V_1)$ is co-finite, therefore $(X \times Y, f_{1,\infty} \times g_{1,\infty})$ is topologically mixing whenever $(X, f_{1,\infty})$ is topologically mixing. □

Theorem 3.23. *Assume $(Y, g_{1,\infty})$ is locally eventually onto. If $(X, f_{1,\infty})$ is strongly transitive, very strongly transitive, exact, fully exact, exact transitive, strongly exact transitive or locally eventually onto, then $(X \times Y, f_{1,\infty} \times g_{1,\infty})$ satisfies the corresponding property.*

Proof. Let $U_1 \subset X$ and $U \subset Y$ be opene subsets. Since $(Y, g_{1,\infty})$ is locally eventually onto, there exists $N \in \mathbb{N}$ such that $g_1^n(U) = Y$ for all $n \geq N$. If $L \subset \mathbb{N}$ so that $\bigcup\{f_1^n(U_1) : n \in L\} = X$ and there exists $N \in L$ with $n \geq N$, then $\bigcup\{(f_1^n \times g_1^n)(U_1 \times U) : n \in L\} = X \times Y$. Using $L = \mathbb{N}$ we obtain the result for strong transitivity. Using $L = \{1, \ldots N_1\}$ with N_1 sufficiently large, we obtain the result for very strong transitivity. Using $L = \{N_1\}$ with N_1 sufficiently large, we obtain the result for locally eventually onto. Remaining properties can be proved easily. □

In the next three results, we provide the sufficient conditions under which the topological transitivity and mixing are preserved by every finite rearrangement and the locally eventually onto property is preserved by every rearrangement of the given system.

Proposition 3.24. *Let $(X, f_{1,\infty})$ be an NDDS where $f_m \circ f_n = f_n \circ f_m$ for all $m, n \in \mathbb{N}$. If $(X, f_{1,\infty})$ is topologically transitive, then $(X, g_{1,\infty})$ is toplogically trasnitive for every finite rearrangement $g_{1,\infty}$ of $f_{1,\infty}$.*

Proof. Let $g_{1,\infty}$ is a finite rearrangement of $f_{1,\infty}$, i.e., there exists $N \in \mathbb{N}$ such that $\{f_1, f_2, \ldots, f_N\} = \{g_1, g_2, \ldots, g_N\}$ and $f_i = g_i$ for all $i \geq N + 1$. Let $U, V \subset X$ be opene. Since $f_m \circ f_n = f_n \circ f_m$ for all $m, n \in \mathbb{N}$, we can see that $f_1^k(U) = g_1^k(U)$ for all $k \geq N$. Since $(X, f_{1,\infty})$ is topologically transitive, there exists $k \geq N$ such that $f_1^k(U) \cap V \neq \emptyset$. Therefore $g_1^k(U) \cap V \neq \emptyset$. Hence, $(X, g_{1,\infty})$ is topologically transitive. □

Proposition 3.25. *Let $(X, f_{1,\infty})$ be an NDDS where $f_m \circ f_n = f_n \circ f_m$ for all $m, n \in \mathbb{N}$. If $(X, f_{1,\infty})$ is topologically mixing, then $(X, g_{1,\infty})$ is toplogically mixing for every finite rearrangement $g_{1,\infty}$ of $f_{1,\infty}$.*

Proof. The proof is similar to that of Proposition 3.24. □

Proposition 3.26. *Let $(X, f_{1,\infty})$ be an NDDS where each f_n is surjective and $f_m \circ f_n = f_n \circ f_m$ for all $m, n \in \mathbb{N}$. If $(X, f_{1,\infty})$ is locally eventually onto, then $(X, g_{1,\infty})$ is locally eventually onto for every rearrangement $g_{1,\infty}$ of $f_{1,\infty}$.*

Proof. Suppose $(X, f_{1,\infty})$ is locally eventually onto. Let $V \subset X$ be opene. Then there exists $k \in \mathbb{N}$ such that $f_1^k(V) = X$. Since $g_{1,\infty}$ is a rearrangement of $f_{1,\infty}$ we can find a natural number N such that $\{f_1, f_2, \ldots, f_k\} \subset \{g_1, g_2, \ldots, g_N\}$. Then $g_1^N(V) = X$, because each f_n is surjective and $f_m \circ f_n = f_n \circ f_m$ for all $m, n \in \mathbb{N}$. Hence, $(X, g_{1,\infty})$ is locally eventually onto. □

4. Conclusions

We studied the variations of topological transitiviy in NDDS. In the case of NDDS, semiconjugacy preserves topological transitivity, strong transitivity, very strong transitivity, exact transitivity, strong exact transitivity, exact and locally eventually onto. The following is the schematic representation of implications among the notions of transitivity for non-autonomous discrete dynamical systems:

$$\text{LEO} \quad \Longrightarrow \quad \text{VST} \quad \Longrightarrow \quad \text{ST} \quad \Longrightarrow \quad \text{TT.}$$

If each f_n is surjective, we have

$$\begin{array}{ccccc} \text{LEO} & & \Longrightarrow & & \text{TM} \\ \Downarrow & & & & \Downarrow \\ \text{SET} & \Longrightarrow & \text{ET} & \Longrightarrow & \text{TT.} \end{array}$$

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