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Location models applied to real-world problems

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Abstract

Este trabajo proporciona nuevas ideas que sirven para modelar y resolver situaciones o procesos de la sociedad actual muy interesantes, como ubicación óptima de un campamento humanitario, de una central nuclear o planta radioctiva de modo que se minimice el riesgo, de dependencias para propósitos específicos en un centro de trabajo (laboratorios, zonas de ocio o de fumadores, etc.); o la determinación del líder de una organización terrorista incluso cuando la información intercambiada entre los miembros de la organización está encriptada, etc.

This work provides new ideas to model and solve situations or processes from the present society that are very interesting, as optimal placement of a humanitarian camp, of nuclear power or radioactive plants, of dangerous or polluting materials in a way that minimizes the risk, of rooms intended for specific purposes in a work center (laboratories, recreational or smoking areas, etc.); or the determination of the leader of a terrorist organization even when the information sent between any two members is encoded, etc.

Keywords: Problemas de localización, problemas reales, optimitzación.
Location models, real-world problems, optimization

1 Introduction

The main purpose of a model is to reproduce aspects of reality in the most faithful way, trying to express (formally) its most important features and behavior that may be expected as a response to certain actions. Engineers, economists, physicists and mathematicians need to build models that allow them to solve reality problems.

In practice, different kinds of models, given by difference equations, differential equations or mathematical programming, are used to model and solve such aspects of reality. Mathematical programming models are generally designed to be used in the process of decision making that can take place in a company or any other type of administration.

One of these types of models are the location models (see [2, 6, 7, 10, 11]). The location problems, in its most general form, can be described as those in which a set of interested parties, distributed especially in a geographic area, are demanding a certain product or service and that demand should be covered by one or more facilities. Then, it has to be decided where facilities should be located, taking into account the requirements of the interested parties and the geographical restrictions, in order to minimize costs.

The study of location models involves, among others, Graph Theory, Network Theory, Operational Research or Geometry (see [1] and [3]). Despite the general approach, these concepts can be applied to very different situations in different areas of knowledge of industrial, technological, economic, sanitary and even criminal type. Likewise, the study of localization models presents some applications that can be useful in the (future) everyday work of the students.

In order to demonstrate this interdisciplinary utility, once the necessary mathematical tools have been introduced, it is shown how to model situations and processes from the real world that are very interesting, as optimal placement of a humanitarian camp, of nuclear power or radioactive plants, of dangerous or polluting materials in a way that minimizes the risk, of rooms intended for specific purposes in a work center (laboratories, recreational or smoking areas, etc.); or the determination of the leader of a terrorist organization even when the information sent between any two members is encoded, etc.

In this context, it is called *absolute median* point of a network [8], the one that minimizes the sum of the lengths to the rest of the nodes of the network; and *absolute antimedian* [9], the point that maximizes such a sum. When what is minimized is the maximum distance to one of the nodes of the network, the optimal point is said to be an *attractive absolute center* [8]. On the contrary, when maximizing the minimum length to one of the nodes of the network, the optimal point is said to be a *repulsive absolute center* [9]. When the point to be located has to be situated in one of the nodes of the network, the adjective “absolute” is removed.

The applications shown in this work will give an idea about the usefulness of *mathematical modeling* as an educational tool to transmit knowledge of Mathematics and to train competent undergraduate students in the capability of modeling. At the same time, the treatment of real problems, that generates more interest in the students, will highlight a greater appreciation of mathematical knowledge as well as an increase in the ability of any student to tackle, now and in the future, this kind of problem, what is usually identified as the lifelong learning in the new educational models.

2 Elements of a location model

Among the essential elements of a location model we can distinguish: user set, (possible) location set, distances or costs, restrictions (care to all users, capacity of locations, etc.) and objective function (to optimize) which measures the effectiveness.

In terms of the user set, we typically distinguish between a finite set located at specific points or a set of areas of demand.

In what refers to the location set, we usually differentiate among discrete models, where the (possible) location set is finite or countable; network models, where the candidates are points in a network, that is, the points that determine all vertices and all the ones in the edges; and continuous models, where the possibilities constitute a continuous set (region of the plane, surface of the space or something similar).

The main goal is to optimize the distances or costs associated to the locations. In this sense, it is made a distinction between attractive models, if the distances or costs are minimized in some sense; and repulsive, if the distances or costs are maximized in some sense. Nevertheless one might encounter in a same location problem several objectives (e.g. minimize time and maximize flow), which requires the use *multiobjective* resolution techniques [12].

The fundamental constraint in a location model will be that the users demand is served by the established locations. This restriction can be determined by a linear equation. Depending on the type of problem it could exist other restrictions, as those associated to the limit of capacity of the locations established. Such restrictions can be set using linear inequations too.

The objective function (to optimize) provides a measure of the effectiveness for each location made. Usually, it is given by a linear function. For this reason, many localization problems can arise as a linear mixed-integer programming problem and can be solved by algorithms of this field.

3 Elements of networks models

A *graph* is the representation of the relations among the elements of a set. When the relation between any two elements has one or more numbers associated, the graph is said to be a *network*. Graphs and networks are very useful to simplify difficult situations, to give schemes that model real situations or to solve problems of the real world.

Mathematically, a graph or simple graph is a pair $G = (V, E)$, where

- V is a nonempty set, whose elements are called *vertices* or *nodes*.
- E is a set, whose elements $e = \{u, v\} \in E$, named *edges*, are determined by two different vertices u and v.

Graphs are represented by graphics in the plane, where the vertices are depicted by points and the edges by segments that join these points.

Two vertices u and v are said to be *adjacent*, if they define an edge, i.e., if there exists $e = \{u, v\} \in E$. Two adjacent vertices are called *terminals* or *extremes* of the edge that they determine. Given a sequence $v_1, v_2, \ldots, v_{n+1}$ of vertices, a *path* is any sequence of edges $e_1, e_2, \ldots, e_n$ such that the terminals of e_i are v_i and v_{i+1} for $i = 1, 2, \ldots, n$.

A graph $G = (V, E)$ is said to be a *network* if every arc (and/or node) has associated one or more numbers. Currently, a positive number is associated to every edge $e \in E$, $\lambda(e) \geq 0$, named *weight* of e and the network is denoted by $R = (V, E, L)$. If a positive number is also associated to every vertex v_i, $w_i \geq 0$, this number is called the *weight* of the node.

In this context, given v_i, v_j vertices of the network $R = (V, E, L)$, an edge $e = \{v_s, v_t\}$ and x a point in such an edge, it is denoted by:

- $w_i \geq 0$ the weight (concerning the demand or simply relative importance) of the vertex v_i.
- l_{st} the weight of such an edge.
- l_{sx} the (proportional) weight of the segment s, x
- d_{ij} the weight of the least weighted path from v_i to v_j, which is often called the *distance* (or *cost*) from v_i to v_j
- $d_{ix} = \min\{d_{is} + l_{sx}, d_{it} + l_{st} - l_{sx}\}$

For every vertex v_i, when the point x varies along e, it can be found a point x_i^e for which

$$d_{is} + l_{sx_i^e} = d_{it} + l_{st} - l_{sx_i^e} \qquad \Rightarrow \qquad l_{sx_i^e} = \frac{d_{it} + l_{st} - d_{is}}{2}$$

(see [1]).

Then the function d_{ix} is defined as

$$d_{ix} = \begin{cases} d_{is} + l_{sx} & \text{if } x \in (s, x_i^e) \\ d_{it} + l_{st} - l_{sx} & \text{if } x \in (x_i^e, t) \end{cases}$$

A *directed graph* or *digraph* is a pair $D = (V, A)$, where

- V is a nonempty set, whose elements are called *nodes* (or *vertices*) .
- A is a set whose elements $a = (u, v) \in A$, named *arcs*, are ordered pairs of nodes (different or identical) u and v.

Digraphs are represented by graphics in the plane, where the nodes are depicted by points and the arcs by arrows between points.

If $a = (u, v)$ is an arc, it is said that u is the initial node and v is the final node. When this distinction is not necessary, we shall refer these two nodes as *terminals* of the arc. Given a sequence $v_1, v_2, \ldots, v_{n+1}$ of vertices, a *(directed) path* is any sequence of arcs $a_1, a_2, \ldots, a_n$ such that the initial node of a_i is v_i and the final node is v_{i+1} for $i = 1, 2, \ldots, n$.

A digraph $D = (V, A)$ is said to be a *directed network* if every arc (and/or node) has associated one or more numbers. Currently, a positive number is associated to every arc $a \in A$, $\lambda(a) \geq 0$, named *weight* of a and the directed network is denoted by $R = (V, A, L)$.

4 Median models applied to real-world problems

4.1 Attractive medians

The median problem consists in finding out a vertex $v^* \in V$ such that the weighted sum of the distances from other vertices to it is minimum, i.e.

$$\min_{v_k \in V} f(v_k) = \min_{v_k \in V} \sum_{v_i \in V} w_i d_{ik}$$

We shall call *median* to any vertex that is an optimal solution of the previous problem.

A median search algorithm consists of the following steps:

1. Estimate the distance matrix $D = (d_{ij})$ using any of the known algorithms (Floyd or Dantzing).
2. Estimate the weighted distance matrix $D_w = (w_i d_{ij})$
3. Add the elements of each column j, so obtaining the sum of the distances from each vertex to vertex j.
4. The vertex corresponding to the column whose sum results the minimum is taken as a median.

When the network is directed the term *in-median* is used to indicate that the displacement is considered from the vertices to the median; and *out-median*, to indicate that the displacement is considered towards the vertices from the median. An in-median search algorithm is the same as the algorithm described for a median, while a search algorithm for an out-median can be obtained as the algorithm described for a median, but changing the word column by the word row.

Below, we give two examples of real-world problems that can be modeled as antimedians:

Optimal location of a computer in an enterprise.

A computer network is represented by a graph in which the weights of the edges indicate the time of transmission of data from one computer to another. The boss wants to be placed in the terminal for which the time of transmission to the rest of computers was minimized, since he often distributes information to the employees. The problem consists in locating which the best location for the boss would be.

Optimal location of a humanitarian camp.

The distribution of villages in an area of Africa is represented by the nodes of a graph in which the weights of the edges indicate the distance from one village to another. In one of the villages is intended to install a humanitarian camp in a way that the amount of time in the distribution of food and medicine to residents of the area was minimized. The problem consists in studying which village would be the best location for such a camp.

The *absolute median* problem consists in finding out a point x^* of an edge of a network for which the weighted sum of the distances to the rest of the vertices is minimum, that is

$$\min_{x \in R} f(x) = \min_{x \in R} \sum_{v_i \in V} w_i d_{ix}$$

We will call absolute median to the point of the network being optimal solution for the above problem.

Before considering an absolute median search algorithm, it is necessary to know two important results which can be found in [4].

Theorem 8.1 *(Hakimi 1972) There is at least one absolute median which is a vertex of the network.*

Corollary 8.2 *The algorithm to search for a median is also valid to find an absolute median.*

When the network is directed the terms *absolute in-median* and *absolute out-median* are used to indicate the direction of the displacement as in the case of the median. Likewise, the search algorithms are the same as the algorithms described for in-median and out-median.

Below, we give two examples of real-world problems that can be modeled as absolute medians:

Optimal location of a warehouse of a company.

A company has a set of lands, indicated by nodes in a graph, where it wants to locate 10 production plants and a warehouse, where it is necessary to transport the manufactured material produced in such plants. The weights of the nodes indicate the level of production relative to each plant. The problem consists in studying where to locate the warehouse, so that the transportation costs from the production plants to it was minimized.

Optimal location of a factory of a company.

A company has a number of clients, located as represented by means of a graph, where the weight of any arc indicates the transportation cost between the two nodes of the arc. The company wants to establish in such a network a factory from which to supply products to its customers. The weights of the nodes indicate the relative supply that is made to each client. The problem consists in deciding where to locate the factory so that the distribution costs was minimized.

4.2 Repulsive medians or antimedians

The *repulsive median* problem consists in finding out a vertex v^* such that the weighted sum of the distances from the rest of the vertices to this one is maximum, that is,

$$\max_{v_k \in V} f(v_k) = \max_{v_k \in V} \sum_{v_i \in V} w_i d_{ik}$$

We will call *repulsive median* or *antimedian* to any vertex being optimal solution of the previous problem.

Thus, the problem is similar to the one before, but substituting the word minimum by maximum and so the algorithm to search for the antimedian can be also obtaining by making such a change in the algorithm for the median.

Next, we give two more real-world problems that can be modeled and solved using antimedians:

Location of the leader of a terrorist organization (even if the information sent between any two members is encoded)

A terrorist organization is modeled by a graph where the nodes represented the members of the organization. The terrorists communicate among themselves via encrypted messages, being impossible to know what they say. However, the secret services have slashed its media and weights of the edges indicate the volume of communication (time, MB interexchanged, etc.). The problem consists in determining who the leader of the terrorist organization is.

Optimal location of a humanitarian camp (under other hypotheses)

The distribution of villages in an area of Africa is represented by the nodes of a graph in which the weights of the edges indicate the level of security to pass from a village to another. In one of the villages is intended to install a humanitarian camp to maximize security in the distribution of food and medicines to the inhabitants of the area. The problem consists in deciding what town would be the best location for the camp.

When the network is directed the terms *in-antimedian* and *out-antimedian* are used to indicate the direction of the displacement. Likewise, the search algorithm for an in-antimedian is the same as the algorithm described for an antimedian, while a search algorithm for an out-antimedian can be obtained as the algorithm described for an antimedian, but changing the word column by the word row.

Below, we give another example of a real-world problem that can be modeled as medians:

Optimal location of computer of an enterprise (under other hypotheses)

A computer network is represented by a graph in which the nodes specify the computers and the weights of the edges indicate the flow of transmission of data from one computer to another. The boss wants to be placed in the terminal for which the flow of transmission to the rest of computers is maximum, since he often distributes information to the employees. The problem consists in discerning which computer would be the best location for the boss.

The *absolute antimedian* problem consists in finding out a point x^* of an edge of a network such that the weighted sum of the distances from other vertices to this one is maximum, i.e.,

$$\max_{x \in R} f(x) = \max_{x \in R} \sum_{v_i \in V} w_i d_{ix}$$

We shall call *absolute antimedian* to any point in the network being optimal solution of the previous problem.

As in the case of the median, the research algorithms for the antimedian can be used in this case, thanks to Hakimi's theorem.

5 Center models applied to real-world problems

5.1 Attractive centers

The *attractive center* problem consists in finding out a vertex $v^* \in V$ such that the distance to the farthest vertex is minimal, i.e.

$$\min_{v_k \in V} g(v_k) = \min_{v_k \in V} \{ \max_{v_i \in V} d_{ik} \}$$

We shall call *(attractive) center* to any vertex being solution of the problem above and *radius* to the value of the function g at this vertex.

The problem so posed does not consider the importance (weighting) of the nodes. This usually occurs for attention centers because it does not seem ethical, for instance, to locate a center of emergencies giving more relevance to a population nucleus than to another for being more numerous. However sometimes the probability tells us that it may be interesting to consider such weighting, what technically does not vary the problem.

A center search algorithm consists of the following steps:

1. Estimate the distance matrix $D = (d_{ij})$ using any of the known algorithms (Floyd or Dantzing).
2. Get the maximum of each column j, so obtaining the maximum of the distances from each vertex to vertex j.
3. The vertex corresponding to the column which has obtained the minimum value is taken as the center.

When the network is directed the term *in-center* is used to indicate that the displacement is considered from the vertices to the center; and *out-center*, to indicate that the displacement is considered towards the vertices from the center. An in-center search algorithm is the same as the algorithm described for a center, while a search algorithm for an out-center can be obtained as the algorithm described for a center, but changing the word column by the word row.

Below, we give two examples of real-world problems that can be modeled as attractive centers:

Optimal location of a kit of first aid

The structure of a company is represented by a graph in which the nodes indicate where the departments are located and the edges represent aisles between them. Some of the corridors present directions, because they are very narrow and they are only used in the indicated direction. The Employment Security Law requires that the company put a kit of first aid in one of the departments of the company. The problem consists in determining which department would be the optimal location for the kit.

Optimal emergency exits

The structure of the headquarters of a public administration is represented by a graph in which the nodes indicate where exits are located and the edges represent the corridors between them. The law of Prevention of Risks obligates the administration to place an emergency exit. The problem consists in studying which exit would be the optimal one.

The *absolute center* problem consists in finding out a point x^* in an edge of a network such that the distance to the farthest vertex is minimum, i.e.,

$$\min_{x \in R} g(x) = \min_{x \in R} \{ \max_{v_i \in V} d_{ix} \}$$

We shall call *absolute center* to any point of the network being solution of the problem above, and *absolute radius* to value of the objective function at this point.

As in the case of centers, the absolute centers problem does not consider the importance (weighting) of the nodes. However, sometimes it may be interesting to consider such weighting, which technically does not vary the problem.

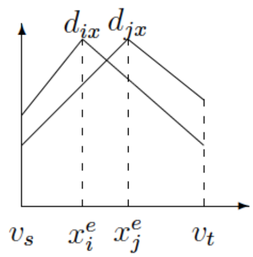
Unfortunately, there is not a theorem for the absolute center similar to Hakimi's one for the absolute median. Actually, as it is easy to check for a network with two nodes, it could occur that there is no absolute center that is vertex. In this sense, the radius and the absolute radius are often different. In this situation, one way to reduce the problem is to find the optimum among a finite number of points called *bottleneck points* (see[1]).

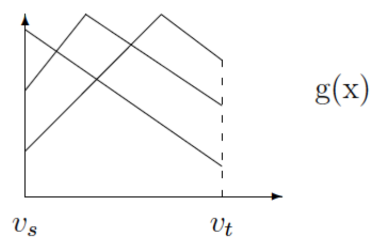
A point x in an edge $e = \{v_s, v_t\}$ is a *bottleneck point* if there exist two vertices v_i, v_j such that $x \in (x_i^e, x_j^e)$ and $d_{ix} = d_{jx}$.

Theorem 8.3 *Every absolute center, which is in the interior of an edge, is a bottleneck point.*

In such a case, the common distance to x is named *bottleneck distance* and it is given by

$$d_{ix} = d_{jx} = \frac{d_{it} + d_{js} + l_{st}}{2}$$





As can be deduced from the graphics, for every edge e, we will be only interested in the bottleneck points such that $g(x) = d_{ix} = d_{jx}$, because they are local minimum points for the objective function g. As there exist only a finite number of such points, it will be relatively easy to find the global minimum among them

Thus, a search algorithm consist of finding the points x that are bottleneck in each edge e, selecting from them those verifying $g(x) = d_{ix} = d_{jx}$ and later extracting among them and the vertices the absolute centers.

When the network is directed the term *absolute in-center* is used to indicate that the displacement is considered from the vertices to the absolute center; and *absolute out-center*, to indicate that the displacement is considered towards the vertices from the absolute center. Search algorithms for them can be obtained from the algorithm described for an absolute center. Nevertheless, in any of the algorithms described before, the directions of the distance have to be taken into account.

Below, we give two examples of real-world problems that can be modeled as absolute centers:

Optimal installation of a fire station

The structure of a forest area is represented by a graph in which the nodes indicate the location of some woodland and bushes areas and the weights of the edges the length of the paths among them. The Administration wants to install a fire station in the network as a first step for prevention of disasters. The problem consists in establishing the optimal location and specifying the distance to this point.

Optimal place of a hospital

The cities of a region are represented by a graph in which the nodes indicate where each of these cities are located and the weights of the edges specify the distances among them. Administration wants to construct a hospital in the region. The problem consists in deciding the optimal place for the hospital.

5.2 Repulsive centers or anticenters

The *anticenter* or *repulsive center* problem consists in finding out a vertex v^* of a network such that the distance to the nearest vertex of the network is maximum, i.e.,

$$\max_{v_k \in V} g(v_k) = \max_{v_k \in V}\{\min_{v_i \in V} d_{ik}\}$$

We shall call *anticenter* or *repulsive center* to any vertex being solution of the problem before and *antiradius* to the value of the objective function g at this vertex.

As for the attractive centers, the problem so posed does not consider the importance (weighting) of the nodes. Thus, the problem is similar to the one of attractive centers, but substituting the word minimum by maximum and so the algorithm to find the anticenter can be also obtaining by making such a change in the algorithm for the attractive center.

When the network is directed the terms *in-anticenter* and *out-anticenter* are used to indicate the direction of the displacement. An in-anticenter search algorithm is the same as the algorithm for an anticenter, while a search algorithm for an out-anticenter can be obtained as the algorithm described for an anticenter, but changing the word column by the word row.

Below, we give two examples of real-world problems that can be modeled as anticenters:

Optimal determination of a smokers room

The structure of a company is represented by a graph in which the nodes specify where the different departments are located and the weights of the edges indicate the length of the aisles between two of them. The Health Law obligates the company to establish a smoking area in one of the departments of the company. The problem consists in determining the department in which the smokers provoke as little nuisances as possible.

Optimal determination of a library

A secondary education school, represented by a graph where the nodes specify their classrooms and the weights of the edges indicate the length of the aisles between two classrooms, aims to establish one of their classrooms as a library. The problem consists in setting its optimal location, so that the noise of the rest of the classrooms disturbs as little as possible to the students who study in that library.

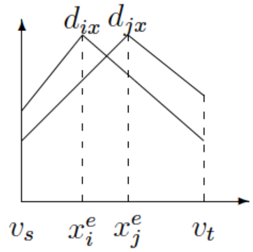
The *absolute anticenter* problem consists in finding out a point x^* of an edge of a network such that the distance to the nearest vertex is maximum, i.e.,

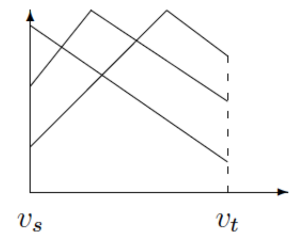
$$\max_{x \in R} g(x) = \max_{x \in R}\{\min_{v_i \in V} d_{ix}\}$$

We shall call *absolute anticenter* to any point of a network being solution of the problem before, and *absolute antiradius* to the value of the objective function g at this point.

As in the case of absolute centers, unfortunately, there is not a theorem for absolute anticenters similar to the Hakimi's one for absolute antimedians. Actually, as it is easy to check for a network with two nodes, it could occur that there is no absolute anticenter that is a vertex. In this sense, the antiradius and the absolute antiradius are often different.

In this situation, similarly to the case of absolute centers, one way to reduce the problem is to find the optimum among a finite number of *bottleneck points* (see [1]).





As can be deduced from the graphics, for every edge e, we will be only interested in the bottleneck points such that $g(x) = d_{ix} = d_{jx}$, because they are local maximum points for the objective function g. As there exist only a finite number of such points, it will be relatively easy to find the global maximum among them. However, the problem is much easier, when are no weights for the nodes. In such a case, it is located at the midpoint of the edge with greater length.

When the network is directed the terms *absolute in-anticenter* and *absolute out-anticenter* are used to indicate the direction of the displacement and search algorithms can be obtained as the algorithms described for absolute in-centers and out-centers respectively.

Below, we give two examples of real-world problems that can be modeled as absolute anticenters:

Optimal location of a nuclear power station

The cities of a certain region are represented through the nodes of a graph and the weights of its edges indicate the distances between these cities. In this region, it is desired to install a nuclear power station. The problem consists in determining which location would be the best one to ensure the lowest social risk.

Optimal location of a landfill

The municipalities of a region are represented through the nodes of a graph and the weights of its edges indicate the distances between these municipalities. In this region, it is intended to locate a landfill. The problem consists in setting its optimum location, so that the discomfort was as little as possible.

6 Conclusions

This paper shows how to model and solve different situations or processes from the present society that are of general interest. As the techniques and resources (graphs, networks, matrices, etc.) are not too much difficult for undergraduate students, this offers a new perspective to introduce maths that will start from interesting, realistic problems for which students can see the point of solving, and to show them how maths can help solve them and could become relevant for real life.
On the other hand, it can help students to reorganize some mental structures in order to improve the capability to model some other real problems in the future, as required by the new educational models and society.

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