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Controlled shadowing property

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Abstract

In this paper we introduce a new notion, named controlled shadowing property and we relate it to some notions in dynamical systems such as topological ergodicity, topologically mixing and specification properties. The relation between the controlled shadowing and chaos in sense of Li-Yorke is studied. At the end we give some examples to investigate the controlled shadowing property.

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1. Introduction

The concept of shadowing is investigated by many authors (see e.g. [4, 5, 11, 13, 14, 15]). Topological mixing and the specification properties are important notions in dynamical systems which has been related to chaos in global sense [10]. To investigate the topological mixing, specification and chaos, the shadowing property along is not useful. So some authors used the another definition of the shadowing property such as average shadowing, ergodic shadowing and d-shadowing to investigate specification property and chaos [6, 7, 12]. In their definitions, there are some large mistakes in jumping of pseudo orbits, but they still obtained some regularity of this mistakes. In this paper we introduce a new notion, named controlled shadowing property and we relate it to some notions in dynamical systems such as topological ergodicity, topologically mixing, specification property and chaos. First, motivated by [8, 9], we show that any surjective map with controlled shadowing property is chain transitive and

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prove that any stable map with controlled shadowing property is topological ergodic which is stronger than transitivity. As well we will relate the controlled shadowing property with the specification property. In fact we will prove that for any surjective map with shadowing property on a compact metric space, the controlled shadowing and the specification properties are equivalent. The relation between the controlled shadowing and chaos in sense of Li-Yorke is the next result of this paper. At the end we give some examples to investigate the controlled shadowing property.

2. Preliminaries

Let (X, d) be a metric space and let $f : X \to X$ be a continuous map. A sequence $\{x_n\}_{n=0}^{\infty}$ is called an orbit of f, denoted by $O(x, f)$, if for each $n \in \mathbb{N}$, $x_{n+1} = f(x_n)$ and we call it a δ-pseudo-orbit of f if

$$d(f(x_i), x_{i+1}) < \delta, \quad \text{for all } i \in \mathbb{N}.$$

A continuous map f is said to have the shadowing property if for each $\varepsilon > 0$ there exists $\delta > 0$ such that every δ-pseudo-orbit $\{x_i\}_{i=0}^{\infty}$ is ε-shadowed by the orbit of some point $y \in X$, i.e

$$d(f^n(y), x_n) < \varepsilon, \quad \text{for all } n \in \mathbb{N}.$$

A map f is called chain transitive if for any $x, y \in X$ and for every $\epsilon > 0$, there exists an ϵ-pseudo orbit (ϵ-chain) from x to y. A point $x \in X$ is stable point if for any $\epsilon > 0$ there is a $\delta > 0$ such that if $d(x, y) < \delta$, then $d(f^i(x), f^i(y)) < \epsilon$ for every $i \in \mathbb{N}$. A surjective continuous map f is stable map if any point of X is stable point.

Let U and V be two nonempty open subsets of X and consider

$$N(U, V) = \{n \in \mathbb{N};\ f^n(U) \cap V \neq \varnothing\}.$$

A map f is called transitive if for each nonempty open subsets U, V of X, $N(U, V) \neq \varnothing$.

f is called topologically ergodic if for every nonempty open subsets U, V of X, $N(U, V)$ has positive upper density, that is

$$D\big(N(U, V)\big) = \limsup_{n\to\infty} \frac{1}{n} \text{card}\big\{N(U, V) \cap \{0, \cdots, n-1\}\big\} > 0,$$

where card(A) denotes the number of members of the finite set A.

f is called topologically mixing if $f \times f$ is transitive.

We say that a sequence $\{x_n\}_{n=0}^{\infty}$ is a controlled-δ-pseudo orbit if

$$\limsup_{n\to\infty} \frac{1}{n} \text{card}\big\{i \in \{0, \cdots, n-1\};\ d(f(x_i), x_{i+1}) \geq \delta\big\} < \delta.$$

We say that a controlled-δ-pseudo-orbit $\{x_n\}_{n=0}^{\infty}$ is control-ε-shadowed by $y \in X$ if

$$\limsup_{n\to\infty} \frac{1}{n} \text{card}\big\{i \in \{0, \cdots, n-1\};\ d(f^i(y), x_i) \geq \varepsilon\big\} < \varepsilon.$$

We say that f has controlled shadowing property, if for every $\epsilon > 0$ there is a $\delta > 0$ such that any controlled-δ-pseudo orbit is control-ε-shadowed by some point of X.

3. Main Results

In this section we obtain chain transitivity, topological ergodicity, topological mixing, specification property and chaos by using the controlled shadowing property.

Theorem 3.1. *Let $f : X \to X$ be a surjective continuous map with the controlled shadowing property. Then f is chain transitive.*

Proof. Let $x, y \in X$ and $0 < \epsilon < \frac{1}{2}$ be given. We show that there is an ϵ-chain from x to y. Suppose that $\delta \geq 0$ is as in the definition of controlled shadowing for $\epsilon > 0$. Choose a positive integer N such that $\frac{1}{N} < \delta$. Since f is surjective hence there is a sequence $\{y_{-3N}, y_{-3N-1}, \cdots, y_0 = y\}$ such that $f(y_i) = y_{i-1}$ for $-3N \leq i < 0$. Consider controlled-δ-pseudo orbit as follows:

$$\{x, f(x), \cdots, f^{3N}(x), y_{-3N}, y_{-3N-1}, \cdots, y, x, \cdots, f^{3N}(x), y_{-3N}, \cdots, y, \cdots\} = \{x_i\}_{i=0}^{\infty}.$$

Therefore we have

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{n}\mathrm{card}\big\{i \in \{0, \cdots, n\} : d\big(f(x_i), x_{i+1}\big) \geq \delta\big\} \\
&\quad = \lim \frac{1}{6kN}\mathrm{card}\big\{i \in \{0, \cdots, 6kN\} : d\big(f(x_i), x_{i+1}\big) \geq \delta\big\} \\
&\leq \frac{3k}{6kN} = \frac{1}{2N} \leq \frac{1}{N} < \delta.
\end{aligned}$$

Consequently $\{x_i\}_{i=0}^{\infty}$ can be control-ϵ-shadowed by some point $z \in X$. This means

$$\lim_{n\to\infty} \frac{1}{n}\mathrm{card}\big\{i \in \{0, \cdots, n\} : d(f^i(z), x_i) \geq \epsilon\big\} < \epsilon.$$

Now, we claim that there are two infinite sequences of positive integers $\{i_1 < i_2 < \cdots\}$ and $\{l_1 < l_2 < \cdots\}$ such that for any $j > 0$, we have

$$x_{i_j} \in \{x, f(x), \cdots, f^{3N}(x)\} \quad \textit{and} \quad d(f^{i_j}(z), x_{i_j}) < \epsilon,$$

and

$$x_{l_j} \in \{y_{-3N}, y_{-3N-1}, \cdots, y\} \quad \textit{and} \quad d(f^{l_j}(z), x_{l_j}) < \epsilon.$$

The reason is that if do not exist such sequences, then we must have

$$\lim_{k\to\infty} \frac{1}{6kN}\mathrm{card}\big\{i \in \{0, \cdots, 6kN\} : d(f^i(z), x_i) \geq \epsilon\big\} \geq \lim_{k\to\infty} \frac{3kN}{6kN} = \frac{1}{2} > \epsilon,$$

which is a contradiction. So we can find $i_0 > l_0 > 1$ and $0 \leq k_0, m_0 \leq 3N$ such that $d\big(f^{i_0}(z), f^{k_0}(x)\big) < \epsilon$ and $d\big(f^{l_0}(z), y_{m_0}\big) < \epsilon$.
Therefore $\big\{x, f(x), \cdots, f^{k_0-1}(x), f^{i_0}(z), \cdots, f^{l_0-1}(z), y_{m_0}, \cdots, y\big\}$ is an ϵ-chain from x to y. This shows that f is chain transitive. □

Theorem 3.2. *Let $f : X \to X$ be a stable map with the controlled shadowing property. Then f is topologically ergodic.*

Proof. Let U and V be two open subsets of X and $x \in U$, $y \in V$. Consider $\epsilon > 0$ such that $N_\epsilon(x) \subset U$, $N_\epsilon(y) \subset V$. Since f is stable, there is a $\delta > 0$ such that if $d(u,v) < \delta$ then $d\big(f^i(u), f^i(v)\big) < \epsilon$ for $i > 0$. Suppose that $\delta_1 \geq 0$ is as in the definition of controlled shadowing for $\delta > 0$. Choose a positive integer N such that $\frac{1}{N} < \delta_1$ and consider controlled-δ_1-pseudo orbit as follows:

$$\{x, f(x), \cdots, f^{3N}(x), y_{-3N}, y_{-3N-1}, \cdots, y, x, \cdots, f^{3N}(x), y_{-3N}, \cdots, y, \cdots\} = \{x_i\}_{i=0}^{\infty}$$

hence we have

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{n}\text{card}\big\{i \in \{0, \cdots, n\} : d\big(f(x_i), x_{i+1}\big) \geq \delta_1\big\} \\
&\quad = \lim \frac{1}{6kN}\text{card}\big\{i \in \{0, \cdots, 6kN\} : d\big(f(x_i), x_{i+1}\big) \geq \delta_1\big\} \\
&\leq \frac{3k}{6kN} = \frac{1}{2N} \leq \frac{1}{N} < \delta_1.
\end{aligned}$$

So $\{x_i\}_{i=0}^{\infty}$ can be control-$\delta-$shadowed by some point $z \in X$. This means

$$\lim_{n\to\infty} \frac{1}{n}\text{card}\big\{i \in \{0, \cdots, n\} : d(f^i(z), x_i) \geq \delta\big\} < \delta.$$

Consider

$$L_x = \big\{i : x_i \in \{x, f(x), \cdots, f^{3N}(x) \text{ } and \text{ } d(f^i(z), x_i) < \delta\big\},$$

$$L_y = \big\{i : x_i \in \{y_{-3N}, y_{-3N-1}, \cdots, y \text{ } and \text{ } d(f^i(z), x_i) < \delta\big\}.$$

We claim that L_x and L_y have positive upper density. that is $D(L_x) > 0$ and $D(L_y) > 0$. We prove for L_x and the proof for L_y is similar. Suppose $D(L_x) = 0$, $\lim_{n\to\infty} \frac{1}{n}\text{card}(L_x \cap \{0, \cdots, n\}) = 0$.

Let

$$L'_x = \big\{i : x_i \in \{x, f(x), \cdots, f^{3N}(x) \text{ } and \text{ } d(f^i(z), x_i) \geq \delta\big\}.$$

We have $D(L_x \cup L'_x) = D(L_x) + D(L'_x)$ and $D(L_x \cup L'_x) \geq \frac{1}{2}$. Since $D(L_x) = 0$ so

$$\lim_{n\to\infty} \frac{1}{n}\text{card}\big(L'_x \cap \{0, \cdots, n\}\big) \geq \frac{1}{2}.$$

But

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{n}\text{card}\big\{i \in \{0, \cdots, n\} : d(f^i(z), x_i) \geq \delta\big\} \geq \\
&\qquad \lim_{n\to\infty} \frac{1}{n}\text{card}\big(L'_x \cap \{0, \cdots, n\}\big) \geq \frac{1}{2}.
\end{aligned}$$

This is a contradiction. So L_x and L_y have positive upper density. We can find $i_0 > 3N$ and $0 \leq k_0 \leq 3N$ such that $d\big(f^{i_0}(z), f^{k_0}(x)\big) < \delta$, so $d\big(f^{i_0-k_0}(z), x\big) < \epsilon$.

Now for any $j \in L_y$ with $j \geq i_0 - k_0 + 3N$, $d\big(f^j(z), f^{m_0}(y)\big) < \delta$ for some $0 \leq m_0 \leq 3N$. So $d\big(f^{j-m_0}(z), y\big) < \epsilon$. put $n_j = (j - m_0) - (i_0 - k_0) > 0$ and therefore $f^{n_j}\big(N_\epsilon(x)\big) \cap N_\epsilon(y) \neq \varnothing$. So $f^{n_j}(U) \cap V \neq \varnothing$.

Hence for n large we have

$$\frac{1}{n}\text{card}\big\{N(U,V) \cap \{0, \cdots, n\}\big\} \geq \frac{1}{n}\text{card}\big\{L_y \cap \{0, \cdots, n\}\big\},$$

so

$$\limsup_{n\to\infty} \frac{1}{n}\text{card}\big\{N(U,V)\cap\{0,\cdots,n\}\big\} > 0.$$

This shows that f is topologically ergodic. $\square$

Remark that in the proof of the above theorems we need that for $y \in X$, $f^{-1}(y)$ is not empty. So we need surjectivity in f.

Theorem 3.3. *If f has the controlled shadowing property, then the same holds for f^k, for any positive integer k.*

Proof. Let $\varepsilon > 0$ be given and $\delta > 0$ be as in the definition of controlled shadowing for f. Suppose that $\{x_n\}_{n=0}^{\infty}$ is a controlled-δ-pseudo orbit for f^k. Consider

$$\{y_i\}_{i=0}^{\infty} = \{x_0, f(x_0), \cdots, f^{k-1}(x_0), x_1, f(x_1), \cdots, f^{k-1}(x_1), x_2, \cdots\}.$$

We can see that $\{y_n\}$ is a controlled-δ-pseudo orbit for f. So there is $z \in X$ such that

$$\limsup_{n\to\infty} \frac{1}{n}\text{card}\big\{i \in \{0,\cdots,n\};\ d(f^i(z), y_i) \geq \varepsilon\big\} < \varepsilon.$$

Therefore

$$\limsup_{n\to\infty} \frac{1}{n}\text{card}\big\{i \in \{0,\cdots,n\};\ d(f^{ki}(z), x_i) \geq \varepsilon\big\} < \varepsilon.$$

So f^k has controlled shadowing property. $\square$

We say that f is totally transitive if all its iterates f^n are transitive; f is topological mixing if for any nonempty open sets U and V in X, there is an $N > 0$ such that $f^n(U) \cap V \neq \varnothing$ for all $n \geq N$.

The specification property was introduced by Bowen in [2].

We say that f has periodic specification property if for any $\varepsilon > 0$, there is an integer $k > 0$ such that for any integer $n \geq 2$, any set $\{y_1, \cdots, y_n\}$ of n poins of X, and any sequence $0 = a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_n \leq b_n$ with $a_{i+1} - b_i \geq k$ for $i = 1, \cdots, n-1$, there is a point $x \in X$ such that for each $1 \leq m \leq n$ and i with $a_m \leq i \leq b_m$, the following conditions hold:

$$d\big(f^i(x), f^i(y_m)\big) < \varepsilon, \tag{3.1}$$

$$f^l(x) = x \text{ where } l = k + b_n. \tag{3.2}$$

If we omit the condition (3.2), then f has the specification property.

Theorem 3.4. *Let X be a compact metric space. If $f : X \to X$ is a continuous surjective map with the controlled shadowing and the shadowing properties, then f is topological mixing and f has the specification property.*

Proof. By Theorems 3.1 and 3.3, since f has the controlled shadowing and the shadowing properties, then f is totally transitive. So by Theorem 1 in [12], the proof is complete. $\square$

Theorem 3.5. *Let $f : X \to X$ be a surjective continuous map on the compact metric space X. If f has the specification property and the shadowing property, then f has the controlled shadowing property.*

Proof. Let $\varepsilon > 0$ be given and k as in the definition of specification for $\varepsilon > 0$. Choose M such that $\frac{k}{M} < \varepsilon$. Suppose that $0 < \delta < \frac{1}{M}$ is as in the definition of shadowing. Take any controlled δ-pseudo-orbit $\{x_n\}_{n=0}^{\infty}$. By the proof of Lemma 12 in [12], we can find infinite sequences of integers $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ such that $0 = a_0 \leq b_0 < a_1 \leq b_1 < \cdots$ and $a_{n+1} - b_n = k$ for $n = 0, 1, 2, \cdots$. Therefore there is a point $z \in X$ such that for any $a_n \leq j \leq b_n$ we have $d(f^j(z), x_j) < \varepsilon$. So we can see

$$\text{card}\{0 \leq j < n;\ d(f^j(z), x_j) \geq \varepsilon\} \leq k\ \text{card}\{0 \leq j \leq n;\ d(f(x_j), x_{j+1}) > \delta\}.$$

Hence we have

$$\limsup_{n\to\infty} \frac{1}{n}\text{card}\{0 \leq j < n;\ d(f^j(z), x_j) \geq \varepsilon\} \leq$$
$$k \limsup_{n\to\infty} \frac{1}{n}\text{card}\{0 \leq j \leq n;\ d(f(x_j), x_{j+1}) \geq \delta\} \leq k\delta < \frac{k}{M} < \varepsilon.$$

So f has controlled shadowing property. □

In the following f is a surjective continuous map on the compact metric space X.

Corollary 3.6. *If f has the shadowing property, then the following conditions are equivalent:*

1) *f has controlled shadowing property,*
2) *f has specification property.*

Proof. By Theorems 3.4 and 3.5 the proof is complete. □

If f is topological mixing, then by [3], it has the specification property. So we have the following corollary.

Corollary 3.7. *If f has the shadowing and topological mixing properties, then f has the controlled shadowing property.*

Theorem 3.8. *If f has the controlled shadowing property, then the same holds for $f \times f$.*

Proof. Suppose $\varepsilon > 0$ is given and $\delta > 0$ is as in the definition of controlled shadowing property for $\frac{\varepsilon}{2}$. Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a controlled-δ-pseudo orbit for $f \times f$. We define metric ρ on $X \times X$ as follows

$$\rho\big((x, y), (x', y')\big) = \max\{d(x, x'), d(y, y')\}.$$

So we can see $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ are two controlled-δ-pseudo orbits for f, then they can be control-$\frac{\varepsilon}{2}$-shadowed by some points z_1 and z_2 respectively. So there is $N \in \mathbb{N}$ such that for any $n \geq N$,

$$\frac{1}{n}\text{card}\big\{i \in \{0, \cdots, n\};\ d(f^i(z_1), x_i) \geq \frac{\varepsilon}{2}\big\} < \frac{\varepsilon}{2},$$
$$\frac{1}{n}\text{card}\big\{i \in \{0, \cdots, n\};\ d(f^i(z_2), y_i) \geq \frac{\varepsilon}{2}\big\} < \frac{\varepsilon}{2}.$$

Let

$$\text{card}\big\{i \in \{0, \cdots, n\};\ d(f^i(z_1), x_i) \geq \frac{\varepsilon}{2}\big\} = k_n,$$
$$\text{card}\big\{i \in \{0, \cdots, n\};\ d(f^i(z_2), y_i) \geq \frac{\varepsilon}{2}\big\} = l_n.$$

So

$$\begin{aligned}&\text{card}\big\{i \in \{0, \cdots, n\};\ \rho\big((f \times f)^i(z_1, z_2), (x_i, y_i)\big) \geq \varepsilon\big\}\\ &= \text{card}\big\{i \in \{0, \cdots, n\};\ d(f^i(z_1), x_i) \geq \varepsilon \text{ or } d(f^i(z_2), y_i) \geq \varepsilon\big\}\\ &\leq k_n + l_n.\end{aligned}$$

Therefore for $n \geq N$ we have

$$\begin{aligned}\frac{1}{n}\text{card}\big\{i \in \{0, \cdots, n\};\ \rho\big((f \times f)^i(z_1, z_2), (x_i, y_i)\big) \geq \varepsilon\big\} &\leq \frac{k_n + l_n}{n}\\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

So $f \times f$ has the controlled shadowing property. □

Theorem 3.9. *If f has the controlled shadowing and the shadowing properties, then f is chaotic in the sense of Li-Yorke.*

Proof. Since f has controlled shadowing property by Theorem 3.8, $f \times f$ has controlled shadowing property. So by Theorem 3.1, $f \times f$ is chain transitive. So the controlled shadowing property implies the chain transitivity of $f \times f$. Also this is well known that the shadowing property of f implies the shadowing property of $f \times f$. Hence by the shadowing property and the chain transitivity, $f \times f$ is transitive. Hence f is topologically weakly mixing. But any weakly mixing map is chaotic in the sense of Li-Yorke (see [10]). Therefore f is chaotic in the sense of Li-Yorke. □

A continuum is a nondegenerate compact connected metric space.
A continuous map f from a compact metric space X to itself is said to be P-chaotic if f has the shadowing property and the periodic points of f are dense in X.

Corollary 3.10. *Every P-chaotic map from a continuum to itself has the controlled shadowing property.*

Proof. By Corollary 3.3 in [1], every P-chaotic map from a continuum to itself is mixing. So by Corollary 3.7, f has the controlled shadowing property. □

Example 3.11. Let $f : [0,1] \to [0,1]$ be the tent map which is defined by

$$f(x) = \begin{cases} 2x & 0 \leq x \leq \frac{1}{2} \\ -2x+2 & \frac{1}{2} \leq x \leq 1 \end{cases}.$$

By [1], f is P-chaotic. By Corollary 3.10 f has the controlled shadowing property.

At the end we explain in the following examples that the shadowing property does not necessarily imply the controlled shadowing property. The first example is in a compact metric space and the second example is in the non-compact metric space.

Example 3.12. Let $f : [0,1] \to [0,1]$

$$f(x) = \begin{cases} 2x & \text{if} \quad x \leq \frac{1}{2} \\ 1 & \text{if} \quad \frac{1}{2} \leq x \leq 1 \end{cases}.$$

f has the shadowing property but does not have the specification property (see Example 2.8 in [1]). Therefore f has not the controlled shadowing property.

Example 3.13. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 4x$. One can see that f has the shadowing property but is not transitive. Indeed for every $\epsilon > 0$ consider $\delta = \frac{\epsilon}{8}$. If $\{x_i\}_{i=0}^{\infty}$ is a δ-pseudo orbit of f, then we can see that $\bigcap f^{-i}(B_{\frac{\epsilon}{2}}(x_i)) \neq \varnothing$. This show that f has the shadowing property. If $U = (1,2)$, $V = (0,1)$, then $f^n(U) \cap V = \varnothing$ for every $n \in \mathbb{N}$. Therefore f is not transitive and so f is not chain transitive (chain transitivity and shadowing property imply transitivity). Hence by Theorem 3.1, f has not the controlled shadowing property.

4. Conclusion

In this paper we have shown that any surjective map with controlled shadowing property is chain transitive and every stable map with controlled shadowing property is topological ergodic which is stronger than transitivity. As well we proved that for any surjective map with shadowing property on a compact metric space, the controlled shadowing and the specification properties are equivalent. The relation between the controlled shadowing and chaos in sense of Li-Yorke has been the next result of this paper. Finally, we gave an example having the controlled shadowing property. At the end by examples we showed that the shadowing property does not necessarily imply the controlled shadowing property.

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