



Selection of the convergence step of the Fx-LMS algorithm[☆]

Miguel Ferrer^a, Maria de Diego^a, Víctor M. García-Mollá^b, Alberto Gonzalez^{a,1,*}

^a Institute of Telecommunications and Multimedia Applications (iTEAM), Universitat Politècnica de València (UPV), Valencia 46022, Spain

^b Department of Information Systems and Computing (DSIC), Universitat Politècnica de València (UPV), Valencia 46022, Spain

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ABSTRACT

The choice of a suitable convergence step that allows the fast convergence of the filtered-x LMS algorithm in practical scenarios is usually carried out by trial and error, since the theoretical developments provide only indicative values. The particularity of the filtered-x structure, which adds a delay between the output of the adaptive filter and the error signal and colors the reference signal, increases the inaccuracy of the theoretical models. Furthermore, the use of band-limited reference signals is beyond the scope of most theoretical studies, which focus on the cases of full-band reference signals, making it difficult to extrapolate these models to band-limited signals. In this paper, we propose a low-cost and meaningful method for selecting the convergence step that is well tuned in most practical cases. It can be used regardless of the type of the reference signal, the system delays, and the nature of the filter required in the filtered-x structure. Numerical simulations are performed to validate the effectiveness of the proposed approach.

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1. Introduction

The filtered-x LMS (Fx-LMS) algorithm [1] is often used in adaptive control systems [2] where the signal generated by the adaptive filter undergoes a modification before exercising control. This effect appears in active noise control (ANC) systems [3] like the one illustrated in Fig. 1. In this figure, the system named S corresponds to the so-called secondary path, which is an acoustic path representing such a transformation in practice. It is necessary to obtain an accurate secondary path estimate in order to modify the reference signal $x(n)$ in accordance with the control algorithm. Such a model is usually implemented using a linear filter [4,5] (system $\tilde{S}$ in Fig. 1) that is applied to the reference signal to transform it into the filtered reference signal, $x_f(n)$.

The controller aims to minimize the instantaneous power of the noise (error signal $e(n)$) at the control point, $J = e^2(n)$, and adjusts iteratively the adaptive filter, $\mathbf{w}$, by means of a steepest descent

strategy that uses the gradient of J with respect to $\mathbf{w}$, which is given by $\nabla J = 2e(n)\mathbf{x}_f$. Hence, the Fx-LMS algorithm updates the adaptive filter coefficients by using

$$\mathbf{w}(n) = \mathbf{w}(n-1) - \mu \mathbf{x}_f(n)e(n), \quad (1)$$

where $\mathbf{w}(n)$ is composed of the L coefficients of the adaptive filter and μ is the convergence step. $\mathbf{x}_f(n)$ is a vector with the last L samples of the filtered reference signal, and $e(n)$ is the error signal (residual noise signal picked up by the error sensor).

Since the filtered reference signal does not exhibit a flat spectrum in common cases, the convergence step parameter that controls the convergence of the Fx-LMS to the optimum filter $\mathbf{w}_{opt}$ by means of Eq. (1) as n increases may be challenging to select. The Fx-LMS algorithm has been extensively studied and analyzed [6–10] as well as the LMS algorithm and its convergence [11–13]. However, the theoretical models proposed to obtain an upper bound for the convergence step to guarantee algorithm stability were derived under many restrictive conditions, and its application is not straightforward. Therefore, the convergence step parameter is usually adjusted by trial and error [14] or using tables [15] in practice.

As a derivation of the theoretical results obtained by the convergence analysis of the LMS algorithm, the following bound is suggested in [16]

$$\mu < \frac{2}{\sigma_{x_f}^2 L} \quad (2)$$

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* Corresponding author.

E-mail addresses: mferrer@dcom.upv.es (M. Ferrer), mdediego@dcom.upv.es (M. de Diego), vmgarcia@dsic.upv.es (V.M. García-Mollá), agonzal@dcom.upv.es (A. Gonzalez).

¹ Eurasip member.

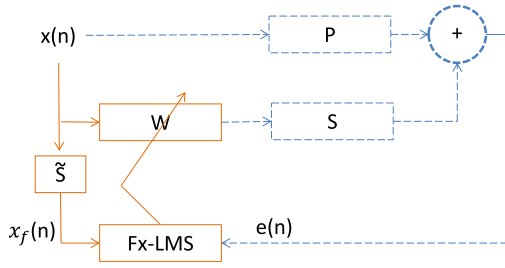


Fig. 1. Active noise control system with filtered-x structure (electrical paths in solid lines, acoustic paths in dashed lines).

with an optimum value of the convergence step given by $\mu_{opt} = \frac{1}{\sigma_{xf}^2 L}$, which is an approximation of

$$\mu_{\lambda} = \frac{1}{\lambda_{max}} \quad (3)$$

where λ_{max} is the largest eigenvalue of the $L \times L$ autocorrelation matrix of the filtered reference signal and σ_{xf}^2 is the average power of the filtered reference signal. Equation (2) is derived by bounding the largest eigenvalue of the autocorrelation matrix by the sum of all its eigenvalues, which coincides with the trace of this matrix. Furthermore, assuming that the noise signal is stationary, this trace is given by $\sigma_{xf}^2 L$.

The maximum convergence step given by Eq. (2) has to be relaxed in practical cases due to the delay introduced by S . Since this bound is very restrictive, it usually works properly when the filtered reference signal exhibits a flat spectrum. However, the spread of the eigenvalues increases when the spectrum of the filtered reference signal is colored or band-limited, and, consequently, the sum of the eigenvalues is closer to the maximum eigenvalue. Thus, the bound stated in Eq. (2) is then no longer so restrictive, and it may even become useless in the case of large delays due to S . Therefore, the stability conditions of the LMS algorithm cannot be applied straightforwardly to the Fx-LMS due to the transformation that the signal undergoes at the output of the adaptive filter (including the delay) before reaching the error sensor. Furthermore, errors in the estimate of the secondary path may also affect the Fx-LMS convergence conditions [17]. Perfect estimation of the secondary path, $\tilde{S} = S$, is considered throughout this work.

The computation of a theoretical upper bound on the convergence step of the Fx-LMS can be computationally costly [18]. Hence, many studies have been carried out to provide computationally efficient bounds. It is shown in [19] that the convergence step could be bounded by

$$\mu < \frac{2}{\sigma_{xf}^2 (L + 2\Delta)} \quad (4)$$

and its optimum value is given by $\mu/2$. Only the simplified case with a pure delay (Δ) on the secondary path is considered. This bound was modified in [20] by considering a secondary path modeled by a FIR filter $s(n)$ of order M , thus obtaining a practical bound based on the optimum step size as

$$\mu \leq \mu_{\Delta_{eq}} = \frac{1}{\sigma_{xf}^2 (L + 2\Delta_{eq})}, \quad (5)$$

where Δ_{eq} is called the equivalent delay of the secondary path and is given by

$$\Delta_{eq} = \frac{\sum_{n=0}^M ns^2(n)}{\sum_{n=0}^M s^2(n)}. \quad (6)$$

Equation (5) coincides with the optimum convergence step from Eq. (4) when the secondary path S is a pure delay. In any other

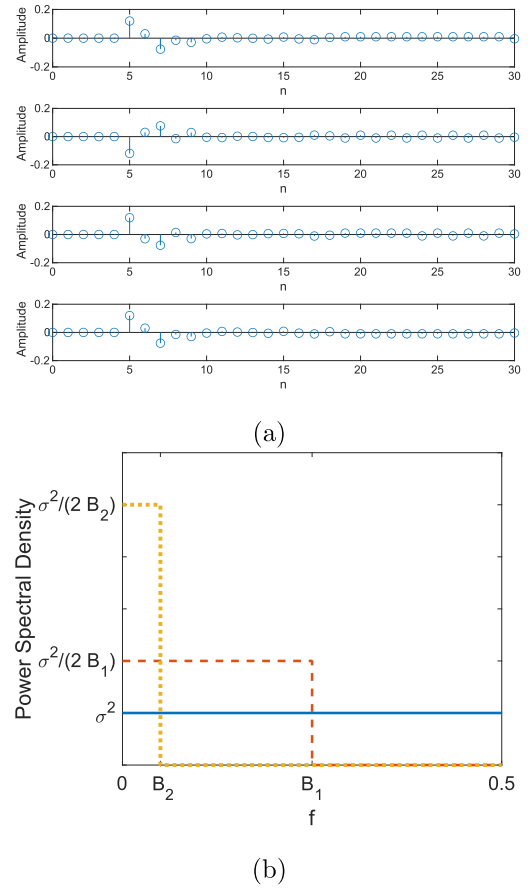


Fig. 2. (a) Different impulse responses of the acoustic path that would provide the same bound for μ according to Eq. (5) when the reference signal is white noise. (b) Different power spectral densities of the reference signal with mean power σ^2 that would provide the same bound for μ according to Eq. (4) when the acoustic path is a pure delay.

case, the calculation given by Eq. (6) intends to consider the effects produced by S . However, such a simple secondary path does not exist in practice, and a convergence step bound that is more feasible than Eq. (5) is required. For example, acoustic paths such as those shown in Fig. 2(a) would have identical convergence step bounds since their reference signal powers and Δ_{eq} values (given by Eq. (5)) coincide, but the maximum convergence step allowed in practical scenarios is different. Similarly, reference signals with the same mean power but different spectral distributions, such as those shown in Fig. 2(b), would have the same bound when the secondary path is a pure delay (given by Eq. (4)). However different bounds have to be used in practice.

The aforementioned bounds only depend on the length of the adaptive filter, the power of the filtered reference signal, and the delay inherent in the secondary path (or an equivalent delay modeled using Δ_{eq}). Nevertheless, those convergence step bounds do not adequately consider the spectral distribution of the reference signal or the frequency response of the secondary path. In [21], the same authors extended the analysis performed in [20] but considering band-limited signals. Moreover, a different mathematical model was derived in [21], and, consequently, the bound Eq. (5) was corrected in [22] by fitting experimentally measured values to a curve that models the behavior of the denominator of Eq. (5), then obtaining:

$$\mu \leq \mu_{\Delta_{eq2}} = \frac{1}{\sigma_{xf}^2 (L + 2 \frac{\Delta_{eq}}{B_w})} \quad (7)$$

where B_w is the bandwidth (normalized with respect to the full bandwidth) where the energy of the filtered reference signal exceeds a certain threshold. Although this result fits well with the experimental data, its extrapolation to other environments is not immediate. In the case of full-band systems or signals, this bound tends to coincide with the bound given by Eq. (5), which is useless in many cases. Moreover, in the case of band-limited signals, it is not always easy to determine the representative bandwidth of the signal because it depends on the selection of the relative-energy threshold.

To address these problems, this work presents a robust and meaningful method to estimate the convergence step bound of the Fx-LMS. The proposed approach is based on a sub-band analysis of the power spectral density for each frequency band. This model is valid for arbitrary spectral energy distribution of the filtered reference signal, provided that the spectrum could be assumed to be roughly flat at each band. Otherwise, it is necessary to know the suitable number and width of the spectral sub-bands. In this regard, we also present a simple algorithm that obtains those parameters. Thus, this approach is a practical way to choose the convergence step parameter, avoiding cumbersome and impractical statistical analyses.

2. Convergence step selection of the Fx-LMS by sub-band analysis

The sub-band Fx-LMS algorithm [23] splits the frequency spectrum of the signals involved into independent frequency bands to make the convergence conditions independent in each band (i.e. working as parallel independent adaptive systems). To achieve this, the algorithm makes use of a critically sampled filter bank. Thus, if we split the spectrum into B bands, the signals in each band could be decimated by a factor of B , leading to a normalized bandwidth of $\frac{1}{2B}$. We can assume that we use ideal filters, which do not introduce delay, and that the number of bands allows us to consider that the power spectral density of the filtered reference signal in each band is constant. Hence, the upper bound of the convergence step of the adaptive filter corresponding to the b th band can be derived from Eq. (5) as:

$$\mu_b = \frac{1}{\sigma_{x_{fb}}^2 \left(\frac{2}{2B}\right)(L + 2\Delta_{eq})} \quad (8)$$

with $\sigma_{x_{fb}}^2$ being the value of the power spectral density of the filtered reference signal corresponding to the b th band. FIR adaptive filters of L coefficients in each band are considered.

On the other hand, the average power of the signal considering the whole band can be written as:

$$\sigma_{xf}^2 = \sum_{b=1}^B \frac{1}{B} \sigma_{x_{fb}}^2. \quad (9)$$

Using Eq. (9) in Eq. (5), we obtain:

$$\mu_{\Delta_{eq}} = \frac{1}{\sum_{b=1}^B \sigma_{x_{fb}}^2 \left(\frac{1}{B}\right)(L + 2\Delta_{eq})}. \quad (10)$$

However, Eq. (10) is only suitable for signals whose spectrum is constant ($\sigma_{x_{fb}}^2 = \sigma_{xf}^2, \forall b$). In any other case, it would be feasible to think that each band should have a different weight in the calculation of the upper bound of the convergence step of the whole system. Thus, we propose the following bound

$$\mu \leq \mu(\text{proposed}) = \frac{1}{\sum_{b=1}^B \sigma_{x_{fb}}^2 \left(\frac{1}{B}\right) F_b (L + 2\Delta_{eq})} \quad (11)$$

where F_b denotes the weighting factor of the b th band. When the reference signal exhibits a flat spectrum, clearly $F_b = 1, \forall b$. In the

opposite case, when all of the energy of the reference signal is contained in a single band ν , the convergence step bound of that decimated system should coincide with the convergence step bound of the ν th band of the sub-band system when decimating the full-band system by a factor of B . It should be noted that when decimating the full-band system, the number of filter coefficients and the equivalent delay are decreased by the decimation factor B . Consequently, the step size satisfies the following inequality for a single-band ν reference signal:

$$\mu \leq \frac{1}{\sum_{b=1}^B \sigma_{x_{fb}}^2 \left(\frac{1}{B}\right) F_b \frac{1}{B} (L + 2\Delta_{eq})} = \frac{1}{\sigma_{x_{fv}}^2 \left(\frac{1}{B}\right) (L + 2\Delta_{eq})}, \quad (12)$$

from which it follows that, in this particular case, $F_b = B$ for $b = \nu$ and $F_b = 0$ for $b \neq \nu$.

The value of F_b that satisfies all of the above conditions is given by

$$F_b = \frac{\sigma_{x_{fb}}^2}{\sigma_{xf}^2} \cdot 1 \quad (13)$$

Thus F_b could be considered a spectral-shape factor. $F_b = 1$ when the energy of the filtered reference signal in this band matches the mean energy of the reference signal. F_b is larger than one when the energy in the band is higher than the mean energy. Finally, F_b is small when the band has low energy compared to the mean energy, becoming zero when it has no energy.

The spectral-shape factor F_b allows to deal with reference signals filtered through any secondary path and with flat spectrum in sub-bands. Eq. (11) is based on the sub-band analysis of the FxLMS algorithm and extends to sub-band analysis previous works presented in [7] and [20], which deal with different reference noises and secondary paths as it was introduced in Section 1.

Equation (11) can be generalized for signals with a flat spectrum in sub-bands with normalized bandwidth given by B_b . The convergence step bound in this case would be given by

$$\mu \leq \mu(\text{proposed}) = \frac{1}{\sum_{b=1}^B \sigma_{x_{fb}}^2 B_b F_b (L + 2\Delta_{eq})}. \quad (14)$$

It must be noted that all shape factors, F_b , become 1 in the case of a full-band flat signal. Thus, in this case, the bound proposed in Eq. (14) would coincide with that of Eq. (5).

The following steps summarized the calculation of the proposed convergence step bound:

1. Select the number of sub-bands B . Appendix A illustrates a proper selection of the number of bands for a general case. Although it could be alternatively fixed without cost for known or common cases depending on the type of the filtered reference signal.
2. Calculate the energy of the reference signal in each sub-band.
3. Calculate Eq. (14).

The calculation of the energy of the reference signal in each sub-band is the most costly operation in practice. It can be carried out using any valid estimator, for instance those based on the FFT. Therefore, the computational cost of the proposed method can be given in practice by $\mathbf{O}(K \log_2 K)$ being K the number of frequency bins used for the spectrum calculation.

3. Results

Several experiments have been carried out to check the validity and accuracy of the parameter proposed in Eq. (14). The results are also compared with the convergence step bounds previously proposed in the literature: μ_λ defined in Eq. (3) (which does not work well in practice due to the delay of the acoustic plant but is used as a comparative reference); $\mu_{\Delta_{eq}}$ from Eq. (5); and $\mu_{\Delta_{eq2}}$ given in

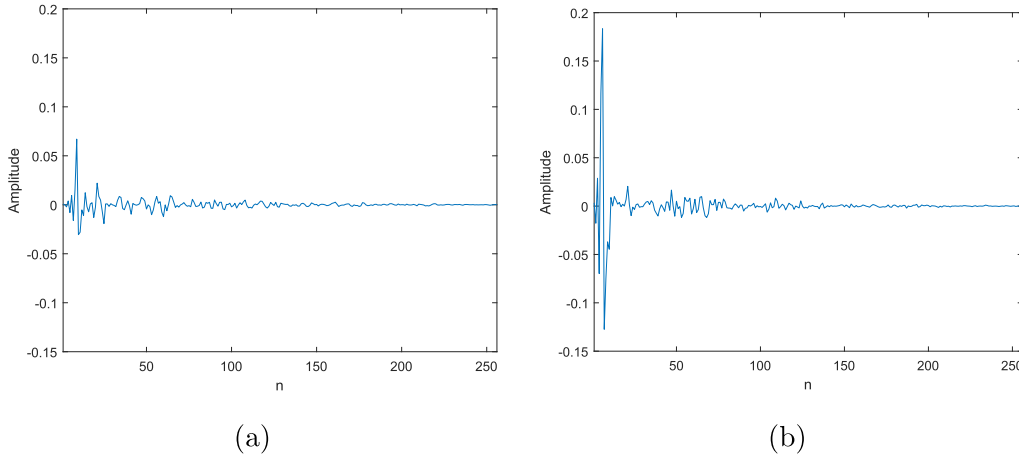


Fig. 3. Acoustic paths used in the experiments (sampling frequency 2 kHz): (a) primary path and (b) secondary path. These paths are available online in [25].

Table 1
Summary of the main characteristics of the experiments.

Exp.	Reference noise signal	Secondary path	Figure
1	full-band White Noise (WN)	delay	Fig. 4
2	low-pass WN/band-pass WN/high-pass WN	delay	Fig. 5a–c
3	low-pass WN (with added full-band WN)	delay/real meas.	Fig. 6(ab)/c
4	single-tone/multi-tone	real meas.	Fig. 7a/b
5	single-tone/multi-tone/low-pass WN	real meas.	Fig. 7c
6	pink noise	delay	Fig. 8

Eq. (7) (for band-limited signals). In addition, the maximum convergence step value that guarantees the algorithm stability in each experiment ('experimental μ ') is also provided. The proposed convergence step bounds should approach (or should at least never be larger than) this 'experimental μ '. This 'experimental μ ' was obtained by means of repeated simulations with increasing values of the step size until the maximum speed of convergence without divergence was reached. The other bounds were calculated according to their mathematical expressions. The estimation of $\sigma_{x_b}^2$ has been calculated from the filtering of the reference signal through the impulse response of the selected secondary path for each band b . This response in each band has been measured using the Swept-Sine Technique [24], which cannot be used for very narrow bandwidths. All of the experiments were performed on a system that is based on the scheme illustrated in Fig. 1. The acoustic paths P and S are modeled with 256 coefficients and correspond to real responses measured in a room and sampled at 2 kHz. They are shown in Fig. 3. A pure delay system is also considered in some experiments to model the secondary path S . Table 1 summarizes the main characteristics of the experiments.

All of the models considered for calculating the convergence step bound except μ_λ (for the reasons previously discussed) should behave properly for reference signals of flat spectrum and pure delays since their expressions will become the same in this case. Consequently, we will first verify this behavior.

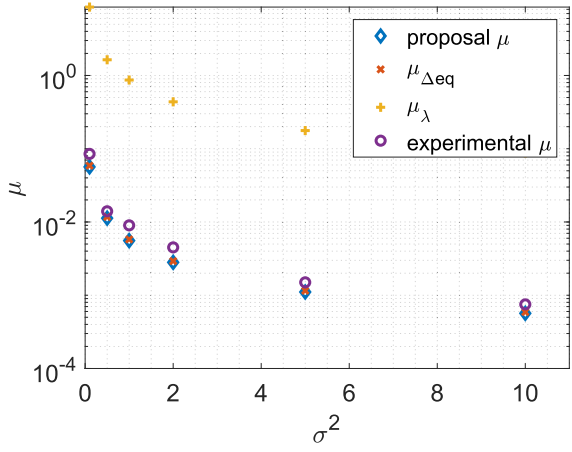
3.1. Step-size bounds for full-band white noise reference signal and a delay in the secondary path

In the first experiment (Fig. 4), the secondary path S is a pure delay system ($\Delta = 10$). An adaptive filter of $L = 150$ coefficients and eight spectral bands of the same width ($B_b = 1/16$,

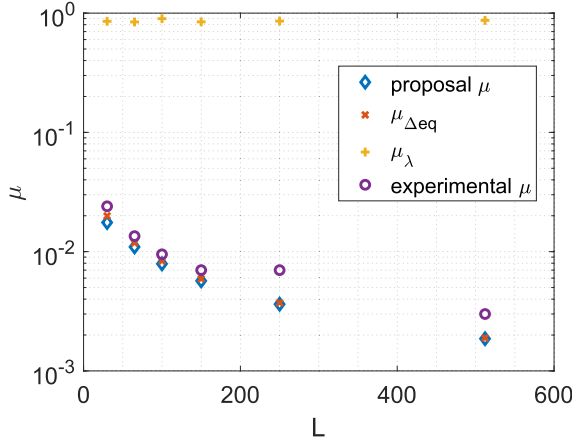
$\forall b$) are considered. The reference signal is white noise with $\sigma^2 \in \{0.1, 0.5, 1, 2, 5, 10\}$. Figure 4(a) illustrates the values of the convergence step bound for each model compared to the corresponding 'experimental μ '. As can be observed, both the proposed bound and $\mu_{\Delta_{eq}}$ offer very similar results, which are close to the experimental one. Note that neither the theoretical value of σ^2 nor the value of the autocorrelation of the reference signal have been used. Alternatively, σ^2 has been estimated from the experimental data since this work aims to provide an upper bound of the convergence step that can be used in practice without resorting to the knowledge of the signal statistics. For this reason, slight differences between the proposed convergence step bound given in Eq. (14) and $\mu_{\Delta_{eq}}$ in Eq. (10) are observed. However, we should theoretically obtain the same values in this experiment. In any case, both values are lower than (but close to) the experimental convergence step. Therefore, they are suitable. On the other hand, it is observed that the values of μ_λ cannot be used in practice since this bound does not consider the delay of the acoustic path, so its value is always much higher than the experimental one. The same conclusions arise when $\sigma^2 = 1$ and the number of filter coefficients or the acoustic path delay changes. This is illustrated by Fig. 4(b) and (c). This means that the bound given by Eq. (5) and the one proposed by Eq. (14) are suitable for flat full-band reference signals and pure delays in the secondary path.

3.2. Step-size bounds for filtered reference signal and a delay in the secondary path

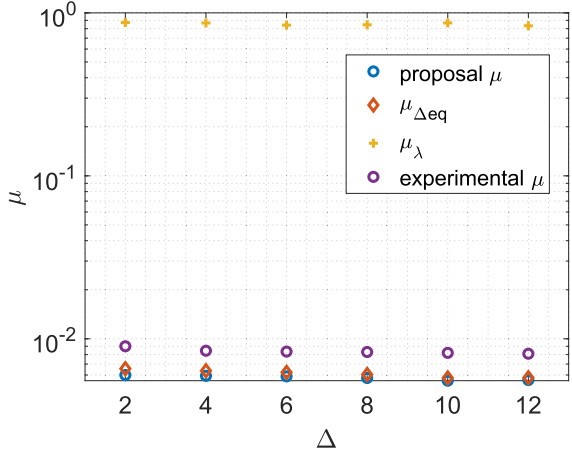
In a second experiment (Fig. 5), several band-limited signals are selected as reference signals using a secondary path with $\Delta = 10$ and an adaptive filter of 150 coefficients. B_w represents the normalized signal bandwidth concerning the total bandwidth. Figure 5 shows the different bounds obtained when filtered white noise is used as a reference signal: Fig. 5(a) for a low-pass filtered signal, Fig. 5(b) for a band-pass filtered signal, and Fig. 5(c) for a high-pass filtered signal. Because the bound $\mu_{\Delta_{eq2}}$ given in Eq. (7) differs from that of $\mu_{\Delta_{eq}}$ for band-limited signals, the parameter $\mu_{\Delta_{eq2}}$ is also computed. As can be observed, the only bound that always remains below the experimental value of the convergence step is the one proposed in this work. On the other hand, $\mu_{\Delta_{eq}}$ and $\mu_{\Delta_{eq2}}$ could be helpful (and close to the proposed approach) only when the reference signals are full-band signals since they exceed the experimental results when B_w decreases.



(a)



(b)

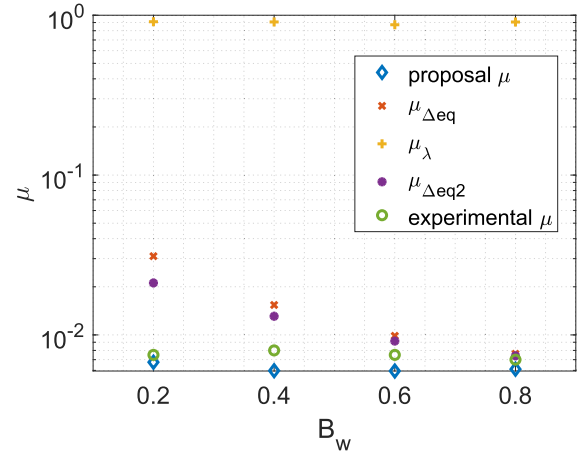


(c)

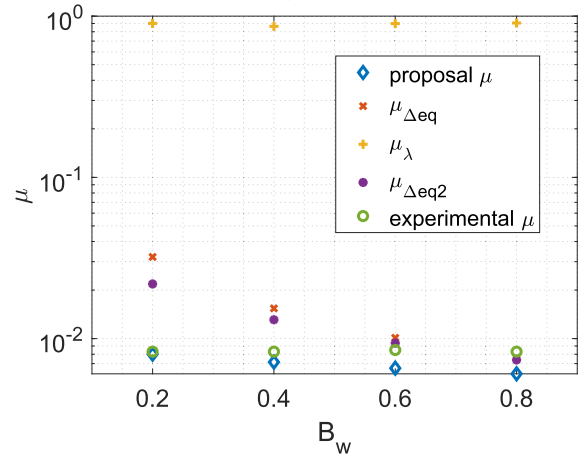
Fig. 4. Convergence step bounds obtained for a white-noise reference signal: (a) varying the average power of the reference signal, σ^2 ; (b) varying the number of adaptive filter coefficients, L ; and (c) varying the acoustic path delay, Δ .

3.3. Step-size bounds for low-pass reference signal with added white noise

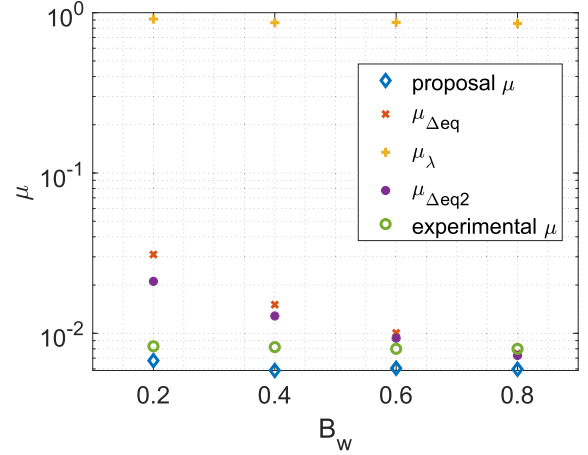
The third experiment, which is illustrated in Fig. 6, aims to verify the validity of the different convergence step bounds when a broadband noise for different signal-to-noise ratio (SNR) is added



(a)



(b)



(c)

Fig. 5. Convergence step bounds obtained for a white-noise reference signal filtered with: (a) a low-pass filter, (b) a band-pass filter, and (c) a high-pass filter, leading to a normalized bandwidth $B_w = [0.2, 0.4, 0.6, 0.8]$.

to a band-limited reference signal. A low-pass reference signal is used. Note that the resulting composed signal cannot be considered band-limited. The SNR varies from -15 to 15 dB in Fig. 6(a) when the reference signal with $B_w = 0.1$ (cut-off frequency $f_c = B_w/2$) and $\Delta = 10$ are considered. Figure 6(b) shows the bounds when SNR=10dB and different B_w (the secondary path is also mod-

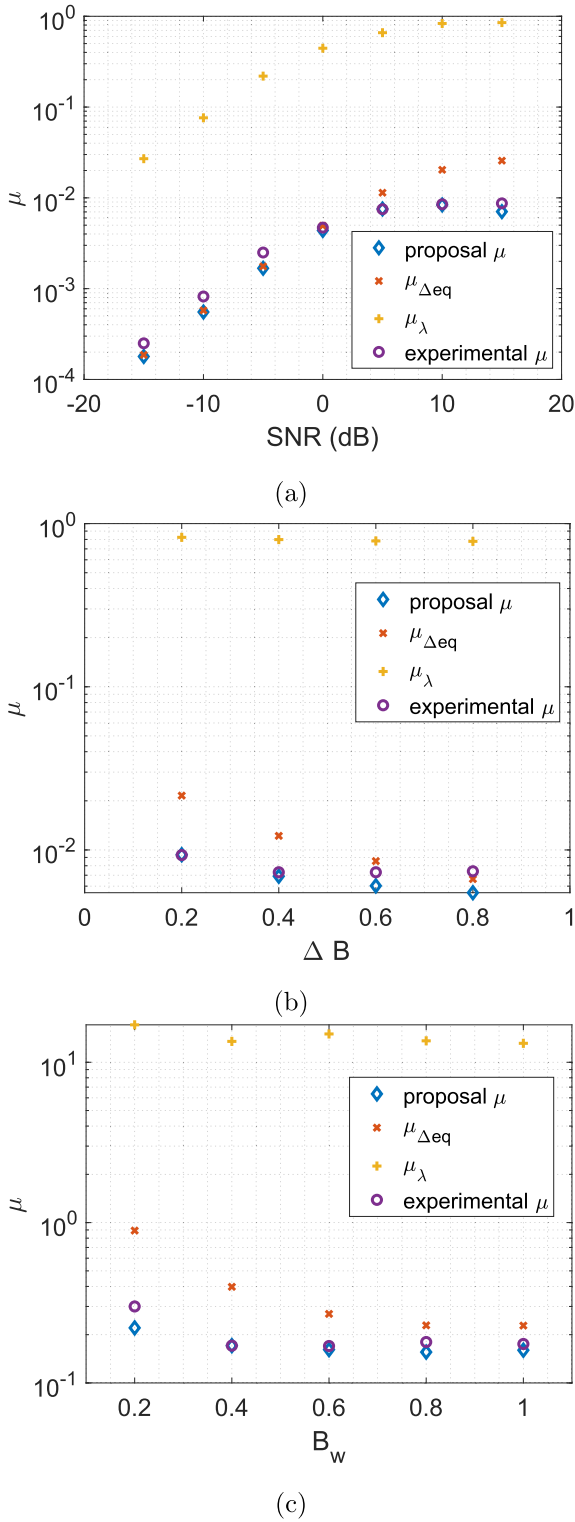


Fig. 6. Convergence step bounds with a band-limited reference signal plus noise: (a) varying SNR, (b) for SNR=10 dB and different bandwidths, and (c) varying the bandwidth and using a real secondary path (not a pure delay system).

eled as a pure delay). When the SNR decreases the broadband noise tends to mask the reference signal. Then the resulting signal tends to be a flat-spectrum signal and the values of both the proposed bound and $\mu_{\Delta eq}$ are similar and very close to the experimental one. However, $\mu_{\Delta eq}$ is no longer useful since it reaches higher values than the experimental ones when the SNR increases,

whereas the proposed bound remains at lower values and close to the experimental ones. On the other hand, a similar behavior is observed when the bandwidth of the reference signal varies. In this case, large bandwidths show similar behavior to the full-band flat-spectrum signals, whereas $\mu_{\Delta eq}$ is no longer valid when the bandwidth decreases.

Figure 6 (c) illustrates the bounds obtained when the secondary path is a real measurement instead of a simulated pure delay system. It is important to note that the reference signal is full-band white noise that is filtered by the acoustic path when $B_w = 1$. Also note that the proposed bound is the only one that shows good agreement with the experimental convergence step in all cases.

3.4. Step-size bounds for single-tone and multi-tone reference signal with a real secondary path

The next study is focused on the effect of using single or multi-tone reference signals instead of white or band-limited noises with the real secondary path commented above. Figure 7(a) illustrates the convergence step bounds obtained for single-tone reference signals. Tones of unit amplitude and discrete frequencies given by [0.05, 0.15, 0.25, 0.35, 0.45] are used. The results show a wide variation of the values due to the effect of the frequency response of the secondary path. As can be observed, the only bound that provides accurate values in each case is, again, the one proposed in this work. However, the rest of the approaches give values that are higher than the experimental convergence parameter. On the other hand, Fig. 7(b) presents the results for a multi-tone reference signal given by

$$x(n) = \sum_{k=0}^{\frac{0.5-\Delta f_0}{\Delta f_0}} \cos[2\pi(\Delta f_0 + k\Delta f_0)n]. \quad (15)$$

Thus, when Δf_0 decreases, the spectral richness of the signal (more tones) increases and the separation between tones decreases. The values of Δf_0 considered are: [0.001 0.005, 0.01, 0.05, 0.1, 0.2]. It can be observed that the proposed model gives values that are higher than the experimental results for the two lowest Δf_0 , so they would not be valid. This is because the reference signal spectrum is split into eight bands of equal width, as in the first experiment. However, the number of bands and their widths should be adjusted so that each one contains a sufficiently flat part of the spectrum (in the case of broadband signals) or only one single-tone (in the case of multi-tone signals). Note that in the case of multi-tone signals with Q tones of amplitude A_q , Eq. (14) can be written as:

$$\mu < \frac{1}{\sum_{b=1}^Q (A_b^2/2) \frac{(A_b^2/2)}{\sum_{q=1}^Q A_q^2/2} (L + 2\Delta eq)}, \quad (16)$$

without the need for an analysis of the signal spectrum.

Consequently, the bands that are used to calculate the proposed bound must be as small as possible when the power spectral density of the filtered reference signal exhibits a large difference between close frequencies or when the multi-tone signal is composed of very close tones. The experiments show that when Δf_0 is very small, there is not enough frequency resolution with only eight bands to obtain a suitable convergence step bound according to the proposed criterion. The same would have happened if values of B_w smaller than 0.2 had been selected in the experiments of Figs. 5 and 6. Thus, the number of analysis bands and their width must be in accordance with the shape of the filtered reference signal spectrum, and their choice becomes more critical as the reference signal spectrum exhibits significant differences between close frequency bands. On the other hand, an accurate estimation of the signal power in each band is needed. Therefore, it turns out that

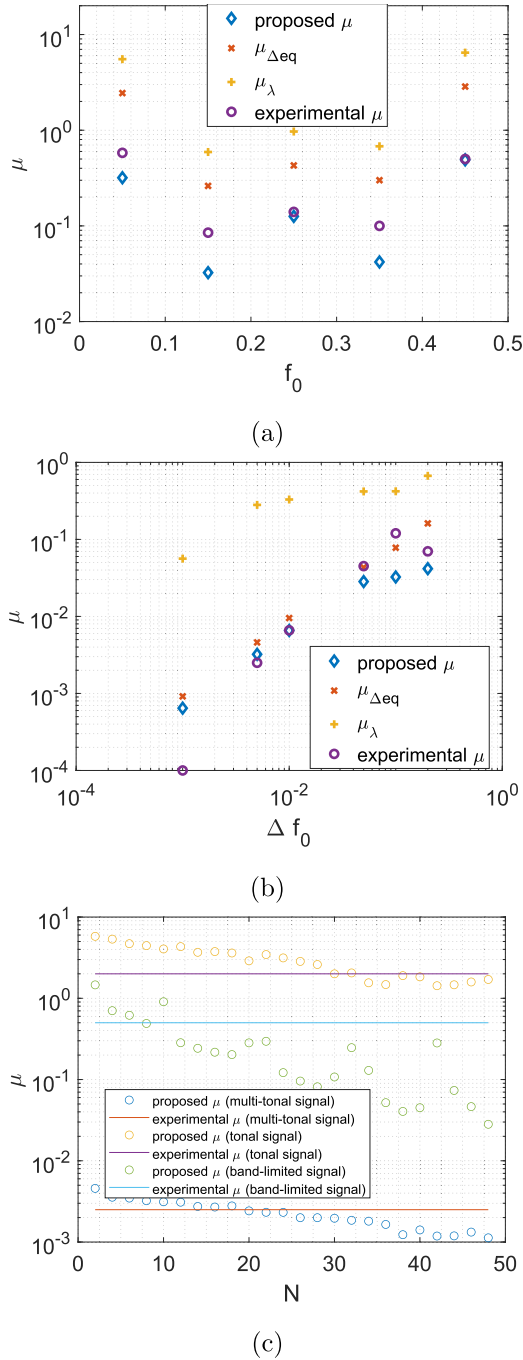


Fig. 7. Convergence step bounds obtained for a real secondary path and: (a) a single-tone reference signal, (b) a multi-tone reference signal, and (c) three different reference signals varying the number of spectral bands B (a single-tone signal, a multi-tone signal with $\Delta f_0 = 0.005$, and a band-limited signal with $B_w = 0.1$).

the frequency resolution for calculating the proposed bound must be chosen as a trade-off between the flatness of the spectrum and the accuracy of the power estimation of the filtered reference signal within each sub-band.

3.5. Step-size bounds for: Single-tone, multi-tone and band-limited reference signals, depending on the number of sub-bands B

In order to determine the influence of the number of analysis bands (for simplicity, of the same bandwidth), the experimental convergence step was compared with the proposed bound for dif-

ferent values of B (number of analysis bands) and different filtered reference signals whose spectra showed large variations (considering in all cases a real acoustic path). Figure 7(c) illustrates the behavior of the proposed bound for different B . In the upper part of the figure, a single-tone reference signal with $\Delta f_0 = 0.005$ is used. As can be observed, suitable bounds are obtained from $B = 40$ analysis bands. Moreover, less than 10 bands is insufficient to guarantee a suitable bound for a single-tone reference signal, although a suitable value is obtained for just eight bands (the case of the previous experiment).

Finally, the results for a band-limited signal and $B_w = 0.1$ (lower than the cases considered in the previous experiments) plus an added noise with SNR=15 dB are also shown in Fig. 7(c). In this experiment, a useful value for the convergence step value is given from 20 analysis bands. An oscillation of the convergence step with the number of bands can be observed. This is due to the relation between the signal energy distribution and the number of sub-bands. Since the signal is band-limited and its transition band is extended between 0.2 and 0.22 (cut-off frequency), when a given sub-band includes this transition band, the power calculation is not accurate enough and the convergence step cannot be accurately estimated. This effect can be observed when the number of bands is far from multiples of 11, whereas the method obtains good results for multiples of 11 in the figure.

Appendix A illustrates the proper selection of the number of bands. Furthermore, an algorithm for the selection of the number and width of the sub-bands based on the spectral flatness is also proposed, which is a practical alternative to the uniform sub-band distribution.

Thus, a meaningful convergence step bound is always found from a certain number of analysis bands. The value of this bound tends to decrease as the number of bands increases and tends to saturate above a given value. It is important to note that the power of the multi-tone signal within each sub-band has been estimated from the samples of the experiment, which means that Eq. (16) has not been used.

3.6. Step-size bounds for pink noise reference signal depending on the number of sub-bands B

Uniform sub-band analysis has been assumed in all of the previous experiments. However, it is not always the most appropriate method. For example, if we consider pink noise signals, where the variation of the power spectral density increases with the frequency, it would be convenient to adjust the bandwidth of each band so that $B_{b+1} = 2B_b$. In this way, we will optimize the number of bands needed to obtain a suitable convergence step bound.

Figure 8 shows the results of the proposed convergence step bound as a function of the number of bands when the reference signal is pink noise and the secondary path is a pure delay system of 10 samples. Uniform and exponentially growing bandwidth resolutions are considered. The adaptive filter has $L = 150$ coefficients in all cases. As can be observed, uniform bandwidth for the spectrum analysis of the filtered reference signal does not provide convergence step bounds that are close to the experimental value in this experiment, even when the number of bands is large. On the contrary, if bands with increasing bandwidth are used, since the spectrum variation is faster where the bandwidth is smaller, we reach the experimental value of the convergence step using fewer bands than with the uniform bandwidth. However, we need more resolution in the narrower (low frequency) bands to estimate the power spectral density in all bands correctly. Thus, for $B > 24$ (where the lower band has a width of $0.5/2^{24}$), we need more resolution than that used in the experiment to obtain an accurate step bound.

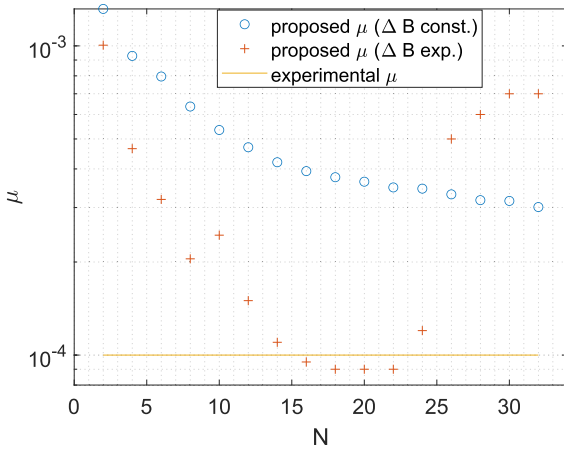


Fig. 8. Convergence step bound for a pink noise signal with different numbers of sub-bands B and constant or exponential bandwidth of the analysis bands.

It should be noted that the nature of this type of signal is the most adverse for the proposed methodology since we are considering a constant spectral density within each band. Thus, we are ‘approximating’ the spectral density of the filtered reference signal. In cases where this approximation does not represent the actual spectrum correctly, the resulting bound may be inaccurate.

Although the proposed bound usually is quite well tuned in practical cases, the convergence step bound proposed in this work can be relaxed by a factor of less than unity (a factor of 0.6 has shown meaningful results) when the shape of the power spectral density of the filtered reference signal changes abruptly with frequency (and therefore a constant value within each band does not correctly represent its shape) or when the estimation of the power spectral density value in each band is not accurate enough.

3.7. Comparison of the proposed convergence step bound with a bound derived from a statistical analysis

Despite the simplicity of the proposed method, its performance is comparable to (and even outperforms) results derived from more complex and computationally expensive models. For example, a statistical analysis is proposed in [18] to model the behavior of the Fx-LMS algorithm applied to ANC, from which a bound for the convergence step could be derived. The computational complexity of this method is very high and increases with the length of the secondary path. The following secondary paths were used in [18] to test the proposed method (where $P(z)$ represents the primary path, $S(z)$ the secondary path, and $\tilde{S}(z)$ the estimate of the secondary path):

- System 1: $P(z) = z^{-22}$, $S(z) = z^{-20}$, $\tilde{S}(z) = S(z)$;
- System 2: $P(z) = 1.5 + z^{-1} - 0.7z^{-2}$, $S(z) = 0.2 + 0.8z^{-1}$, $\tilde{S}(z) = S(z)$;
- System 3: $P(z) = 1.5 + z^{-1} - 0.7z^{-2}$, $S(z) = 0.2 + 0.8z^{-1}$, $\tilde{S}(z) = 0.4 + 0.9z^{-1} \neq S(z)$;
- System 4: $P(z)$ and $S(z)$ are represented in Fig. 3 of [18] (with 20 coefficients), and $\tilde{S}(z) = S(z)$.

The reference signal (and noise) used in [18] is given by $x(n) = \eta x(n-1) + \phi(n)$, where $\phi(n)$ is a Gaussian random variable and $\eta = 0$ when $x(n)$ is white noise, while $\eta = 0.8$ when $x(n)$ is colored noise (modeled by an AR system). The adaptive filters have four coefficients for the first three systems and eight coefficients for the last one.

Table 2 shows the comparative results obtained. The experiments that provided the results of Table III of [18] have been reproduced. The results of the convergence step bound proposed in

Table 2

Comparative table between the proposed method and the results of [18].

Ref. signal	system ner.	experimental	[18]	proposed
White noise	System 1	0.071	0.061	0.0691
	System 2	0.56	0.384	0.7685
	System 4	0.43	0.360	0.43
AR noise	System 1	0.0093	0.0081	0.0087
	System 2	0.087	0.06	0.0986
	System 3	0.065	0.047	0.0902

this work fit the experimental ones quite well for the systems with Paths 1 and 4 (even better than those proposed in [18]). However, the proposed bound is somewhat higher than the experimental one (but of the same order) for the systems with Paths 2 and 3. Therefore, this bound would have to be relaxed in practice to assure convergence. Note that the solutions for the adaptive filter in these two systems (Paths 2 and 3) are non-causal, so the ‘causal’ solution reached by the adaptive filter in practice does not really cancel the noise (only slightly decrease the power of the noise signal). Consequently, the convergence analysis of these systems (Paths 2 and 3) should not be considered equivalent to the previous ones (Paths 1 and 4).

4. Multichannel extension

The extension of this work to a multichannel ANC system comprised of J actuators and K error sensors (as it is depicted in Fig. 9) arises almost straightforwardly when the following considerations are taken into account:

- The multichannel system has $J \times K$ secondary paths that provide $J \times K$ filtered reference signals ($x_{jk}(n)$).
- J adaptive filters of L coefficients are required.
- The update equation for the j th filter (following the multiple error filtered-x LMS algorithm proposed in [26]) is:

$$\mathbf{w}_j(n) = \mathbf{w}_j(n-1) - \mu \sum_{k=1}^K \mathbf{x}_{jk}(n) e_k(n), \quad (17)$$

- In the case of more than one reference signal, a single reference signal resulting from combining all of them can be considered for the purpose of calculating the convergence step bound.
- The value of the trace of the multichannel autocorrelation matrix of the filtered reference signals that would allow bounding the highest eigenvalue of this matrix is given by $\sigma_{\mathbf{mc}}^2 = L \sum_{k=1}^K \sum_{j=1}^J \sigma_{jk}^2$, where σ_{jk}^2 is the mean power of the reference signal filtered through the acoustic path between the j th actuator and the k th error sensor.
- The equivalent delay of the multichannel system, $\Delta_{\mathbf{mc}}$, would be chosen as the most restrictive (the largest) among the equivalent delays of all of the acoustic (secondary) paths.

Therefore, Eq. (5) can be modified for the multichannel case as

$$\mu < \mu_{1\Delta_{\mathbf{mc}}} = \frac{1}{\sigma_{\mathbf{mc}}^2 (L + 2\Delta_{\mathbf{mc}})}, \quad (18)$$

which is again valid only for full-band filtered reference signals. On the other hand, Eq. (18) can be directly applied or, following the work in [26], the most restrictive single channel bound for this system can be used, which means:

$$\mu < \mu_{2\Delta_{\mathbf{mc}}} = \frac{1}{(JK)\sigma_{jk,\max}^2 (L + 2\Delta_{\mathbf{mc}})}. \quad (19)$$

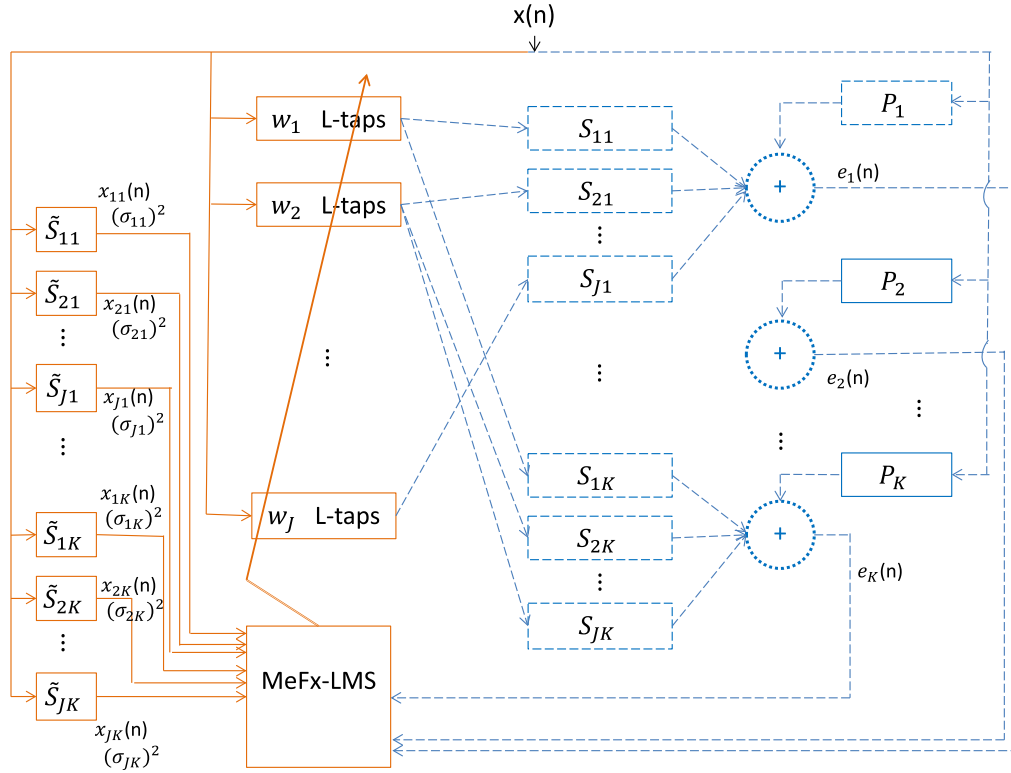


Fig. 9. Multichannel ANC system (electrical paths in solid lines, acoustic paths in dashed lines).

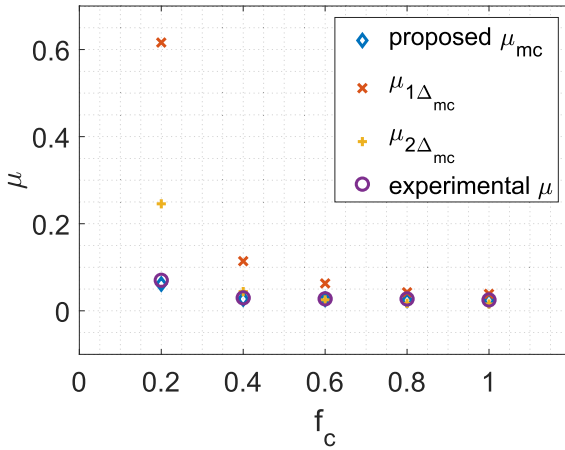


Fig. 10. Convergence step bound of a multichannel (2×2) system with a reference signal that was low-pass filtered at different cutoff frequencies (f_c).

However, if the filtered reference signal is analysed in B bands of B_b bandwidth, the following expression derived from the single channel case can be proposed

$$\mu < \mu_{mc} = \frac{1}{\sum_{b=1}^B \sigma_{mc_b}^2 B_b F_b (L + 2\Delta_{mc})}, \quad (20)$$

where $\sigma_{mc_b}^2$ is the power spectrum value of the filtered signal within the b band, $F_b = \frac{\sigma_{mc_b}^2}{\hat{\sigma}_{mc}^2}$, and, $\hat{\sigma}_{mc}^2 = \sum_{b=1}^B \sigma_{mc_b}^2 \Delta B_b$.

Figure 10 shows the convergence step bounds obtained for each of the three multichannel convergence step bounds mentioned above, together with the (experimental) maximum convergence step that guarantees system stability. A multichannel system with

two actuators ($J = 2$) and two error sensors ($K = 2$) has been used. The reference noise signal was unit power white noise that was filtered through a low-pass filter whose cutoff frequency varied from 0.2 to 1 (full-band signal). As can be observed, the proposed convergence step fits very well with the experimental one. When the reference signal is full-band, all of the proposals are close to the experimental value, although $\mu_{1\Delta_{mc}}$ is higher than the experimental value and so useless. As the reference signal becomes band-limited, the alternative bounds move further away from the experimental one and become useless in almost any case because they provide values that are higher than the experimental bound. The bound proposed in this work guarantees stability and is the closest to the experimental bound of the convergence step in all cases.

5. Conclusion

Although the analysis of the stability of the Fx-LMS algorithm has been performed in several previous studies, these studies are too restrictive, provide theoretical bounds of the convergence step that are mainly only valid for simple cases (acoustic paths consisting of pure delays, tonal signals, etc.), or they depend on statistical parameters of inaccurate (and not always simple) estimation in practical cases. In this work, we have proposed a convergence parameter bound that guarantees the stability of the Fx-LMS algorithm, providing values that are very close to those achieved in practice, even for multichannel systems. The calculation of this bound is performed by using an estimation of the spectral density per frequency band of the filtered reference signal (where this spectral density can be considered constant). It has been proved that the bound proposed in this work is suitable for signals and filters (acoustic paths) of different types, and it notably improves the results of the theoretical bounds used so far.

Declaration of Competing Interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests:

Alberto Gonzalez reports financial support, equipment, drugs, or supplies, and travel were provided by Spain Ministry of Science and Innovation. Alberto Gonzalez reports financial support, article publishing charges, travel, and writing assistance were provided by Government of Valencia.

CRediT authorship contribution statement

Miguel Ferrer: Conceptualization, Methodology, Formal analysis, Software, Writing – review & editing. **Maria de Diego:** Conceptualization, Methodology, Formal analysis, Writing – review & editing. **Víctor M. García-Mollá:** Conceptualization, Methodology, Formal analysis, Writing – review & editing. **Alberto Gonzalez:** Conceptualization, Methodology, Formal analysis, Funding acquisition, Writing – review & editing.

Data availability

Data will be made available on request.

Appendix A. Selection of the number and widths of the analysis sub-bands

The advantage of the proposed method lies in its simplicity and accuracy. However, this accuracy depends on two main factors:

- The estimation of the power spectral density of the reference signal must be sufficiently representative, which leads us to use methods with sufficient frequency resolution concerning the considered bandwidths.
- The power spectral density of each band can be considered constant and approximately equal to the value of the average power in that band.

Therefore, the number of sub-bands B and their widths B_b must be suitably chosen. The reliability of the estimated power spectral density in each band depends on the spectral analysis method used. The bandwidths B_b used to calculate the proposed bound should not be lower than the frequency resolution of the spectral analysis method employed, which can be denoted by $B_{\min}$. On the other hand, assuming that the power spectral density in each band is correctly estimated, the reliability of the proposed convergence step bound requires the power spectral density to be almost constant within each band. Note that the power spectral density value used in the calculation of the proposed bound is an estimation that is closer to the average value of the power spectral density within each band. Hence, if the band does not exhibit a constant spectral density, this indicator could underestimate the weight of that band in the calculation of the bound. A safer bound arises when the maximum value of the power spectral density is selected, instead of its mean value. However, this could lead to obtaining a bound that is too restrictive when the power spectral density is very variable within any band.

We suggest the Spectral Flatness parameter to calculate the sub-band widths. This parameter is defined for sub-band b that encompasses b_l to b_u frequency bins as [27]:

$$\Gamma_b = \frac{\left(\prod_{k=b_l}^{b_u} s_k \right)^{\frac{1}{b_u-b_l}}}{\frac{1}{b_u-b_l} \sum_{k=b_l}^{b_u} s_k}, \quad (\text{A.1})$$

where s_k is the value of the estimated spectral power corresponding to the b_k frequency bin.

The Spectral Flatness is closer to 1 for flat bands and obtains lower values for spikier bands ($0 < \Gamma < 1$). A Spectral Flatness that is lower than a given threshold Γ_{th} can warn us that the signal spectrum in the selected sub-band is not flat enough, and, consequently, its estimated power spectral density is not representative. Bands with a Spectral Flatness lower than the required threshold Γ should be split into narrower bands.

Generally speaking, the use of bands of constant width and suitable spectral resolution is enough for most applications. However, Algorithm 1, which makes use of a Spectral Flatness thresh-

Algorithm 1 Algorithm to select $B_b \forall b \in \{1, \dots, B\}$ (normalized by $B_{\min}$).

```

1: Input:  $\mathbf{S} = [s_0, s_1, \dots, s_{K-1}]$ ,  $\Gamma_{th}$ 
2: Output:  $B_b$ ,  $1 \leq b \leq B$ 
3:  $b = 1$ ;  $b_l = 0$ ;  $b_u = 0$ ;  $\Gamma_b = 1$ 
4:  $K = \text{length}(\mathbf{S})$ 
5: while  $b_u < K$  do
6:   while  $\Gamma_b > \Gamma_{th}$  do
7:      $b_u = b_u + 1$ 
8:      $\Gamma_b = \frac{\left( \prod_{k=b_l}^{b_u} s_k \right)^{\frac{1}{b_u-b_l}}}{\frac{1}{b_u-b_l} \sum_{k=b_l}^{b_u} s_k}$ ,  $b_u - b_l + 1$  sums and  $(b_u - b_l)^2 + 1$ 
                        products *
9:   end while
10:   $B_b = b_u - b_l$ 
11:   $b = b + 1$ 
12:   $b_l = b_u$ 
13:   $\Gamma_b = 1$ 
14: end while
15: if  $b_u > K$  then
16:    $B_b = K - 1 - b_l$ 
17: end if

```

*The computational cost of this algorithm depends mainly on the computation of the spectral flatness in line 8. This calculation has to be performed several times for each band selection. The total number of operations for each band selection also depends on the value of the threshold Γ_{th} .

old, provides suitable bandwidths for the calculation of the proposed convergence step bound in all cases, being K , the number of frequency bins.

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