Procrustes problem for the inverse eigenvalue problem of normal (skew) J-Hamiltonian matrices and normal J-symplectic matrices

S. Gigola* L. Lebtahi† N. Thome‡

Abstract

A square complex matrix A is called (skew) J-Hamiltonian if AJ is (skew) Hermitian where J is a real normal matrix such that $J^2 = -I$, where I is the identity matrix. In this paper, we solve the Procrustes problem to find normal (skew) J-Hamiltonian solutions for the inverse eigenvalue problem. In addition, a similar problem is investigated for normal J-symplectic matrices.

Keywords: Inverse eigenvalue problem, (skew) J-Hamiltonian matrix, J-symplectic matrix, Moore-Penrose inverse, Procrustes problem.
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1 Introduction

Inverse eigenvalue problems arise as important tools in several research subjects such as structural design, parameter identification, modeling, etc. [5, 13, 30]. Inverse eigenvalue problems consist of the construction of a matrix A with a determined structure and a specified spectrum. In the literature, these problems were studied under certain constraints on A as in [1, 7, 10, 21, 22, 26, 27, 28, 30, 31]. For instance, by using Hermitian generalized Hamiltonian matrices, the associated inverse eigenvalue problem was solved in [31]. Moreover, in [1], the general solution of the inverse eigenvalue problem for Hermitian generalized skew Hamiltonian matrices was given and the expression of the solution for the optimal approximation problem was obtained. For symplectic matrices and several factorizations related to them, we refer the reader to [29].

*Departamento de Matemática. Facultad de Ingeniería. Universidad de Buenos Aires. Buenos Aires, Argentina. E-mail: sgigola@fi.uba.ar.

†Facultat de Ciències Matemàtiques, Departament de Matemàtiques, Universitat de València. Valencia, Spain. E-mail: leila.lebtahi@uv.es.

‡Instituto Universitario de Matemática Multidiscilpinar. Universitat Politècnica de València. Valencia, Spain. E-mail: njthome@mat.upv.es.

Our interest is to study the Procrustes problem associated with a specific class of matrices (J-Hamiltonian, skew J-Hamiltonian, and J-symplectic matrices) as we will detail in what follows.

It is remarkable that Hamiltonian matrices play an important role in several engineering areas such as optimal quadratic linear control [20, 24], H_∞ optimization [33] and the solution of algebraic Riccati equations [2, 16], among others. In [17], symplectic matrices are studied from a point of view of their connection with predetermined left eigenvalues.

Specifically, for given $k \times k$ real matrices A, B, and C with B and C being symmetric matrices, the algebraic Riccati equation $XA + A^T X - XBX + C = O$ has solution $X = -VU^{-1}$, where the block matrix $\begin{bmatrix} U \\ V \end{bmatrix}$ spans an H-invariant subspace of the corresponding subspace for the Hamiltonian matrix given by $H = \begin{bmatrix} A & B \\ C & -A^T \end{bmatrix}$ provided that U is nonsingular [15]. We recall that a $2k \times 2k$ real matrix G is Hamiltonian if the product $G \begin{bmatrix} O & I_k \\ -I_k & O \end{bmatrix}$ is symmetric, where I_k will stand for the $k \times k$ identity matrix. Moreover, H determines the form of all the Hamiltonian matrices.

On the other hand, important applications for symplectic matrices arise mainly in Mathematical Physics in areas such as quantum mechanics. For instance, symplectic matrices give the phase-space representation of Gaussian transformations in quantum photonics [4]. In addition, Linear Quadratic Control Problem [20] requires this kind of matrices in their investigations.

The symbols $\mathbb{C}^{m\times n}$ and $\mathbb{R}^{m\times n}$ stand for the sets of $m \times n$ complex real matrices. The symbols M^* and $M^\dagger$ will denote the conjugate transpose and the Moore-Penrose inverse of a matrix M, respectively. We remind that for a given rectangular complex matrix $M \in \mathbb{C}^{m\times n}$, its Moore-Penrose inverse is the (unique) matrix $M^\dagger \in \mathbb{C}^{n\times m}$ that satisfies $MM^\dagger M = M$, $M^\dagger MM^\dagger = M^\dagger$, $(MM^\dagger)^* = MM^\dagger$, and $(M^\dagger M)^* = M^\dagger M$. This matrix always exists [3]. We also need the following notation for both specified orthogonal projectors: $Q_M = I_n - M^\dagger M$ and $P_M = I_m - MM^\dagger$. It is worth noting that $M^\dagger P_M = O$ and $P_M(M^\dagger)^* = O$. We will use the inner product given by $\langle A, B\rangle = \mathrm{trace}(AB^*)$ for $A, B \in \mathbb{C}^{n\times n}$ and the Frobenius norm of A given by $\|A\|_F = \sqrt{\langle A, A\rangle}$. The symbol $\sigma(A)$ denotes the spectrum of the matrix $A \in \mathbb{C}^{n\times n}$.

Let $J \in \mathbb{R}^{n\times n}$ be a normal matrix such that $J^2 = -I_n$. It is well known that a matrix $A \in \mathbb{C}^{n\times n}$ is called J-Hamiltonian if $(AJ)^* = AJ$ [11]. Moreover, $A \in \mathbb{C}^{n\times n}$ is called skew J-Hamiltonian if $(AJ)^* = -AJ$.

On the other hand, a matrix $A \in \mathbb{C}^{n\times n}$ is called J-symplectic if $A^*JA = J$. Every J-symplectic matrix A is nonsingular, with determinant satisfying $|\det(A)| = 1$ and its inverse is given by $A^{-1} = J^{-1}A^*J = -JA^*J$ since J is skew-Hermitian. Even more, it is well-known [23, 6, 19] that $\det(A) = 1$, which means that the algebraic multiplicities of the eigenvalues 1 and -1 are both even. For some applications of symplectic matrices, we refer the reader for example to [18].

Throughout all this paper, we will consider a fixed real normal matrix J such that $J^2 = -I$, where I denotes the identity matrix of adequate size.

Observe that, if $A \in \mathbb{C}^{n\times n}$ then the spectral condition for A to be J-Hamiltonian is that $\lambda \in \sigma(A)$ if and only if $-\bar{\lambda} \in \sigma(A)$, that is, $\lambda \in \sigma(A)$ implies $\{\lambda, -\bar{\lambda}\} \subseteq \sigma(A)$. Even more, when $A \in \mathbb{R}^{n\times n}$, the condition $\lambda \in \sigma(A)$ implies $\{\lambda, -\lambda, \bar{\lambda}, -\bar{\lambda}\} \subseteq \sigma(A)$, since in real polynomials non-real roots appear in pairs of the type $a \pm bi$. This fact forces the values of the diagonal of the given matrix D have to obey these symmetry properties. Notice that for

$$A = \begin{bmatrix} 2i & i \\ -i & 2i \end{bmatrix} \qquad \text{and} \qquad J = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

we have that the normal matrix A is J-Hamiltonian and $\sigma(A) = \{1 + 2i, -1 + 2i\}$. Thus, $\lambda \in \sigma(A)$ does not imply $\bar{\lambda} \in \sigma(A)$.

Otherwise, if $A \in \mathbb{C}^{n\times n}$ is skew J-Hamiltonian then $\lambda \in \sigma(A)$ if and only if $\bar{\lambda} \in \sigma(A)$, that is, if $\lambda \in \sigma(A)$ then $\{\lambda, \bar{\lambda}\} \subseteq \sigma(A)$. However, in general, $\lambda \in \sigma(A)$ does not imply $-\lambda \in \sigma(A)$ as the following example shows:

$$A = \begin{bmatrix} 3/2 & i/2 \\ -i/2 & 3/2 \end{bmatrix} \qquad \text{and} \qquad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

This skew J-Hamiltonian matrix A satisfies $1 \in \sigma(A) = \{1, 2\}$ but $-1 \notin \sigma(A)$.

On the other hand, the spectral condition for A to be J-symplectic is that $\lambda \in \sigma(A)$ implies $\left\{\lambda, \bar{\lambda}, \frac{1}{\lambda}, \frac{1}{\bar{\lambda}}\right\} \subseteq \sigma(A)$.

For a given full-column rank matrix $X \in \mathbb{C}^{n\times m}$ and a given diagonal matrix $D \in \mathbb{C}^{m\times m}$, we introduce the set $\mathcal{S}$ of all solutions of the inverse eigenvalue problem $AX = XD$, that is,

$$\mathcal{S} = \{A \in \mathbb{C}^{n\times n} : AX = XD\}. \tag{1}$$

The set $\mathcal{S}$, with the additional assumption that A is a normal J-Hamiltonian matrix, is studied in [11]; the set $\mathcal{S}$, with the additional assumption that A is a normal skew J-Hamiltonian matrix, is studied in [32]. However, to our knowledge, the set $\mathcal{S}$ for A being a normal J-symplectic matrix, has not yet been studied.

We will consider the following sets:

$$\mathcal{NJH} := \{A \in \mathbb{C}^{n\times n} : AA^* = A^*A, (AJ)^* = AJ\},$$

$$\mathcal{NJSH} := \{A \in \mathbb{C}^{n\times n} : AA^* = A^*A, (AJ)^* = -AJ\},$$

and

$$\mathcal{NJS} := \{A \in \mathbb{C}^{n\times n} : AA^* = A^*A, A^*JA = J\}.$$

Statement problems: This paper solves the Procrustes problem associated with normal (respectively, skew) J-Hamiltonian matrices, as well as, to normal J-symplectic matrices. That is, assuming that $\mathcal{S} \cap \mathcal{T} \neq \emptyset$, for a given matrix $\widetilde{A} \in \mathbb{C}^{n\times n}$, we study the existence and uniqueness of matrices $\widehat{A} \in \mathcal{S} \cap \mathcal{T}$ such that

$$\min_{A\in\mathcal{S}\cap\mathcal{T}} \|\widetilde{A} - A\|_F = \|\widetilde{A} - \widehat{A}\|_F, \tag{2}$$

for each one of the following cases: $\mathcal{T} = \mathcal{NJH}$ or $\mathcal{T} = \mathcal{NJSH}$, or $\mathcal{T} = \mathcal{NJS}$.

This paper is organized as follows. Section 2 solves the Procrustes problem for normal J-Hamiltonian matrices. Section 3 is dedicated to solve the Procrustes problem corresponding to normal skew J-Hamiltonian matrices. Finally, in Section 4, we present and solve the inverse eigenvalue problem and the Procrustes problem associated with normal J-symplectic matrices, as well as a numerical example is provided.

2 Optimization problem for normal J-Hamiltonian matrices

Throughout this section, for a given full-column rank matrix $X \in \mathbb{C}^{n\times m}$, a given diagonal matrix $D \in \mathbb{C}^{m\times m}$, and a given matrix $\widetilde{A} \in \mathbb{C}^{n\times n}$, assuming that $\mathcal{S} \cap \mathcal{NJH} \neq \emptyset$ in (1), we treat the existence and uniqueness problem of finding $\widehat{A} \in \mathcal{S} \cap \mathcal{NJH}$ such that

$$\min_{A\in\mathcal{S}\cap\mathcal{NJH}} \|\widetilde{A} - A\|_F = \|\widetilde{A} - \widehat{A}\|_F. \tag{3}$$

Remark 1 *We note that diagonalizable (even singular) matrices give rise to a wide class of matrices that may appear in the inverse eigenvalue problem (1), which can be constructed by different matrices X and D. For instance, $\mathcal{S} \neq \emptyset$ whenever $m > n$ and $X = \begin{bmatrix} X_1 \\ O \end{bmatrix}$ where $X_1 \in \mathbb{C}^{n\times n}$ is a fixed diagonal matrix (because the system $AX_1 = X_1D$ has, trivially, a diagonal solution). Hence, we can assure that infinitely many solutions are found (since all matrices B such that $B(PX) = (PX)D$ belong to $\mathcal{S}$, for any nonsingular matrix P). Given that the set of diagonalizable matrices is dense in the set of all matrices [12], we conclude that, in practice, it is quite common to find situations where matrices solving (1) do appear (for example, all matrices of index 1). In addition, $\mathcal{S}$ may contains matrices satisfying $AA^* = A^*A$ and $(AJ)^* = AJ$, for example, matrices having the form of the direct sum $D_1 \oplus (-D_1^*)$, for D_1 being a diagonal matrix of adequate size, that is, $\mathcal{S} \cap \mathcal{NJH} \neq \emptyset$.*

Uniqueness of solution of Procrustes problem for $\mathcal{T} = \mathcal{NJH}$

Once the existence of solution of (3) is proven, for a non-empty set $\mathcal{S} \cap \mathcal{NJH}$, we have that

$$\mathcal{S} \cap \mathcal{NJH} = A_0 + \{A \in \mathbb{C}^{n\times n} : AX = 0\}$$

is an affine subspace of $\mathbb{C}^{n\times n}$ (with the canonical affine structure [9]) for any $A_0 \in \mathcal{S}\cap\mathcal{NJH}$. By [9, p. 245], the uniqueness of solution of (3) is guaranteed since that the distance of a point to an affine subspace is always attained in a single point.

From now on, we focus our efforts on the proof of the existence problem.

The general expression for matrices $A \in \mathcal{S} \cap \mathcal{NJH}$

Since $J^2 = -I_n$, it is clear that $n = 2k$ for some positive integer k. Then, there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that

$$J = U \begin{bmatrix} iI_k & O \\ O & -iI_k \end{bmatrix} U^*. \tag{4}$$

Let us consider the following partition

$$X = U \begin{bmatrix} X_1 \\ X_2 \end{bmatrix},$$

where $X_1, X_2 \in \mathbb{C}^{k\times m}$. The general normal J-Hamiltonian solution of the inverse eigenvalue problem (1) is given by (see [11, Theorem 2])

$$A = U \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{bmatrix} U^*, \tag{5}$$

where

$$A_{12} = X_1 D Q_{X_1} \left(X_2 Q_{X_1}\right)^\dagger + Y_{12} P_{X_2 Q_{X_1}}, \tag{6}$$

for arbitrary $Y_{12} \in \mathbb{C}^{k\times k}$,

$$\begin{aligned} A_{11} &= \left[X_1 D - X_1 D Q_{X_1} \left(X_2 Q_{X_1}\right)^\dagger X_2 - Y_{12} P_{X_2 Q_{X_1}} X_2\right] X_1^\dagger + \\ &\quad + (X_1^\dagger)^* X_2^* P_{X_2 Q_{X_1}} Y_{12}^* P_{X_1} + P_{X_1} Y_{11} P_{X_1}, \end{aligned} \tag{7}$$

and

$$\begin{aligned} A_{22} &= \left[X_2 D - \left(Q_{X_1} X_2^*\right)^\dagger Q_{X_1} D^* X_1^* X_1 - P_{X_2 Q_{X_1}} Y_{12}^* X_1\right] X_2^\dagger + \\ &\quad + (X_2^\dagger)^* \left[X_1^* X_1 D Q_{X_1} \left(X_2 Q_{X_1}\right)^\dagger - X_1^* Y_{12} P_{X_2 Q_{X_1}}\right] P_{X_2} + P_{X_2} Y_{22} P_{X_2}, \end{aligned} \tag{8}$$

for arbitrary skew-Hermitian $Y_{11}, Y_{22} \in \mathbb{C}^{k\times k}$ under the following conditions:

$$X_1 D Q_{X_1} Q_{X_2 Q_{X_1}} = O, \tag{9}$$

$$\left[X_1 D - X_1 D Q_{X_1} (X_2 Q_{X_1})^\dagger X_2 - Y_{12} P_{X_2 Q_{X_1}} X_2\right] Q_{X_1} = O, \tag{10}$$

$$\left[X_2 D - (Q_{X_1} X_2^*)^\dagger Q_{X_1} D^* X_1^* X_1 - P_{X_2 Q_{X_1}} Y_{12}^* X_1\right] Q_{X_2} = O, \tag{11}$$

$A_{11}A_{12} = A_{12}A_{22}$, and the matrices

$$X_1^* \left(X_1 D - A_{12} X_2\right) \quad \text{and} \quad X_2^* \left(X_2 D - A_{12}^* X_1\right) \tag{12}$$

are skew-Hermitian.

Note that the condition (12) is equivalent to $X_1^* X_1 D - X_2^* X_2 D$ is skew-Hermitian.

In order to find an optimal solution $\widehat{A}$, as we have stated in (3), we first transform the problem into a more simple one.

The simplified problem

Let $\widetilde{A} \in \mathbb{C}^{n\times n}$ be an arbitrary given matrix such that $U^*\widetilde{A}U$ is partitioned as

$$U^*\widetilde{A}U = \left[\begin{array}{cc} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & \widetilde{A}_{22} \end{array}\right], \tag{13}$$

where $\widetilde{A}_{ij} \in \mathbb{C}^{k\times k}$ and U is the same matrix as in (4). Since Frobenius norm is unitarily invariant, by using the matrix $A \in \mathcal{S} \cap \mathcal{NJH}$ given in (5), we obtain

$$\begin{array}{rcl} \|\widetilde{A} - A\|_F^2 &=& \|U^*\widetilde{A}U - U^*AU\|_F^2 \\ &=& \|\widetilde{A}_{11} - A_{11}\|_F^2 + \|\widetilde{A}_{12} - A_{12}\|_F^2 + \|\widetilde{A}_{21} - A_{12}^*\|_F^2 + \|\widetilde{A}_{22} - A_{22}\|_F^2. \end{array}$$

Now, we study each of these four norms separately.

Explicit form of the solution

Next results are needed in order to minimize the norm of some matrices with special structures. We quote Lemma 1 in [27] with a slight modification.

Lemma 1 *Let $B \in \mathbb{C}^{q\times m}$, $P_1 \in \mathbb{C}^{q\times q}$, and $P_2 \in \mathbb{C}^{m\times m}$ where $P_i^2 = P_i = P_i^*$ for $i = 1, 2$. Then the following statements hold:*

(i) there exists the minimum of $\|B - P_1EP_2\|_F$ for E ranging the set $\mathbb{C}^{q\times m}$.

(ii) This minimum is attained in E if and only if E satisfies $P_1(B - E)P_2 = O$.

(iii) $\min\limits_{E\in\mathbb{C}^{q\times m}} \|B - P_1EP_2\|_F = \|B - P_1BP_2\|_F$.

Lemma 2 *Let $M \in \mathbb{C}^{m\times n}$. The equation $P_MYP_M = O$ has always a (nontrivial) skew-Hermitian solution $Y \in \mathbb{C}^{n\times m}$.*

Proof. It follows as an immediate consequence of applying [14, Theorem 2.4] to iX with $C = O$ and $A = B = P_M$. ■

Now, we are ready to give the explicit solution to problem (3). Assume that $\mathcal{S}\cap\mathcal{NJH} \neq \emptyset$.

We start by studying $\|\widetilde{A}_{12} - A_{12}\|_F^2$ since it contains the arbitrary matrix Y_{12} that also appears in the expressions of A_{11} and A_{22}.

Under the condition (9), by using expression (6) of A_{12}, Lemma 1 assures that

$$\|\widetilde{A}_{12} - A_{12}\|_F^2 = \left\|\widetilde{A}_{12} - X_1DQ_{X_1}\left(X_2Q_{X_1}\right)^\dagger - Y_{12}P_{X_2Q_{X_1}}\right\|_F^2$$

attains its minimum in Y_{12} if and only if Y_{12} satisfies

$$\left[\widetilde{A}_{12} - X_1DQ_{X_1}\left(X_2Q_{X_1}\right)^\dagger - Y_{12}\right]P_{X_2Q_{X_1}} = O.$$

Since $(X_2Q_{X_1})^\dagger P_{X_2Q_{X_1}} = O$, the last equality simplifies to

$$(\widetilde{A}_{12} - Y_{12})P_{X_2Q_{X_1}} = O. \tag{14}$$

In this case,

$$\min_{A_{12}\in\mathbb{C}^{k\times k}} \|\widetilde{A}_{12} - A_{12}\|_F^2 = \left\| \left[\widetilde{A}_{12}X_2 - X_1D\right] Q_{X_1} \left(X_2Q_{X_1}\right)^\dagger \right\|_F^2,$$

since $R - RP_M = RMM^\dagger$ for arbitrary matrices R and M of adequate sizes.

Recalling that the Frobenius norm is invariant under taking adjoint and applying the last result to

$$\|\widetilde{A}_{21} - A_{12}^*\|_F^2 = \|((\widetilde{A}_{21})^* - A_{12})^*\|_F^2 = \|(\widetilde{A}_{21})^* - A_{12}\|_F^2$$

we have

$$\min_{A_{12}\in\mathbb{C}^{k\times k}} \|(\widetilde{A}_{21})^* - A_{12}\|_F^2 = \left\| \left[(\widetilde{A}_{21})^*X_2 - X_1D\right] Q_{X_1} \left(X_2Q_{X_1}\right)^\dagger \right\|_F^2$$

if and only if Y_{12} satisfies

$$((\widetilde{A}_{21})^* - Y_{12})P_{X_2Q_{X_1}} = O. \tag{15}$$

From (6) we obtain

$$\widehat{A}_{12} = X_1DQ_{X_1} \left(X_2Q_{X_1}\right)^\dagger + \widetilde{A}_{12}P_{X_2Q_{X_1}}. \tag{16}$$

It is clear that from (14) and (15) we get

$$\widetilde{A}_{12}P_{X_2Q_{X_1}} = Y_{12}P_{X_2Q_{X_1}} = (\widetilde{A}_{21})^*P_{X_2Q_{X_1}}.$$

Conversely, if $\widetilde{A}_{12}P_{X_2Q_{X_1}} = (\widetilde{A}_{21})^*P_{X_2Q_{X_1}} =: Z$, post-multiplying both sides by $P_{X_2Q_{X_1}}$ we get

$$\widetilde{A}_{12}P_{X_2Q_{X_1}} = (\widetilde{A}_{21})^*P_{X_2Q_{X_1}} = ZP_{X_2Q_{X_1}}.$$

Now, $Y_{12} := Z$ satisfies (14) and (15).

Since $Y_{12}P_{X_2Q_{X_1}} = \widetilde{A}_{12}P_{X_2Q_{X_1}} = (\widetilde{A}_{21})^*P_{X_2Q_{X_1}}$, the conditions (10) and (11) are equivalent to

$$\left[X_1D - X_1DQ_{X_1}(X_2Q_{X_1})^\dagger X_2 - \widetilde{A}_{12}P_{X_2Q_{X_1}}X_2\right] Q_{X_1} = O \tag{17}$$

and

$$\left[X_2D - (Q_{X_1}X_2^*)^\dagger Q_{X_1}D^*X_1^*X_1 - P_{X_2Q_{X_1}}\widetilde{A}_{21}X_1\right] Q_{X_2} = O, \tag{18}$$

respectively. Moreover, expression (7) gives

$$\begin{aligned}\|\widetilde{A}_{11} - A_{11}\|_F^2 \;\; = \;\; & \Big\|\widetilde{A}_{11} - \left[X_1D - X_1DQ_{X_1} \left(X_2Q_{X_1}\right)^\dagger X_2 - \widetilde{A}_{12}P_{X_2Q_{X_1}}X_2\right] X_1^\dagger - \\ & -(X_1^\dagger)^*X_2^*P_{X_2Q_{X_1}}\widetilde{A}_{21}P_{X_1} - P_{X_1}Y_{11}P_{X_1}\Big\|_F^2.\end{aligned}$$

Denoting by R the fixed terms in the last norm, that is,

$$\begin{aligned} R &= \widetilde{A}_{11} - \left[X_1 D - X_1 D Q_{X_1} (X_2 Q_{X_1})^{\dagger} X_2 - \widetilde{A}_{12} P_{X_2 Q_{X_1}} X_2\right] X_1^{\dagger} - \\ &\quad -(X_1^{\dagger})^* X_2^* P_{X_2 Q_{X_1}} \widetilde{A}_{21} P_{X_1}, \end{aligned}$$

Lemma 1 ensures that $\|\widetilde{A}_{11} - A_{11}\|_F^2$ attains its minimum in Y_{11} if and only if $P_{X_1}(R - Y_{11})P_{X_1} = O$, which is equivalent to

$$P_{X_1}(\widetilde{A}_{11} - Y_{11})P_{X_1} = O, \tag{19}$$

since $X_1^{\dagger} P_{X_1} = O$ and $P_{X_1}(X_1^{\dagger})^* = O$. In this case,

$$\min_{A_{11} \in \mathbb{C}^{k \times k}} \|\widetilde{A}_{11} - A_{11}\|_F^2 = \|R - P_{X_1} \widetilde{A}_{11} P_{X_1}\|_F^2.$$

Then, by using (7) we obtain

$$\begin{aligned} \widehat{A}_{11} &= \left[X_1 D - X_1 D Q_{X_1} (X_2 Q_{X_1})^{\dagger} X_2 - \widetilde{A}_{12} P_{X_2 Q_{X_1}} X_2\right] X_1^{\dagger} + \\ &\quad +(X_1^{\dagger})^* X_2^* P_{X_2 Q_{X_1}} \widetilde{A}_{21} P_{X_1} + P_{X_1} \widetilde{A}_{11} P_{X_1}. \end{aligned} \tag{20}$$

By Lemma 2, (19) has always a skew-Hermitian solution $\widetilde{A}_{11} - Y_{11}$ for a skew-Hermitian Y_{11}. Then, $\widetilde{A}_{11}$ is skew-Hermitian as well.

Conversely, if $\widetilde{A}_{11}$ is skew-Hermitian then equation $P_{X_1} C P_{X_1} = O$ has a skew-Hermitian solution C. Setting $Y_{11} := \widetilde{A}_{11} - C$, we get that Y_{11} is a skew-Hermitian matrix that satisfies (19).

A similar reasoning allows us to say that the minimum of $\|\widetilde{A}_{22} - A_{22}\|_F^2$ is attained in Y_{22} if and only if $P_{X_2}(\widetilde{A}_{22} - Y_{22})P_{X_2} = O$ which is equivalent to $\widetilde{A}_{22}$ is skew-Hermitian. Finally, from (8) the last norm is minimized by

$$\begin{aligned} \widehat{A}_{22} &= \left[X_2 D - \left(Q_{X_1}) X_2^*\right)^{\dagger} Q_{X_1} D^* X_1^* X_1 - P_{X_2 Q_{X_1}} \widetilde{A}_{21} X_1\right] X_2^{\dagger} + \\ &\quad +(X_2^{\dagger})^* \left[X_1^* X_1 D Q_{X_1} (X_2 Q_{X_1})^{\dagger} - X_1^* \widetilde{A}_{12} P_{X_2 Q_{X_1}}\right] P_{X_2} + P_{X_2} \widetilde{A}_{22} P_{X_2}. \end{aligned} \tag{21}$$

The results obtained can be summarized in the following theorem.

Theorem 1 *Let $\widetilde{A} \in \mathbb{C}^{n \times n}$ be a partitioned matrix as in (13) and assume that $\mathcal{S} \cap \mathcal{NJH} \neq \emptyset$. Then the problem (3) has a unique solution $\widehat{A} \in \mathcal{S} \cap \mathcal{NJH}$ if and only if $\widetilde{A}_{11}$, $\widetilde{A}_{22}$, and $X_1^* X_1 D - X_2^* X_2 D$ are skew-Hermitian, (9), (17), (18), and $(\widetilde{A}_{12} - (\widetilde{A}_{21})^*) P_{X_2 Q_{X_1}} = O$ are satisfied and if we define $\widehat{A}_{11}$, $\widehat{A}_{12}$, and $\widehat{A}_{22}$ by (20), (16) and (21), respectively, the condition $\widehat{A}_{11} \widehat{A}_{12} = \widehat{A}_{12} \widehat{A}_{22}$ must be fulfilled. In this case, the (unique) optimal solution is given by*

$$\widehat{A} = U \begin{bmatrix} \widehat{A}_{11} & \widehat{A}_{12} \\ (\widehat{A}_{12})^* & \widehat{A}_{22} \end{bmatrix} U^*.$$

By using the condition given in the statement of Theorem 1, we notice that

$$\widehat{A}_{12} = X_1 D Q_{X_1} \left(X_2 Q_{X_1}\right)^{\dagger} + (\widetilde{A}_{21})^{*} P_{X_2 Q_{X_1}}$$

gives an alternative expression for $\widehat{A}_{12}$.

Remark 2 *It is well known that the set $\mathcal{S}$ is wide enough for Hermitian and generalized skew-Hamiltonian matrices as it was established in [1], and for normal skew J-Hamiltonian matrices as showed in [11, 32], for the particular matrix $J = \begin{bmatrix} O & I_k \\ -I_k & O \end{bmatrix}$. It is clear that all these cases are particular ones of normal matrices. So, it is natural that the set $\mathcal{S}$ for $\mathcal{NJH}$ must be wider.*

Algorithm and numerical examples

The algorithm presented below indicates a procedure that solves the Procrustes problem associated with the inverse eigenvalue problem stated in this subsection.

Algorithm 1

Input: Matrices $\widetilde{A}, J, X$, and D such that $J^2 = -I_n$ and X and D constrained to the condition $\mathcal{S} \cap \mathcal{NJH} \neq \emptyset$.

Output: The optimal matrix $\widehat{A}$ that solves Problem (3).

Step 1 Diagonalize $J = U(iI_k \oplus -iI_k)U^*$.

Step 2 Partition $U^*X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$.

Step 3 Partition $U^*\widetilde{A}U = \begin{bmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & \widetilde{A}_{22} \end{bmatrix}$.

Step 4 Compute $P = I_m - X_1^{\dagger} X_1$ and $T = I_m - X_2^{\dagger} X_2$.

Step 5 Compute $L = I_m - (X_2 P)^{\dagger} X_2 P$ and $Q = I_k - X_2 P (X_2 P)^{\dagger}$.

Step 6 Compute $M = X_1 D - X_1 D P \left(X_2 P\right)^{\dagger} X_2 - \widetilde{A}_{12} Q \, X_2$ and $N = X_2 D - \left(P X_2^*\right)^{\dagger} P D^* X_1^* X_1 - Q \widetilde{A}_{21} X_1$.

Step 7 If $\widetilde{A}_{11}$ is not skew-Hermitian or $\widetilde{A}_{22}$ is not skew-Hermitian, go to Step 18.

Step 8 If $X_1^* X_1 D - X_2^* X_2 D$ is not skew-Hermitian, go to Step 18.

Step 9 If $(\widetilde{A}_{12} - (\widetilde{A}_{21})^*)Q \neq O$ or $X_1 D P L \neq O$, go to Step 18.

Step 10 If $MP \neq O$ or $NT \neq O$, go to Step 18.

Step 11 Compute $R = I_k - X_1 X_1^\dagger$ and $S = I_k - X_2 X_2^\dagger$.

Step 12 Compute $\widehat{A}_{11} = M X_1^\dagger + (X_1^\dagger)^* X_2^* Q \widetilde{A}_{21} R + R \widetilde{A}_{11} R$.

Step 13 Compute $\widehat{A}_{12} = X_1 D P \left(X_2 P\right)^\dagger + \widetilde{A}_{12} Q$.

Step 14 Compute $\widehat{A}_{22} = N X_2^\dagger + (X_2^\dagger)^* \left[X_1^* X_1 D P \left(X_2 P\right)^\dagger - X_1^* \widetilde{A}_{12} Q\right] S + S \widetilde{A}_{22} S$.

Step 15 If $\widehat{A}_{11}\widehat{A}_{12} - \widehat{A}_{12}\widehat{A}_{22} \neq O$, go to Step 18.

Step 16 Compute $\widehat{A} = U \begin{bmatrix} \widehat{A}_{11} & \widehat{A}_{12} \\ (\widehat{A}_{12})^* & \widehat{A}_{22} \end{bmatrix} U^*$.

Step 17 Go to End.

Step 18 "There is not a normal J-Hamiltonian matrix $\widehat{A}$ such that $\min\limits_{A \in \mathcal{S}} \|\widetilde{A} - A\|_F = \|\widetilde{A} - \widehat{A}\|_F$".

End

Remark 3 *We have used MATLAB R2022a and MATHEMATICA 13.1 packages. Our worked examples have been considered for matrices of small sizes by using symbolic computations.*

In what follows, we present some illustrative examples.

Example 1 *Let consider the matrix*

$$J = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = U \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{bmatrix} U^*,$$

where

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} i & 0 & -i & 0 \\ 0 & i & 0 & -i \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Moreover, we consider the matrices

$$X = \begin{bmatrix} -\sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & -\frac{ia}{\sqrt{2}} \\ \sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & \frac{a}{\sqrt{2}} \end{bmatrix}, \text{ with } a \in \mathbb{C}, \qquad D = \begin{bmatrix} 1+i & 0 & 0 \\ 0 & -1+i & 0 \\ 0 & 0 & i \end{bmatrix},$$

and infinitely many data depending on parameters $b, h, k, z \in \mathbb{C}$ given by

$$\widetilde{A} = \frac{1}{2}\begin{bmatrix} 5i - 2\mathrm{Re}(h) & -z - \bar{b} - i\sqrt{3} & 9 - 2\mathrm{Im}(h) & -iz + i\bar{b} + 3\sqrt{3} \\ -b - \bar{z} - i\sqrt{3} & 7i - 2k & 3\sqrt{3} - ib + i\bar{z} & 3 \\ -9 - \mathrm{Im}(h) & -3\sqrt{3} - iz + i\bar{b} & 5i + 2\mathrm{Re}(h) & z + \bar{b} - i\sqrt{3} \\ -3\sqrt{3} - ib + i\bar{z} & -3 & b + \bar{z} - i\sqrt{3} & 7i + 2k \end{bmatrix}.$$

Applying Algorithm 1 we get

$$X_1 = \begin{bmatrix} 1+i & 1-i & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 1-i & 1+i & 0 \\ 0 & 0 & a \end{bmatrix},$$

$$P = \frac{1}{2}\begin{bmatrix} 1 & i & 0 \\ -i & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \qquad T = \frac{1}{2}\begin{bmatrix} 1 & -i & 0 \\ i & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$L = \frac{1}{2}\begin{bmatrix} 1 & -i & 0 \\ i & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad Q = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$M = \begin{bmatrix} -1+i & 1+i & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad N = \begin{bmatrix} 1+i & -1+i & 0 \\ 0 & 0 & ia \end{bmatrix},$$

$$R = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad S = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Step 7 is satisfied since

$$\widetilde{A}_{11} = \begin{bmatrix} -2i & -2i\sqrt{3} \\ -2i\sqrt{3} & 2i \end{bmatrix} \quad \text{and} \quad \widetilde{A}_{22} = \begin{bmatrix} 7i & i\sqrt{3} \\ i\sqrt{3} & 5i \end{bmatrix}$$

are skew-Hermitian. Now,

$$X_1^* X_1 D - X_2^* X_2 D = \begin{bmatrix} 0 & 4+4i & 0 \\ -4+4i & 0 & 0 \\ 0 & 0 & -ia\bar{a} \end{bmatrix}$$

is skew-Hermitian, so Step 8 is satisfied.

Moreover, Step 9 is valid because $(\widetilde{A}_{12} - (\widetilde{A}_{21})^*)Q = O$ and $X_1 DPL = O$ where

$$\widetilde{A}_{12} = \begin{bmatrix} h & \bar{b} \\ \bar{z} & k \end{bmatrix} \quad \text{and} \quad \widetilde{A}_{21} = \begin{bmatrix} \bar{h} & z \\ b & k \end{bmatrix}.$$

Furthermore, $MP = O$, $NT = O$. Hence, Step 10 is satisfied.

After doing the computations of $\widehat{A}_{11}$, $\widehat{A}_{12}$, and $\widehat{A}_{22}$ we have that $\widehat{A}_{11}\widehat{A}_{12} = \widehat{A}_{12}\widehat{A}_{22}$ and

$$\widehat{A} = \frac{1}{2}\begin{bmatrix} 2i & 0 & -2 & 0 \\ 0 & 3i & 0 & -1 \\ -2 & 0 & 2i & 0 \\ 0 & 1 & 0 & 3i \end{bmatrix}.$$

We can easily check that the matrix $\widehat{A}$ is normal J-Hamiltonian and satisfies the equation $AX = XD$.

Example 2 *Let consider the matrices J and U given in the previous example and consider also the matrices*

$$\widetilde{A} = \frac{1}{2}\begin{bmatrix} 5i - 2\mathrm{Re}(h) & -i\sqrt{3} - z + i & 9 - 2\mathrm{Im}(h) & 3\sqrt{3} - iz + 1 \\ -i\sqrt{3} - i - \bar{z} & -4 + 7i & 3\sqrt{3} + 1 + i\bar{z} & 3 \\ -9 - 2\mathrm{Im}(h) & -3\sqrt{3} - iz + 1 & 5i + 2\mathrm{Re}(h) & -i\sqrt{3} + z - i \\ -3\sqrt{3} + 1 + i\bar{z} & -3 & -i\sqrt{3} + i + \bar{z} & 4 + 7i \end{bmatrix}, \quad h, z \in \mathbb{C},$$

$$X = \begin{bmatrix} -\sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 0 \\ \sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & i\sqrt{2} \end{bmatrix}, \qquad D = \begin{bmatrix} 1+i & 0 & 0 \\ 0 & -1+i & 0 \\ 0 & 0 & 2i \end{bmatrix}.$$

Steps 7, 8, and 9 are satisfied, $MP = O$ but since $NT = \begin{bmatrix} 0 & 0 & 0 \\ 1-i & -1-i & 0 \end{bmatrix} \neq O$, Step 10 is not satisfied. Hence, the Algorithm ensures that the problem has no solution.

3 The normal skew J-Hamiltonian case

This section deals with the optimization problem for normal skew J-Hamiltonian matrices. For a given full-column rank matrix $X \in \mathbb{C}^{n\times m}$ and a diagonal matrix $D \in \mathbb{C}^{m\times m}$, throughout this section, we denote by $\mathcal{S} \cap \mathcal{NJSH}$ the set of all the solutions $A \in \mathbb{C}^{n\times n}$ of $AX = XD$ for A being a normal skew J-Hamiltonian matrix.

By using the same notation as in Section 2, the structure of the normal skew J-Hamiltonian matrices is given by (see [32, Lemma 1])

$$A = U \begin{bmatrix} A_{11} & A_{12} \\ -A_{12}^* & A_{22} \end{bmatrix} U^*,$$

where A_{11} and A_{22} are Hermitian, $A_{11}A_{12} = A_{12}A_{22}$ and U is a unitary matrix diagonalizing matrix J.

Notice that for a given normal skew J-Hamiltonian matrix $A \in \mathbb{C}^{n\times n}$, it is easy to see that the matrix $B := iA$ is also normal and, moreover, B is J-Hamiltonian. Furthermore, the matrix equation $AX = XD$ is equivalent to $BX = X\widetilde{D}$, where $\widetilde{D} = iD$. Observe that, if D satisfies the restriction on the spectrum of the skew J-Hamiltonian matrix A (that is, if $\lambda \in \sigma(A)$ then $\{\lambda, \bar{\lambda}\} \subseteq \sigma(A)$), the restriction on the spectrum of $\widetilde{D}$ is also satisfied (that is, if $\lambda \in \sigma(B)$ then $\{\lambda, -\lambda, \bar{\lambda}, -\bar{\lambda}\} \subseteq \sigma(B)$).

Then, the general form of matrices in $\mathcal{S} \cap \mathcal{NJSH}$ is obtained by applying [11, Theorem 2] to $B = iA$ and $\widetilde{D} = iD$. The following result gives necessary and sufficient conditions that ensure the existence of solutions of the inverse eigenvalue problem for matrices having the structure considered in this section.

Theorem 2 *Let $X \in \mathbb{C}^{n\times m}$ be a full-column rank matrix, $D \in \mathbb{C}^{m\times m}$ be a diagonal matrix and $J \in \mathbb{R}^{n\times n}$ be a normal matrix such that $J^2 = -I_n$, where $n = 2k$. Consider the partition $X = U \begin{bmatrix} X_1^* & X_2^* \end{bmatrix}^*$ where $X_1, X_2 \in \mathbb{C}^{k\times m}$. Then there exists $A \in \mathcal{S} \cap \mathcal{NJSH}$ if and only if the conditions: $(X_1 D - A_{12} X_2) Q_{X_1} = O$,*

$$\left[X_1 D - X_1 D Q_{X_1} (X_2 Q_{X_1})^\dagger X_2 - Y_{12} P_{X_2 Q_{X_1}} X_2\right] Q_{X_1} = O,$$

$$\left[X_2 D + (Q_{X_1} X_2^*)^\dagger Q_{X_1} D^* X_1^* X_1 + P_{X_2 Q_{X_1}} Y_{12}^* X_1\right] Q_{X_2} = O,$$

$X_1^* X_1 D - X_2^* X_2 D$ is Hermitian, and $A_{11} A_{12} = A_{12} A_{22}$ hold. In this case, the general solution is given by

$$A = U \begin{bmatrix} A_{11} & A_{12} \\ -A_{12}^* & A_{22} \end{bmatrix} U^*,$$

with

$$A_{11} = \left[X_1 D - X_1 D Q_{X_1} (X_2 Q_{X_1})^\dagger X_2 - Y_{12} P_{X_2 Q_{X_1}} X_2\right] X_1^\dagger + (X_1^\dagger)^* X_2^* P_{X_2 Q_{X_1}} Y_{12}^* P_{X_1} + P_{X_1} Y_{11} P_{X_1},$$

$$A_{12} = X_1 D Q_{X_1} (X_2 Q_{X_1})^\dagger + Y_{12} P_{X_2 Q_{X_1}},$$

and

$$\begin{aligned} A_{22} &= \left[X_2 D + (Q_{X_1} X_2^*)^\dagger Q_{X_1} D^* X_1^* X_1 + P_{X_2 Q_{X_1}} Y_{12}^* X_1\right] X_2^\dagger + \\ &\quad + (X_2^\dagger)^* \left[X_1^* X_1 D Q_{X_1} (X_2 Q_{X_1})^\dagger + X_1^* Y_{12} P_{X_2 Q_{X_1}}\right] P_{X_2} + P_{X_2} Y_{22} P_{X_2}, \end{aligned}$$

where $Y_{11}, Y_{22} \in \mathbb{C}^{k\times k}$ are arbitrary Hermitian matrices and $Y_{12} \in \mathbb{C}^{k\times k}$ is an arbitrary matrix.

Now, the Procrustes problem associated with the matrices in $\mathcal{S} \cap \mathcal{NJSH}$ is solved by applying Theorem 1 to $B = iA$ and $\widetilde{D} = iD$. Summarizing, we establish the following theorem.

Theorem 3 *Let $\widetilde{A} \in \mathbb{C}^{n\times n}$ be a partitioned matrix as in (13) and assume that $\mathcal{S} \cap \mathcal{NJSH} \neq \emptyset$. Then the problem stated in (2) has a unique solution $\widehat{A} \in \mathcal{S} \cap \mathcal{NJSH}$ if and only if the matrices $\widetilde{A}_{11}$, $\widetilde{A}_{22}$, and $X_1^* X_1 D - X_2^* X_2 D$ are Hermitian, $(X_1 D - A_{12} X_2) Q_{X_1} = O$,*

$$\left[X_1 D - X_1 D Q_{X_1} (X_2 Q_{X_1})^\dagger X_2 - \widetilde{A}_{12} P_{X_2 Q_{X_1}} X_2\right] Q_{X_1} = O,$$

$$\left[X_2 D + (Q_{X_1} X_2^*)^\dagger Q_{X_1} D^* X_1^* X_1 - P_{X_2 Q_{X_1}} \widetilde{A}_{21} X_1\right] Q_{X_2} = O,$$

$(\widetilde{A}_{12} + (\widetilde{A}_{21})^*) P_{X_2 Q_{X_1}} = O$ hold; and if we define

$$\widehat{A}_{11} = \left[X_1 D - X_1 D Q_{X_1} (X_2 Q_{X_1})^\dagger X_2 - \widetilde{A}_{12} P_{X_2 Q_{X_1}} X_2\right] X_1^\dagger + (X_1^\dagger)^* X_2^* P_{X_2 Q_{X_1}} \widetilde{A}_{21} Q_{X_1} + P_{X_1} \widetilde{A}_{11} P_{X_1},$$

$$\widehat{A}_{12} = X_1 D Q_{X_1} (X_2 Q_{X_1})^\dagger + \widetilde{A}_{12} P_{X_2 Q_{X_1}},$$

and

$$\widehat{A}_{22} = \left[X_2 D + \left(Q_{X_1}) X_2^*\right)^\dagger Q_{X_1} D^* X_1^* X_1 + P_{X_2 Q_{X_1}} \widetilde{A}_{21} X_1\right] X_2^\dagger + \\ +(X_2^\dagger)^* \left[X_1^* X_1 D Q_{X_1} \left(X_2 Q_{X_1}\right)^\dagger - X_1^* \widetilde{A}_{12} P_{X_2 Q_{X_1}}\right] P_{X_2} + P_{X_2} \widetilde{A}_{22} P_{X_2},$$

then the condition $\widehat{A}_{11}\widehat{A}_{12} = \widehat{A}_{12}\widehat{A}_{22}$ *must also be fulfilled.*

In this case, the unique optimal solution is given by

$$\widehat{A} = U \begin{bmatrix} \widehat{A}_{11} & \widehat{A}_{12} \\ -(\widehat{A}_{12})^* & \widehat{A}_{22} \end{bmatrix} U^*.$$

Algorithm and numerical examples

This subsection presents an algorithm to solve the Procrustes problem related to the inverse eigenvalue problem stated. For this case, it reads more simplified. We also present some numerical examples.

Algorithm 2

Input: Matrices $\widetilde{A}, J, X$, and D such that $J^2 = -I_n$ and X and D constrained to the condition $\mathcal{S} \cap \mathcal{NJSH} \neq \emptyset$.

Output: The optimal matrix $\widehat{A}$ that solves Problem (2) for $\mathcal{T} = \mathcal{NJSH}$.

Step 1 Compute $\widetilde{D} = iD$.

Step 2 Apply Algorithm 1 to $\widetilde{A}, J, X$, and $\widetilde{D}$ obtaining the normal J-Hamiltonian $\widehat{B}$.

Step 3 The optimal solution is $\widehat{A} = -i\widehat{B}$ provided that $\mathcal{S} \cap \mathcal{NJSH} \neq \emptyset$

End

Now, we provide some numerical examples.

Example 3 *Let consider the same matrices J, U, and X as in Example 1. Moreover, we consider the matrices*

$$D = \begin{bmatrix} 1+i & 0 & 0 \\ 0 & 1-i & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \widetilde{A} = \frac{1}{4}\begin{bmatrix} -2h+7-2i & 1-2i & -2-3i-2ih & 2-i \\ -3-2i & -7 & -2+3i & -i \\ -2+7i-2ih & 2+i & 2h+11+2i & 1+10i \\ -2+5i & i & 5-6i & 17 \end{bmatrix},$$

with $h \in \mathbb{C}$. Since matrix X remains unchanged, by applying Algorithm 2 we get the same values for the matrices X_1, X_2, P, T, L, Q, R, and S. Now, we have

$$M = \begin{bmatrix} 1+i & 1-i & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad N = \begin{bmatrix} 1-i & 1+i & 0 \\ 0 & 0 & a \end{bmatrix}.$$

Steps 7 and 9 are satisfied since

$$\widetilde{A}_{11} = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} \quad \text{and} \quad \widetilde{A}_{22} = \frac{1}{4}\begin{bmatrix} 11+2\sqrt{3} & -2+\sqrt{3}+4i \\ -2+\sqrt{3}-4i & 9-2\sqrt{3} \end{bmatrix}$$

are Hermitian and, moreover, $(\widetilde{A}_{12} + (\widetilde{A}_{21})^*)Q = O$ *and* $X_1DPL = O$ *where*

$$\widetilde{A}_{12} = \begin{bmatrix} 1+i & i \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad \widetilde{A}_{21} = \begin{bmatrix} h & 2i \\ -i & 3 \end{bmatrix}.$$

Furthermore, $MP = O$*,* $NT = O$*, and*

$$X_1^*X_1D - X_2^*X_2D = \begin{bmatrix} 0 & -4-4i & 0 \\ -4+4i & 0 & 0 \\ 0 & 0 & -a\bar{a} \end{bmatrix}$$

is Hermitian. Thus, Steps 8 and 10 are satisfied.

After doing the computations of $\widehat{A}_{11}$*,* $\widehat{A}_{12}$*, and* $\widehat{A}_{22}$ *we have that* $\widehat{A}_{11}\widehat{A}_{12} = \widehat{A}_{12}\widehat{A}_{22}$ *and*

$$\widehat{A} = \begin{bmatrix} 1 & 0 & -i & 0 \\ 0 & 1 & 0 & 0 \\ -i & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We can easily check that the matrix $\widehat{A}$ *is normal skew J-Hamiltonian and satisfies the equation* $AX = XD$*. Furthermore, the value of the norm is*

$$\|\widetilde{A} - \widehat{A}\|_F = \sqrt{|h|^2 - 2\mathrm{Re}(h) + 45},$$

and its minimum value is attained for $h = 1$ *and equals* $2\sqrt{11} = 6.63325$*.*

Example 4 *Let consider the matrices* J*,* U*, and* $\widetilde{A}$ *given in Example 3 and also consider the matrices*

$$X = \begin{bmatrix} -\sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 0 \\ \sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & i\sqrt{2} \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1+i & 0 & 0 \\ 0 & 1-i & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Step 7 is satisfied, $X_1DPL = O$ *but since* $(\widetilde{A}_{12} + (\widetilde{A}_{21})^*)Q = \begin{bmatrix} 0 & 2i \\ 0 & 6 \end{bmatrix} \neq O$*, Step 9 is not satisfied. Hence, Algorithm 2 ensures that the Procrustes problem has no solution.*

4 The normal J-symplectic case

This section deals with the inverse eigenvalue problem for normal J-symplectic matrices and the associated optimization problem. In order to solve them, we need to recall some results about the Cayley transform.

Let $A \in \mathbb{C}^{n\times n}$ such that $\det(I_n + A) \neq 0$, that is, -1 is not an eigenvalue of A. The Cayley transform of A is the matrix given by [8]

$$A^C := (I_n - A)(I_n + A)^{-1}. \tag{22}$$

We recall some useful properties:

(i) if A^C is the Cayley transform of some matrix $A \in \mathbb{C}^{n\times n}$, then the matrix $I_n + A^C$ is nonsingular.

(ii) $(I_n - A)(I_n + A)^{-1} = (I_n + A)^{-1}(I_n - A)$.

(iii) A^C satisfies (22) if and only if $A = \left(I_n - A^C\right)\left(I_n + A^C\right)^{-1}$.

(iv) The Cayley transform, $A \mapsto A^C$, is an involution on the set of matrices $A \in \mathbb{C}^{n\times n}$ such that $I_n + A$ is nonsingular, that is, $M = A^C$ if and only if $A = M^C$.

(v) A and A^C commute.

(vi) $(A^C)^* = (A^*)^C$.

In this section, we will consider a fixed normal matrix $J \in \mathbb{R}^{2n\times 2n}$ such that $J^2 = -I_{2n}$.

Lemma 3 *Let $A \in \mathbb{C}^{2n\times 2n}$ such that $\det(I_{2n}+A) \neq 0$. Then A is normal and J-symplectic if and only if A^C is normal and J-Hamiltonian.*

Proof. It is easy to prove that A is normal if and only if $(I_{2n} + A)^{-1}$ is normal. By using (22), we get the equality $(I_{2n} + A^C)(I_{2n} + A) = 2I_{2n}$. Then, A normal is equivalent to $I_{2n} + A^C$ is normal. And, finally, A normal if and only if A^C is normal.

Now, for a given J-symplectic $A \in \mathbb{C}^{2n\times 2n}$, substituting $A = \left(I_{2n} - A^C\right)\left(I_{2n} + A^C\right)^{-1}$ in $A^*JA = J$, we get

$$\left(\left(I_{2n} + A^C\right)^*\right)^{-1}\left(I_{2n} - A^C\right)^* J\left(I_{2n} - A^C\right)\left(I_{2n} + A^C\right)^{-1} = J.$$

Premultiplying by $\left(I_{2n} + A^C\right)^*$ and postmultiplying by $I_{2n} + A^C$ we have

$$\left(I_{2n} - A^C\right)^* J\left(I_{2n} - A^C\right) = (I_{2n} + A^C)^* J\left(I_{2n} + A^C\right).$$

Operating and using that $J^{-1} = J^* = -J$, we arrive at $JA^C = \left(JA^C\right)^*$, that is, A^C is J-Hamiltonian.

On the other hand, let A^C be J-Hamiltonian, that is, $JA^C = \left(JA^C\right)^*$. Since $A = \left(I_{2n} - A^C\right)\left(I_{2n} + A^C\right)^{-1}$, and by using that $J^{-1} = J^* = -J$, we have

$$\begin{aligned} A^*JA &= \left(\left(I_{2n} + A^C\right)^{-1}\right)^* \left(I_{2n} - A^C\right)^* J \left(I_{2n} - A^C\right)\left(I_{2n} + A^C\right)^{-1} \\ &= -\left(\left(I_{2n} + A^C\right)^*\right)^{-1} \left(I_{2n} - A^C\right)^* J^* \left(I_{2n} - A^C\right)\left(I_{2n} + A^C\right)^{-1} \\ &= -\left(\left(I_{2n} + A^C\right)^*\right)^{-1} \left(J^* - (JA^C)^*\right) \left(I_{2n} - A^C\right)\left(I_{2n} + A^C\right)^{-1} \\ &= -\left(\left(I_{2n} + A^C\right)^*\right)^{-1} \left(-J - JA^C\right) \left(I_{2n} - A^C\right)\left(I_{2n} + A^C\right)^{-1} \\ &= \left(\left(I_{2n} + A^C\right)^*\right)^{-1} J \left(I_{2n} + A^C\right) \left(I_{2n} - A^C\right)\left(I_{2n} + A^C\right)^{-1}. \end{aligned}$$

Now, by using property (ii), we get

$$\begin{aligned} A^*JA &= \left(\left(I_{2n} + A^C\right)^*\right)^{-1} J \left(I_{2n} - A^C\right) = \left(\left(I_{2n} + A^C\right)^*\right)^{-1} \left(J - JA^C\right) \\ &= \left(\left(I_{2n} + A^C\right)^*\right)^{-1} \left(-J^* - (JA^C)^*\right) = -\left(\left(I_{2n} + A^C\right)^*\right)^{-1} \left(J + JA^C\right)^* \\ &= -\left(\left(I_{2n} + A^C\right)^*\right)^{-1} \left(I_{2n} + A^C\right)^* J^* = -J^* = J. \end{aligned}$$

■

For a given normal J-symplectic matrix $A \in \mathbb{C}^{2n\times 2n}$ such that $\det(I_{2n} + A) \neq 0$, we have that matrix A^C is also normal and J-Hamiltonian. There exists a unitary matrix $U \in \mathbb{C}^{2n\times 2n}$ such that

$$J = U \begin{bmatrix} iI_n & O \\ O & -iI_n \end{bmatrix} U^*. \tag{23}$$

For that matrix U, we consider the partition of matrix A^C as follows

$$A^C = U \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} U^*.$$

Applying [11, Theorem 1] to matrix A^C and using that $A = \left(I_{2n} - A^C\right)\left(I_{2n} + A^C\right)^{-1}$, the general expression of A is given by

$$A = U \begin{bmatrix} I_n - A_{11} & -A_{12} \\ -A_{12}^* & I_n - A_{22} \end{bmatrix} \begin{bmatrix} I_n + A_{11} & A_{12} \\ A_{12}^* & I_n + A_{22} \end{bmatrix}^{-1} U^*,$$

where $A_{11}^* = -A_{11}$, $A_{22}^* = -A_{22}$, and $A_{11}A_{12} = A_{12}A_{22}$.

Furthermore, for a given full-column rank matrix $X \in \mathbb{C}^{2n\times 2m}$ and a diagonal matrix $D \in \mathbb{C}^{2m\times 2m}$, throughout this section, we denote by $\mathcal{S} \cap \mathcal{NJS}$ the set of all the solutions $A \in \mathbb{C}^{2n\times 2n}$ of $AX = XD$ for A being normal J-symplectic such that $\det(I_{2n} + A) \neq 0$. By substituting matrix A by $\left(I_{2n} - A^C\right)\left(I_{2n} + A^C\right)^{-1}$ in the matrix equation $AX = XD$, we get $X(I_{2m} - D) = A^C X(I_{2m} + D)$. Since $\det(I_{2n} + A) \neq 0$, -1 is not an eigenvalue of A, so $I_{2m} + D$ is nonsingular. Thus

$$AX = XD \iff A^C X = XD^C.$$

As observed in the previous case, if D satisfies the restriction on the spectrum of the J-symplectic matrix A, the restriction on the spectrum of $D^C = (I_{2m} - D)(I_{2m} + D)^{-1}$ is also satisfied, that is, if $\lambda \in \sigma(A)$ such that $\lambda \neq -1$ then $\left\{ \frac{1-\lambda}{1+\lambda}, -\frac{1-\bar{\lambda}}{1+\bar{\lambda}} \right\} \in \sigma(A^C)$.

Then, the general form of normal J-symplectic matrices (such that -1 is not an eigenvalue), solution of the inverse eigenvalue problem given by (1), is obtained by applying [11, Theorem 2] to A^C and D^C. The necessary and sufficient conditions that assure the existence of solutions of the inverse eigenvalue problem are given in the following result.

Theorem 4 *Let $X \in \mathbb{C}^{2n\times 2m}$ be a full-column rank matrix, $D \in \mathbb{C}^{2m\times 2m}$ be a diagonal matrix and $J \in \mathbb{R}^{2n\times 2n}$ be a normal matrix such that $J^2 = -I_{2n}$. Consider the partition $X = U \begin{bmatrix} X_1^* & X_2^* \end{bmatrix}^*$ where $X_1, X_2 \in \mathbb{C}^{n\times 2m}$. Then there exists a normal J-symplectic matrix A, solution of the problem (1), if and only if the conditions: $X_1 D^C Q_{X_1} Q_{X_2 Q_{X_1}} = O$,*

$$\left[X_1 D^C - X_1 D^C Q_{X_1} (X_2 Q_{X_1})^{\dagger} X_2 - Y_{12} P_{X_2 Q_{X_1}} X_2\right] Q_{X_1} = O,$$

$$\left[X_2 D^C - (Q_{X_1} X_2^*)^{\dagger} Q_{X_1} (D^C)^* X_1^* X_1 - P_{X_2 Q_{X_1}} Y_{12}^* X_1\right] Q_{X_2} = O,$$

$X_1^* X_1 D^C - X_2^* X_2 D^C$ is skew-Hermitian, and $A_{11}A_{12} = A_{12}A_{22}$ hold. In this case, the general solution is given by

$$A = U \begin{bmatrix} I_n - A_{11} & -A_{12} \\ -A_{12}^* & I_n - A_{22} \end{bmatrix} \begin{bmatrix} I_n + A_{11} & A_{12} \\ A_{12}^* & I_n + A_{22} \end{bmatrix}^{-1} U^*,$$

with

$$\begin{aligned} A_{11} &= \left[X_1 D^C - X_1 D^C Q_{X_1}(X_2 Q_{X_1})^{\dagger} X_2 - Y_{12} P_{X_2 Q_{X_1}} X_2\right] X_1^{\dagger} + (X_1^{\dagger})^* X_2^* P_{X_2 Q_{X_1}} Y_{12}^* P_{X_1} + \\ &+ P_{X_1} Y_{11} P_{X_1}, \end{aligned}$$

$$A_{12} = X_1 D^C Q_{X_1} (X_2 Q_{X_1})^{\dagger} + Y_{12} P_{X_2 Q_{X_1}},$$

and

$$\begin{aligned} A_{22} &= \left[X_2 D^C - (Q_{X_1} X_2^*)^{\dagger} Q_{X_1} (D^C)^* X_1^* X_1 - P_{X_2 Q_{X_1}} Y_{12}^* X_1\right] X_2^{\dagger} + \\ &+ (X_2^{\dagger})^* \left[X_1^* X_1 D^C Q_{X_1} (X_2 Q_{X_1})^{\dagger} - X_1^* Y_{12} P_{X_2 Q_{X_1}}\right] P_{X_2} + P_{X_2} Y_{22} P_{X_2}, \end{aligned}$$

where $Y_{11}, Y_{22} \in \mathbb{C}^{n\times n}$ are arbitrary skew-Hermitian matrices and $Y_{12} \in \mathbb{C}^{n\times n}$ is an arbitrary matrix.

Now, let $\widetilde{A} \in \mathbb{C}^{2n\times 2n}$ be an arbitrary matrix such that $U^* \widetilde{A} U$ is partitioned as

$$U^* \widetilde{A} U = \begin{bmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & \widetilde{A}_{22} \end{bmatrix}, \tag{24}$$

where $\widetilde{A}_{ij} \in \mathbb{C}^{n\times n}$ and U is the same matrix as in (23). Assuming that $\mathcal{S} \cap \mathcal{NJH} \neq \emptyset$ and applying Theorem 1 to A^C and D^C, the Procrustes problem associated with the matrices in $\mathcal{S} \cap \mathcal{NJS}$ can be completely solved. The results are summarized in the following theorem.

Theorem 5 *Let $\widetilde{A} \in \mathbb{C}^{2n\times 2n}$ be a partitioned matrix as in (24) and assume that $\mathcal{S} \cap \mathcal{NJS} \neq \emptyset$. There exists a unique solution $\widehat{A}$ normal and J-symplectic of (2) that also satisfies problem (1) if and only if the matrices $\widetilde{A}_{11}$, $\widetilde{A}_{22}$, and $X_1^* X_1 D^C - X_2^* X_2 D^C$ are skew-Hermitian, and the following conditions: $X_1 D^C Q_{X_1} Q_{X_2 Q_{X_1}} = O$,*

$$\left[X_1 D^C - X_1 D^C Q_{X_1}(X_2 Q_{X_1})^\dagger X_2 - \widetilde{A}_{12} P_{X_2 Q_{X_1}} X_2\right] Q_{X_1} = O,$$

$$\left[X_2 D^C - (Q_{X_1} X_2^*)^\dagger Q_{X_1} (D^C)^* X_1^* X_1 - P_{X_2 Q_{X_1}} \widetilde{A}_{21} X_1\right] Q_{X_2} = O,$$

$(\widetilde{A}_{12} - (\widetilde{A}_{21})^*) P_{X_2 Q_{X_1}} = O$ hold; and if we define

$$\begin{aligned}\widehat{A}_{11} &= \left[X_1 D^C - X_1 D^C Q_{X_1} (X_2 Q_{X_1})^\dagger X_2 - \widetilde{A}_{12} P_{X_2 Q_{X_1}} X_2\right] X_1^\dagger + \\ &\quad +(X_1^\dagger)^* X_2^* P_{X_2 Q_{X_1}} \widetilde{A}_{21} P_{X_1} + P_{X_1} \widetilde{A}_{11} P_{X_1},\end{aligned}$$

$$\begin{aligned}\widehat{A}_{22} &= \left[X_2 D^C - (Q_{X_1} X_2^*)^\dagger Q_{X_1} (D^C)^* X_1^* X_1 - P_{X_2 Q_{X_1}} \widetilde{A}_{21} X_1\right] X_2^\dagger + \\ &\quad +(X_2^\dagger)^* \left[X_1^* X_1 D^C Q_{X_1} (X_2 Q_{X_1})^\dagger - X_1^* \widetilde{A}_{12} P_{X_2 Q_{X_1}}\right] P_{X_2} + P_{X_2} \widetilde{A}_{22} P_{X_2},\end{aligned}$$

and

$$\widehat{A}_{12} = X_1 D^C Q_{X_1} (X_2 Q_{X_1})^\dagger + \widetilde{A}_{12} P_{X_2 Q_{X_1}},$$

then the condition $\widehat{A}_{11}\widehat{A}_{12} = \widehat{A}_{12}\widehat{A}_{22}$ must be fulfilled. In this case, the unique optimal solution is given by

$$\widehat{A} = U \begin{bmatrix} I_n - \widehat{A}_{11} & -\widehat{A}_{12} \\ -\widehat{A}_{12}^* & I_n - \widehat{A}_{22} \end{bmatrix} \begin{bmatrix} I_n + \widehat{A}_{11} & \widehat{A}_{12} \\ \widehat{A}_{12}^* & I_n + \widehat{A}_{22} \end{bmatrix}^{-1} U^*.$$

Algorithm and numerical example

The following algorithm allows to solve the Procrustes problem related to the inverse eigenvalue problem for normal J-symplectic matrices.

Algorithm 3

Input: Matrices $\widetilde{A}, J, X$, and D such that $J^2 = -I_{2n}$ and X and D constrained to the condition $\mathcal{S} \cap \mathcal{NJS} \neq \emptyset$.

Output: The optimal matrix $\widehat{A}$ that solves Problem (2) for $\mathcal{T} = \mathcal{NJS}$.

Step 1 Compute $D^C = (I_{2m} - D)(I_{2m} + D)^{-1}$.

Step 2 Apply Algorithm 1 to $\widetilde{A}, J, X$, and D^C obtaining the normal J-Hamiltonian matrix $\widehat{B}$.

Step 3 The optimal solution is $\widehat{A} = (I_{2n} - \widehat{B})(I_{2n} + \widehat{B})^{-1}$.

End

Example 5 *Let consider the matrix*

$$J = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = U \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{bmatrix} U^*,$$

where

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & i & -i & 0 \\ i & 0 & 0 & -i \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

Moreover, we consider the matrices

$$X = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & ia & 0 & ig \\ 0 & ia & 0 & -ig \\ 0 & -a & 0 & g \\ 0 & a & 0 & g \end{bmatrix}, \text{ with } a, g \in \mathbb{C}, \quad D = \begin{bmatrix} 1-i & 0 & 0 & 0 \\ 0 & 1+i & 0 & 0 \\ 0 & 0 & \frac{1}{2} - \frac{i}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} + \frac{i}{2} \end{bmatrix},$$

$$\text{and} \quad \widetilde{A} = \begin{bmatrix} 3i & -\frac{1}{5} - i & 3 & 3 \\ -\frac{1}{5} - i & 3i & 3 & 3 \\ -3 & -3 & 3i & \frac{1}{5} - i \\ -3 & -3 & \frac{1}{5} - i & 3i \end{bmatrix}.$$

It can be easily checked that

$$D^C = \frac{1}{5} \begin{bmatrix} -1+2i & 0 & 0 & 0 \\ 0 & -1-2i & 0 & 0 \\ 0 & 0 & 1+2i & 0 \\ 0 & 0 & 0 & 1-2i \end{bmatrix}.$$

Applying Algorithm 1 to $\widetilde{A}, J, X$, and D^C we get

$$X_1 = \begin{bmatrix} 0 & a & 0 & 0 \\ 0 & 0 & 0 & g \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 & -a & 0 & 0 \\ 0 & 0 & 0 & g \end{bmatrix},$$

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad M = \frac{2}{5}\begin{bmatrix} 0 & -ia & 0 & 0 \\ 0 & 0 & 0 & -ig \end{bmatrix},$$

$$N = \frac{2}{5}\begin{bmatrix} 0 & ia & 0 & 0 \\ 0 & 0 & 0 & -ig \end{bmatrix}, \quad R = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad and \quad S = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Step 7 of Algorithm 1 is satisfied since

$$\widetilde{A}_{11} = \begin{bmatrix} 0 & -4i \\ -4i & 0 \end{bmatrix} \quad and \quad \widetilde{A}_{22} = \begin{bmatrix} 6i & 2i \\ 2i & 6i \end{bmatrix}$$

are both skew-Hermitian.
Since $X_1^* X_1 D^C - X_2^* X_2 D^C = O$, this matrix is skew-Hermitian, so Step 8 is satisfied.
As

$$\widetilde{A}_{12} = \widetilde{A}_{21} = \frac{1}{5}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

Step 9 is fullfilled because $(\widetilde{A}_{12} - (\widetilde{A}_{21})^*)Q = O$ and $X_1 D^C P L = O$.
Moreover, $MP = O$ and $NT = O$, so Step 10 is satisfied.
Carrying out the calculation of $\widehat{A}_{11}$, $\widehat{A}_{12}$, and $\widehat{A}_{22}$ we have that $\widehat{A}_{11}\widehat{A}_{12} = \widehat{A}_{12}\widehat{A}_{22}$ and

$$\widehat{B} = \frac{1}{5}\begin{bmatrix} -2i & -1 & 0 & 0 \\ -1 & -2i & 0 & 0 \\ 0 & 0 & -2i & i \\ 0 & 0 & 1 & -2i \end{bmatrix}.$$

This matrix $\widehat{B}$ is normal J-Hamiltonian and satisfies the equation $AX = XD^C$.
Finally, the optimal solution is

$$\widehat{A} = \frac{1}{4}\begin{bmatrix} 3+3i & 1+i & 0 & 0 \\ 1+i & 3+3i & 0 & 0 \\ 0 & 0 & 3+3i & -1-i \\ 0 & 0 & -1-i & 3+3i \end{bmatrix},$$

which is normal J-symplectic and satisfies the equation $AX = XD$.

The matrix equation for an undamped mass-spring system with n degrees of freedom looks like

$$M\ddot{X} + KX = O,$$

where Y is a time-dependent vector that describes the motion, $M = \mathrm{diag}(m_1, m_2, \cdots, m_n)$

is the mass matrix and

$$K = \begin{bmatrix} k_1+k_2 & -k_2 & 0 & \cdots & 0 & 0 & 0 \\ -k_2 & k_2+k_3 & -k_3 & \dots & 0 & 0 & 0 \\ 0 & -k_3 & k_3+k_4 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & k_{n-2}+k_{n-1} & -k_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & -k_{n-1} & k_{n-1}+k_n & -k_n \\ 0 & 0 & 0 & \cdots & 0 & -k_n & k_n \end{bmatrix}$$

is the stiffness matrix which is tridiagonal. Both are symmetric and the mass and stiffness parameters must be positive.

Since we seek harmonic solutions for Y, it has the form $Y = [y_1 \ y_2 \ \cdots \ y_n]$ where $y_j = b_j \sin(\omega t)$. Substituting into the motion equation we get

$$M(-\omega^2)Y + KY = 0.$$

Given that the matrix M is nonsingular, we can rewrite the last equation as

$$M^{-1}KY = \omega^2 Y.$$

This problem can be regarded as an eigenvector and eigenvalue problem where the unknown matrix to find is K.

Let X be the matrix whose columns are the eigenvectors and D the diagonal matrix whose elements are the eigenvalue. In this case, the general matrix equation becomes $AX = XD$ where $A = M^{-1}K$.

Let consider the input matrices

$$M = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}, \ X = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}, \ D = \frac{1}{2}\begin{bmatrix} \sqrt{6}-\sqrt{2} & 0 \\ 0 & \sqrt{6}+\sqrt{2} \end{bmatrix}, \ J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The eigenvalues of J are i and $-i$, $U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ i & -1 \end{bmatrix}$, and

$$A = \frac{1}{2}\begin{bmatrix} \sqrt{6} & -\sqrt{2} \\ -\sqrt{2} & \sqrt{6} \end{bmatrix}.$$

Then, $K = MA = \frac{1}{2}\begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{3} \end{bmatrix}$.

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