Contributions to analysis and control of Takagi-Sugeno systems via piecewise, parameter-dependent, and integral Lyapunov functions

Author: IVÁN TEMOATZIN GONZÁLEZ GERMÁN

Supervisors: D. ANTONIO SALA PIQUERAS
D. MIGUEL ÁNGEL BERNAL REZA

April 11, 2018

This work has been supported by the Consejo Nacional de Ciencia y Tecnología (CONACyT) and the Consejo Estatal de Ciencia y Tecnología (COECyT) Sonora, Mexico under Grant 383252.

This work was also supported by the Spanish ministry of Economy and European Union, grant DPI2016-81002-R (AEI/FEDER, UE), and by the Proyecto de Ciencia Básica SEP-CONACYT 168406, PFCE 2015-18, and Project ITSON-PROFAPI CA-18.

Abstract

This thesis considers a Lyapunov-based approach for analysis and control of nonlinear systems whose dynamical equations are rewritten as a Takagi-Sugeno model or a convex polynomial one. These structures allow solving control problems via convex optimisation techniques, more specifically linear matrix inequalities and sum-of-squares, which are efficient tools from the computational point of view. After providing a basic overview of the state of the art in the field of Takagi-Sugeno models, this thesis address issues on piecewise, parameter-dependent and line-integral Lyapunov functions, with the following contributions:

An improved algorithm to estimate the domain of attraction of nonlinear systems for continuous-time systems. The results are based on piecewise Lyapunov functions, linear matrix inequalities, and geometrical argumentations; level-set approaches in prior literature are significantly improved.

A generalised parameter-dependent Lyapunov function for synthesis of controllers for Takagi-Sugeno systems. The approach proposed a multi-index control law that feeds back the time derivative of the membership function of the Takagi-Sugeno model to cancel out the terms that cause a priori locality in the Lyapunov analysis.

A new integral Lyapunov function for stability analysis of nonlinear systems. These results generalise those based on line-integral Lyapunov functions to the polynomial framework; it turns out path-independency requirements can be overriden by an adequate definition of a Lyapunov function with integral terms.

Resumen

Esta tesis considera un enfoque basado en Lyapunov para el análisis y control de sistemas no lineales cuyas ecuaciones dinámicas son reescritas como un modelo Takagi-Sugeno o uno polinomial convexo. Estas estructuras permiten resolver problemas de control mediante técnicas de optimización convexa, más concretamente desigualdades matriciales lineales y suma de cuadrados, que son eficientes herramientas desde un punto de vista computacional. Después de proporcionar una visión general básica del estado actual en el campo de los modelos Takagi-Sugeno, esta tesis aborda cuestiones sobre las funciones de Lyapunov por trozos, dependiente de parámetros e integral de línea, con las siguientes contribuciones:

Un algoritmo mejorado para estimaciones del dominio de atracción de sistemas no lineales para sistemas de tiempo continuo. Los resultados se basan en funciones de Lyapunov por trozos, desigualdades matriciales lineales y argumentaciones geométricas; enfoques basados en conjuntos de nivel en la literatura previa se han mejorado significativamente.

Una función Lyapunov generalizada dependiente de parámetros para la síntesis de controladores para sistemas Takagi-Sugeno. El enfoque propone una ley de control multi-índice que retroalimenta la derivada del tiempo de las funciones de membresía del modelo Takagi-Sugeno para anular los términos que causan localidad a priori en el análisis de Lyapunov.

Una nueva función integral de Lyapunov para el análisis de estabilidad de sistemas no lineales. Estos resultados generalizan aquellos basados en funciones de Lyapunov integral de línea al marco polinomial; resulta que los requisitos de independencia del camino pueden ser anulados por una definición adecuada de una función Lyapunov con términos integrales.

Resum

Aquesta tesi considera un enfocament basat en Lyapunov per a l'anàlisi i control de sistemes no lineals les equacions dinàmiques dels quals són reescrites com un model Takagi-Sugeno o un de polinomial convex. Aquestes estructures permeten resoldre problemes de control mitjançant tècniques d'optimització convexa, més concretament desigualtats matricials lineals i suma de quadrats, que són eines eficients des d'un punt de vista computacional. Després de proporcionar una visió general bàsica de l'estat actual en el camp dels models Takagi-Sugeno, aquesta tesi aborda qüestions sobre les funcions de Lyapunov per trossos, dependent de paràmetres i integral de línia, amb les següents contribucions:

Un algoritme millorat per a estimar el domini d'atracció de sistemes no lineals per a sistemes de temps continu. Els resultats es basen en funcions de Lyapunov per trossos, desigualtats matricials lineals i argumentacions geomètriques; enfocaments basats en conjunts de nivell en la literatura prèvia s'han millorat significativament.

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Una nova funció integral de Lyapunov per a l'anàlisi d'estabilitat de sistemes no lineals. Aquests resultats generalitzen aquells basats en funcions de Lyapunov integral de línia al marc polinomial; resulta que els requisits d'independència del camí poden ser anul·lats per una definició adequada d'una funció Lyapunov amb termes integrals.

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Chapter 1

Introduction

This chapter introduces the main ideas of the thesis, which is concerned with the generalisation of previous results on stability analysis and controller design of nonlinear systems based on convex optimisation techniques. In order to motivate this study, we begin with a brief historical review on some closely related convex structures such as linear parameter varying systems, Takagi-Sugeno models, and convex polynomial models. While these structures have certain advantages when used for nonlinear control schemes, they present a number of limitations; some of them are presented in the second section, which will be helpful for understanding the main results of this work. The chapter concludes with a brief overview of the contents and the publications derived from this research.

1.1 Motivation and background

A mathematical model of a dynamical system describes its behaviour along time. Within the field of control systems, it usually adopts a state-space representation, which is a set of multivariable differential equations (or difference equations for the case of discrete-time systems) which contains the states (minimal information to determine the future behavior of the dynamical system) obtained through certain known physical laws. Usually, the equations that represent the dynamics are nonlinear functions (polynomial, exponential, logarithmic, sinusoidal, etc), which induce a variety of phenomena which is hard to analyse. Some examples of nonlinear phenomena are finite-escape time, multiple isolated equilibrium points, limit cycles, chaos, etc (Khalil 2002).

Analysis and control of linear time-invariant (LTI) systems has been well developed long ago (Kailath 1980). Some of these developments can be straightforwardly extended to the linear time-varying (LTV) case (C. T. Chen 2012). Nevertheless, if approaches based on linearisation are put aside, nonlinear systems cannot be treated with linear techniques, which has motivated a variety of frameworks such as backstepping for strict feedback systems (Khalil 2002), sliding modes which eliminate matched disturbances via discontinuous terms (Utkin 1992), geometric control for exact feedback linearization (Isidori 1995), and passivity-based design which relies on the ability of finding an energy-like Lyapunov function (Ortega et al. 1998), etc. Nonlinear methods remain thus limited to a number of systems with low dimension and special structures, lacking the level of generality and systematicity linear methods have, let alone its numerical computability and implementation.

A different route for analysis and control of nonlinear systems has been developed from the field of linear parameter varying (LPV) systems, first introduced in the Ph.D. thesis of Shamma (J. Shamma 1988). The origins of LPV systems can be traced back to (classical) gain scheduling control (Safonov 1980), which consists in a collection of linear controllers, each of them “stabilising" at different operation points and indexed by a measurable parameter or “scheduling variable” (J. Shamma 1999). Likewise, an LPV model consists on a family of linear systems blended together by a *scheduling parameter*; this parameter is unknown a priori, but it is available to be measured online or, at least, bounded. Whereas the *scheduling parameter* in gain-scheduling is a function of the states, in the LPV framework, the *scheduling parameter* is independent of the states, i.e., its possible explicit dependence on the system states or time is neglected. Usually, functions of the scheduling variables were defined to hold the convex sum property, so the model could be subsumed into a linear polytope, i.e., a convex sum of linear systems (Apkarian and Gahinet 1995).

The resemblance between LPV systems $\dot{x} = A(\theta)x$ and nonlinear ones $\dot{x} = A(x)x$, as well as the concurrent appearance of Takagi-Sugeno (TS) models which already have a polytopic form depending on time, states, or parameters (Takagi and Sugeno 1985), encouraged researchers to go further in using the direct Lyapunov method and the convex sum property to formally derive analysis and design conditions based on some convex representation of such systems (Tanaka and H. Wang 2001). In recent years, such convex form has become known as a quasi-LPV system, as its scheduling variables may contain states, parameters, or uncertainties (J. S. Shamma and Cloutier 1992). Obtaining a convex model from a nonlinear one can be done by approximation (Ohtake, Tanaka, and H. Wang 2001) or exact rewriting (Taniguchi, Tanaka, and H. Wang 2001); such representation is not unique (Sala 2009). It turns out, convexity plays a great deal in adapting linear techniques to the nonlinear context, although mild assumptions and slight modifications need to be made: necessity is lost, which implies some level of conservativeness is introduced (Z. Lendek, T.M Guerra, et al. 2010).

Besides aiding the designer to mimic linear approaches in nonlinear contexts, subsuming a nonlinear model into a TS one has a very important advantage: conditions thus derived usually lead to linear matrix inequalities (LMIs), which belong to the realm of semidefinite programming (SDP). SDP problems are solved in polynomial time via convex optimisation techniques (Boyd et al. 1994); a variety of commercial software tools are available that can be readily used to solve them: the LMI Toolbox (Gahinet et al. 1995), the SeDuMi (Sturm 1999), and the Mosek solver (E. D. Andersen and K. D. Andersen 2000), the latter two usually used along with the Yalmip interface (Löofberg 2004). Thus, thanks to convexity, TS models along with LMIs gave birth to new control techniques such as parallel distributed compensation (PDC) (H. Wang, Tanaka, and Griffin 1996) and a variety of solutions for observation (Tanaka, Ikeda, and H. Wang 1998), delay systems (Y. Cao and Frank 2000), output feedback (Yoneyama et al. 2000), generalisations for descriptor forms (Taniguchi, Tanaka, Yamafuji, et al. 1999), etc. Note that there exist multiple practical application of the TS-LMI framework, for instance, (García-Nieto et al. 2009; Precup and Hellendoorn 2011; Cazarez-Castro et al. 2017). Similarly, nonlinear generalisations of TS systems known as convex or fuzzy polynomial models (Sala and C. Ariño 2009) have been successfully used along with sum-of-squares (SOS) tools (Prajna, Papachristodoulou, Seiler, et al. 2004) which, happily, also belong to the SDP sort of optimization problems.

Although the sector nonlinearity methodology facilitates the analysis of nonlinear systems via the direct Lyapunov Methods and LMIs (conclusions drawn on the TS model are directly valid for the nonlinear one), there are problems for which the standard TS-LMI framework is not able to find a solution, i.e., it is conservative (Sala, T.M. Guerra, and Babuska 2005; Sala 2009; L.A. Mozelli et al. 2009).

This conservatism comes from three main sources:

1. *The way MFs are taken into account in nested convex sums.*

   In order to obtain LMI conditions, the MFs should be dropped off from signed nested convex sums. Since the MFs are all positive within the modelling area, an easy way to do so is to ask every term in the sum to have the desired sign (Tanaka and H. Wang 2001), but of course that might be quite conservative.

For that reason, a variety of results –referred as "sum relaxations"– have been proposed in order to tackle this problem. For example, in (Tanaka and Sugeno 1992), they take into account that there are terms in a nested convex sum that share the same MFs; in (Tuan et al. 2001) a partial solution of the co-positivity problem was proposed; in (Liu and Zhang 2003), slack matrices are added in the LMI conditions to relax the results; whereas in (Sala and Ariño 2007; Kruszewski et al. 2009) asymptotically necessary and sufficient conditions are proposed through a complexity parameter.

Since the TS models have all the nonlinearities grouped together in the MFs, only the vertex (linear) models are considered in the LMI conditions. This is to say, the MFs are considered as independent variables that only hold the convex sum property and their dependence on the states is neglected, introducing the so called *shape-independent* conservatism. Thus, a nested convex sum may be positive even if some of its terms are not (Sala and Ariño 2007), a condition that has been tackled with some shape-dependent results such as (Bernal, T. M. Guerra, and Kruszewski 2009).

2. *The non-uniqueness of the TS model.*

   The sector-nonlinearity approach provides a methodology to rewrite a nonlinear system into a convex model. Nevertheless, this representation is not unique (Sala, T.M. Guerra, and Babuska 2005; Feng 2006), i.e., depending on the chosen TS model, different conclusions can be reached with shape-independent LMIs for the same nonlinear system. In (Robles et al. 2017; Robles et al. 2016) different approaches were proposed to obtain an "optimal" TS model with respect to some performance measure. The same goes for polynomials fuzzy models which can be differently chosen. Moreover, if the original nonlinear model is considered, it might be better expressed (with less number of vertexes, for instance) if a descriptor form is adopted (T. M. Guerra, Estrada-Manzo, and Zs. Lendek 2015).

3. *The family of Lyapunov function which is employed.*

   The standard TS-LMI framework is based on quadratic Lyapunov functions, thus neglecting the fact that a system may be stable but not quadratically stable (Khalil 2002). Thus, larger classes of Lyapunov functions have been proposed, all of which include the quadratic one as a particular case. Some of them are: piecewise (PWLFs) continuous (Johansson, Rantzer, and Arzen 1999) and discrete (Feng 2004), where the state space is partitioned according to the activation of the linear or piecewise models, allowing the Lyapunov function to change from one region to another; parameter-dependent

(PDLF), also known as non-quadratic, fuzzy, or convex, first appeared in (Blanco, Perruqueti, and Borne 2001), makes use of the MFs of the TS model in order to share the flexibility and structure of the latter, also available in continuous (Tanaka, Hori, and H. Wang 2003; Bernal and T. M. Guerra 2010) and discrete versions (T.M. Guerra and Vermeiren 2004; T.M. Guerra, Kruszewski, and Bernal 2009); the fuzzy line-integral (LILF) first proposed in (Rhee and Won 2006) and refined in (Marquez, T.M. Guerra, et al. 2014), which employs line integrals to avoid dealing with the time derivative of the MFs in the continuous-time context. Similarly to the classical TS arena, that of polynomial/sum-of-squares (SOS) have enriched its set of Lyapunov functions by employing polynomial ones (Tanaka, Ohtake, and H. Wang 2009).

As it can be concluded from the discussion above, a system may be proven stable if more information on the MFs is taken into account and more general classes of Lyapunov functions are used (Z. Lendek, T.M Guerra, et al. 2010). Several results have achieved the so-called *asymptotical exactness*, i.e., the conservatism is reduced as the computational resources increase (conditions depend on a *complexity parameter*); in theory, when the complexity parameter increase to infinity, conservatism (from that source) is reduced to zero. For instance, the use of multiple nested convex structures in the Lyapunov function have provided a way to simultaneously tackle the co-positivity problem and the use of more general Lyapunov functions, namely the homogenous polynomially parameter-dependent (HPPD) Lyapunov functions (Chesi et al. 2007; R.C.L. Oliveira and P.L. Peres 2007; R.C.L.F Oliveira, C. de Oliveira, and P.L. Peres 2008; Chesi 2010; Ding 2010); nonetheless, most of these results are shape-independent. In the case of polynomial Lyapunov functions, if the degree of the polynomial is increased at will, the results become asymptotically exact (up to the gap between positive and SOS polynomials (Chesi 2007)). Nevertheless, the increase of the complexity parameter, usually leads to an exponential increase of the computational resources; in other words, these approaches quickly reach their computational limits. Additionally, the use of PWLFs for the analysis of nonlinear systems is still conservative and there is room for improvement; the time derivatives of the MFs when PDLFs are employed is still a problem that needs refinement; in (Rhee and Won 2006) the problem of the time derivative of the MFs is avoided only for a limited class of TS models. This thesis provides some answers to these questions that actually improve over existing results.

1.2 Objectives

The main objective of this thesis is to reduce conservatism when convex optimisation techniques are applied for the analysis and control of nonlinear systems. In particular, the use of different classes of Lyapunov functions is explored, all within an LMI framework.

The Lyapunov function studied in this work are:

1. *Piecewise Lyapunov Function:*

   In the piecewise framework, this thesis provides three results on the use of piecewise Lyapunov functions for the stability analysis of nonlinear systems: (a) an affine piecewise modelling techniques that generalise the sector-nonlinearity methodology via easily implementable optimisation-based affine modelling which produces ordinary TS models if the modelling region contains the origin; (b) some geometric properties of the state space are taking into account via Positivstellensatz (S-procedure) argumentations; (c) a new methodology to determine the "largest" estimate of the domain of attraction of the origin of a nonlinear system, within an LMI framework.

2. *Parameter-dependent Lyapunov Functions*

   As mentioned before, there has been a number of works tackling the problem of the time derivative of the MFs when continuous-time TS models are analysed or synthesised with PDLFs. Some of them simply assume that the time derivative has a known bound (Blanco, Perruqueti, and Borne 2001; Tanaka, Hori, and H. Wang 2003); others relate this time derivative with the information arising from the modeling area (T.M. Guerra and Bernal 2009; Bernal and T. M. Guerra 2010; T.M. Guerra, Bernal, et al. 2012; T.M. Guerra and Bernal 2012); some others provide LMIs to guarantee the time derivative to be bounded under certain assumptions (Pan et al. 2012; Jaadari et al. 2012). In this thesis, a new generalised PDLF is proposed along with a generalised multi-index control law that cancels out the terms that cause *a priori* locality in the Lyapunov analysis; moreover, the resulting conditions are purely LMI.

3. *Integral Lyapunov Functions*

   The widely-cited work (Rhee and Won 2006) proposed an interesting fuzzy line-integral Lyapunov function, presenting global LMI stability conditions that avoided involving the time derivatives of the MFs. This thesis shows a new polynomial Lyapunov function with integral terms that generalise the work in (Rhee and Won 2006) for cases on which the later cannot be directly applied; it also goes beyond the TS framework including the polynomial one: it turns out that path independency conditions for line integrals are automatically verified if the integral is expressed as a sum of single-variable terms.

1.3 Structure of the thesis

This thesis is divided in two parts:

- Part I summarises the most relevant results in the literature related to the objectives of this thesis. In Chapter 2, some stability concepts are reviewed to correctly understand the direct Lyapunov method. In the same Chapter, an overview of the concept of an LMI and its use for the stability analysis of linear models is conducted. Additionally, some common matrix properties to transform matrix inequalities into LMIs are given in Chapter 2.

  Chapter 3 presents the sector nonlinearity methodology to rewrite a nonlinear model as a TS one in order to perform stability analysis and controller design. It shows how the direct Lyapunov method is employed altogether with the convex structure of the TS model to express stability and stabilisation conditions in terms of LMIs. This chapter concludes by introducing the use of convex Lyapunov functions (PDLF and LILF) as well as PWLF; some problems related with these Lyapunov functions are commented.

  Chapter 4 presents a review on the standard polynomial fuzzy framework. It begins by explaining what are SOS polynomials and their relationship with LMIs. It then follows with the presentation of a systematic methodology to obtain an exact convex polynomial model of a nonlinear model via the Taylor-series approach (generalisation of the sector nonlinearity approach); these models can reduce conservatism with respect to the TS approach. At the end of the chapter, the *dynamical extension* approach is presented as an alternative to the convex polynomial models. This approach allows to model a nonlinear non-polynomial system as a polynomial one with algebraic restrictions.

- Part II contains the contributions of this work. The first contribution is presented in Chapter 5, where a new procedure for an exact piecewise affine Takagi-Sugeno modelling is explained. This models will later prove to be useful for stability analysis when some geometric restrictions are added in the LMI conditions. With both results, an iterative LMI-based algorithm is proposed for the estimation of the Domain of Attraction (DA) of a nonlinear system. Putting all these results together, Chapter 5 conclude with the important subject of asymptotic exactness for the proposed procedure.

Chapter 6 deals with the design of feedback control. The proposed approach makes use of a generalised PDLF and a generalised multi-index control law that employs the time derivative of the MFs, avoiding the problem of dealing with the time derivatives of the MFs and providing a simplified and easier alternative to recent results on this matter; moreover, the resulting conditions are purely LMI.

Chapter 7 presents a new Polynomial-Integral Lyapunov Function (PILF) for the stability analysis of nonlinear systems. This new PILF generalise earlier results in the LMI/Line-integral framework (Rhee and Won 2006) to the polynomial case. Additionally, the new approach allows using the line-integral approach to a larger class of nonlinear systems.

- This thesis ends in Chapter 8, drawing some concluding remarks and providing some ideas for future work.

Note that most of the content of part II is a verbatim copy of published material (indicated at the beginning of each chapter). Thus, there may be repetitions of preliminary material and notation changes. On page 165, a full list of publications by the PhD candidate is presented.

Part I

State of the Art

Chapter 2

Lyapunov stability and linear matrix inequalities

This chapter presents an overview on the indirect Lyapunov method for stability analysis and controller design of nonlinear systems. Such method is based on the linearisation of a nonlinear system on an equilibrium point, conveniently placed at the origin via a straightforward transformation. It is shown that the linearisation method leads to conditions in the form of linear matrix inequalities (LMIs), which are efficiently solved via convex optimisation techniques. LMIs are discussed in some detail since they are the main computational tool used in this thesis.

2.1 Lyapunov stability

One of the most important results in the analysis of control systems was the theory proposed by the Russian mathematician Aleksandr Lyapunov at the end of the 19th century. In his original thesis "The General Problem of Stability of Motion" (1892), Lyapunov proposed two methods to establish the stability of an equilibrium point of a dynamical system. The first method says that if the linearisation on such point is stable, there exists a "neighborhood" around the equilibrium point where all the trajectories of the nonlinear system go to zero as time tends to infinity, i.e., the equilibrium point is asymptotically stable. The second method (also known as Lyapunov's direct method) basically says that the stability of a nonlinear system could be proved if there exists a positive energy-like function of the state which monotonically decreases over time.

The following definitions introduce different types of stability:

Definition 2.1.1. *(Haddad and Chellaboina 2008) Consider an autonomous nonlinear dynamical system*

$$\dot{x}(t) = f(x(t)) \text{ with } x(0) = x_0, \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ *denotes the state space vector and* $f(\cdot) : \Omega \to \mathbb{R}^n$ *is a locally Lipschitz map from a domain* $\Omega \subseteq \mathbb{R}^n$ *into* $\mathbb{R}^n$*. The solution of* (2.1) *for initial condition* x_0 *will be denoted as* $\psi(t, x_0)$*.*

- *An* ***isolated equilibrium point*** $\bar{x}$ *is a state value such that* $f(\bar{x}) = 0$ *and* $f(x) \neq 0$ *for some neighbourhood of* $\bar{x}$*, i.e., the system will remain on it for all future time once it happens to be there.*
- *The equilibrium point* $\bar{x}$ *is said to be* ***Lyapunov stable****, if, for every* $\varepsilon > 0$*, there exists a* $\delta = \delta(\varepsilon)$ *such that, if* $\|x(0) - \bar{x}\| < \delta$*, then for every* $t \geq 0$ *we have* $\|x(t) - \bar{x}\| < \varepsilon$*.*
- *The equilibrium point* $\bar{x}$ *is said to be* ***asymptotically stable*** *if it is Lyapunov stable and there exists* $\delta > 0$ *such that if* $\|x(0) - \bar{x}\| < \delta$*, then* $\lim_{t\to\infty} \|x(t) - \bar{x}\| = 0$*.*
- *The equilibrium point* $\bar{x}$ *is said to be* ***exponentially stable*** *if it is asymptotically stable and there exist* $\alpha > 0$*,* $\beta > 0$*,* $\delta > 0$ *such that if* $\|x(0) - \bar{x}\| < \delta$*, then* $\|x(t) - \bar{x}\| < \alpha\|x(0) - \bar{x}\|e^{-\beta t}$*, for* $t \geq 0$*.*
- *An equilibrium point* $\bar{x}$ *is* ***unstable*** *if it is not Lyapunov stable.*

Basically, Lyapunov stability means that solutions starting "close enough" of the equilibrium (with a distance δ) remain "close enough" forever (within a distance ε from it). Asymptotic stability in the sense of Lyapunov means that solutions starting close enough to an equilibrium point will eventually converge to it. Exponential stability is asymptotic stability with the extra property of having its solutions bounded by an exponential decay rate $\alpha\|x(0) - \bar{x}\|e^{-\beta t}$. For further explanation the reader is referred to (Khalil 2002).

Whereas linear systems of the form $\dot{x} = Ax$ can have only one isolated equilibrium point at the origin $x = 0$, nonlinear systems may have multiple equilibria as well as a number of exclusively nonlinear phenomena such as limit cycles, finite-time escape, chaos, etc. Therefore, the definitions above provide a formal framework to attach stability concepts to the properties of isolated equilibrium points. Without loss of generality, in the sequel we assume that the equilibrium point under analysis is at the origin, i.e., $\bar{x} = 0$.

2.1.1 Lyapunov's direct method

Stability of an equilibrium point $x = 0$ of a nonlinear system $\dot{x} = f(x)$ can be established via a Lyapunov function candidate, i.e., a *positive-definite function* of the state, $V(x)$, which is often related to the energy of the system. If the time derivative of such function monotonically decreases to zero along time, it implies that the total "energy" of the system goes to zero and that the referred equilibrium point is therefore asymptotically stable. In other words, if $\dot{V}(x)$ is a *negative-definite function*, the Lyapunov function candidate $V(x)$ becomes a *Lyapunov function* for this system, a sufficient condition for establishing the stability properties of the origin. The usefulness of the method relies on the fact that no solution of the differential or difference equations needs to be known. The Lyapunov's stability theorem can be stated as follows:

Theorem 2.1.1. *(Lyapunov 1992) Consider the system* (2.1) *having the origin as equilibrium point, i.e.* $x(0) = 0 \Leftarrow f(0) = 0$*, and let* $\Omega \subset \mathbb{R}^n$ *be a domain containing the origin. Let* $V : \Omega \to \mathbb{R}$ *be a continuously differentiable function in* Ω *such that the following conditions are fulfilled:*

$$V(0) = 0 \tag{2.2}$$

$$V(x) > 0 \quad \forall x \in \Omega,\, x \neq 0 \tag{2.3}$$

$$\dot{V}(x) = \frac{dV(x)}{dt} < 0 \quad \forall x \in \Omega,\, x \neq 0 \tag{2.4}$$

then the origin is asymptotically stable in the sense of Lyapunov. If $\Omega \equiv \mathbb{R}^n$ *and* $V(x)$ *being radially unbounded, i.e.,* $\|x\| \to \infty \Rightarrow V(x) \to \infty$*, then the origin is globally asymptotically stable.*

This method is called "direct" because it does not require the system to be transformed in any way: it is supposed that the time derivative of the Lyapunov function will eventually involve the system equations. The existence of a Lyapunov function is a sufficient condition for the stability of an equilibrium point; conversely, for every stable equilibrium point there must exist a Lyapunov function (W. Hahn 1967). Despite its power and generality, this result has a major drawback: there is no general methodology for searching Lyapunov functions for nonlinear systems. Some forms, such as the quadratic one, have been used for simplicity because they work fine in the linear case.

Indeed, in the case of linear time-invariant (LTI) systems, the existence of a quadratic Lyapunov function $V(x) = x^T Px$ is a sufficient and necessary condition for the global asymptotic stability of $\dot{x} = Ax$. In particular, to apply theorem 2.1.1 to $\dot{x} = Ax$, consider the Lyapunov function candidate $V(x) = x^T Px$, where $P = P^T > 0$ to satisfy condition (2.3). The time derivative of $V(x)$ is given by:

$$\dot{V}(x) = x^T P\dot{x} + \dot{x}^T Px = x^T \left(PA + A^T P\right) x. \tag{2.5}$$

Now, (2.4) is guaranteed if and only if $PA + A^T P < 0$ as it coincides with the definition of a negative-definite matrix. The inequalities $P > 0$ and $PA+A^T P < 0$ are linear matrix expressions; determining whether or not there is an instance of P such that the inequalities hold is an LMI problem. As we will see in Section 2.2, LMIs can be efficiently computationally solved, i.e, if an optimal solution exists it will be found.

The set of all initial conditions from which the trajectories of a system converge to a given equilibrium point is called its *domain of attraction* (DA). Clearly, given multiple equilibrium points, their respective DAs must be disjoint; moreover, if a an equilibrium point is unstable its DA reduces to itself. More formally, a definition of the DA of an equilibrium point at the origin $x = 0$ is the following:

Definition 2.1.2. *(Khalil 2002) The* ***domain of attraction*** *of the system* (2.1)*, denoted as $\mathcal{D}$, is the set of points belonging to the state space whose trajectory $x(t) = \psi(t, x_0)$ ends in the asymptotically stable equilibrium point $x(t) = 0$.*

$$\mathcal{D} := \left\{x \in \mathbb{R}^n : \psi(t, x) \in \Omega\, \forall t \geq 0,\ \lim_{t\to\infty} \psi(t, x) = 0\right\}. \tag{2.6}$$

In general, computing the domain of attraction is extremely difficult. Nevertheless, Lyapunov functions can be used to estimate the region of attraction. From 2.1.1, if there exist a Lyapunov function $V(x)$ that satisfies the conditions of asymptotic stability over a domain Ω and, $\mathcal{E}_c := \{x \in \mathbb{R}^n : V(x) \leq c\}$ being a bounded set such that $\mathcal{E}_c \subset \Omega$, then every trajectory staring in $\mathcal{E}_c$ remains in $\mathcal{E}_c$ and approaches the origin as $t \to \infty$. Therefore, $\mathcal{E}_c$ is an estimate of the DA, i.e, $\mathcal{E}_c \subset \mathcal{D}$. Nevertheless, this estimate may be much smaller than the actual DA. In Chapter 5, a new methodology for asymptotically estimate the DA will be presented.

2.1.2 Comments on nonautonomous and time-delay systems

Lyapunov theory for autonomous systems can be extended to nonautonmous systems, i.e, systems in the form:

$$\dot{x} = f(x, t), \tag{2.7}$$

where $f : [0, \infty) \times \Omega \to \mathbb{R}^n$ is piecewise continuous in t and locally Lipschitz in x on $[0, \infty) \times \Omega$, and $\Omega \subset \mathbb{R}^n$ is a domain that contains the origin $x = 0$. In this class of systems, the expressions for the time dependence in f are assumed to be known beforehand. There are plenty of extensions for this class of systems, for more details see (Khalil 2002; Malisoff and Mazenc 2009).

Beside the autonomous and nonautonomous systems, Lyapunov function theory can also be develop for *retarded functional differential equations*, which have the form

$$\dot{x} = f(t, x(t), x(t-h)),\ x(t_0 + \theta) = \phi(\theta),\ \theta \in [-h, 0], \tag{2.8}$$

where $x(t) \in \mathbb{R}^n$, f is continuous in all arguments and locally satisfies Lipschitz condition with respecto to the second argument, where ϕ is a continuous vector-valued initial function. Equations of thus type are also called time delayed differential equations.

Lyapunov related functions are key for the stability analysis and control design for systems with time-delay. Two importan theorems for delayed systems are the Razumikhin Theorem and the Lyapunov-Krasovski Theorem. Both rely on delayed Lyapunov functions or functionals, which are often constructed by first building Lyapunov functions for the corresponding undelayed systems, i.e., setting the delayed equal to zero. For a more detailed background of the stability of time-delay systems see for instance (Gu, Kharitonov, and J. Chen 2003).

Over the last two decades, Lyapunov-Krasovski functionals have been used extensively for the analysis of linear systems. For linear systems, Lyapunov- Krasovski functionals give stability criteria in terms of linear matrix inequalities, which can be analyzed through numerical methods; see for instance (Fridman 2014). Mostly, delay analysis involves use of Lyapunov-Krasovskii functionals in the form $V = x^T P x + \int_a^b x^T Q x + \int_c^d \dot{x}^T R \dot{x} + \int_e^f \int_g^h \dot{x} S \dot{x} + \cdots$ for some delay-bound related integration limits. Nevertheless, the motivation of this thesis is focus on autonomous systems without delays, although all the results presented can also be extended to the nonautonomous or time-delay case.

2.1.3 Lyapunov's indirect method

The Lyapunov's indirect method establishes the properties of an equilibrium point by studying the behaviour of the *linearised* system which, under certain conditions, locally preserves the stability properties of the original nonlinear system. Since the method requires transforming the nonlinear equations and examining the eigenvalues of the linearised system matrix instead of looking for a Lyapunov function, it is referred to as indirect. Nevertheless, it should be kept in mind that the proof of the criteria in the following theorem is based on a quadratic Lyapunov function associated to the linearised system:

Theorem 2.1.2. *(Khalil 2002) Let $x = 0$ be an equilibrium point for the nonlinear system $\dot{x}(t) = f(x(t))$, where $f(\cdot) : \Omega \to \mathbb{R}^n$ is continuously differentiable and Ω is a neighborhood of the origin. Let*

$$A = \left.\frac{\partial f(x)}{\partial x}\right|_{x=0}$$

be the Jacobian matrix *of $f(x)$ at $x = 0$. Then,*

1. *The origin is* ***asymptotically stable*** *if $Re(\lambda_i) < 0$ for all eigenvalues of A.*
2. *The origin is* ***unstable*** *if $Re(\lambda_i) > 0$ for one or more of the eigenvalues of A.*
3. *if $Re(\lambda_i) \leq 0\ \forall i$ with $Re(\lambda_i) = 0$ for some i, linearisation* ***fails*** *to determine the stability of the equilibrium point.*

λ_i, $i \in \{1, 2, \ldots, n\}$ are the eigenvalues of the matrix A.

Theorem 2.1.2 provides a simple procedure to analyse the stability of an equilibrium point at the origin of a nonlinear system. Moreover, the quadratic Lyapunov function $V(x) = x^T P x$ with $P > 0$, $PA + A^T P < 0$, is also a Lyapunov function for the nonlinear system in some neighborhood of the origin. The Lyapunov function is the quadratic form as in the linear case shown in the previous section.

2.1.4 Stabilisation via linearisation

The linearisation method can also be used to "solve" the stabilisation problem. This method allow us to obtain a *local* control law for a nonlinear model; local in the sense of that the feedback control law stabilize in a neighborhood of the origin. To this end, consider the system

$$\dot{x}(t) = f(x(t), u), \tag{2.9}$$

where $f(0,0) = 0$ and $f(x,u)$ is continuously differentiable function in a domain $\Omega_x \times \Omega_u \subset \mathbb{R}^n \times \mathbb{R}^m$ that contains the origin ($x = 0$, $u = 0$). Linearisation of the (2.9) at the origin ($x = 0$, $u = 0$) results in the linear system

$$\dot{x} = Ax + Bu \tag{2.10}$$

where

$$A = \left.\frac{\partial f(x,u)}{\partial x}\right|_{x=0,u=0} \qquad B = \left.\frac{\partial f(x,u)}{\partial u}\right|_{x=0,u=0}.$$

If the pair (A, B) is controllable, or at least stabilisable, we can continue with the control design. Consider a linear state feedback control $u(x) = Kx$, where $K \in \mathbb{R}^{m\times n}$. The closed-loop system yields as:

$$\dot{x} = f(x, Kx). \tag{2.11}$$

Since the origin remains an equilibrium point, it follows from theorem 2.1.2 that the origin is locally asymptotically stable if the linearisation of the closed-loop system (2.11) is stable. If a gain K is given, the linearised closed-loop system is stable if and only if there exist $P > 0$ such that

$$P(A + BK) + (A + BK)^T P < 0. \tag{2.12}$$

Thanks to the Lyapunov's methods, a Lyapunov function can always be found for the closed-loop system. Thus, the quadratic Lyapunov function $V(x) = x^T Px$ is a Lyapunov function for the closed-loop nonlinear system in the neighborhood of the origin.

Again, the inequality (2.12) is a matrix one, but it seems nonlinear as the variables K and P appeared multiplied. Nevertheless, straightforward matrix manipulations and properties can be used to show that the previous conditions are indeed convex, i.e., LMIs. Some of these properties are shown in the following section.

2.2 Linear matrix inequalities

As mentioned in the sections above, this thesis pursue LMI conditions for the analysis and synthesis of controllers for nonlinear systems via exact convex representations. Thus, a brief introduction on the LMI theory is presented in this section. LMIs are a fundamental tool for analysis and synthesis of convex nonlinear control systems and can be easily implemented with convex optimisation techniques. More details can be found in (Boyd et al. 1994; Gahinet et al. 1995; C. Scherer 2004).

Before going any further, some definitions follow concerning signed matrix expressions:

Definition 2.2.1. *Consider two symmetric matrices M_1, $M_2 \in \mathbb{R}^{n\times n}$, i.e., $M_1 = M_1^T$ and $M_2 = M_2^T$. Then:*

1. *$\sigma(M_1)$ denotes the spectrum of M_1, i.e., the set of all its eigenvalues.*
2. *M_1 is **positive semidefinite** ($M_1 \geq 0$) if $x^T M x \geq 0$ $\forall x \in \mathbb{R}^n$, i.e., $Re(\sigma(M_1)) \geq 0$.*
3. *M_1 is **positive definite** ($M_1 > 0$) if $x^T M x > 0$, $\forall x \in \mathbb{R}^n$, $x \neq 0$, i.e., $Re(\sigma(M_1)) > 0$.*
4. *$M_1 \succ M_2$ means that each entry in matrix M_1 is greater than its corresponding one in M_2, i.e., $[M_1]_{ij} > [M_2]_{ij}$, $\forall i, j$.*
5. *$M_1 > M_2$ means that $M_1 - M_2 > 0$.*

Similar definitions can be made for $M_1 < 0$, $M_1 \leq 0$, $M_1 \prec M_2$, and $M_1 < M_2$.

Since the appearance of *semidefinite programming* (SDP), a number of problems from control theory has been solved numerically by expressing them as convex optimisation tasks with a linear objective function subject to a constraint that is an affine combination of symmetric matrices (Vandenberghe and Boyd 1996). In practice, SDP is typically expressed using LMI notation. Reiterating, it is convenient expressing a result as an LMI because it can be efficiently solved *numerically* using interior-point methods; moreover, an optimal solution is guaranteed. Several software toolboxes are available today that implement interior-point algorithms to solve LMIs; for instance, the LMI Toolbox for MATLAB (Gahinet et al. 1995), the solver SeDuMi (Sturm 1999), and the solver MOSEK (ApS 2015), the later two are usually employed along with the YALMIP interface (Löofberg 2004). A formal definition follows.

Definition 2.2.2. *A **linear matrix inequality** (Boyd et al. 1994) has the following form*

$$F(x) = F_0 + \sum_{i=1}^{m} x_i F_i > 0, \tag{2.13}$$

*where $x \in \mathbb{R}^m$ is a vector of m real numbers called as **decision variables**; $F_i = F_i^T \in \mathbb{R}^{n\times n}$, $i \in \{0, 1, \ldots, m\}$ are given real symmetric matrices; the inequality $>$ means that $F(x)$ is positive-definite, or equivalently, $Re(\lambda(F(x))) > 0$ where $\lambda(F(x))$ denotes the spectrum of $F(x)$, i.e., the set of all its eigenvalues. Thereby, (2.13) is called an LMI for x.*

Generally, the variables in an LMI are matrices, for example, the Lyapunov inequality $PA + A^T P < 0$ where A is given and $P = P^T$ is the decision variable. In this case the LMI is not written explicitly in the form (2.13) above, but the equivalence become clear by taking $F_0 = 0$, $F_i = -A^T - A$, and x_i, $i \in \{1, \ldots, m\}$ as each unknown entry of $P \in \mathbb{R}^{n \times n}$, finally $m = n\,(n+1)\,/2$. The definition in (2.13) is closer to the spirit of the LMI toolboxes as they search for a feasible instance of the decision vector with entries x_i.

The following three standard problems are relevant in the LMI framework (Boyd et al. 1994; C. Scherer 2004):

1. *Feasibility problem* (FP): Consists in finding a solution instance x to the LMI system $F(x) > 0$. If x exists, the LMI $F(x) > 0$ is called *feasible*, otherwise it is said to be *infeasible*.

2. *Eigenvalues problem* (EVP): Consists in minimising the maximum eigenvalue of a matrix that depends affinely on a variable, subject to an LMI constraint (or determine that the constraint is infeasible), i.e.,

$$\begin{array}{ll} \text{minimize} & \lambda \\ \text{subject to} & \lambda I - F(x) > 0, \quad G(x) > 0 \end{array}$$

   where F and G are symmetric matrices that depend affinely on the optimisation variable x.

3. *Generalised eigenvalue problem* (GEVP): Consists in minimising the eigenvalues of a pair of matrices which depend affinely on a variable, subject to a set of LMI-constraints. The general form of a GEVP is:

$$\begin{array}{ll} \text{minimize} & \lambda \\ \text{subject to} & \lambda G(x) - F(x) > 0, \quad F(x) > 0, \quad H(x) > 0 \end{array}$$

   where F, G, and H are symmetric matrices that are affine functions of x. The problem can be rewritten as

$$\begin{array}{ll} \text{minimize} & \lambda_{\max}\left(F(x), G(x)\right) \\ \text{subject to} & G(x) > 0, \quad H(x) > 0 \end{array}$$

   where $\lambda_{\max}(F, G)$ denotes the largest generalised eigenvalue of $\lambda G - F$ with $G > 0$, i.e., the largest eigenvalue of the matrix $G^{-1/2} F G^{-1/2}$.

The examples and results in this thesis were obtained using MOSEK as the LMI/-SOS solver with YALMIP interface.

It can be noted that GEVP is a quasiconvex optimisation problem (Boyd et al. 1994) because the constrains are convex but the objective is not. Nevertheless, the minimum objective can be obtained by Iterative LMI (ILMI) methods, for example, bisection search.

Recalling the conditions in equation (2.12) (stability of the closed-loop system) are not expressed as LMIs. Nevertheless, there are some properties which are commonly used to transform matrix expressions into LMIs. Some of these are summarised below.

Property 2.2.1 (System of LMIs). *A set of LMIs $F_1 > 0, \cdots, F_k > 0$ is equivalent to the single LMI:*

$$F = \begin{bmatrix} F_1 & 0 & \cdots & 0 \\ 0 & F_2 & \cdots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & \cdots & 0 & F_k \end{bmatrix} > 0$$

.

Property 2.2.2 (Congruence). *Let $P = P^T > 0$ and Q be a full-column rank matrix, the expression $Q^T PQ$ is also positive-definite. Indeed, if $P > 0$ then $x^T Px > 0$ hold for all $x \neq 0$. In particular, if $x = Qv$ and $v^T Q^T PQv > 0$, hence $Q^T PQ > 0$.*

Property 2.2.3 (Schur Complement). *Consider the LMI*

$$M = \begin{bmatrix} A & B^T \\ B & C \end{bmatrix} > 0 \tag{2.14}$$

where $A \in \mathbb{R}^{m\times m} > 0$, $B \in \mathbb{R}^{m\times n}$, and $C \in \mathbb{R}^{n\times n} > 0$ are full-rank matrices. Thus, M is equivalent to

$$A - B^T C^{-1} B > 0, \tag{2.15}$$

$$C - B^T A^{-1} B > 0 \tag{2.16}$$

Property 2.2.4 (S-procedure). *Let $F_i = F_i^T \in \mathbb{R}^{n\times n}$, $x \in \mathbb{R}^n$, being such that $x^T F_i x \geq 0$, $i \in \{1, \ldots, p\}$, and the quadratic inequality condition*

$$x^T F_0 x \geq 0 \tag{2.17}$$

$x \neq 0$. There exist positive real scalars $s_1, \ldots, s_p$ such that

$$F_0 - \sum_{i=1}^{p} s_i F_i \geq 0. \tag{2.18}$$

Property 2.2.5 (Finsler's Lemma). *Let $x \in \mathbb{R}^n$, $Q = Q^T \in \mathbb{R}^{n\times n}$, and $R \in \mathbb{R}^{m\times n}$ such that rank$(R) < n$; the following expressions are equivalent:*

- $x^T Q x < 0,\ \forall x \in \{x \in \mathbb{R}^n : x \neq 0, Rx = 0\}$.
- $\exists X \in \mathbb{R}^{n\times m} : Q + XR + R^T X^T < 0$.

Resuming conditions in equation (2.12), they can be expressed as LMIs in order to find the gain K and the Lyapunov matrix P by applying some of the previous properties.

Consider again the expression (2.12), to which the property of congruence with $X = P^{-1}$ is applied to obtain:

$$AX + XA^T + BKX + XK^T B^T < 0. \tag{2.19}$$

Thus, taking the change of variable $M = KX$, the following equivalent inequality is obtained:

$$AX + XA^T + BM + M^T B^T < 0. \tag{2.20}$$

Note that a solution of X and M to the previous inequality guarantees a *unique* pair P and K; the state-feedback gain K is recovered as $K = MP$. This means that the expression was an LMI all along and underlines the fact that an LMI is such because of its feasibility set and may lie hidden within an apparently non-convex problem. Now, we can investigate the stability of nonlinear systems and design controllers for the stabilisation problem, although only locally. Consider the following examples:

Example 2.2.1. *The ball and beam system is one of the most popular and important laboratory models for studying control system engineering, which control goal is calculated the torque u at the pivot of the beam, such that the ball can roll moving towards the center of the beam. For this sake, consider the following state space representation of the ball and beam system shown schematically in Fig. 2.1.*

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ \dfrac{-mx_3(2x_4x_2 - g\cos(x_1))}{mx_3^2 + I_b} + \dfrac{u}{mx_3^2 + I_b} \\ x_4 \\ -\dfrac{5}{7}(g\sin(x_1) - x_3x_2^2) \end{bmatrix}, \tag{2.21}$$

with x_1 being the beam angle with respect to the horizontal line (rad), x_2 being the velocity of the beam angle (rad/s), x_3 being the distance of the ball from the beam

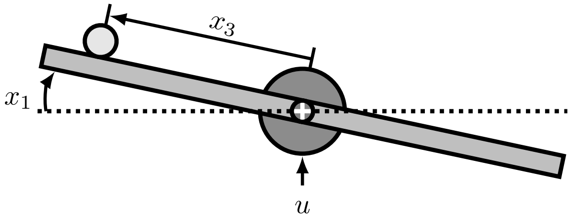

Figure 2.1: The ball and beam system.

center (m), x_4 being the linear velocity of the ball (m/s), u being the torque applied to the beam (N·m), $I_b = \frac{Ma^2}{12}$ is beam's moment of inertia, $M = 1kg$ is the mass of beam, $a = 1m$ is the length of the beam, and $m = 0.05kg$ is the mass of the ball.

We are interesting in study the equilibrium point at ($x = 0$, $u = 0$), which is the point when the ball is static at the center of the beam. Linearisation of the system at the origin results in:

$$\dot{x} = Ax + Bu \tag{2.22}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 5.8860 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{9.81}{1.4} & 0 & 0 & 0 \end{bmatrix}, \quad B \begin{bmatrix} 0 \\ 12 \\ 0 \\ 0 \end{bmatrix}.$$

The eigenvalues of A are $\lambda_{1,2,3,4} = \pm 1.79195 \pm 1.79194i$. Hence, the origin is unstable. Additionally, via Lyapunov's method and LMIs we can look for a Lyapunov function for the linearised system. If we programm the LMI conditions for stability ($P > 0$ such that $PA + A^TP < 0$), the LMI solver will tell us that the problem is infeasible and therefore the system is unstable.

Nevertheless, we can design a linear feedback control in the form $u = Kx$ using conditions in (2.20). If we use the solver MOSEK (ApS 2015) in MATLAB, the following Lyapunov matrix P and control gain K is obtained:

$$P = \begin{bmatrix} 41.1430 & 5.4698 & -16.1926 & -15.9487 \\ 5.4698 & 1.0783 & -2.0526 & -2.1632 \\ -16.1926 & -2.0526 & 9.1622 & 7.5583 \\ -15.9487 & -2.1632 & 7.5583 & 7.2815 \end{bmatrix},$$

$$K = \begin{bmatrix} -5.3692 & -0.8037 & 1.2713 & 1.8018 \end{bmatrix}.$$

Note that the quadratic Lyapunov function $V(x) = x^TPx$ prove asymptotical stability for the closed-loop nonlinear system with $u = Kx$. However, since it is local stability, the DA of the origin is unknown.

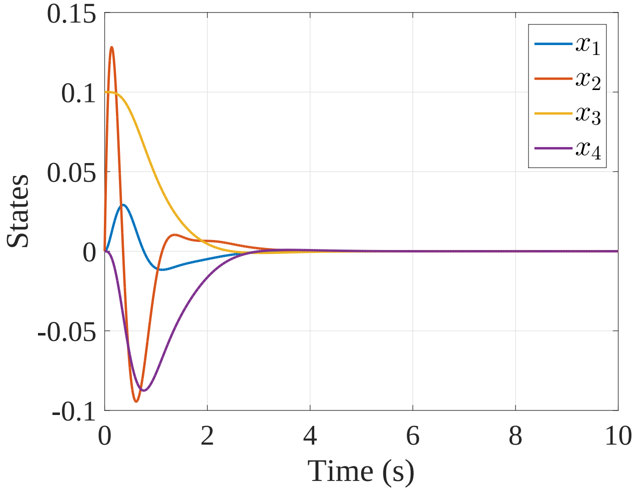

Figure 2.2: Time evolution of the states of the ball and beam model under the control law $u = Kx$.

In Fig. 2.2, some trajectories of the closed-loop system are shown from the initial condition $x(0) = \begin{bmatrix} 0 & 0 & 0.1 & 0 \end{bmatrix}$ *which converge to the origin.*

The following is the MATLAB code for solving the current example.

```
% Define decision variables:
X=sdpvar(4);
M=sdpvar(1,4);
% Define the know matrices:
A=[0 1 0 0; 0 0 5.8860 0; 0 0 0 1; -9.81/1.4 0 0 0];
B=[0; 12; 0; 0];
eps=0.0001;
% Define LMI constraints:
LMI=[X>=eps*eye(4) A*X+X*A^T+B*M+M'*B'<=-eps*eye(4)];
% Call the solver:
sol=optimize(LMI);
```

Notice that in the example above, we just guaranteed that there exist a neighborhood of the origin where the nonlinear model (2.21) is stabilisable. How large is this neighbourhood? This of course an important question as we would like to know which initial conditions lead to stable solutions and how far we can go from the origin without losing stability. In other words, an estimation of the DA of the closed-loop system would come at hand, but provided linearisation is a result of existence, we are unable to use it for this purpose. The direct Lyapunov method and the original nonlinear setup should be used to fulfill this requirement. But can we preserve the LMI approach we just presented? Indeed, we can: in the following section, stability analysis and controller design with a estimation of DA will be proven in an LMI framework via exact convex representations of the nonlinear model, namely, Takagi-Sugeno models.

Chapter 3

Takagi-Sugeno models

This chapter gives a brief overview on the analysis and synthesis of nonlinear systems via Takagi-Sugeno (TS) models. First, it is shown how a TS model can be obtained from a nonlinear one via the sector nonlinearity approach. If the convex structure of the TS model and the Lyapunov's direct method are combined, we can obtain sufficient LMI conditions both for stability analysis and controller design. The gap between sufficiency and necessity of conditions, i.e., conservativeness, arise, among other factors, from the choice of Lyapunov function, which is quadratic in the standard TS-LMI framework. Since the contributions in this thesis are focused on richer classes of Lyapunov functions –parameter-dependent, line-integral, and piecewise–, this chapter concludes presenting them as well as discussing unsolved issues which will be the subject of the improvements later proposed in this work.

3.1 Takagi-Sugeno modelling

Takagi-Sugeno (TS) models have attracted the interest of researchers in the field of control systems because they are able to exactly represent a large class of nonlinear systems in a compact set of their state space by means of a convex structure which proves useful when combined with the direct Lyapunov method. A TS model is a convex blending of linear models weighted by nonlinear membership functions (MFs); these models arise from linearisation (approximate approach) (Ohtake, Tanaka, and H. Wang 2001) or from sector nonlinearity (exact approach) (Taniguchi, Tanaka, and H. Wang 2001). Since this thesis is focused on the latter, a procedure to construct a TS model from a nonlinear one using the sector nonlinearity approach is presented in the following. The idea of using sector-nonlinearity in fuzzy model construction first appeared in (Kawamoto et al. 1992): it allows obtaining an exact representation of a nonlinear model in a TS form inside a compact set of the state space.

3.1.1 Sector nonlinearity

Consider an affine in-control continuous-time nonlinear system of the form

$$\dot{x}(t) = A(x)x(t) + B(x)u(t), \tag{3.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the input vector, $A(\cdot)$ and $B(\cdot)$ are smooth matrix possibly nonlinear functions of appropriate dimensions. Assume there are p different non-constant terms $z_i(x)$, $i \in \{1, 2, \dots, p\}$, in $A(x)$ and $B(x)$ which are bounded in a compact set $\Omega \subset \mathbb{R}^n$ such that $0 \in \Omega$; they will be the entries of the so-called premise vector $z(x) \in \mathbb{R}^p$.

Let $z_j(x) \in \left[\underline{z}_j, \overline{z}_j\right]$, $j \in \{1, 2, \dots, p\}$ be the set of bounded non-constant terms in $A(x)$ and $B(x)$ belonging to Ω. Clearly, each of these terms can be written as a convex sum of its bounds, i.e., $z_j(x) = w_0^j(x)\underline{z}_j + w_1^j(x)\overline{z}_j$ with $w_0^j(x)$, $w_1^j(x)$, $j \in \{1, 2, \dots, p\}$, weighting functions (WFs) of the form:

$$w_0^j(z_j) = \frac{\overline{z}_j - z_j(x)}{\overline{z}_j - \underline{z}_j}, \quad w_1^j(z_j) = 1 - w_0^j(z_j), \quad j \in \{1, 2, \dots, p\}. \tag{3.2}$$

Convex sums can be stacked together as nested ones at the leftmost side of expressions, which implies that (3.1) can be exactly rewritten as the following *tensor-product Takagi-Sugeno model*:

$$\dot{x}(t) = \sum_{i_1=0}^{1} \sum_{i_2=0}^{1} \cdots \sum_{i_p=0}^{1} w_{i_1}^1 w_{i_2}^2 \cdots w_{i_p}^p \left(A_{(i_1, i_2, \dots, i_p)} x(t) + B_{(i_1, i_2, \dots, i_p)} u(t)\right) \tag{3.3}$$

$$= \sum_{\mathbf{i} \in \mathbb{B}^p} \mathbf{w_i} \left(A_{\mathbf{i}} x(t) + B_{\mathbf{i}} u(t)\right) = A_w x(t) + B_w u(t), \tag{3.4}$$

where $\mathbf{i} = (i_1, i_2, \dots, i_p)$, $\mathbb{B} \in \{0, 1\}$, $\mathbf{w_i} = w_{i_1}^1 w_{i_2}^2 \cdots w_{i_p}^p$, $A_{\mathbf{i}} = A(x)|_{\mathbf{w_i}=\mathbf{1}}$, $B_{\mathbf{i}} = B(x)|_{\mathbf{w_i}=\mathbf{1}}$, $\mathbf{1} = \underbrace{(1, 1, \dots, 1)}_{p \text{ ones}}$.

More classically and attending their fuzzy origins, TS models used to be written in terms of *membership functions* (MFs):

$$h_i = h_{1+i_1+i_2\times 2+\cdots+i_p\times 2^{p-1}} = \prod_{j=1}^{p} w^j_{i_j}(z_j), \tag{3.5}$$

with $i \in \{1, 2, \ldots, r\}$, $r = 2^p$, $i_j \in \{0, 1\}$. As the WFs, MFs (3.5) hold the convex-sum property in Ω:

$$\sum_{i=1}^{r} h_i(\cdot) = 1,\ h_i(\cdot) \geq 0,\ i \in \{1, 2, \ldots, r\}. \tag{3.6}$$

Based on the previous definitions, an exact representation of (3.1) in Ω is given by the following *classical Takagi-Sugeno model*:

$$\dot{x}(t) = \sum_{i=1}^{r} h_i\left(z(x)\right)\left(A_i x(t) + B_i u(t)\right) = A_h x(t) + B_h u(t), \tag{3.7}$$

with $(A_i, B_i) = (A(x), B(x))\left.\right|_{h_i=1}$, $i \in \{1, 2, \ldots, r\}$. Importantly, this model is an exact rewriting of the nonlinear model (3.1); so it is the equivalent tensor-product model (3.3): they are *not* approximations.

The next example illustrates how to build a TS model from a given nonlinear dynamical system by the sector nonlinearity methodology.

Example 3.1.1. *Consider the nonlinear model of an inverted pendulum on a cart (Tanaka and H. Wang 2001)(cf. Fig. 3.1):*

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \dfrac{g\sin x_1}{x_1\left(1.33l - alm\cos^2(x_1)\right)} & -\dfrac{amlx_2\sin(2x_1)}{2\left(1.33l - alm\cos^2(x_1)\right)}\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix} 0 \\ -\dfrac{a\cos x_1}{1.33l - alm\cos^2(x_1)}\end{bmatrix} u(t), \tag{3.8}$$

where $x_1(t)$ denotes the angle of the pendulum measured from the vertical upward position, $x_2(t)$ the angular velocity, $m = 2$ the mass of the pendulum, $M = 8$ the mass of the cart, $g = 9.81$ the acceleration due to gravity, $l = 0.5$ the length of the pendulum, and $a = (m + M)^{-1}$ a parameter. From the physical setup, it is clear that a realistic assumption is that $x_1(t) \in [-0.25\pi, 0.25\pi]$ and $x_2(t) \in [-1, 1]$. Nonlinearities can be chosen in a variety of ways; a natural choice is:

$$z_1(x) = \frac{\sin x_1}{1.33lx_1 - almx_1\cos^2(x_1)},$$

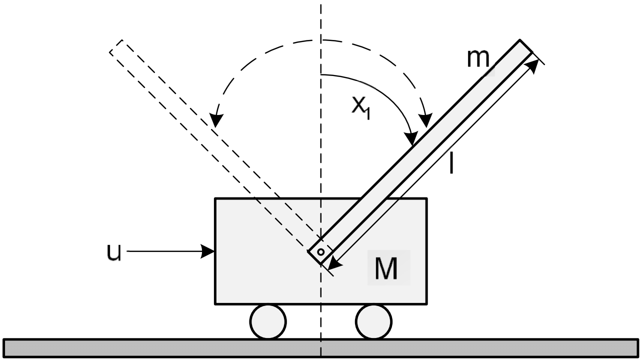

Figure 3.1: Inverted pendulum.

$$z_2(x) = \frac{x_2 \sin(2x_1)}{1.33l - alm\cos^2(x_1)},$$
$$z_3(x) = \frac{\cos x_1}{1.33l - alm\cos^2(x_1)}.$$

Then, the nonlinearities belong to the following intervals:

$$z_1(x) \in [1.46, 1.7647], \quad z_2(x) \in [-1.6216, 1.6216], \quad z_3(x) \in [1.1467, 1.7647].$$

The WFs are: $w_0^1(x) = \dfrac{1.7647 - z_1(x)}{0.3047}$, $w_0^2(x) = \dfrac{1.6216 - z_2(x)}{3.2432}$, $w_0^3(x) = \dfrac{1.7647 - z_3(x)}{0.6180}$, $w_1^1(x) = 1 - w_0^1(x)$, $w_1^2(x) = 1 - w_0^2(x)$, $w_1^3(x) = 1 - w_0^3(x)$. *Thus, we get can rewrite* (3.8) *as the following tensor-product TS model:*

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ g\left(w_0^1(z_1)\underline{z}_1 + w_1^1(z_1)\overline{z}_1\right) & -0.5aml\left(w_0^2(z_2)\underline{z}_2 + w_1^2(z_2)\overline{z}_2\right)\end{bmatrix} x(t) \\ &\quad + \begin{bmatrix} 0 \\ -a\left(w_0^3(z_3)\underline{z}_3 + w_1^3(z_3)\overline{z}_3\right)\end{bmatrix} u(t). \\ &= \sum_{i_1=0}^{1}\sum_{i_2=0}^{1}\sum_{i_3=0}^{1} w_{i_1}^1(z_1) w_{i_2}^2(z_2) w_{i_3}^3(z_3)\left(A_{(i_1,i_2,i_3)}x(t) + B_{(i_1,i_2,i_3)}u(t)\right), \end{aligned} \tag{3.9}$$

where

$$A_{000} = A_{001} = \begin{bmatrix} 0 & 1 \\ 14.3226 & 0.0811 \end{bmatrix}, \quad A_{010} = A_{011} \begin{bmatrix} 0 & 1 \\ 14.3226 & -0.0811 \end{bmatrix},$$
$$A_{100} = A_{101} = \begin{bmatrix} 0 & 1 \\ 17.3117 & 0.0811 \end{bmatrix}, \quad A_{110} = A_{111} = \begin{bmatrix} 0 & 1 \\ 17.3117 & -0.0811 \end{bmatrix},$$

$$B_{000}=B_{010}=B_{100}=B_{110}=\begin{bmatrix}0\\-0.1147\end{bmatrix},\ B_{001}=B_{011}=B_{101}=B_{111}=\begin{bmatrix}0\\-0.1765\end{bmatrix}.$$

Using the same WFs as above, the following MFs are obtained:

$h_1(z(x))=w_0^1w_0^2w_0^3$, $h_2(z(x))=w_0^1w_0^2w_1^3$, $h_3(z(x))=w_0^1w_1^2w_0^3$, $h_4(z(x))=w_0^1w_1^2w_1^3$, $h_5(z(x))=w_1^1w_0^2w_0^3$, $h_6(z(x))=w_1^1w_0^2w_1^3$, $h_7(z(x))=w_1^1w_1^2w_0^3$, $h_8(z(x))=w_1^1w_1^2w_1^3$,

based on which, the following classical TS model can be found:

$$\dot{x}(t)=\sum_{i=1}^{8}h_i(z(x))\left(A_ix(t)+B_iu(t)\right), \tag{3.10}$$

where the corresponding matrices of the linear local models are:

$$A_1=A_2=\begin{bmatrix}0&1\\14.3226&0.0811\end{bmatrix},\quad A_3=A_4=\begin{bmatrix}0&1\\14.3226&-0.0811\end{bmatrix},$$
$$A_5=A_6=\begin{bmatrix}0&1\\17.3117&0.0811\end{bmatrix},\quad A_7=A_8=\begin{bmatrix}0&1\\17.3117&-0.0811\end{bmatrix},$$
$$B_1=B_3=B_5=B_7=\begin{bmatrix}0\\-0.1147\end{bmatrix},\quad B_2=B_4=B_6=B_8=\begin{bmatrix}0\\-0.1765\end{bmatrix}.$$

Obviously, the two TS models above are equivalent to (3.8); moreover, note that they have the same vertex models. Nevertheless, they may serve differently depending on the context as nested convex sums may lead to polynomial expressions of WFs or MFs, which by association may produce different sets of LMIs. These characteristics are exploited in this work, but keep in mind that sometimes the choice of TS model is only made to keep up with the historical background: for instance, piecewise contexts have usually recurred to classical representations (Johansson, Rantzer, and Arzen 1999) while parameter-dependent Lyapunov functions are usually associated with tensor-product-related relaxations (D. Lee and D. Kim 2014).

Note that while the righthand side of the TS models above are algebraically equivalent to the original nonlinear setup, the convex sum property only holds within the compact set $\Omega=\{x:|x_1|\le 0.25\pi, |x_2(t)|\le 1\}$*; outside it, some MFs* $h_i(\cdot)$ *become negative or greater than one, which will turn relevant for stability analysis.*

3.2 Quadratic Lyapunov function

Stability analysis of nonlinear systems can be performed via any of their exact TS representations. For the latter, stability is traditionally investigated using a quadratic Lyapunov function, which is among the reasons why conditions thus obtained are only sufficient, i.e., if they fail nothing can be concluded. It turns out that quadratic Lyapunov functions can be appropriately combined with the convex structure of TS models to produce conditions that mimic the linear case presented before; such conditions are in the form of linear matrix inequalities (LMIs) (Boyd et al. 1994; Tanaka and H. Wang 2001).

3.2.1 Stability analysis of TS models

Consider the following quadratic Lyapunov function candidate

$$V(x) = x^T(t)Px(t), \quad P = P^T > 0 \tag{3.11}$$

along with the continuous-time autonomous TS model (which corresponds to TS model (3.7) with $u(t) = 0$):

$$\dot{x}(t) = \sum_{i=1}^{r} h_i\left(z(x)\right) A_i x(t) = A_h x(t), \tag{3.12}$$

where $A_i \in \mathbb{R}^{n\times n}$ and h_i, $i \in \{1, 2, \ldots, r\}$, have the usual meanings, the latter being MFs that hold the convex sum property in a compact set Ω. As shown in section 3.1.1, this TS model may be the result of applying the sector nonlinearity approach to a continuous-time nonlinear model to obtain an equivalent convex representation.

This TS model is quadratically stable if there exists a quadratic Lyapunov function (3.11) such that its time-derivative is negative definite. The derivative of (3.11) is given by:

$$\begin{aligned}
\dot{V}(t) &= \dot{x}^T(t)Px(t) + x^T(t)P\dot{x}(t) \\
&= \left(\sum_{i=1}^{r} h_i(z)A_i x(t)\right)^T Px(t) + x^T(t)Px(t)\left(\sum_{i=1}^{r} h_i(z)A_i x(t)\right) \\
&= \sum_{i=1}^{r} h_i(z)x^T(t)\left(PA_i + A_i^T P\right)x(t),
\end{aligned} \tag{3.13}$$

where the fact that $\sum_{i=1}^{r} h_i(\cdot) = 1$ has been used to put this sum at the leftmost side of the expression above. Now, since $h_i(\cdot) \geq 0$, $i \in \{1, 2, \ldots, r\}$, a sufficient condition to guarantee $\dot{V}(x) < 0$ is $PA_i + A_i^T P < 0$. Thus, this reasoning just proved the following:

Theorem 3.2.1. *(Tanaka and H. Wang 2001) The origin $x = 0$ of the autonomous model (3.12) is asymptotically stable if there exists a matrix $P = P^T > 0$ such that the following LMIs are satisfied:*

$$PA_i + A_i^T P < 0$$

for $i \in \{1, 2, \ldots, r\}$.

Since the convex sum property holds only in the compact set Ω, any trajectory starting in the outermost Lyapunov level $V(x) = x(t)^T Px(t) = k$, $k \in \mathbb{R}$ within Ω goes to zero. Note that if $\Omega = \mathbb{R}^n$, i.e., if the convex sum property of the MFs hold everywhere, the origin is globally asymptotically stable; this is the case of the TS models in the fuzzy context Tanaka and H. Wang 2001.

Example 3.2.1. *Consider the following continuous-time nonlinear model*

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 + x_1^2 & -1 \\ \sin x_2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{3.14}$$

which is assumed to operate within the compact set $\Omega = \{x : |x_1(t)| \le 1, |x_2(t)| \le 0.5\pi\}$. The following TS model can be constructed from (3.14):

$$\dot{x}(t) = \sum_{i=1}^{4} h_i(z(x)) \left(A_i x(t) + B_i u(t) \right), \tag{3.15}$$

with $A_1 = \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix}$, $A_2 = \begin{bmatrix} -2 & -1 \\ 1 & -2 \end{bmatrix}$, $A_3 = \begin{bmatrix} -1 & -1 \\ -1 & -2 \end{bmatrix}$, $A_4 = \begin{bmatrix} -1 & -1 \\ 1 & -2 \end{bmatrix}$, $z_1(x) = x_1^2(t)$, $z_2(x) = \sin x_2(t)$, $w_0^1 = 1 - x_1^2(t)$, $w_0^2 = 0.5 - 0.5 \sin x_2(t)$, $w_1^1 = 1 - w_0^1$, $w_1^2 = 1 - w_0^2$, $h_1(z(x)) = w_0^1 w_0^2$, $h_2(z(x)) = w_0^1 w_1^2$, $h_3(z(x)) = w_1^1 w_0^2$, and $h_4(z(x)) = w_1^1 w_1^2$. Recall that TS model (3.15) is an exact representation of the nonlinear model (3.14) in the compact set Ω, whose boundaries are shown in Fig. 3.2 with a solid borderline rectangle. For this example, the following P satisfies theorem 3.2.1:

$$P = \begin{bmatrix} 0.4076 & -0.0985 \\ -0.0985 & 0.3232 \end{bmatrix}, \tag{3.16}$$

i.e., it satisfies the inequalities:

$$P = P^T > 0, \; PA_1 + A_1^T P < 0,$$
$$PA_2 + A_2^T P < 0, \; PA_3 + A_3^T P < 0, \; PA_4 + A_4^T P < 0.$$

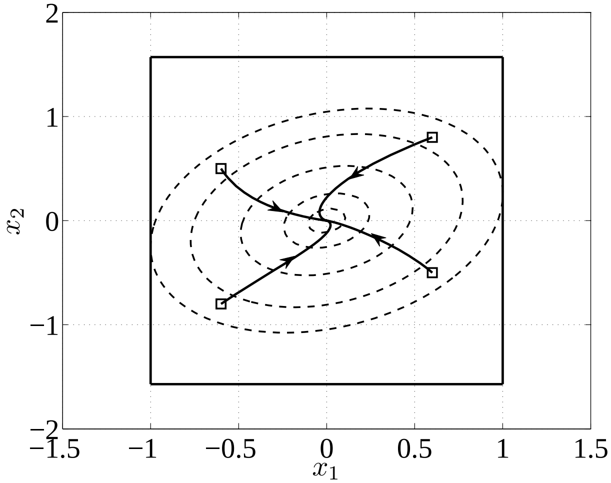

Figure 3.2: Lyapunov levels and model states trajectories (Theorem 3.2.1)

As mentioned before, trajectories starting in the outermost Lyapunov level $V(x) = x(t)^T Px(t) = k$, $k \in \mathbb{R}$ *within* Ω *are guaranteed to converge to the origin since the convex sum property on which the Lyapunov analysis is based only holds within* Ω. *Fig. 3.2 shows in dashed lines some Lyapunov levels corresponding to* (3.11)*; four system trajectories are also shown from different initial conditions: as expected, they all converge to the origin.*

It is important to notice that having all local matrices Hurwitz (i.e. matrices whose eigenvalues have strictly negative real parts) is not enough for ensuring the stability of a TS model, because the domain of Hurwitz matrices is non-convex. Consider the following example:

Example 3.2.2. *Consider the matrices:*

$$A_1 = \begin{bmatrix} -2 & 30 \\ 0 & -1 \end{bmatrix}, A_2 = \begin{bmatrix} -2 & 0 \\ 30 & -1 \end{bmatrix}.$$

These matrices are Hurwitz stable, and they have the eigenvalues at -2 *and* -1. *Now, consider the convex combination:*

$$A = 0.5 \times A_1 + 0.5 \times A_2 = \begin{bmatrix} -5 & 15 \\ 15 & -1 \end{bmatrix},$$

whose eigenvalues are -16.5083 *and* 13.5083, *therefore* A *is non-Hurwitz.*

Due to the fact that the Lyapunov function (3.11) is quadratic in x, we speak of *quadratic stability*; similarly, when a system is *quadratically stable*, it implies that it is stable via a quadratic Lyapunov function. Nevertheless, if a system is stable, it is *not necessarily* quadratically stable. Hence, conditions obtained using the Lyapunov function (3.11) are only sufficient from the point of view of choice of Lyapunov function. There are of course other sources of conservatism when TS models and LMIs are employed for establishing the stability properties of an equilibrium point of a nonlinear system. Such conservativeness means that failure to meet the conditions above does not establish stability nor instability of the TS model (Z. Lendek, T.M Guerra, et al. 2010). A test of non-existence of a common matrix $P = P^T$ is given in (Johansson, Rantzer, and Arzen 1999); it excludes quadratic stability of a given TS model:

Proposition 3.2.1. *If there exists positive definite matrices R_i satisfying*

$$R_i = R_i^T > 0, \quad i \in \{1, 2, \dots, r\}$$
$$\sum_{i=1}^{m} \left(A_i^T R_i + R_i A_i\right) > 0$$

then there is no matrix $P = P^T > 0$ such that LMIs conditions in theorem 3.2.1 hold.

3.2.2 Stabilisation

To perform controller synthesis for a TS model using state feedback, several control laws can be used. Besides ordinary state feedback $u(t) = Fx(t)$, a common solution which includes it as a particular case is the *parallel distributed compensation* (PDC), first appeared in (Sugeno and Kang 1988) without stability analysis as a simple convex blending of local feedback gains. The LMI stability analysis was done and the corresponding control law named PDC in (H. Wang, Tanaka, and Griffin 1995). The PDC controller is composed of linear state feedbacks blended together using the same MFs $h_i(z(x))$ as the TS model, which assumes that the state and the MFs are available:

$$u(t) = \sum_{i=1}^{r} h_i(z(x)) K_i x(t) = K_h x(t), \tag{3.17}$$

with K_i being gains of adequate size to be determined.

Substituting the control law (3.17) in the TS model (3.7) gives the following closed-loop system:

$$\begin{aligned}\dot{x}(t) =& \sum_{i=1}^{r} h_i(z(x))A_i x(t) + \left(\sum_{i=1}^{r} h_i(z(x))B_i\right)\left(\sum_{j=1}^{r} h_j(z(x))K_j x(t)\right)\\ =& \sum_{i=1}^{r} h_i(z(x)) \underbrace{\left(\sum_{j=1}^{r} h_j(z(x))\right)}_{=1} A_i x(t) + \left(\sum_{i=1}^{r} h_i(z(x))B_i\right)\left(\sum_{j=1}^{r} h_j(z(x))K_j x(t)\right)\end{aligned}$$

and finally, it is clear that the closed-loop model is composed of r^2 linear models:

$$\dot{x}(t) = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(x))h_j(z(x))\left(A_i + B_i K_j\right)x(t) = (A_h + B_h K_h)x(t), \tag{3.18}$$

where, again, the fact that $\sum_{i=1}^{r} h_i(\cdot) = 1$ has been taken into account. The next result considers that the gains are already given; a good initial guess is to stabilise each pair (A_i, B_i) via gain K_i.

Theorem 3.2.2. (Tanaka and H. Wang 2001) *The origin $x = 0$ of the TS model* (3.18) *is asymptotically stable if $\exists P = P^T > 0$ such that:*

$$PA_i + A_i^T P + PB_i K_j + K_j^T B_i^T P < 0, \tag{3.19}$$

hold for $i, j \in \{1, 2, \dots, r\}$. The Lyapunov function is given by $V(x) = x^T P x$ and any trajectory starting in the outermost Lyapunov level inside Ω goes asymptotically to zero.

Proof. Consider a quadratic Lyapunov function candidate $V(x) = x^T P x$, $V(0) = 0$ and $V(x) > 0$, $\forall x \neq 0$ with $P = P^T > 0$, then,

$$\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x} \tag{3.20}$$

$$= x^T\left((A_h + B_h K_h)^T P + P(A_h + B_h K_h)\right)x \tag{3.21}$$

$$= \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(x))h_j(z(x))x^T\left((A_i + B_i K_j)^T P + P(A_i + B_i K_j)\right)x < 0. \tag{3.22}$$

Sufficient conditions to guarantee $\dot{V}(x) < 0$ are thus

$$PA_i + A_i^T P + PB_i K_j + K_j^T B_i^T P < 0, \quad \forall i, j \in \{1, 2, \dots, r\}. \tag{3.23}$$

□

Recall that we are concerned with analysis and synthesis of nonlinear systems via TS models. Since the latter is an exact rewriting of the former, the control feedback (3.17) stabilizes the original nonlinear system with TS model (3.7) within the outermost Lyapunov level in the modeling region Ω.

There are two issues in the previous result:

1. *Convex sum relaxations:* The use of the same MFs both in the control law and the system produces, once the direct Lyapunov method is applied, a signed double convex sum; getting LMIs from it can be done in a variety of ways, each of them called a *sum relaxation* and associated with the *co-positivity problem* (Murty and Kabadi 1987). The set of inequalities (3.19) is not the only way to guarantee the double convex sum in $\dot{V}(x)$ to be negative. In the next section, other ways to drop off the MFs involved in double convex sums will be given.

2. *LMI synthesis:* Given a set of control gains K_j, $j \in \{1, 2, \ldots, r\}$, the model (3.18) is (quadratically) stable if the conditions on theorem 3.2.2 are feasible. However, these conditions assume the set of gains is already given; i.e., they are in fact a stability test. In order to provide LMI conditions for controller design (synthesis), i.e., to determine gains K_j, $j \in \{1, 2, \ldots, r\}$, along with the Lyapunov matrix P it is necessary to apply some of the LMI properties in section 2.2.

The first issue is now considered in some detail before proceeding with the second one.

Convex sum relaxation

As seen above, when the direct Lyapunov method is applied to closed-loop TS models, it leads to expressions containing double convex sums, from which MFs should be removed in order to obtain LMIs. There are several ways to perform this task, some more or less conservative, some more or less complex. The scheme employed to perform this task is called *sum relaxation*. Relaxations help reducing the gap that separates sufficient LMI conditions from the convex expressions they guarantee. The fact that there is room for improvement comes from the absence of the MFs in the LMI conditions used for analysis and control design of TS models, let alone their shape (C. Ariño and Sala 2007). Some relaxation lemmas follow:

LEMMA 3.2.1. (Tanaka and Sano 1994) *Let* Υ_{ij}, $i,j \in \{1,\ldots,r\}$ *being a collection of matrices of the same size; then, the double convex-sum*

$$\sum_{i=1}^{r}\sum_{j=1}^{r} h_i\left(z(x)\right) h_j\left(z(x)\right) \Upsilon_{ij} < 0 \tag{3.24}$$

is verified if

$$\begin{aligned} \Upsilon_{ii} < 0, \quad & \forall i \in \{1,2,\ldots,r\}, \\ \Upsilon_{ij} + \Upsilon_{ji} < 0, \quad & \forall (i,j) \in \{1,2,\ldots,r\}^2, \quad i<j. \end{aligned} \tag{3.25}$$

LEMMA 3.2.2. (Tuan et al. 2001) *Let* Υ_{ij}, $i,j \in \{1,2,\ldots,r\}$ *being a collection of matrices of proper size. The inequality* (3.24) *is verified if the following conditions hold:*

$$\begin{aligned} \Upsilon_{ii} < 0, \quad & \forall i \in \{1,2,\ldots,r\}, \\ \tfrac{2}{r-1}\Upsilon_{ii} + \Upsilon_{ij} + \Upsilon_{ji} < 0, \quad & \forall (i,j) \in \{1,2,\ldots,r\}^2, \quad i \neq j. \end{aligned} \tag{3.26}$$

Notice that these conditions are only sufficient; however, there are other relaxations that become necessary through a complexity parameter (Sala and Ariño 2007; Kruszewski et al. 2009). Despite the fact that these approaches "close" the relaxation issue, they quickly become intractable for the actual LMI solvers due to the enormous growth in the number of LMIs which is a function of the system order and the desired closeness to the necessity. The relaxations here presented are considered more convenient since they make a good compromise between numerical complexity and quality of solutions; moreover: they do not add slack matrices (E. Kim and H. Lee 2000).

LMI synthesis

Resuming conditions in theorem 3.2.2, we are now ready to prove that they can actually be expressed as LMIs in order to find the gains K_j, $j \in \{1,2\ldots,r\}$ of the PDC-control law (3.17) instead of "guessing" them and verifying a posteriori if they produced a stabilised system. This is called synthesis as we intend to synthesise a controller via an LMI which will be obtained by applying some of the properties listed in section 2.2.

Consider again the expression (3.21), to which the property of congruence with $X = P^{-1}$ is applied to obtain:

$$A_h X + X A_h^T + B_h K_h X + X K_h^T B_h^T < 0. \tag{3.27}$$

Thus, taking the change of variable $M_h = K_h X$, the following equivalent inequality is obtained.

$$\Upsilon_{hh} = A_h X + X A_h^T + B_h M_h + M_h^T B_h^T < 0. \tag{3.28}$$

Note that having a solution X, M_h guarantees a *unique* pair P and K_h as $K_h = M_h X^{-1}$; therefore, the solution space has not been altered by the transformations above, yet, the result is ready to be cast as an LMI once a relaxation scheme is applied. Indeed, to guarantee the double convex sum in (3.28) to be negative-definite, a sum relaxation allows us establishing the following theorem.

Theorem 3.2.3. *The origin $x = 0$ of the TS model* (3.18) *is asymptotically stable if there exist matrices $X = X^T > 0$ and M_i, $i \in \{1, 2, \dots, r\}$ such that conditions* (3.25) *or* (3.26) *hold with*

$$\Upsilon_{ij} = A_i X + X A_i^T + B_i M_j + M_j^T B_i^T, \tag{3.29}$$

for $i, j \in \{1, 2, \dots, r\}$. In such case, the control gains are given by $K_i = M_i X^{-1}$, the Lyapunov function is $x^T P x$ with $P = X^{-1}$, and every trajectory of the system within the outermost Lyapunov level inside Ω goes asymptotically to zero.

Example 3.2.3. *Consider the TS model* (3.10) *of an inverted pendulum on a cart obtained in the example 3.1.1, reproduced here for convenience:*

$$\dot{x}(t) = \sum_{i=1}^{8} h_i(z(x)) \left(A_i x(t) + B_i u(t) \right), \tag{3.30}$$

where

$$A_1 = A_2 = \begin{bmatrix} 0 & 1 \\ 14.3226 & 0.0811 \end{bmatrix}, \quad A_3 = A_4 = \begin{bmatrix} 0 & 1 \\ 14.3226 & -0.0811 \end{bmatrix},$$
$$A_5 = A_6 = \begin{bmatrix} 0 & 1 \\ 17.3117 & 0.0811 \end{bmatrix}, \quad A_7 = A_8 = \begin{bmatrix} 0 & 1 \\ 17.3117 & -0.0811 \end{bmatrix},$$
$$B_1 = B_3 = B_5 = B_7 = \begin{bmatrix} 0 \\ -0.1147 \end{bmatrix}, \quad B_2 = B_4 = B_6 = B_8 = \begin{bmatrix} 0 \\ -0.1765 \end{bmatrix}.$$

Using theorem 3.2.3 with conditions (3.26) *the following results are obtained using the solver MOSEK and the YALMIP interface:*

$$P = \begin{bmatrix} 0.5139 & 0.1206 \\ 0.1206 & 0.0438 \end{bmatrix}, K_1 = \begin{bmatrix} 407.4190 & 70.6658 \end{bmatrix}, K_2 = \begin{bmatrix} 270.7068 & 47.7490 \end{bmatrix},$$
$$K_3 = \begin{bmatrix} 406.7666 & 69.5068 \end{bmatrix}, K_4 = \begin{bmatrix} 271.4284 & 46.7314 \end{bmatrix}, K_5 = \begin{bmatrix} 423.7512 & 70.1878 \end{bmatrix},$$
$$K_6 = \begin{bmatrix} 293.6729 & 48.1255 \end{bmatrix}, K_7 = \begin{bmatrix} 423.0828 & 69.0443 \end{bmatrix}, K_8 = \begin{bmatrix} 294.2176 & 47.1462 \end{bmatrix}.$$

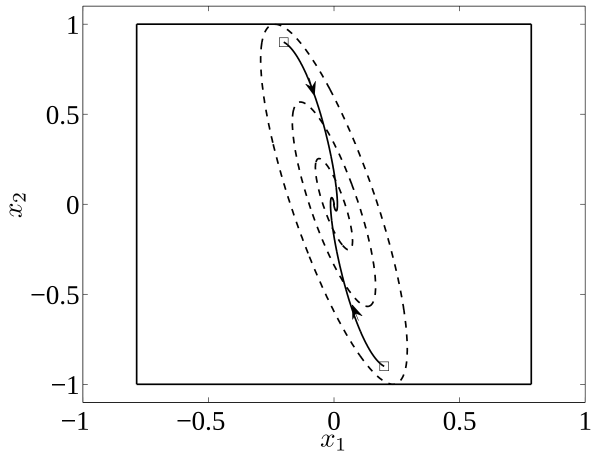

Figure 3.3: Lyapunov levels and model states trajectories (Theorem 3.2.3).

Figure 3.3 shows with a solid borderline rectangle the boundaries of the compact Ω *where the convex sum property holds, and due to the fact that TS model* (3.10) *is an exact representation of the nonlinear model* (3.8)*, this model under the PDC control law* (3.17) *is asymptotically stable inside the outermost Lyapunov curve level* $V(x) = x(t)^T Px(t) = k$*,* $k \in \mathbb{R}$ *within* Ω*. Figure 3.3 shows in dashed lines some Lyapunov curve levels; it also shows two state trajectories from different initial conditions which converge to the origin as it was expected.*

Note that the outermost Lyapunov level within the modeling region Ω gives an estimate of the DA. Nevertheless, determining the maximum k such that $\{V(x) \leq k\} \subseteq \Omega$ is a task that require additional LMIs. Suppose that the region Ω is a symmetric polytope containing the origin $x = 0$:

$$\Omega = \{x \in \mathbb{R}^n : |a_i^T x| \leq 1, \ i \in \{1, 2, \ldots, n_p\}\}$$

LEMMA 3.2.3. *(S. Cao, Rees, and Feng 1999)* $\Theta = \{x \in \mathbb{R}^n | x^T Px \leq 1\}$*,* $P = P^T > 0$ *is an ellipsoid contained in* Ω *which itself contains the maximum volume sphere centered of radius* $\lambda^{\frac{1}{2}}$ *at* $x = 0$ *if the following LMI problem is feasible:*

$$\begin{aligned} &\textit{minimize } \lambda \\ &\textit{subject to } \lambda I \geq P > 0, \\ &\qquad\qquad \begin{bmatrix} P & a_i \\ a_i^T & 1 \end{bmatrix} \geq 0, i \in \{1, 2, \ldots, n_p\}. \end{aligned}$$

Then, no other ellipsoid in Ω *contains a larger centered sphere.*

Notice that adding LMIs to former results does not require any adaptation as they appear as further convex constraints on convex solution sets. This "modularity" of LMI results is one of their most valuable characteristics.

As mentioned in the introduction, the results above are conservative, i.e., (a) the origin of a nonlinear system with a TS model (3.12) might be asymptotically stable while theorem 3.2.1 fails to establish this fact, (b) there exists a PDC control law of the form (3.17) that makes the origin of the closed-loop system (3.18) asymptotically stable while theorem 3.2.3 fails to provide it. Conditions are only sufficient, not necessary, due to several reasons, some of which have already been listed in the introduction: the sum relaxation, the choice of TS model, and the kind of Lyapunov function. The following sections explore some of the answers researchers have provided to tackle the latter source of conservativeness.

3.3 Parameter-dependent Lyapunov functions

In general, there is no systematic method to find a Lyapunov function associated to a stable equilibrium point of a system. Specifically, when a TS model is under consideration, it is apparent that quadratic stability does not involve information of the MFs; therefore, including somehow the MFs can eliminate some drawbacks. Pursing this idea, the TS-LMI framework has been expanded by using parameter-dependent Lyapunov function (PDLF) candidates (Blanco, Perruqueti, and Borne 2001; Tanaka, Hori, and H. Wang 2003), which share the structure of the TS model they are applied to:

$$V(x(t)) = \sum_{i=1}^{r} h_i(z(x))x^T(t)P_i x(t) = x^T(t)P_h x(t) \tag{3.31}$$

where $P_i = P_i^T > 0$ and $h_i(z(x))$ are the same MFs of the associated TS model, for $i \in \{1, 2, \ldots, r\}$. PDLFs are also known as non-quadratic Lyapunov functions (which, of course, is quite an unspecific name), convex Lyapunov functions (which, again, may refer to a larger family of Lyapunov functions involving, for instance, convex sums of polynomials), and fuzzy Lyapunov functions (which is outmoded and misleading, as we focus on nonlinear systems with known model, not a fuzzy one). PDLFs are not quadratic since the MFs $h_i(z(x))$ depend on the states.

PDLFs accomplish the task of including the MFs in their definition. Nonetheless, this inclusion leads to some problems in the continuous-time case. To see this, consider the following continuous-time TS model:

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(z(x))A_i x(t) = A_h x(t), \tag{3.32}$$

which is asymptotically stable if there exists a PDLF (3.31) such that its time derivative is negative in some vicinity of $x = 0$. The time derivative of (3.31) is calculated as

$$\begin{aligned}
\dot{V}(x) &= x^T \left(P_h A_h + A_h^T P_h + \underbrace{\frac{dP_h}{dt}}_{\dot{P}_h} \right) x \\
&= \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(x))h_j(z(x))x^T \left(A_j^T P_i + P_i A_j + \sum_{i=1}^{r} \dot{h}_i(z(x))P_i \right) x
\end{aligned}$$

The term $\dot{P}_h$ involves the time derivatives of the MFs $h_i(z)$, which, by the chain rule have the form:

$$\frac{dh_i}{dt} = \frac{\partial h_i}{\partial z}\dot{z}(t).$$

Although $\partial h_i(z)/\partial z$ can be easily calculated and bounded, $\dot{z}(t)$ is a priori unknown and may depend on the states or exogenous signals (Blanco, Perruqueti, and Borne 2001). A first solution to overcome this problem for stability purposes has been to directly bound the time derivative of the MFs, i.e., $\left|\dot{h}_i\right| < \phi_i$ (Blanco, Perruqueti, and Borne 2001; Tanaka, Hori, and H. Wang 2001; Tanaka, Hori, and H. Wang 2003; L.A. Mozelli et al. 2009), something customary in the LPV field (F. Wu and Dong 2006), but of little realism when a nonlinear system with the full state available is concerned. The main drawback of doing this is the need of verifying a posteriori that the system trajectories do not escape from the specified boundaries.

Theorem 3.3.1. *(Tanaka, Hori, and H. Wang 2003) Assume that* $\left|\dot{h}_\rho(z(x))\right| < \phi_\rho$*, where* $\phi_\rho \geq 0$*. The TS model* (3.32) *is stable if there exist* $\phi_1, \phi_2, \ldots, \phi_\rho$ *such that*

$$\begin{aligned}
&P_i > 0, \quad i \in \{1, 2, \ldots, r\} \\
&\sum_{\rho=1}^{r} \phi_\rho P_\rho + \frac{1}{2}\left(P_i A_j + P_j A_i + (*)\right), \quad i \leq j
\end{aligned}$$

The corresponding PDLF is thus given by (3.31).

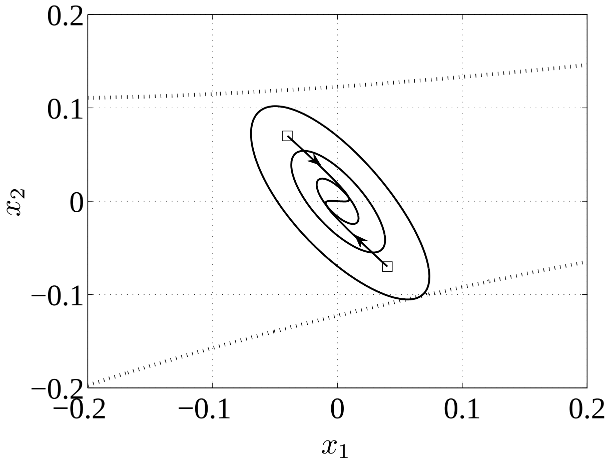

Figure 3.4: Outermost Lyapunov level and system trajectories (Example 3.3.1)

Example 3.3.1. *Consider a continuous-time TS system with two local models*

$$A_1 = \begin{bmatrix} -0.23 & 0.30 \\ -2 & -0.75 \end{bmatrix} \qquad A_2 = \begin{bmatrix} -2 & 10.4 \\ 1.3 & -14.3 \end{bmatrix}$$

Quadratic stability of this system cannot be proven, as the LMIs from theorem 3.2.1 are infeasible. However, we can prove asymptotic stability by using theorem 3.3.1 with $\phi_1 = \phi_2 = 0.3279$*, which produces the following feasible solution:*

$$P_1 = \begin{bmatrix} 57.8285 & 6.1677 \\ 6.1677 & 18.7128 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 98.4874 & 73.4775 \\ 73.4775 & 55.5888 \end{bmatrix}.$$

As mentioned before, directly bounding the time-derivative of the MFs may have a very negative effect: in figure 3.4, bounds $\dot{h}_1 = \phi_1$ *and* $\dot{h}_2 = \phi_2$ *are shown with dotted lines. Note that the local stability result is only valid for the outermost Lyapunov level shown in this figure* inside $\dot{h}_i \leq \phi_i$, $i = 1, 2$. *This outermost Lyapunov level is the solid ellipsoid in the same figure. System trajectories from two different initial conditions are also included for illustration purposes.*

Another approach based on further analysis of the properties of the time-derivative of the MFs $\dot{h}_i$ appeared in (T.M. Guerra and Bernal 2009); it is based on the following fact (T.M. Guerra and Bernal 2009; Bernal and T. M. Guerra 2010):

Theorem 3.3.2. *The TS model* (3.32) *is asymptotically stable if there exist* $P_i = P_i^T > 0$, $i \in \{1, 2, \ldots, r\}$, *such that* $P_h A_h + A_h^T P_h < 0$.

What does this result mean? It states a sufficient condition for *local* stability of a TS model by means of a PDLF; it does not guarantee global stability nor stability in the outermost Lyapunov level inside the modeling area Ω. It simply states that there exists a vicinity of the origin for which all trajectories converge to the origin, making it locally asymptotically stable. It is a result of existence which does not provide any constructive method to know the size of such vicinity, let alone the biggest one. Nevertheless, this result is the departure point for several algorithms that employ PDLFs to get successively better estimates of the region of attraction for TS models whose origin is locally asymptotically stable (T.M. Guerra and Bernal 2009; Bernal and T. M. Guerra 2010; T.M. Guerra, Bernal, et al. 2012).

Theorem 3.3.3. *(T.M. Guerra and Bernal 2009) If there exist matrices $P_i = P_i^T > 0$, $i \in \{1, 2, \ldots, r\}$, such that LMIs*

$$\frac{2}{r-1}\Upsilon^m_{\alpha\alpha} + \Upsilon^m_{\alpha\beta} + \Upsilon^m_{\beta\alpha} < 0, \tag{3.33}$$

hold for $(\alpha, \beta) \in \{1, \ldots, r\}^2$, $m \in \{1, 2, \ldots, 2^{p\times n}\}$ with

$$\Upsilon^m_{\alpha\beta} = P_\alpha A_\beta + A_\beta^T P_\alpha + \sum_{k=1}^{p}\sum_{u=1}^{n}(-1)^{d^m_{ku}}\lambda_{ku}\left(LA_\beta\right)_{ku}\left(P_{g_1(\alpha,k)} - P_{g_2(\alpha,k)}\right),$$

$z = Lx$, d^m_{ku} defined from the binary representation of $m - 1 = d^m_{pn} + d^m_{p(n-1)} \times 2 + \cdots + d^m_{11} \times 2^{p\times(n-1)}$, and $g_1(\alpha, k)$, $g_2(\alpha, k)$ defined as:

$$g_1(\alpha, k) = \left\lfloor (\alpha - 1)/2^{p+1-k} \right\rfloor \times 2^{p+1-k} + 1 + (\alpha - 1) \mod 2^{p-k},$$
$$g_2(a, k) = g_1(\alpha, k) + 2^{p-k},$$

then $x(t)$ tends to zero exponentially for any trajectory satisfying (3.32) in the outermost Lyapunov level contained in $R = \bigcap_{k,u}\left\{x : \left|\frac{\partial w_0^k}{\partial z_k}x_u\right| \leq \lambda_{ku}\right\}$.

A more recent alternative to deal with the time-derivatives of the MFs when PDLFs are used, consists in mixing the previous methodologies by bounding the terms $\dot{w}_0^k$ that appear after developing $\dot{P}_h$, i.e., the expression

$$\dot{P}_h = \sum_{k=1}^{p}\dot{w}_0^k\left(P_{g_1(z,k)} - P_{g_2(z,k)}\right)$$

is replaced by

$$\dot{P}_h = \sum_{k=1}^{p}(-1)^{d^m_k}\beta_k\left(P_{g_1(z,k)} - P_{g_2(z,k)}\right)$$

in the inequality $P_hA_h + A_h^TP_h + \dot{P}_h < 0$ in order to guarantee it, where $g_1(\cdot,\cdot)$ and $g_2(\cdot,\cdot)$ are defined as above, $m-1 = d_p^m + d_{p-1}^m \times 2 + \cdots + d_1^m \times 2^{p-1}$, $m \in \{1,2,\ldots,2^p\}$, and $|\dot{w}_0^k| \leq \beta_k$, $k \in \{1,2,\ldots,p\}$ is guaranteed by some extra-LMIs, i.e., these bounds are *not* assumed a priori as in the first approach presented.

PDLFs have succeeded noticeably in the discrete-time framework as they are not faced with the problem of the time derivative of the MFs (T.M. Guerra and Vermeiren 2004; T.M Guerra, Kruszewski, and Lauber 2009). In the continuous framework, with the limitations described above, they have produced controllers (T.M. Guerra, Bernal, et al. 2012; Pan et al. 2012; Nachidi, Tadeo, and Benzaouia 2012), observers (Aguiar, Márquez, and Bernal 2016), descriptors, etc. In this thesis, the contributions on this area are focused on stability and stabilisation, so the previous references are only given out of completeness.

3.4 Line-integral Lyapunov functions

Although originally considered as a parameter-dependent Lyapunov function (fuzzy in the context of its original appearance), the proposal in (Rhee and Won 2006) constituted a breakthrough as it employed the MFs without dealing with their time derivatives. The proposed line-integral Lyapunov function is:

$$V(x) = \int_{\Gamma(0,x)} f(\psi)d\psi, \tag{3.34}$$

with $\Gamma(0,x)$ being a path from the origin 0 to the current state x, ψ as a dummy vector for the integral, and $f(x) = \sum_{i=1}^r h_i(z(x))P_ix = P_hx$. Calculating the time derivative of (3.34) as:

$$\dot{V}(x) = x^T\left(P_hA_h + A_h^TP_h\right)x, \tag{3.35}$$

we can see that the time derivative of the MFs does not appear, unlike the previous approaches using a PDLF; this idea will be resumed later for one of the main contributions of this thesis.

Nevertheless, to be a Lyapunov function candidate, $V(x)$, has to satisfy necessary path independent conditions. These path-independent conditions are presented in the following lemma.

LEMMA 3.4.1. *Let $f(x) = [f_1(x), f_2(x), \ldots, f_n(x)]^T$. A necessary and sufficient conditions for $V(x)$ to be path-independent function is*

$$\frac{\partial f_i(x)}{\partial x_j} = \frac{\partial f_j(x)}{\partial x_i}, \tag{3.36}$$

for $i,j, \in \{1,2,\ldots,n\}$.

Due to the these necessary conditions in (3.34), the approach in (Rhee and Won 2006) was only applicable to a specify class of TS models where the number of MFs were at most n and each of them depends exactly on one state variable.

In the previous results some sum relaxation schemes have been chosen, but the reader should keep in mind that *any* sum relaxation scheme can be used instead of those hereby proposed, for instance lemma 3.2.1 (Tanaka and Sano 1994), lemma 3.2.2 (Tuan et al. 2001), or the asymptotically sufficient and necessary conditions in (Sala and Ariño 2007; Kruszewski et al. 2009).

3.5 Piecewise Lyapunov function

Yet another way of involving the MFs into the LMI conditions for stability and stabilisation of nonlinear systems via TS models, consists in dividing the state space in regions within which the MFs induce a different –perhaps simplified– convex model of the system. If the Lyapunov function candidate is allowed to change according to this partition, it may increase the chances of becoming an actual Lyapunov function not only because it will provide more flexibility (different Lyapunov matrices per partition (Johansson, Rantzer, and Arzen 1999)), but also because there are several ways of including the geometric information of the partition (Yakubovich 1977). To illustrate these points, consider the following example adapted from (Johansson and Rantzer 1998):

Example 3.5.1. *Consider the system:*

$$\dot{x}(t) = \begin{cases} A_1 x(t), & \text{if } x_1 < 0 \\ A_2 x(t), & \text{if } x_1 \geq 0 \end{cases} \tag{3.37}$$

with $A_1 = \begin{bmatrix} -10 & -8 \\ -2 & -4 \end{bmatrix}$, $A_2 = \begin{bmatrix} -1 & -2 \\ 10 & -1 \end{bmatrix}$.

Quadratic stability reduces to finding $P = P^T > 0$ *such that* $A_1^T P + PA_1 < 0$ *and* $A_2^T P + PA_2 < 0$. *Nevertheless, if*

$$R_1 = \begin{bmatrix} 0.85 & -0.7 \\ -0.7 & 0.67 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 2.62 & 1.01 \\ 1.01 & 0.57 \end{bmatrix},$$

then

$$\sum_{i=1}^{2} \left(A_i^T R_i + R_i A_i \right) = \begin{bmatrix} 0.74 & 0.04 \\ 0.04 & 0.53 \end{bmatrix} > 0,$$

which, according to property 3.2.1, proves that no such P exists. Hence, quadratic stability fails to demonstrate the stability of (3.37).

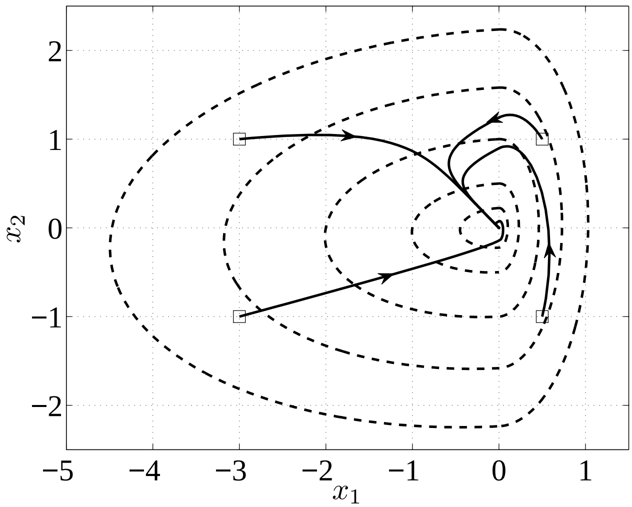

Figure 3.5: Trajectories in the phase plane of the model states (3.37).

Now consider the piecewise Lyapunov function:

$$V(x) = \begin{cases} x^T P x, & \text{if } x_1 < 0 \\ x^T \left(P + \eta C^T C\right) x, & \text{if } x_1 \geq 0 \end{cases} \tag{3.38}$$

with $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$, *the LMI problem for stability is find* $P = P^T > 0$ *and* η *such that:*

$$P + \eta C^T C > 0, \; PA_1 + A_1^T P < 0,$$
$$(P + \eta C^T C)A_2 + A_2(P + \eta C^T C) < 0,$$

with $P = \begin{bmatrix} 0.05 & -0.01 \\ -0.01 & 0.20 \end{bmatrix}$ *and* $\eta = 0.89$ *a solution to the previous inequalities is obtained, since*

$$PA_1 + A_1^T = \begin{bmatrix} -0.95 & -0.68 \\ -0.68 & -1.5 \end{bmatrix} < 0,$$
$$(P + \eta C^T C)A_2 + A_2^T(P + \eta C^T C) = \begin{bmatrix} -2.04 & 0.16 \\ 0.16 & -0.37 \end{bmatrix} < 0.$$

Thus, the origin of (3.37) *is an asymptotically stable equilibrium point. The level surfaces of the computed Lyapunov function are indicated in figure 3.5 along with some system trajectories.*

The previous example shows that although quadratic stability cannot be proven, introducing some "knowledge" into the Lyapunov function can eliminate some drawbacks. An approach for systematically using the MFs to induce a state space partition according to the scheduling variables for continuous-time Takagi-Sugeno models first appeared in (Johansson and Rantzer 1998). In a subsequent work, the same authors provided piecewise analysis of Takagi-Sugeno models with polyhedral partitions; piecewise Lyapunov functions (PWLFs) were naturally incorporated (Johansson, Rantzer, and Arzen 1999). Discrete-time counterparts can be found in (Feng 2003; Feng et al. 2005).

This subsection is based on (Johansson, Rantzer, and Arzen 1999), where MF-induced polyhedral partitions of the state space where used to define a PWLF for affine TS models, i.e., models of the form

$$\dot{x}(t) = \sum_{l=1}^{r} h_l(z(x))\left(A_l x(t) + a_l\right), \tag{3.39}$$

where $A_l \in \mathbb{R}^{n\times n}$ and $a_l \in \mathbb{R}^{n\times 1}$ are the local matrices and affine terms, respectively, and MFs h_l, $l \in \{1, 2, \ldots, r\}$ depend on the premise vector $z(\cdot)$, which in turn is assumed to depend linearly on the system state $x(t)$, i.e., $z(x) = Cx$, with $C \in \mathbb{R}^{p\times n}$.

Regions where $h_l(x) = 1$ for some l will be called *operating regimes* since only the l-th subsystem $\dot{x}(t) = A_l x(t) + a_l$ is active on them. Otherwise, in between operating regimes, regions will be called *interpolation regimes*. Both these regions have also a geometrical interpretation, provided that the premise vector $z(x(t))$ depends linearly on the states $x(t)$: they form a polyhedral collection $\{X_i\}_{i\in I} \subseteq \mathbb{R}^n$, where I is the set of cell indices. It is important to underline that this specifications restrict the class of TS models to which this approach can be applied to those whose linear consequents are *not simultaneously activated*; this impedes applying this methodology to TS models which are obtained via sector nonlinearity.

For each cell X_i a set $K(i)$ will be defined as the set of indices of the system matrices used in the interpolation within that cell. Naturally, for operating regimes, $K(i)$ contains only a single element. Since this approach investigates exponential stability of the origin, I will be divided in two sets: I_0 which will contain the indices of cells that contain the origin and I_1 which will have the indices of cells that do not contain the origin.

Defining

$$\overline{A}_k = \begin{bmatrix} A_k & a_k \\ 0 & 0 \end{bmatrix}, \quad \overline{x}(t) = \begin{bmatrix} x(t) \\ 1 \end{bmatrix} \tag{3.40}$$

where it is assumed that $a_k = 0$ for all $k \in K(i)$ with $i \in I_0$, the system (3.39) can be rewritten as

$$\dot{\overline{x}} = \sum_{k \in K(i)} h_k(x)\overline{A}_k\overline{x}, \quad x \in X_i.$$

Traditionally, this approach partitions the state space according to the activation of the linear models, allowing the Lyapunov function to change from one region to another, for instance

$$V(x) = \begin{cases} x^T(t)P_i x(t), & x \in X_i, \quad i \in I_0 \\ \begin{bmatrix} x \\ 1 \end{bmatrix}^T \overline{P}_i \begin{bmatrix} x \\ 1 \end{bmatrix}, & x \in X_i, \quad i \in I_1 \end{cases} \tag{3.41}$$

The above partition is natural for those TS models that do not have all their linear models activated at once. Nevertheless, we insist that this assumption does not hold for TS models built by using the sector nonlinearity approach, which are our focus in this thesis.

In order to guarantee continuity of the PWLF across the borders between regimes, this function is parameterised by matrices $\overline{F}_i = [F_i \ f_i]$, $i \in I$, with $F_i \in \mathbb{R}^{n\times n}$, $f_i \in \mathbb{R}^{n\times 1}$, $f_i = 0$ for $i \in I_0$, such that:

$$\overline{F}_i \begin{bmatrix} x \\ 1 \end{bmatrix} = \overline{F}_j \begin{bmatrix} x \\ 1 \end{bmatrix}, \quad x \in \{X_i \cap X_j\}, \ i,j \in I. \tag{3.42}$$

A systematic procedure for constructing these matrices is given in (Johansson, Rantzer, and Arzen 1999). Then, P_i and $\overline{P}_i$ in (3.41) are parameterised as follows:

$$\begin{aligned} P_i &= F_i^T T F_i, \quad i \in I_0 \\ \overline{P}_i &= \overline{F}_i^T T \overline{F}_i, \quad i \in I_1. \end{aligned} \tag{3.43}$$

with T being a symmetric matrix of appropriate dimensions which collects the free parameters of the Lyapunov function. Note that this arrangement is a compromise between continuity and LMI formulation of the results, which is hereby possible.

Since matrices P_i or $\overline{P}_i$ are only used to describe the Lyapunov function in cell X_i, it is possible to use the S-procedure in property 2.2.4 to reduce conservativeness by constructing matrices $\overline{E}_i = [E_i \ e_i]$ with $E_i \in \mathbb{R}^{n\times n}$, $e_i \in \mathbb{R}^{n\times 1}$, $e_i = 0$ for $i \in I_0$, that satisfy

$$\overline{E}_i \begin{bmatrix} x \\ 1 \end{bmatrix} \succeq 0, \quad x \in X_i, \ i \in I \tag{3.44}$$

where for every matrix W_i with nonnegative entries $W_i \succeq 0$, condition (3.44) implies that

$$\begin{bmatrix} x \\ 1 \end{bmatrix}^T \overline{E}_i^T W_i \overline{E}_i \begin{bmatrix} x \\ 1 \end{bmatrix} > 0, \quad \forall x \in X_i, \ i \in I.$$

As with matrices $\overline{F}_i$, these $\overline{E}_i$ can also be systematically constructed (Johansson, Rantzer, and Arzen 1999). Moreover, there is a toolbox for MATLAB that automatically produces the set of matrices $\overline{F}_i$ and $\overline{E}_i$ for a given partition (Hedlund and Johansson 1999).

Then, we have the following result on piecewise quadratic stability of continuous-time affine TS models:

Theorem 3.5.1. *(Johansson, Rantzer, and Arzen 1999) If there exist symmetric matrices T, $U_i \succeq 0$ and $W_{ik} \succeq 0$ such that*

$$\begin{aligned} P_i &= F_i^T T F_i, \quad i \in I_0 \\ \overline{P}_i &= \overline{F}_i^T T \overline{F}_i, \quad i \in I_1 \end{aligned} \tag{3.45}$$

satisfy

$$\begin{aligned} P_i - E_i^T U_i E_i > 0 \\ A_k^T P_i + P_i A_k + E_i^T W_{ik} E_i < 0 \end{aligned} \tag{3.46}$$

for $i \in I_0,\ k \in K(i)$, and

$$\begin{aligned} \overline{P}_i - \overline{E}_i^T U_i \overline{E}_i > 0 \\ \overline{A}_k^T \overline{P}_i + \overline{P}_i \overline{A}_k + \overline{E}_i^T W_{ik} \overline{E}_i < 0 \end{aligned} \tag{3.47}$$

for $i \in I_1,\ k \in K(i)$, then $x(t)$ tends to zero exponentially for every continuous piecewise $\mathcal{C}^1$ trajectory in $\cup_{i \in I} X_i$ satisfying (3.39).

We do not intend to reproduce the details given in (Johansson, Rantzer, and Arzen 1999), but it is important to notice that LMIs (3.46) and (3.47) guarantee the PWLF candidate (3.41) to be positive everywhere and its time derivative negative in every state-space partition. Auxiliary matrices $\overline{F}_i$ and $\overline{E}_i$ guarantee continuity of the PWLF as well as the inclusion of specific partition information via an adaptation of the S-procedure, respectively. Once the system trajectory enters the regions containing the origin, the following theorem guarantee exponential convergence to the origin.

Theorem 3.5.2. Piecewise exponential stability: *Let $V(t)$ be decreasing and piecewise $\mathcal{C}^1$. If there exist positive scalars α, β, and $\gamma > 0$ such that*

$$\alpha \|x(t)\|^2 \leq V(t) \leq \beta \|x(t)\|^2 \tag{3.48}$$

$$\frac{d}{dt} V(t) \leq -\gamma \|x(t)\|^2 \ \textbf{a.e.} \tag{3.49}$$

then $\|x(t)\|^2 \leq \beta \alpha^{-1} e^{-\gamma t/\beta} \|x(0)\|^2$.

The above partition is natural for those TS models that do not have all their linear models activated at once. Unfortunately, this assumption does not hold for TS models built by using the sector nonlinearity approach. In (Gonzalez and Bernal 2016) a first step towards this adaptation has been made; we revisit it in chapter 5 where it is generalised to piecewise affine TS models. Stabilisation based on PWLFs remains unfortunately a BMI problem as shown in (Feng et al. 2005); other tasks such as observer design (Qiu, Feng, and Gao 2012) or output feedback (Qiu, Feng, and Gao 2013) also remain open as they face the philosophical problem of depending on where an estimated state lies.

Comments on other developments

Although out of the scope of this thesis, it is important to know that, based on quadratic Lyapunov functions and TS models, a whole framework has been developed that goes beyond stability analysis and controller design of nonlinear systems: it covers observers (Tanaka, Ikeda, and H. Wang 1998), output feedback (Yoneyama et al. 2000; Nachidi, Benzaouia, et al. 2008), descriptors (Taniguchi, Tanaka, Yamafuji, et al. 1999), delay systems (Y. Cao and Frank 2000), etc. Perhaps because the original results happened to be LMIs, researchers in this area have always been concerned about expressing their results in this numerically efficient form, a fact that became a distinctive feature of the field.

One additional advantage on the use of TS-LMI framework for the analysis of nonlinear systems is that via LMI we can impose some performance criterios for the dynamical system. For example, decay-rate (Taniguchi, Tanaka, and H. Wang 2001), $\mathcal{H}_2$ guarantee cost (H. Wu and Cai 2006), $\mathcal{H}_\infty$ robust stabilization (K. Lee, Jeung, and H. Park 2001), etc. Additionally, apart from stability or $\mathcal{H}_\infty$ bounds, there are other problems of interest, such as bounding the deviations from the origin under disturbances (Salcedo, Martínez, and García-Nieto 2008; Pitarch, Sala, and C.V. Ariño 2015). Moreover, the use of TS-LMI can be applied to solve two or more different control objectives at the same time, i.e., a set of the LMI can solve two or more problems, for example: output feedback robust for time-delay TS systems (K. Lee, J. Kim, and Jeung 2001), robust control for TS systems with time-delay (Zaidi et al. 2016), etc.

Chapter 4

Convex-polynomial models

This chapter surveys results on stability analysis and stabilisation of nonlinear systems by using polynomial models and sum-of-squares (SOS) techniques. It begins with by introducing the SOS conditions and their relations with the LMI framework; then, the fuzzy polynomial modelling methodology is presented. Similarly to the TS-LMI approach, it is shown that polynomial models and SOS tools can be combined for analysis and design of nonlinear control systems. Moreover, as an alternative to the fuzzy polynomials models, the dynamical extension approach is briefly introduced at the end of the chapter.

4.1 Introduction

The sum-of-squares (SOS) technique was introduced by Parrilo in his thesis (Parrilo 2000), allowing an algorithmic analysis of polynomial nonlinear systems (i.e., systems consisting on polynomials of the states on its righthand side) using Lyapunov methods (Papachristodoulou and Prajna 2002). As its name suggests, sum of squares is another way to imply that an expression is positive, though a positive expression may not be SOS (Chesi 2007). The SOS approach is a generalisation of the LMI framework as, in fact, proving that a polynomial has a SOS decomposition (i.e., that it can be written as a sum of squares) is a convex problem; in other words, polynomial positivity can be tested via LMIs. As products of polynomials are also polynomials, Lyapunov stability analysis of a polynomial nonlinear system can be performed via a polynomial Lyapunov function candidate (i.e., a positive polynomial of the states amenable to SOS) and SOS techniques.

Convex polynomial models were first introduced as a direct generalisation of classical TS ones, i.e., no attention was paid to the origin of the MFs as they were assumed to hold the convex sum property everywhere; they were referred to as *fuzzy polynomial models*. Since they came from the fuzzy framework, the consequents in each fuzzy rule, i.e., the righthand side of the model, was a matrix of polynomials multiplied by the state (Tanaka, Yoshida, et al. 2007a; Tanaka, Ohtake, and H. Wang 2007) or a polynomial vector field (Sala 2007). Clearly, via the SOS approach, these fuzzy polynomial models could be systematically analysed in a similar way as with the TS-LMI framework.

If a TS model is to be obtained as an exact representation of a given nonlinear system, the sector nonlinearity approach comes at hand; similarly, the generalisation in (Sala and C. Ariño 2009), based on the Taylor series, allows rewriting a given system as a convex sum of polynomials of arbitrary order within a compact set of the state space: they are referred in this work as convex polynomial models. These models can reduce the conservatism –both in analysis and synthesis– caused by convex model construction with respect to TS approaches. The SOS-fuzzy polynomial approach has given some successfully results for the stability and stabilisation of nonlinear models (Sala and C. Ariño 2009; Tanaka, Yoshida, et al. 2007b; Tanaka, Ohtake, and H. Wang 2009; Tanaka, Yoshida, et al. 2009); all of them as an extension of the seminal methodologies in (Prajna, Papachristodoulou, and F. Wu 2004).

The SOS-convex polynomial framework have also achieved the so-called asymptotical exactness for smooth nonlinear systems: if there exists a smooth Lyapunov function (so that its Taylor series converges to it), there will exist a polynomial Lyapunov function and a fuzzy polynomial model with a finite degree, which will allow proving stability of the original system with some extra assumptions (Sala and C. Ariño 2009).

Although the SOS-convex polynomial framework is a powerful tool for the analysis and control of nonlinear systems, it has some limitations, not only from the fact that there are positive polynomials that are not SOS, but also from the computational point of view which rapidly exhausts the available resources; moreover, control synthesis requires an affine-in-control structure as well as some additional artificial variables, which introduce some conservativeness (Sala 2009).

4.2 Sum-of-square decomposition

The SOS framework is based on the idea that any $2d$-degree polynomial $p(x) : \mathbb{R}^n \to \mathbb{R}$, $d \in \mathbb{N}$, can be written in the form $Z^T(x)QZ(x)$ with $Z(x)$ being a vector of all the monomials up to degree d and Q a non unique matrix built from the polynomial coefficients. If $Q \geq 0$ then the Cholesky factor of it ($Q = L^T L$), allows expressing $p(x) = Z^T(x)L^T LZ(x)$, i.e., a sum of squares. Note that the problem of finding a $Q \geq 0$ can be cast as a LMI problem, so finding SOS decompositions of a polynomial is a convex problem and LMI Lyapunov results (degree 2 polynomials) can be easily extended to higher degree polynomials in both Lyapunov functions and nonlinear models. The basic ideas of the SOS approach are summarized in this section.

Definition 4.2.1. *The set of **SOS** polynomials in the variables x is the set defined by*

$$\Sigma_x := \left\{ p(x) \in \mathbb{R}^n \to \mathbb{R} \,\middle|\, p(x) = \sum_{i=1}^{M} f_i^2(x) \right\}, \tag{4.1}$$

with $M \in \mathbb{Z}^+$.

An equivalent characterisation of SOS polynomials is given in the following proposition.

Proposition 4.2.1. *(Parrilo 2000) A polynomial $p(x)$ of degree $2d$ is a SOS if and only if there exists a positive semidefinite matrix Q and a vector of monomials $Z(x)$ containing monomials in x of degree less or equal to d such that*

$$p(x) = Z^T(x)QZ(x). \tag{4.2}$$

In general, the monomials in $Z(x)$ are not algebraically independent. Expanding $Z(x)^T QZ(x)$ and equating the coefficients of the resulting monomials to the ones in $p(x)$, we obtain a set of affine constraints in the elements of Q. Since $p(x)$ being SOS is equivalent to $Q \geq 0$, the problem of finding a Q which proves that $p(x)$ is SOS can be cast as an LMI problem. For the sake of clarity, consider the following example:

Example 4.2.1. *Suppose that we want to know if the following polynomial is SOS:*

$$p(x_1, x_2) = x_1^4 + 2x_2^4 - 2.5x_1^2x_2^2$$

For this purpose, define $Z(x) = \begin{bmatrix} x_2^2 & x_1^2 & x_1x_2 \end{bmatrix}^T$ and consider the following form:

$$p(x_1, x_2) = x_1^4 + 2x_2^4 - 2.5x_1^2x_2^2$$

$$= Z^T(x) \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & q_{33} \end{bmatrix} Z(x)$$
$$= q_{11}x_2^4 + q_{22}x_1^4 + 2q_{23}x_1^3x_2 + 2q_{12}x_1x_2^3 + (2q_{12} + q_{33})\, x_1^2x_2^2,$$

from which we obtained the constraints:

$$q_{11} = 2,\ 2q_{12} + q_{33} = -2.5,\ q_{13} = 0,\ q_{22} = 1,\ q_{23} = 0.$$

Then, $p(x)$ is SOS if and only if there exist $Q \geq 0$ satisfying the last equations. The following matrix Q satisfy the above equations and it is positive definite:

$$Q = \begin{bmatrix} 2 & -1.4 & 0 \\ -1.4 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix},$$

which Cholesky decomposition $Q = L^TL$ with

$$L = \begin{bmatrix} \sqrt{2} & -\frac{1.4}{\sqrt{2}} & 0 \\ 0 & \sqrt{0.02} & 0 \\ 0 & 0 & \sqrt{0.3} \end{bmatrix},$$

yields the following SOS decomposition:

$$p(x) = \left(2x_2^2 - \frac{1.4}{\sqrt{2}}x_1^2\right)^2 + \left(\sqrt{0.02}x_1^2\right)^2 + \left(\sqrt{0.3}x_1x_2\right).$$

The following lines of code present an implementation in MATLAB/YALMIP to check if a polynomial is SOS.

```
% Define problem variables and the polynomial p(x):
sdpvar x1 x2
p=x1^4+2*x2^4-2.5*x1^2*x2^2;

% Call the solver to check if p(x) is SOS:
[sol,z,Q]=solvesos(sos(p));
```

There are instances where $p(x)$ being SOS is equivalent to $p(x) \geq 0$: (i) when $n = 2$; (ii) when $d = 2$; (iii) when $n = 3$ and $d = 4$. Checking if a polynomial $p(x)$ is nonnegative is an NP-hard problem when the degree of $p(x)$ is at least 4 (Murty and Kabadi 1987). On the other hand, checking whether a polynomial $p(x)$ is SOS is computationally tractable; indeed, it is a LMI problem, which has worst-case polynomial time complexity.

For the case of Lyapunov's stability, one can only be interested in proving local positivity of a polynomial Lyapunov function. Since, SOS polynomials are globally nonnegative ($p(x) \in \Sigma_x \Rightarrow p(x) \geq 0, \forall x$), the *Positivstellensatz* theorem comes in handy, which is recalled in the next subsection.

4.2.1 Positivstellensatz

As originally explained in (Parrilo 2000), the *Positivstellensatz* argumentation extends the use of Lagrange multipliers and S-procedure in the LMI framework to the polynomial-SOS case, thus allowing local information to be included as constraints in SOS conditions.

Consider a region Ω defined by known polynomials restrictions as follows:

$$\Omega = \{x \in \mathbb{R}^n : g_1(x) \geq 0, g_2(x) \geq 0, \ldots, g_q(x) \geq 0, h_1(x) = 0, h_2(x) = 0, \ldots, h_r(x) = 0\}. \tag{4.3}$$

Then, a sufficient condition for a polynomial $p(x)$ being positive in Ω is stated in the following theorem.

Theorem 4.2.1. *If SOS polynomials $s_i(x) \in \Sigma_x$ and arbitrary ones $t_j(x)$ can be found fulfilling:*

$$p(x) - \varepsilon(x) - \sum_{i=1}^{q} s_i(x) g_i(x) + \sum_{j=1}^{r} t_j(x) h_j(x) \in \Sigma_x, \tag{4.4}$$

then $p(x)$ is locally greater than or equal to $\varepsilon(x)$ in the region Ω.

Proof. For all $x \in \Omega$, the term $\sum_{i=1}^{q} s_i(x) g_i(x) \geq 0$ (it is nonnegative) and $\sum_{j=1}^{r} t_j(x) h_j(x) = 0$ (it is zero), so $p(x) - \varepsilon(x) \geq \sum_{i=1}^{q} s_i(x) g_i(x) \geq 0$ for all $x \in \Omega$. □

The polynomials $s_i(x)$ and $t_j(x)$ are denoted as *Positivstellensatz multipliers*, analogous to Lagrange and Karush-Kuhn-Tucker (KKT) multipliers in constrained optimisation (Bertsekas 1999).

Theorem 4.2.1 is a simplified version of the original Positivstellensatz result, in which less conservative expression can be stated by setting higher degree multipliers ($s_i(x), t_j(x)$), products of $p(x)$ with new multipliers or by adding more terms involving products of the $p(x)$, $g_i(x)$, and $h_j(x)$ belonging to the respective cone and ideal. However, more complex statements are avoided in practice because some of them lead to nonconvex problems and also the computational complexity increase considerably. For more details, refer to (Jarvis-Wloszek et al. 2005; Stengle 1974).

4.2.2 SOS matrices

Via SOS techniques, we can also solve state dependent LMIs which will appear in Subsection 4.5 for nonlinear control synthesis.

A state-dependent LMI is an infinite dimensional convex optimisation problem of the form

$$\text{minimize } \sum_{i=1}^{m} a_i c_i \tag{4.5}$$

$$\text{subject to } F_0(x) + \sum_{i=1}^{m} c_i F_i(x) \geq 0, \tag{4.6}$$

where a_i are some fixed real coefficients, c_i are the decision variables, and $F_i(x)$ are some symmetric matrix functions of the indeterminate $x \in \mathbb{R}^n$. The matrix inequality (4.6) basically means that the left hand side of the inequality is positive semidefinite for all $x \in \mathbb{R}^n$. Solving the above optimisation problem amounts to solving an infinite set of LMIs and hence is computationally hard. However, when all $F_i(x)$ are symmetric *polynomial* matrices in x, the sum of squares decomposition can provide a computational relaxation for the conditions (4.6). This relaxation is stated in the following proposition.

Proposition 4.2.2. *(Prajna, Papachristodoulou, and F. Wu 2004) Let $F(x) \in \mathbb{R}^{N \times N}$ be a symmetric polynomial matrix of degree $2d$ in $x \in \mathbb{R}^n$. Furthermore, let $Z(x)$ be a column vector whose entries are all monomials in x with degree no greater than d, and consider the following conditions.*

1. *$F(x) \geq 0$ for all $x \in \mathbb{R}^n$.*
2. *$v^T F(x) v$ is a sum of squares, where $v \in \mathbb{R}^N$.*
3. *There exists a positive semidefinite matrix Q such that*

$$v^T F(x) v = (v \otimes Z(x))^T Q (v \otimes Z(x)),$$

where $\otimes$ denotes the Kronecker product.

Then (1) $\Leftarrow$ (2) *and* (2) $\Leftrightarrow$ (3).

The proof of this proposition is based on the Cholesky decomposition (Higham 1990) and the eigenvalue decomposition (Parrilo 2000). In this way, the classical LMI-framework (positive-definiteness of matrices with linear expressions as elements (Boyd et al. 1994)) is extended to the polynomial case.

It can be noticed that the above proposition increase the complexity due to the introduction of the auxiliary variables v. However, there exist another equivalent ways to deal with polynomials SOS matrices with less computational cost, for instance:

Proposition 4.2.3. *(C.W. Scherer and Hol 2006) Let $F(x) \in \mathbb{R}^{N\times N}$ a symmetric polynomial matrix of degree $2d$ in $x \in \mathbb{R}^n$. $F(x)$ is a SOS polynomial matrix if and only if there exist a constant matrix $Q \geq 0$ satisfying*

$$F(x) = (I \otimes z(x))^T Q (I \otimes z(x)), \forall x \in \mathbb{R}^n, \tag{4.7}$$

with $z(x)$ being a column vector whose entries are all monomials in x with degree no greater than d.

4.3 Convex-polynomial modelling

The Taylor-based modelling techniques is a generalisation of the well-know TS sector nonlinearity methodology, but on this new case each non-polynomial expression is rewritten as a convex sum of polynomials (TS models are convex sums of linear terms). Furthermore, this convex polynomial modelling techniques allows us to progressively obtain more precise models as the degrees of the involved polynomials increase; they are *precise* in the sense of the polynomial vertexes will fit more closely the nonlinearity being modelled. This methodology is detailed in this section.

Consider the following dynamical system:

$$\dot{x}(t) = f(x(t)) + g(x)u(t), \tag{4.8}$$

with $x \in \mathbb{R}$ being the state vector, $u(t)$ is the input vector, and $x = 0$, $u = 0$ being an equilibrium point, i.e., $f(0) = 0$. Assume that $f(\cdot)$ can be expressed in the form:

$$\dot{x}(t) = \tilde{f}(\eta(x), x) + \tilde{g}(\eta(x), x)u(t), \tag{4.9}$$

being $\eta(x) = \begin{bmatrix} \eta_1(x) & \eta_2(x) & \cdots & \eta_p(x) \end{bmatrix}^T$ a set of continuous functions which collects all *non-polynomial* nonlinearities present in $f(\cdot)$ and $g(\cdot)$ in (4.8). Thus, once all the nonpolynomials functions $\eta_j(x)$, $j \in \{1, 2, \ldots, p\}$ have been identified, they will be rewritten as a convex sums of polynomials of arbitrary order, following a Taylor-series approach first described in (Sala and C. Ariño 2009) and detailed below.

LEMMA 4.3.1. *(Sala and C. Ariño 2009; Chesi 2009) Consider a sufficiently smooth function of one real variable, $\eta(x)$, so that its Taylor expansion of degree N exists (Apostol 1967), i.e., there exists an intermediate point $\psi(x) \in [0, x]$, so that:*

$$\eta(x) = \sum_{i=0}^{N-1} \frac{\eta^{[i]}(0)}{i!} x^i + \frac{\eta^{[N]}(\psi(x))}{N!} x^N, \tag{4.10}$$

where $\eta^{[i]}(x)$ denotes the i-th derivative of $\eta(\cdot)$ and $\eta^{[0]}(x)$ is defined, plainly, as $\eta(x)$. Additionally, assume that $\eta^{[N]}(x)$ is continuous in a compact region of interest Ω. Denoting the Taylor approximation of order N of the function $\eta(x)$ by:

$$\eta_N(x) = \sum_{i=0}^{N-1} \frac{\eta^{[i]}(0)}{i!} x^i,$$

and let

$$T_N(x) = \frac{\eta(x) - \eta_N(x)}{x^N}.$$

In the region Ω, $T_N(x)$ is bounded; therefore, the following bounds are well defined:

$$\psi_0 := \sup_{x\in\Omega} T_N(x), \quad \psi_1 := \inf_{x\in\Omega} T_N(x),$$

based on which the following convex rewriting of $T_N(x)$ arises:

$$T_N(x) = w_0(x) \cdot \psi_0 + w_1(x) \cdot \psi_1, \tag{4.11}$$

with weighting functions (WFs) defined as:

$$w_0(x) = \frac{T_n(x) - \psi_1}{\psi_0 - \psi_1}, \quad w_1(x) = 1 - w_0(x). \tag{4.12}$$

Then, an equivalent convex representation of (4.10) exists in the form:

$$\eta(x) = w_0(x) \cdot p_0(x) + w_1(x) \cdot p_1(x) = \sum_{i=0}^{1} w_i(x) \cdot p_i(x) = p_w(x), \forall x \in \Omega, \tag{4.13}$$

where $p_0(x) = \eta_N(x) + \psi_0 x^N$ and $p_1(x) = \eta_N(x) + \psi_1 x^N$ are polynomials of degree N, and $w_0(x)$, $w_1(x)$ are weighting functions which hold the convex sum property in the compact region Ω.

If every $\eta_j(x)$ in (4.9) is rewritten as in (4.13), then (4.9) can be rewritten as the following tensor product convex polynomial model:

$$\begin{aligned}\dot{x}(t) &= \sum_{i_1=0}^{1}\sum_{i_2=0}^{1}\cdots\sum_{i_q=0}^{1} w_{i_1}^1 w_{i_2}^2 \cdots w_{i_q}^q \left(F_{(i_1,i_2,\cdots,i_q)}(x) + B_{(i_1,i_2,\cdots,i_q)}(x)u(t)\right),\\ &= \sum_{\mathbf{i}\in\mathbb{B}^q} \mathbf{w_i}\left(F_{\mathbf{i}}(x) + B_{\mathbf{i}}(x)\right) = F_w(x) + B_w(x)u(t),\end{aligned} \tag{4.14}$$

where $\mathbf{i} = (i_1, i_2, \ldots, i_q)$,$\mathbb{B} \in \{0,1\}$, $\mathbf{w_i} = w_{i_1}^1 w_{i_2}^2 \cdots w_{i_q}^q$, $F_{\mathbf{i}}(x) = \tilde{f}(\eta(x), x)|_{\mathbf{w_i=1}}$, $B_{\mathbf{i}}(x) = \tilde{g}(\eta(x), x)|_{\mathbf{w_i=1}}$, $\mathbf{1} = \underbrace{(1, 1, \ldots, 1)}_{p \text{ ones}}$. As in the TS models, the classical representation used to be written in terms of *membership functions* (MFs):

$$h_i = h_{1+i_1+i_2\times 2+\cdots+i_q\times 2^{q-1}} = \prod_{j=1}^{q} w_{i_j}^j(z_j), \tag{4.15}$$

with $i \in \{1, 2, \ldots, r\}$, $r = 2^q$, $i_j \in \{0, 1\}$. As in the TS case, each of the r MFs h_i represents a combination instance of extreme values of the nonpolynomials expressions η_j; the full polynomial convex model stems from evaluating the state functions $f(\cdot)$ and $g(\cdot)$ in each of these combinations i.e: $F_i(x) = f(\cdot)|_{h_i=1}$ and $B_i(x) = g(\cdot)|_{h_i=1}$, Then, a polynomial convex representation of (4.8) is given by:

$$\begin{aligned}\dot{x}(t) &= \sum_{i=1}^{r} h_i(\eta(x))\left(F_i(x) + B_i(x)u(t)\right),\\ &= F_h(x) + B_h(x)u(t).\end{aligned} \tag{4.16}$$

Owing to the way this model is constructed, the functions $F_i(x)$ $B_i(x)$ are vectors of polynomials resulting from polynomials already present in $f(\cdot)$ as well as from products of polynomials p_0^j, p_1^j, $j \in \{1, 2, \ldots, q\}$ produced by the convex rewritten of nonpolynomials terms. Furthermore, all the nonlinearities, which cannot be described as polynomials of a prescribed degree, are captured in the MFs (h_i, $i \in \{1, 2, \ldots, r\}$) with convex structures, a key property for Lyapunov-based stability analysis and design.

Remark 4.3.1. *If $f(0) = 0$, setting $N = 1$ in the developments in lemma 4.3.1 we obtain the usual sector-nonlinearity methodology that bounds a function between two first degree polynomials.*

Example 4.3.1. *Consider again the ball and beam system in Example 2.2.1 whose model is reproduce here for convenience:*

$$\dot{x} = \begin{bmatrix} x_2 \\ \dfrac{-mx_3(2x_4x_2 - g\cos x_1)}{mx_3^2 + I_b} + \dfrac{u}{mx_3^2 + I_b} \\ x_4 \\ -\dfrac{5}{7}(g\sin x_1 - x_3x_2^2) \end{bmatrix}, \tag{4.17}$$

with the variables in the compact set $\Omega = \{x : |x_1| \leq 2,\ |x_3| \leq 0.5, x_2, x_4 \in \mathbb{R}\}$, *and* $m = 0.05$, $I_b = 1/12$, *and* $g = 9.81$. *The objective of this example is to illustrate the Taylor-series modelling approach for a nonlinear model. For this sake, consider three nonpolynomial nonlinearities in* (4.17) *and their Taylor series around* $x = 0$*:*

$$\begin{aligned} \sin x_1 &= x_1 - \frac{x_1^3}{3!} + \frac{x_1^5}{5!} - \frac{x_1^7}{7!} + \ldots \\ \cos x_1 &= 1 - \frac{x_1^2}{2!} + \frac{x_1^4}{4!} - \frac{x_1^6}{6!} + \ldots \\ \left(mx_3^2 + I_b\right)^{-1} &= \frac{1}{I_b} - \frac{mx_3^2}{I_b^2} + \frac{m^2x_3^4}{I_b^3} - \frac{m^3x_3^6}{I_b^4} + \ldots \end{aligned}$$

If only the first-degree terms from the Taylor-series expansion are used to rewrite $\eta^1(x_1) = \sin x_1$, $\eta^2(x_1) = \cos x_1$, *and* $\eta^3(x_3) = \left(mx_3^2 + I_b\right)^{-1}$ *as convex expressions, we will obtain the same outcome as if sector nonlinearity were used, i.e., based on the terms* $\eta_1^1(x_1) = 0$, $\eta_1^2(x_1) = 1$, *and* $\eta_1^3(x_3) = 12$, *the following expressions and their bounds are found:*

$$\begin{aligned} T_1^1(x_1) &= \frac{\sin x_1 - 0}{x_1},\ 0.04546 = \psi_0^1 \leq T_1^1(x_1) \leq \psi_1^1 = 1, \\ T_1^2(x_1) &= \frac{\cos x_1 - 1}{x_1},\ -0.7081 = \psi_0^2 \leq T_1^2(x_1) \leq \psi_1^2 = 0.7081, \\ T_1^3(x_3) &= \frac{\left(mx_3^2 + I_b\right)^{-1} - I_b^{-1}}{x_3},\ -3.1304 = \psi_0^3 \leq T_1^3(x_3) \leq \psi_1^3 = 3.1304. \end{aligned}$$

Thus, we can construct bounds the nonlinearities by linear (TS) terms:

$$\begin{aligned} 0.4546x_1 \leq \sin x_1 &\leq x_1, \\ 1 - 0.7081x_1 \leq \cos x_1 &\leq 1 + 0.7081x_1, \\ 12 - 3.1304x_3 \leq \left(mx_3^2 + I_b\right)^{-1} &\leq 12 + 3.1304x_3. \end{aligned} \tag{4.18}$$

If now we use the cubic term in the Taylor series, another polynomials bounds would be obtained by considering

$$\begin{aligned}\eta_3^1(x_1) = x_1,\, T_3^1(x_1) &= \frac{\sin x_1 - x_1}{x_1^3},\\ \eta_3^2(x_1) = 1 - 0.5x_1^2,\, T_3^2(x_1) &= \frac{\cos x_1 - 1 + 0.5x_1^2}{x_1^3},\\ \eta_3^3(x_3) = 12 - 7.2x_3^2,\, T_3^3(x_3) &= \frac{\left(0.05x_3^2 + 1/12\right)^{-1} - 12 + 7.2x_3^2}{x_3^3}.\end{aligned} \tag{4.19}$$

Thus, $T_3^1(x_1) \in [-0.1667, -0.1363]$, $T_3^2(x_1) \in [-0.0730, 0.0730]$, *and* $T_3^3(x_3) \in [-1.8783, 1.8783]$ *are bounded in* Ω. *Hence, now the nonlinearities are bounded by:*

$$\begin{aligned}x_1 - 0.1667x_1^3 x_1 \le \sin x_1 \le x_1 - 0.1363x_1^3,\\ 1 - 0.5x_1^2 - 0.0730x_1^3 \le \cos x_1 \le 1 - 0.5x_1^2 + 0.0730x_1^3\\ 12 - 7.2x_3^2 - 1.8783x_3^3 \le \left(mx_3^2 + I_b\right)^{-1} \le 12 - 7.2x_3^2 + 1.8783x_3^3.\end{aligned} \tag{4.20}$$

If we proceed to fifth order:

$$\begin{aligned}\eta_5^1(x_1) = x_1 - \frac{x_1^3}{6},\, T_5^1(x_1) &= \frac{\sin x_1 - x_1 + \frac{x_1^3}{6}}{x_1^5},\\ \eta_5^2(x_1) = 1 - \frac{x_1^2}{2} + \frac{x_1^4}{24},\, T_5^2(x_1) &= \frac{\cos x_1 - 1 + \frac{x_1^2}{2} - \frac{x_1^4}{24}}{x_1^5},\\ \eta_5^3(x_3) = 12 - 7.2x_3^2 + 4.32x_3^4,\, T_5^3(x_3) &= \frac{\left(0.05x_3^2 + \frac{1}{12}\right)^{-1} - 12 + 7.2x_3^2 - 4.32x_3^4}{x_3^5},\end{aligned}$$

we found the following bounds:

$$\begin{aligned}x_1 - \frac{x_1^3}{6} + 0.0076x_1^5 \le \sin x_1 \le x_1 - \frac{x_1^3}{6} + 0.0083x_1^5,\\ 1 - \frac{x_1^2}{2} + \frac{x_1^4}{24} - 0.0026x_1^5 \le \cos x_1 \le 1 - \frac{x_1^2}{2} + \frac{x_1^4}{24} + 0.0026x_1^5,\\ 12 - 7.2x_3^2 + 4.32x_3^4 - 1.1270x_3^5 \le \left(mx_3^2 + I_b\right)^{-1} \le 12 - 7.2x_3^2 + 4.32x_3^4 + 1.1270x_3^5.\end{aligned}$$

Once we have the polynomials vertexes of a desired degree, we can rewrite (4.17) *as a convex polynomial model, for example, if we desire an exact 3-th degree poly-*

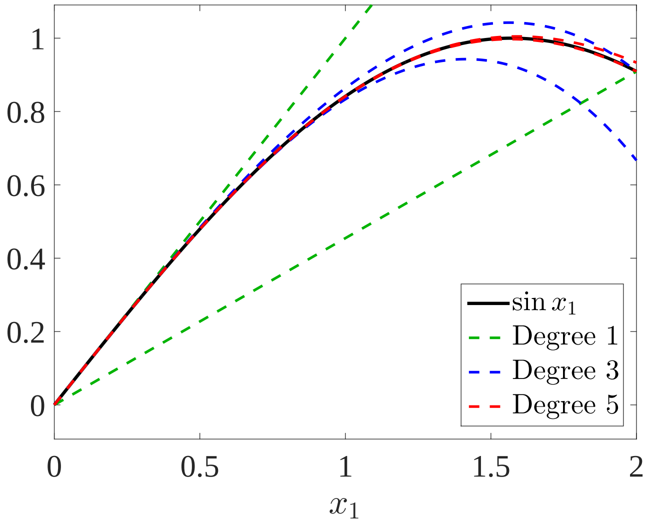

Figure 4.1: Polynomial bounds of $\sin x_1$ for $x_1 \in [-2, 2]$.

nomial convex rewriting of (4.17) *comes as:*

$$\dot{x} = \sum_{i_1=0}^{1} \sum_{i_2=0}^{1} \sum_{i_3=0}^{1} w_{i_1}^1(x_1) w_{i_2}^2(x_1) w_{i_3}^3(x_3) \left(F_{(i_1,i_2,i_3)}(x) + B_{(i_1,i_2,i_3)}(x) u(t) \right), \tag{4.21}$$

where

$$\begin{aligned} w_0^1(x_1) &= \frac{T_3^1(x_1) + 0.1667}{-0.1363 + 0.1667}, \quad w_1^1(x_1) = 1 - w_0^1(x_1), \\ w_0^2(x_1) &= \frac{T_3^2(x_1) + 0.0730}{0.0730 + 0.0730}, \quad w_1^2(x_1) = 1 - w_0^2(x_1), \\ w_0^3(x_3) &= \frac{T_3^3(x_3) + 1.8783}{1.8783 + 1.8783}, \quad w_1^3(x_3) = 1 - w_0^3(x_3), \end{aligned}$$

$$F_{(i_1,i_2,i_3)}(x) = \begin{bmatrix} x_2 \\ -p_{i_3}^3(x_3) m x_3 (2x_4 x_2 - g p_{i_2}^2(x_1)) \\ x_4 \\ -\frac{5}{7}(g p_{i_1}^1(x_1) - x_3 x_2^2) \end{bmatrix}, \; B_{(i_1,i_2,i_3)}(x) = \begin{bmatrix} 0 \\ p_{i_3}^3(x_3) \\ 0 \\ 0 \end{bmatrix},$$

with $i_j \in 0, 1$, $j \in \{1, 2, 3\}$, *and* $p_{i_j}^j(\cdot)$ *are the polynomials bounds in* (4.20). *The same procedure is applicable for any desired degree of the convex polynomial.*

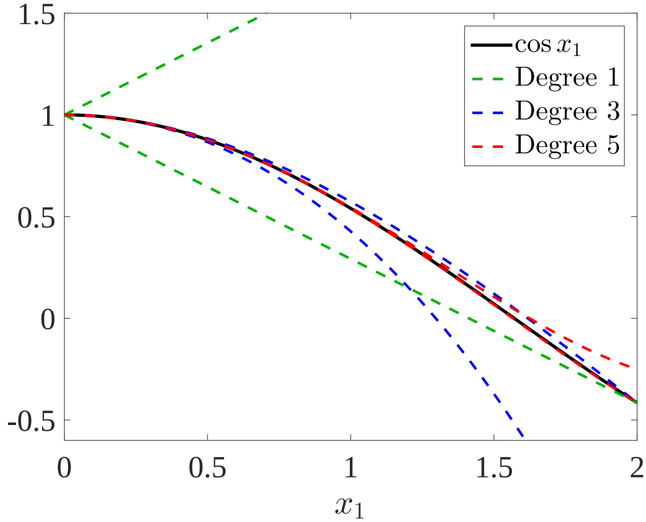

Figure 4.2: Polynomial bounds of $\cos x_1$ for $x_1 \in [-2, 2]$.

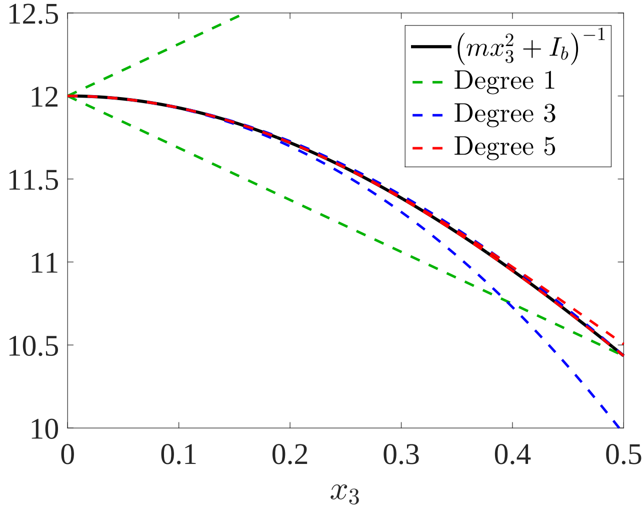

Figure 4.3: Polynomial bounds of $\left(mx_3^2 + I_b\right)^{-1}$ for $x_3 \in [-0.5, 0.5]$.

Figures 4.1, 4.2, and 4.3 illustrate the fact that the bounding polynomials get progressively closer to the nonlinearity as their degree increases.

The Taylor-series approach can also be applied to multivariable functions that can be written as an expression tree with functions of one variable, i.e., addition and multiplication. This idea is illustrated in the following example.

Example 4.3.2. *Consider the function*

$$f(x_1, x_2) = \sin\left(\frac{x_2}{x_1^2+1}\right) \tag{4.22}$$

to be modeled in the region $\Omega = \{x : |x_i| \leq 1, i \in \{1,2\}\}$. *In* Ω, *the argument of the sinusoid* $\upsilon = \dfrac{x_2}{x_1^2+1}$ *satisfy* $\upsilon \in [-1,1]$. *Thus,* $\sin\upsilon$ *may be modeled with the following third-order convex polynomial model:*

$$\sin\upsilon = w_0^1\left(\upsilon - 0.1585\upsilon^3\right) + w_1^1\left(\upsilon - 0.1667\upsilon^3\right), \tag{4.23}$$

where

$$w_0^1 = \frac{\sin\upsilon - \upsilon + 0.1667\upsilon^3}{0.0082\upsilon^3}, \quad w_1^1 = 1 - w_0^1.$$

Additionally, $\left(x_1^2+1\right)^{-1}$ *may be modeled in* Ω *as:*

$$\left(x_1^2+1\right)^{-1} = w_0^2\left(1+0.5x_1\right) + w_1^2\left(1-0.5x_1\right) \tag{4.24}$$

with $w_0^2 = -x_1/\left(x_1^2+1\right)+0.5$, $w_1^2 = 1-w_0^2$. *Now, replacing in* (4.22) *the sinusoid by* (4.23), $\upsilon = x_2/\left(x_1^2+1\right)$, *and later using* (4.24), *we get a convex polynomial model in the form:*

$$\begin{aligned} f(x_1,x_2) =& x_2\left(w_0^2\left(1+0.5x_1\right) + w_1^2\left(1-0.5x_1\right)\right) \\ &- \left(0.1585w_0^1 + 0.1667w_1^1\right)x_2^3\left(w_0^2\left(1+0.5x_1\right) + w_1^2\left(1-0.5x_1\right)\right)^3. \end{aligned}$$

4.4 Stability analysis via SOS

Once we have a polynomial nonlinear model, either by a already polynomial one or a convex polynomial representation, we can apply SOS techniques for the stability analysis of nonlinear systems. For this sake, consider the convex polynomial system of the form:

$$\dot{x} = \sum_{i=1}^{r} h_i(x)F_i(x), \tag{4.25}$$

the following well-known results are derived from stability theory.

Theorem 4.4.1. *(Prajna, Papachristodoulou, Seiler, et al. 2005; Sala and C. Ariño 2009; Tanaka, Yoshida, et al. 2009) The origin $x = 0$ of the convex polynomial model* (4.25) *is asymptotically stable if there exists polynomial Lyapunov function $V(x)$ such that $V(0) = 0$, and*

$$V(x) - \varepsilon(x) \in \Sigma_x \tag{4.26}$$

$$-\frac{\partial V}{\partial x} F_i(x) - \varepsilon(x) \in \Sigma_x, , \tag{4.27}$$

for $i \in \{1, 2, \ldots, r\}$. $\varepsilon(x)$ is a radially unbounded positive polynomial.

Since $V(x)$ needs to be positive *definite*, not just positive semidefinite, the following proposition will help to chose $\varepsilon(x)$.

Proposition 4.4.1. *Given a polynomial $V(x)$ of degree $2d$, let $\varepsilon(x) = \sum_{i=1}^{n} \sum_{j=1}^{d} \epsilon_{ij} x_i^{2j}$ such that:*

$$\sum_{j=1}^{d} \epsilon_{ij} > \gamma \, \forall i \in \{1, 2, \ldots, n\},$$

with γ being a positive number, and $\varepsilon_{ij} \geq 0$ for all i and j. Then the condition

$$V(x) - \varepsilon(x) \in \Sigma_x \tag{4.28}$$

guarantees the positive definiteness of $V(x)$.

Proof. The function $\varepsilon(x)$ as defined above is positive definite if $\epsilon_{i,j}$'s satisfy the conditions mentioned in the proposition. Then $V(x) - \varepsilon(x)$ being SOS implies that $V(x) \geq \varepsilon(x)$, and therefore $V(x)$ is positive definite. □

As in the TS case, the Taylor-series convex polynomial models are only valid locally in most cases, i.e., stability is not proved in the whole state space where the SOS conditions hold (unless $\Omega = \mathbb{R}^n$). Due to the WFs only hold the convex sum property in the compact set Ω, the actually proven DA is the largest invariant set $V(x) \leq k$, $k \in \mathbb{R}$ contained in Ω.

4.4.1 Local stability via SOS

Due to the fact that many nonlinear systems of interest are not globally stable, or proving global stability would require high-degree polynomial Lyapunov function exhausting the available computational resources, some refinements to the above stability conditions are need in order to obtain a DA estimate.

If condition in theorem 4.4.1 fails to prove global stability, the *Positivstellensatz* theorem 4.2.1 allow us to posed local stability conditions. For this sake, consider a region of the state space Ω defined by:

$$\Omega = \{x \in \mathbb{R}^n : g_1(x) \geq 0, g_2(x) \geq 0, \dots, g_\gamma(x) \geq 0, h_1(x) = 0, h_2(x) = 0, \dots, h_\iota(x) = 0\}, \tag{4.29}$$

where $g_j(x)$ and h_k a set of γ and ι known polynomials respectively. Then, the following results is derived for local stability in Ω.

LEMMA 4.4.1. *If a polynomial function $V(x)$ such that $V(0) = 0$, SOS polynomials $s_j^1(x) \in \Sigma_x$,$s_j^2(x) \in \Sigma_x$, and arbitrary ones $t_k^1(x)$, $t_k^2(x)$, can be found fulfilling:*

$$V(x) - \varepsilon(x) - \sum_{j=1}^{\gamma} s_j^1(x) g_j(x) + \sum_{k=1}^{\iota} t_k^1(x) h_k(x) \in \Sigma_x, \tag{4.30}$$

$$-\frac{\partial V(x)}{\partial x} F_i - \varepsilon(x) - \sum_{j=1}^{\gamma} s_j^2(x) g_j(x) + \sum_{k=1}^{\iota} t_k^2(x) h_k(x) \in \Sigma_x, \tag{4.31}$$

for $i \in \{1, 2, \dots, r\}$, begin $\varepsilon(x)$ defined as in proposition 4.4.1, then the origin $x = 0$ of the convex polynomial model (4.25) is asymptotically stable. Furthermore, an estimate for the DA of the origin $x = 0$ is $\mathcal{D} = \{x \in \mathbb{R}^n : V(x) \leq \alpha\}$, where $\alpha = \min_{x \in \partial\Omega} V(x)$ and $\partial\Omega$ is the boundary of Ω.

In order to obtain the Lyapunov function level set containing the largest region with a particular predefined shape, i.e., an sphere or an hypercube, additional SOS constrains may be added.

EXAMPLE 4.4.1. *Consider the polynomial system:*

$$\dot{x}(t) = \begin{bmatrix} -x_2 \\ x_1 + \left(x_1^2 - 1\right) x_2 \end{bmatrix}. \tag{4.32}$$

For the above system, linearisation shows that the origin is stable: there is a neighborhood of it belonging to its DA provable with a Lyapunov function. However, phase plane simulation shows that it has an unstable limit cycle so there is not a global Lyapunov function for the system.

Let us consider a region of interest characterised as $\Omega = \{x \in \mathbb{R}^2 | \alpha - x_1^2,\ \alpha - x_2^2 \geq 0 \geq 0$. Applying lemma 4.4.1 with a quadratic Lyapunov function and proposition 4.4.1 with $\gamma = 0.0001$, the maximum α feasible is less to one, i.e., $\alpha < 1$. If lemma

4.4.1 is applied with a polynomial Lyapunov function of degree 4, we can get an $\alpha = 1.8610$*. For different degrees polynomial Lyapunov functions are summarized in the following table. The lines of code below show an implementation of the*

degree $V(x)$	2	4	6	8	12
α	0.99	1.8610	2.3121	2.5483	2.6201

current example if $\alpha = 1.8610$ *with a polynomial Lyapunov function of degree 4.*

```
% Define independet variables
sdpvar x1 x2
% Define the system equation
dx=[-x2; x1+(x1^2-1)*x2];
% Tolerance epsilon (Proposition 4.4.1)
epsi1=0; epsi2=0;
CX=[]; or=4; x=[x1;x2];
for i=1:2
    epsum1{i}=0; epsum2{i}=0;
    for j=1:or/2
        ep1{i,j}=sdpvar(1); ep2{i,j}=sdpvar(1);
        epsum1{i}=epsum1{i}+ep1{i,j};
        epsum2{i}=epsum2{i}+ep2{i,j};
        epsi1=epsi1+ep1{i,j}*x(i)^(2*j);
        epsi2=epsi2+ep2{i,j}*x(i)^(2*j);
        CX=[CX ep1{i,j}>=0 ep2{i,j}>=0];
    end
    CX=[CX epsum1{i}>=10^(-4) epsum2{i}>=10^(-4)];
end
% Lyapuno function degree 4 and partial derivative
[V,cp,vp]=polynomial([x1,x2],4,2);
dV=jacobian(V,[x1,x2]);
% Create the Positivstellensatz multiplier degree 4
coef=cp';
for i=1:4
    for j=1:2
        [s{i,j},cs{i,j},vs{i,j}]=polynomial([x1,x2],4,2);
        coef=[coef cs{i,j}'];
        CX=[CX sos(s{i,j})];
    end
end
alp=1.8610^2; %
% Define SOS constraints
CX=[CX sos(V-epsi1-s{1,1}*(alp-x1^2)-s{1,2}*(alp-x2^2))];
CX=[CX sos(-dV*dx-epsi2-s{2,1}*(alp-x1^2)-s{2,2}*(alp-x2^2))];
% Solve SOS problem
sol=solvesos(CX,[],[],coef);
```

4.5 Stabilisation via SOS

Consider the following affine-in-control convex polynomial model:

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(x)(A_i(x)Z(x) + B_i(x)u(t)) = A_h(x)Z(x) + B_h(x)u(t), \tag{4.33}$$

where $Z(x) \in \mathbb{R}^N$ is a vector of monomials in x such that $Z(x) = 0$ if and only if $x = 0$. Let $M(x) \in \mathbb{R}^{N\times n}$ be a polynomial matrix whose (i, j)-th entry is given by

$$M_{ij}(x) = \frac{\partial Z_i(x)}{\partial x_j},$$

for $i \in \{1, 2, \ldots, N\}$, $j \in \{1, 2, \ldots, n\}$. Finally, let $A_j(x)$ denotes the j-th row of $A(x)$, $J = \{j_1, j_2, \ldots, j_m\}$ denotes the row indices of $B(x)$ whose corresponding row is equal to zero, and define $\tilde{x} = \left[x_{j_1}, x_{j_2}, \cdots, x_{j_m}\right]^T$.

A first approach to design a stabilizing control law could be extending the well-known ideas of parallel-distributed compensator (PDC) to the polynomial framework (Tanaka, Yoshida, et al. 2007b) (which it adaptation of (Prajna, Papachristodoulou, and F. Wu 2004) to the fuzzy case):

$$u = \sum_{i=0}^{r} h_i K_i(x)Z(x) = K_h(x)Z(x), \tag{4.34}$$

where K_i, $i \in \{1, 2, \ldots, r\}$ are polynomial matrices to be found.

Define a polynomial candidate Lyapunov function in the form:

$$V(x) = Z(x)^T P(\tilde{x})Z(x), \tag{4.35}$$

then, the following theorem can be used to design a polynomial PDC control law.

Theorem 4.5.1. *The origin $x = 0$ of the system* (4.33) *is asymptotically stable, if there exist symmetric polynomial matrix $P(\tilde{x}) \in \mathbb{R}^{n\times n}$, and polynomial matrices $K_j \in \mathbb{R}^{n\times N}$, $j \in \{1, 2, \ldots, r\}$, a constant $\varepsilon_1 > 0$, and $\varepsilon_2(x) > 0$ for $x \neq 0$, such that:*

$$v^T \left(P(\tilde{x}) - \varepsilon_1 I\right) v \in \Sigma_{x,v}, \tag{4.36}$$

$$-v^T \Big(M(x)A_h(x)P(\tilde{x}) + M(x)B(x)K(x) + P(\tilde{x})A_h^T(x)M^T(x)$$
$$+K^T(x)B^T(x)M^T(x) - \sum_{j\in J} \frac{\partial P(\tilde{x})}{\partial x_j} \left(A_j(x)Z(x)\right) + \varepsilon_2(x)I\Big) v \in \Sigma_{x,v}, \tag{4.37}$$

for $(i,j) \in \{1,2,\ldots,r^2\}$, $v \in \mathbb{R}^N$. *Then, the controller* (4.34) *stabilizes the system* (4.33) *in a region of the state space* $V_c \subset \Omega$ *begin* Ω *the modelling region and* $V_c = \{x : V(x) = Z^T(x)P^{-1}(\tilde{x})Z(x) < c\}$. *Controllers gains can be obtained by* $K_h(x) = M_h(x)P^{-1}(\tilde{x})$.

It can be notice that conditions above are shown in their most general form: any sum relaxation scheme can be applied to them in order to obtain SOS conditions.

4.5.1 Dynamical Extension

Another approach to obtain a pure polynomial model from a nonlinear one is recasting the non polynomial nonlinearities to new auxiliary state variables.

The following algorithm is an adaptation from the one explained in (Savageau and Voit 1987), and it is applicable to a very large class of nonpolynomial systems, namely those whose vector field is composed of sums and productos of elementary functions, or nested elementary functions of elementary function (exponential (e^x), logarithm ($\ln x$), power (x^a), trigonometric ($\sin x$, , $\cos x$, etc.), etc).

Consider the nonpolynomial system in the form $\dot{x} = f(x)$, which has an equilibrium at the origin.

1. Create new state variables x_{n+i} for each elemental nonpolynomial nonlinear function (sinus, cosines, logarithm, exponential, etc.) $f_i(x)$, or combination of them, and assign $x_{n+i} = f_i(x)$.
2. Compute, using the chain rule, the derivative of the new state variables $\dot{x}_{n+i} = \frac{df_i(x)}{dt}$ and replace each $f_i(x)$ by the new x_{n+i} in the whole system's model.
3. As a results of the above step, new nonlinearities might appear in $\dot{x}_{n+i}$. Then, repeat the above steps with the new extended dynamical equations until obtaining a totally polynomial model.
4. Additional information, if provided, can be added as algebraic constraints over the new variables x_{n+i}.

The following extended polynomial model is obtained:

$$\begin{aligned} \dot{\tilde{x}}_1 &= f_1(\tilde{x}_1, \tilde{x}_2), \\ \dot{\tilde{x}}_2 &= f_2(\tilde{x}_1, \tilde{x}_2), \end{aligned} \tag{4.38}$$

where $\tilde{x}_1 = [x_1, x_2, \ldots, x_n]$ are the original sate variables and $\tilde{x}_2 = [x_{n+1}, x_{n+2}, \ldots, x_{n+m}]$ are the new variables introduced in the recasting process. Additionally, some constraints will aries directly from the recasting process, denoted by:

$$G_1(\tilde{x}_1, \tilde{x}_2) = 0, \tag{4.39}$$

$$G_2(\tilde{x}_1, \tilde{x}_2) \geq 0. \tag{4.40}$$

It is best to illustrate the application of the above algorithm by an example.

Example 4.5.1. *Consider a reaction wheel pendulum (Spong, Corke, and Lozano 2001), whose dynamics are given by:*

$$\dot{x}(t) = \begin{bmatrix} x_2 \\ D^{-1} I_2 \left(\bar{m} g \sin x_1 - u(t) \right) \\ D^{-1} I_2 \left(-\bar{m} g \sin x_1 + a u(t) \right) \end{bmatrix}, \tag{4.41}$$

being x_1 the pendulum angle, x_2 the pendulum velocity, x_3 the disk velocity, u motor torque input applied on the disk, $m_1 = 0.02$ mass of the pendulum, $m_2 = 0.3$ mass of the wheel, $l_1 = 0.125$ length of the pendulum, $l_{c1} = 0.063$ distances to the center of mass of the pendulum, $T_1 = 47 \times 10^{-6}$ moment of inertia of the pendulum, $T_2 = 32 \times 10^{-6}$ moment of inertia of the wheel, $a = m_1 l_{c1}^2 + m_2 l_1^2 + T_1 + T_2$, $D = aI_2 - I_2^2$, and $\bar{m} = m_1 l_{c1} + m_2 l_1$.

We want to recast as a system with polynomial vector field. Define $x_4 = \sin x_1$ and compute its derivative by the chain rule $\dot{x_4} = \cos x_1 \dot{x_1}$. Notice that $\dot{x_4}$ is not yet in a polynomial form, thus we need to define another new variable $x_5 = \cos x_1$. Using the chain rule of differentiation again, we obtain:

$$\dot{x}(t) = \begin{bmatrix} x_2 \\ D^{-1} I_2 \left(\bar{m} g x_4 - u(t) \right) \\ D^{-1} I_2 \left(-\bar{m} g x_4 + a u(t) \right) \\ -x_2 x_5 \\ x_2 x_4 \end{bmatrix}.$$

At this point, we terminate the recasting process, since the equations are in a polynomial form. In addition, the trigonometric constrain $\sin^2 x_1 + \cos^2 x_1 = 1$ can be added by the algebraic constraint: $x_4^2 + x_5^2 - 1 = 0$. A more detailed description can be found in (Papachristodoulou and Prajna 2005).

The extended model (4.38) is *not* a convex model. Nonetheless, this technique of recasting can be used as an alternative or can be combined with the sector nonlinearity (4.3) in order to obtain a convex representation of a new non-polynomial nonlinearity involving any x_{n+i}. This avoids the introduction of a new variable x_{n+i} with its corresponding dynamical equation. In this way, an extended convex polynomial model is obtained.

4.6 Polynomial parameter-dependent Lyapunov function

The SOS approach has been explored along with a polynomial parameter-dependent Lyapunov function (PPDLF). Following the same ideas as in the TS-PDLF case, a PPDLF share the structure of the convex polynomial model applied:

$$V(x(t)) = \sum_{i_1=0}^{1}\sum_{i_2=0}^{1}\cdots\sum_{i_q=0}^{1} w_{i_1}^1 w_{i_2}^2 \cdots w_{i_q}^q p_{(i_1,i_2,\cdots,i_q)}(x) = \sum_{\mathbf{i}\in\mathbb{B}^q} \mathbf{w_i} p_{\mathbf{i}}(x), \qquad (4.42)$$

where $\mathbf{i} = (i_1, i_2, \ldots, i_q)$,$\mathbb{B} \in \{0,1\}$, $\mathbf{w_i} = w_{i_1}^1 w_{i_2}^2 \cdots w_{i_q}^q$, $p_{\mathbf{i}}(x) \in \mathbb{R}$ are polynomials to be determined, and the MFs $w_{i_j}^j$ are those in the convex polynomial model (4.14). This function is a generalisation of the PDLF in section 3.3 where $p_{\mathbf{i}}(x)$ are restricted to be homogeneous quadratic polynomials in the states. However, the inclusion of the MFs in their definition leads to the same problems as in the TS-PDLF case. A solution of the time-derivative of the MFs was proposed in (Bernal, Sala, et al. 2011) via incorporating locality and membership-shape information (bounds on the partial derivatives). For this sake, consider the time-derivative of $\mathbf{w_i}$ in (4.42):

$$\dot{\mathbf{w}}_{\mathbf{i}} = \frac{\partial \mathbf{w_i}}{\partial \eta}\dot{\eta} = \sum_{k=1}^{q} \frac{\partial \mathbf{w_i}}{\partial \eta_k}\dot{\eta}_k = \sum_{k=1}^{q} \frac{\partial}{\partial \eta_k}\left(\prod_{j=1}^{q} w_{i_j}^j(\eta_j)\right)\dot{\eta}_k = \sum_{k=1}^{q} \frac{\partial w_{i_k}^k}{\partial \eta_k}\left(\prod_{\substack{j=1\\ j\neq k}}^{q} w_{i_j}^j(\eta_j)\right)\dot{\eta}_k.$$

Multiplying by $w_{i_k}^k + \left(1 - w_{i_k}^k\right) = 1$ gives

$$\begin{aligned}\dot{\mathbf{w}}_{\mathbf{i}} &= \sum_{k=1}^{q} \frac{\partial w_{i_k}^k}{\partial \eta_k}\left(w_{i_k}^k \prod_{\substack{j=1\\ j\neq k}}^{q} w_{i_j}^j(\eta_j) + \left(1 - w_{i_k}^k\right) \prod_{\substack{j=1\\ j\neq k}}^{q} w_{i_j}^j(\eta_j)\right)\dot{\eta}_k \\ &= \sum_{k=1}^{q} \frac{\partial w_{i_k}^k}{\partial \eta_k}\left(\mathbf{w_i} + \mathbf{w}_{\bar{\mathbf{i}}(k)}\right)\dot{\eta}_k, \qquad (4.43)\end{aligned}$$

where $\bar{\mathbf{i}}(k)$ is defined as the q-bit binary index resulting from changing the kth bit of $\mathbf{i}$ to its complement. This form allows to recover convex expressions form the Lyapunov analysis.

Continuing with the Lyapunov method, consider the time-derivative of the PPDLF (4.42) along the trajectories of the polynomial convex model (4.14) and taking

(4.43) into account gives:

$$\dot{V}(x)=\sum_{\mathbf{i}\in\mathbb{B}^q}\left(\mathbf{w_i}\dot{p}_\mathbf{i}(x)+\dot{\mathbf{w}}_\mathbf{i}p_\mathbf{i}(x)\right)=\sum_{\mathbf{i}\in\mathbb{B}^q}\left(\mathbf{w_i}\dot{p}_\mathbf{i}(x)+\sum_{k=1}^{q}\frac{\partial w_{i_k}^k}{\partial\eta_k}\Big(\mathbf{w_i}+\mathbf{w}_{\bar{\mathbf{i}}(k)}\Big)\,\dot{\eta}_k p_\mathbf{i}(x)\right),$$
$$=\sum_{\mathbf{i}\in\mathbb{B}^q}\mathbf{w_i}\left(\dot{p}_\mathbf{i}(x)+\sum_{k=1}^{q}\frac{\partial w_{i_k}^k}{\partial\eta_k}\dot{\eta}_k\left(p_\mathbf{i}(x)-p_{\bar{\mathbf{i}}(k)}(x)\right)\right), \tag{4.44}$$

where the identity $\sum_{\mathbf{i}\in\mathbb{B}^q}\mathbf{w}_{\bar{\mathbf{i}}(k)}p_\mathbf{i}=\sum_{\mathbf{i}\in\mathbb{B}^q}\mathbf{w_i}p_{\bar{\mathbf{i}}(k)}$ has been used to obtain the above expression. Since η_k and $p_\mathbf{i}$ are polynomials and $\dot{x}$ is taken from its convex polynomial representation in (4.14), substituting the expressions $\dot{\eta}_k=\dfrac{\partial\eta_k}{\partial x}\dot{x}$ and $\dot{p}_\mathbf{i}=\dfrac{\partial p_\mathbf{i}}{\partial x}\dot{x}$ in (4.44), yields:

$$\dot{V}(x)=\sum_{\mathbf{i}\in\mathbb{B}^q}\mathbf{w_i}\left(\frac{\partial p_\mathbf{i}}{\partial x}\sum_{\mathbf{j}\in\mathbb{B}^q}\mathbf{w_j}F_\mathbf{i}(x)+\sum_{k=1}^{q}\frac{\partial w_{i_k}^k}{\partial\eta_k}\frac{\partial\eta_k}{\partial x}\sum_{\mathbf{j}\in\mathbb{B}^q}\mathbf{w_j}F_\mathbf{i}(x)\Big(p_\mathbf{i}(x)-p_{\bar{\mathbf{i}}(k)}(x)\Big)\right),$$
$$=\sum_{\mathbf{i}\in\mathbb{B}^q}\sum_{\mathbf{j}\in\mathbb{B}^q}\mathbf{w_i}\mathbf{w_j}\left(\frac{\partial p_\mathbf{i}}{\partial x}F_\mathbf{i}(x)+\sum_{k=1}^{q}\frac{\partial w_{i_k}^k}{\partial\eta_k}\frac{\partial\eta_k}{\partial x}F_\mathbf{i}(x)\left(p_\mathbf{i}(x)-p_{\bar{\mathbf{i}}(k)}(x)\right)\right). \tag{4.45}$$

Note that all the terms in the above expression are MFs or polynomials, except for $\dfrac{\partial w_0^k}{\partial\eta_k}$, which can be rewritten as a convex sum of polynomials in the same way as the convex polynomial model (4.14) was obtained. For this sake, consider the polynomial vector $\dfrac{\partial\eta_k}{\partial x}\in\mathbb{R}^{n\times 1}$ and the convex polynomial representation of $\dfrac{\partial w_0^k}{\partial\eta_k}$, i.e.,

$$\frac{\partial w_0^k}{\partial\eta_k}\cdot\frac{\partial\eta_k}{\partial x}=\sum_{\mathbf{v}_k\in\mathbb{B}^{s_k}}\mu_{\mathbf{v}_k}^k(x)\mathbf{r}_{\mathbf{v}_k}^k(x),\ k\in\{1,2,\ldots,q\}, \tag{4.46}$$

with s_k being the number of possible nonpolynomial nonlinearities in $\dfrac{\partial w_0^k}{\partial\eta_k}$, and $\mu_{\mathbf{v}_k}^k=\mu_{v_k^1}^k\mu_{v_k^2}^k\cdots\mu_{v_k^{s_k}}^k$, $\sum_{v_k^i=0}^{1}\mu_{v_k^i}^k(\cdot)=1$, $\mu_{v_k^i}^k(\cdot)\geq 0$ being the MFs associated with each modeled nonlinearity, and $\mathbf{r}_{\mathbf{v}_k}^k(x)\in\mathbb{R}^{n\times 1}$ being the resulting polynomial vector.

Substituting (4.46) in (4.45) yields

$$\begin{aligned}\dot{V}(x) &= \sum_{\mathbf{i}\in\mathbb{B}^q}\sum_{\mathbf{j}\in\mathbb{B}^q}\mathbf{w_i w_j}\left(\frac{\partial p_{\mathbf{i}}}{\partial x}F_{\mathbf{i}}(x) + \sum_{k=1}^{q}\sum_{\mathbf{v}_k\in\mathbb{B}^{s_k}}\mu_{\mathbf{v}_k}^k \mathbf{r}_{\mathbf{v}_k}^k F_{\mathbf{i}}(x)\left(p_{\mathbf{i}}(x) - p_{\bar{\mathbf{i}}(k)}(x)\right)\right),\\ &= \sum_{\mathbf{i}\in\mathbb{B}^q}\sum_{\mathbf{j}\in\mathbb{B}^q}\sum_{\mathbf{v}_1\in\mathbb{B}^{s_1}}\sum_{\mathbf{v}_2\in\mathbb{B}^{s_2}}\cdots\sum_{\mathbf{v}_q\in\mathbb{B}^{s_q}}\mathbf{w_i w_j}\mu_{\mathbf{v}_1}^1\mu_{\mathbf{v}_2}^2\cdots\mu_{\mathbf{v}_q}^q\\ &\quad\times\left(\frac{\partial p_{\mathbf{i}}}{\partial x}F_{\mathbf{i}}(x) + \sum_{k=1}^{q}\mathbf{r}_{\mathbf{v}_k}^k F_{\mathbf{i}}(x)\left(p_{\mathbf{i}}(x) - p_{\bar{\mathbf{i}}(k)}(x)\right)\right).\end{aligned}$$

Defining the polynomial vector

$$\hat{\mathbf{p}}_{\mathbf{i}} = \begin{bmatrix} p_{\mathbf{i}} - p_{\hat{\mathbf{i}}(1)} \\ p_{\mathbf{i}} - p_{\hat{\mathbf{i}}(2)} \\ \vdots \\ p_{\mathbf{i}} - p_{\hat{\mathbf{i}}(q)} \end{bmatrix} \in \mathbf{R}^{1\times q}, \tag{4.47}$$

the polynomial matrix

$$\mathbf{R_v} = \begin{bmatrix} \mathbf{r}_{\mathbf{v}_1}^1 \\ \mathbf{r}_{\mathbf{v}_2}^2 \\ \vdots \\ \mathbf{r}_{\mathbf{v}_q}^q \end{bmatrix} \in \mathbf{R}^{q\times n}, \tag{4.48}$$

and the multi-index $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p)$, the previous expression can be rewritten as

$$\dot{V}(x) = \sum_{\mathbf{i}\in\mathbb{B}^q}\sum_{\mathbf{j}\in\mathbb{B}^q}\sum_{\mathbf{v}\in\mathbb{B}^\sigma}\mathbf{w_i w_j}\mu_{\mathbf{v}}\left(\frac{\partial p_{\mathbf{i}}}{\partial x}F_{\mathbf{i}}(x) + \hat{\mathbf{p}}_{\mathbf{i}}^T\mathbf{R_v}F_{\mathbf{i}}(x)\right). \tag{4.49}$$

with $\sigma = s_1 + s_2 + \cdots + s_q$. Then, the result can be summarized in the following theorem.

Theorem 4.6.1. *The convex polynomial model* (4.14) *with MF-derivatives as in* (4.46) *is asymptotically stable if there exist polynomials $p_{\mathbf{i}}(x)$, and non-negative, radially unbounded polynomials $\varepsilon_1(x), \varepsilon_2(x) > 0$ such that:*

$$\begin{gathered} p_{\mathbf{i}}(x) - \varepsilon_1(x) \in \Sigma_x \\ -\frac{\partial p_{\mathbf{i}}}{\partial x}F_{\mathbf{i}}(x) - \hat{\mathbf{p}}_{\mathbf{i}}^T\mathbf{R_v}F_{\mathbf{i}}(x) - \varepsilon_2(x) \in \Sigma_x, \end{gathered}$$

for all $\mathbf{i}, \mathbf{j} \in \mathbb{B}^q$, $\mathbf{v} \in \mathbb{B}^\sigma$ with $\hat{\mathbf{p}}_{\mathbf{i}}$ and $\mathbf{R_v}$ defined as in (4.47)-(4.48).

Note that this class of Lyapunov functions can reduce conservatism for the stability analysis of nonlinear systems. Nevertheless, as in the TS-LMI case, the MFs need to be *a priori* bounded, but in this case, by polynomials of the state.

Part II

Contributions

Chapter 5

Piecewise Lyapunov function

This chapter generalises recent results on stability analysis and estimation of the domain of attraction of nonlinear systems via exact piecewise affine Takagi-Sugeno models. Algorithms in the form of linear matrix inequalities are proposed that produce progressively better estimates which are proved to asymptotically render the actual domain of attraction; regions already proven to belong to such domain of attraction can be removed and the estimate can contain significant portions of the modelling region boundary; in this way, level-set approaches in prior literature can be significantly improved. Illustrative examples and comparisons are provided.

The contents of this chapter appeared in the journal article:

- T. Gonzalez, A. Sala, and M. Bernal (2017). "Piecewise-Takagi-Sugeno asymptotically exact estimation of the domain of attraction of nonlinear systems". In: *Journal of the Franklin Institute* 354.3, pp. 1514–1541.

5.1 Introduction

Takagi-Sugeno (TS) models, systematically obtained via the sector nonlinearity approach (Taniguchi, Tanaka, and H. Wang 2001), have proved to be suitable for generalisation of linear techniques to handle nonlinear stability issues (H. Wang, Tanaka, and Griffin 1996), since they are convex sums of linear systems weighted by membership functions (MFs). When combined with the direct Lyapunov method, TS models naturally lead to linear matrix inequalities (LMIs) (Boyd et al. 1994), which can be efficiently solved via convex optimization techniques already implemented in commercially available software (Sturm 1999). The TS modelling approach has been also extended to distributed-parameter systems governed by partial differential equations (F. Wu and H. Li 2008; Qiu, Feng, and Gao 2016); nevertheless, this class of systems are out of the scope of this work.

Though the TS and nonlinear models are locally equivalent in some compact Ω, also known as the modelling region, the LMI stability analysis is conservative (Sala 2009; J. Chen et al. 2016; Marquez, T.M Guerra, et al. 2016). This is mainly due to the fact that only vertex (linear) models are considered, i.e., MFs are ignored, thus introducing the so called *shape-independent* conservatism (Sala 2009).

Within shape-independent approaches, piecewise analysis is known for reducing conservatism by lowering the separation among the vertex models via a partition of Ω. Moreover, affine terms can be introduced in TS models if the region under consideration does not include the origin (Gonzalez, Sala, Bernal, and Robles 2015). This allows considering more general piecewise-quadratic Lyapunov functions (PWQLF) (Johansson, Rantzer, and Arzen 1999); other piecewise options are considered in (C. Ariño, Perez, et al. 2014; Guo et al. 2014; Y. Chen et al. 2015) for stability analysis. Piecewise TS approaches for control design have also been reported but they usually are in BMI form (Hu and Blanchini 2010); the work (Qiu, Feng, and Gao 2013) presents a piecewise control synthesis procedure keeping the LMI structure, at the cost of conservatism in some steps; as we discuss a non-conservative stability-analysis setup, the issues in (Qiu, Feng, and Gao 2013) will not be considered here. Practical applications of affine TS models appear in, for instance, (Schulte and H. Hahn 2004), and those of piecewise models have been reported in (Cuesta and Ollero 2004).

The problem to be addressed in this chapter is the determination of the "largest" estimate of the domain of attraction (DA) of the origin of a nonlinear system $\dot{x} = f(x)$ in a modelling region Ω. To be precise, considering every conceivable $\mathscr{C}^2$ Lyapunov function which might exist for a system with continuous $f(\cdot)$, with enough computational resources, the proposal will prove any point in the interior of the union of all level sets (see below) in Ω to be part of the DA.

The problem of estimating the DA has been partially addressed in prior literature. Indeed, if $0 \in \Omega$, level sets of Lyapunov functions for which $\dot{V} < -\gamma x^T x$, $\gamma > 0$, for all $x \in \Omega$, $x \neq 0$, belong to the DA; this is the approach pursued in most stability analysis proposals in literature (Khalil 2002); these level sets are usually "tangent" to the boundary of Ω and have been already extended to the piecewise case

(Gonzalez and Bernal 2016). However, the DA can contain significant portions of the boundary of Ω if the trajectories "point" towards its interior; hence, standard level-set results can be expanded (Pitarch, Sala, C.V. Ariño, and Bedate 2012). Also, a related approach was pursued in (Pitarch, Sala, and C.V. Ariño 2014) in the polynomial-fuzzy arena, introducing the idea of getting progressively better estimates of the domain of attraction by subtracting already-proven estimates. More recently, with non-piecewise models but piecewise Lyapunov functions, a shape-independent approach for maximal DA computation for TS systems has been presented in (C. Ariño, Perez, et al. 2014); in (Hu and Blanchini 2010; Y. Chen et al. 2015) a piecewise Lyapunov function defined by the minimum or maximum of quadratics (or higher-order polynomials) is considered. However, in such cases the delimitation of the regions is not fixed a priori and the problem ends up being a bilinear matrix inequality (BMI).

The most related prior-literature work on the ideas here is (Gonzalez and Bernal 2016), based on exact piecewise affine TS models (PWATS) and iteratively changing the modelling region Ω. The work here presented generalises (Gonzalez and Bernal 2016), by considering the fact that level sets can exit Ω, introducing more general multipliers, exploiting previously proven DA estimates (lifting decrescence and continuity constraints inside them), and modifying the above-mentioned iterations on the modelling region shape accounting for the more powerful results, within an LMI framework. The proposal in this investigation, based on the Farkas lemma, is asymptotically exact; hence, if a particular point belongs to the interior of the "true" DA, a suitable fine enough partition will prove it to belong to the DA.

This work is organized as follows: extensive preliminaries are introduced in section 5.2, covering the definition of DA, the different TS piecewise modelling options, basic results on piecewise stability, and the relevance of the Positivstellensatz (S-procedure) argumentation; in section 5.3 new results and algorithms are inferred that generalise previous approaches for estimation of the DA; the important subject of asymptotic exactness of the proposed results is treated in section 5.4; illustrative examples are given along the contents of the chapter. Conclusions in section 5.5 gather some final remarks, and an appendix collects the proofs of the main results.

5.2 Preliminaries

Consider an autonomous nonlinear model

$$\dot{x}(t) = f(x(t)) \tag{5.1}$$

with $x(t) \in \mathbb{R}^n$ as the state vector and $f(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ being a $\mathscr{C}^2$ nonlinear vector field, i.e., with continuous second partial derivatives. By assumption, the origin will be an equilibrium point, i.e., $f(0) = 0$. The solution of (5.1) for initial condition x_0 will be denoted as $\phi(t, x_0)$.

The *domain of attraction* (Khalil 2002) of $x = 0$ for (5.1) is the set

$$\mathcal{D} := \{x \in \mathbb{R}^n : \lim_{t\to\infty} \phi(t, x) = 0\}. \tag{5.2}$$

5.2.1 Affine Fuzzy Modelling

The well-known sector nonlinearity technique (Taniguchi, Tanaka, and H. Wang 2001) allows finding an equivalent Takagi-Sugeno model in a compact set Ω of the state space including the origin. This work considers regions which do not contain the origin; the sector-nonlinearity ideas can be generalised to such a case, following (Gonzalez, Sala, Bernal, and Robles 2015).

Indeed, as f is linearisable at the origin, denoting as A its Jacobian, we can rewrite f in (5.1) as

$$f(x) = Ax + \sum_{j=1}^{p} M_j \rho_j(x) \tag{5.3}$$

with $\rho_j : \mathbb{R}^n \mapsto \mathbb{R}$, for $j = \{1, 2, \ldots, p\}$, being some nonlinearities whose linearisation is zero[1], and M_j being column vectors indicating how nonlinearity ρ_j enters in each of the equations of (5.1). As Ω is compact and f is $\mathscr{C}^2$, each ρ_j can be bounded in Ω by two affine functions:

$$\underline{z}_j(x) \le \rho_j(x) \le \bar{z}_j(x) \tag{5.4}$$

where:

$$\bar{z}_j(x) = a_1^j H_j x + b_1^j, \quad \underline{z}_j(x) = a_0^j H_j x + b_0^j, \tag{5.5}$$

being a_i^j, b_i^j scalars, and H_j row vectors, configuring arbitrarily tight linear bounds on $\rho_j(x)$. Once the bound (5.4) is available, we can express:

$$\rho_j(x) = \sum_{i=0}^{1} w_i^j(x) \left(a_i^j H_j x + b_i^j \right) \tag{5.6}$$

[1] There is no loss of generality, as the Jacobian (first-derivatives) can be embeeded in A; for instance, $\sin(x) = x + g(x)$, with $g(x) = \sin(x) - x$, $\partial g/\partial x = 0$.

being the memberships given by the well-known interpolation expression:

$$w_0^j(x) := \frac{\overline{z}_j(x) - \rho_j(x)}{\overline{z}_j(x) - \underline{z}_j(x)}, \quad w_1^j(\cdot) := 1 - w_0^j(\cdot). \tag{5.7}$$

Operating with all ρ_j, for $j \in \{1, 2, \ldots, p\}$, then $r = 2^p$ membership functions can be defined as

$$h_i(x) := \prod_{j=1}^{p} w_{i_j}^j(x), \tag{5.8}$$

with $i \in \{1, 2, \ldots, r\}$, building a binary-digit expression of i as $i = i_p \times 2^{p-1} + \ldots + i_2 \times 2 + i_1 + 1$, $i_j \in \{0, 1\}$. Obviously, the MFs hold the convex sum property, i.e., $\sum_{i=1}^{r} h_i(x) = 1$, $h_i(x) \geq 0$. Using such memberships, (5.1) can be expressed as:

$$\dot{x} := \sum_{i=1}^{r} h_i(x) \left(Ax + \sum_{j=1}^{p} M_j \left(a_{i_j}^j H_j x + b_{i_j}^j \right) \right) \tag{5.9}$$

If the standard shorthand notation $\Upsilon_h := \sum_{i=1}^{r} h_i(z(t)) \Upsilon_i$ is adopted, from (5.9), denoting $A_i := A + \sum_{j=1}^{p} M_j a_{i_j}^j H_j$ and $b_i := \sum_{j=1}^{p} M_j b_{i_j}^j$, the nonlinear model (5.1) in Ω can be compactly written as the following affine-TS model:

$$\dot{x}(t) = A_h x(t) + b_h, \quad x(t) \in \Omega, \tag{5.10}$$

Several options for affine piecewise TS modelling are available; the examples worked out in this chapter used the minimum-weighted area approach in described in the following subsection.

5.2.2 Minimum-Weighted Area Piecewise Affine Takagi-Sugeno Models

Sector-nonlinearity TS models come from bounding a single-variable nonlinearity $\rho(x)$ between two sectors defined by lines crossing the origin $y = a_1 x$ and $y = a_2 x$, in such a way that

$$a_1 x \leq \rho(x) \leq a_2 x, \quad x \geq 0 \tag{5.11}$$
$$a_2 x \leq \rho(x) \leq a_1 x, \quad x \leq 0 \tag{5.12}$$

Given that different inequalities hold for either side of the origin, as we are considering "piecewise" models, we will restrict our modelling proposal to regions in which the origin is *not* in their interior, in order to propose affine modelling with just one of the conditions above, i.e., either (5.11) or (5.12) but not both.

In contrast with ordinary PWTS models, affine modelling is based on bounding nonlinearities between hyperplanes that do not necessarily pass by the origin. In the above scalar case, affine modelling will require bounding the nonlinearity as:

$$a_1 x + b_1 \leq \rho(x) \leq a_2 x + b_2 \tag{5.13}$$

for some a_1, a_2, b_1, b_2, for all $x \in \Omega_k$, generalising (5.11).

Obviously, any two linear functions bounding the nonlinearity as (5.13) cannot intersect within the 2D region

$$R = \{(x, y) : x \in \Omega_k,\ y = \rho(x)\} \tag{5.14}$$

so, given that the bounding is made with straight lines, evidently, such bounds cannot intersect with $co(R)$ where $co(\cdot)$ denotes the convex hull of an arbitrary set. Hence, for the PWATS model to be non-conservative, the lines $a_1 x + b_1$ and $a_2 x + b_2$ should be chosen between those delimiting the convex hull of R.

However, there are many of those possible lines. So, in order to generate a systematic way of obtaining them, an optimisation criteria should be chosen. One possible option would be to choose the two lines in which the covered area is smallest. However, given that LMIs are somehow using “linear” system results implicitly, a “weighted area” is proposed as:

$$A(a_1, b_1, a_2, b_2) = \int_{x_{min}}^{x_{max}} \frac{(a_2 - a_1)\, x + (b_2 - b_1)}{x} dx \tag{5.15}$$

because in this way, the closer we are to the origin, the more important the accuracy of the model is. So, the “optimal” piecewise affine TS model proposed is the one that minimises $A(a_1, b_1, a_2, b_2)$ subject to constraints (5.13). Actually, the integral above can be easily carried out, resulting in:

$$A(a_1, b_1, a_2, b_2) = (a_2 - a_1)\,(x_{max} - x_{min}) + (b_2 - b_1)\,(\ln|x_{max}| - \ln|x_{min}|)\,.$$

Note that the formula (5.15) is undefined if $x_{min} \leq 0 \leq x_{max}$, so the modelled region cannot contain the origin. However, the following result gives an interesting insight on the proposed affine modelling criterion:

LEMMA 5.2.1. *If $0 < x_{min} < x_{max}$, then, if $x_{min} \to 0$, the obtained PWATS model tends to the piecewise sector-nonlinearity TS model.*

Proof outline. As the weight of the points close to the origin tends to infinity (in fact, the integral does not converge for $x_{min} \to 0$, –this is intentional–), the optimal model tends to the one closing the sector the most possible, i.e., the sector nonlinearity one. □

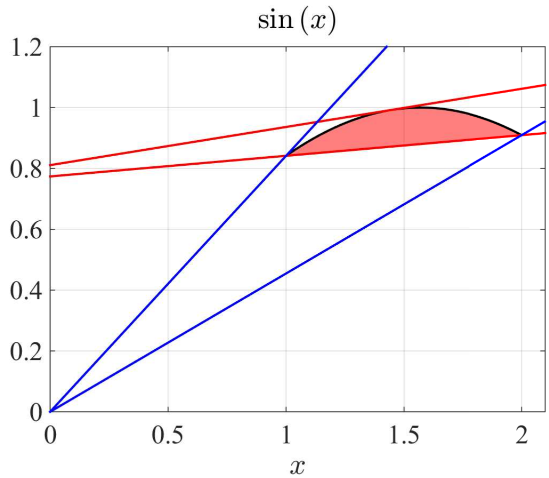

Figure 5.1: Sine bounded by two sectors in [1,2].

Once we have the optimal parameters, the PWATS is the one given by:

$$\rho(x) = w_1 \bar{a}_1(x) + (1 - w_1)\bar{a}_2(x) \tag{5.16}$$

where

$$w_1 = \frac{a_2 x + b_2 - \rho(x)}{a_2 x + b_2 - (a_1 x + b_1)}, \quad w_2 = 1 - w_1$$

$$\bar{a}_1(x) = a_1 x + b_1, \quad \bar{a}_2(x) = a_2 x + b_2$$

Example 5.2.1. *Consider the nonlinearity $\rho(x) = \sin(x)$, which must be modeled by an affine TS model in $\Omega = [1, 2]$. The Figure 5.1 depicts the nonlinearity (in black solid line), i.e. the region R, as well as the convex hull of R in red filling. Bounding the set by two lines is, of course, not unique. For instance, if we selected two lines intersecting at the origin we would get a TS model, whose bounding vertices would be given by the equations of the blue lines:*

$$\bar{a}_1(x) = \sin(1) \cdot x, \quad \bar{a}_2(x) = \sin(2)/2 \cdot x$$

If we select the "minimum weighted area" affine model, we would get the two bounds depicted with red lines, given by:

$$\bar{a}_1(x) = 0.0678 \cdot x + 0.7736, \quad \bar{a}_2(x) = 0.1255 \cdot x + 0.8108,$$

as in (5.16).

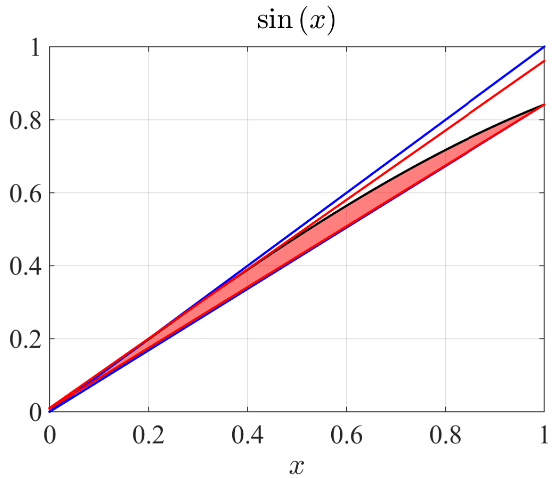

Figure 5.2: Sine bounded by two sectors in [0.05,1].

For comparison, if the modelling region were closer to the origin, for instance $\Omega = [0.05, 1]$*, then the TS model and the optimal weighted-area affine one would be much closer (see Figure 5.2), as expected by Lemma 5.2.1: the red lines would match the blue ones when the lower bound of* Ω *tends to zero.*

Of course, while getting an n-th order PWATS model with p nonlinearities, each of them should be rewritten as in (5.16). Once this is done for each nonlinearity and each region, an structure of the sort (5.10) arises, whose validity will hold for a partitioned region of interest Ω. The number of rules will be a power of two, as usual in standard TS modelling, too.

Piecewise Affine TS models

Consider a connected modelling region Ω, which is partitioned into q subregions with disjoint interiors, Ω_k, $k \in \{1, 2, \ldots, q\}$, i.e.,

$$\bigcup_{k=1}^{q} \Omega_k = \Omega, \qquad \text{int}(\Omega_k) \cap \text{int}(\Omega_l) = \emptyset.$$

If the above-discussed affine fuzzy modelling techniques are used, we can express the original nonlinear dynamics as a piecewise affine TS model (PWATS) (Johansson, Rantzer, and Arzen 1999) in the form[2]:

$$\begin{aligned} \dot{x}(t) &= A_h^k x(t), \quad x(t) \in \Omega_k, \quad k \in K_0, \\ \dot{x}(t) &= A_h^k x(t) + b_h^k, \quad x(t) \in \Omega_k, \quad k \in K_1, \end{aligned} \tag{5.17}$$

where $K_0 := \{k : 0 \in \Omega_k\}$ is the set of indexes of those regions Ω_k that include the origin and $K_1 := \{k : 0 \notin \Omega_k\}$ is the set of indexes of the remaining ones (not containing the origin).

For later analysis, each of the regions Ω_k will be described by a set of constraints $\Omega_k := \{\sigma_j^k(x) \geq 0,\ j \in \{1, 2, \ldots, n_k\}\}$. If $\sigma_j^k(x)$ are affine functions of x, the partition of Ω is a so-called *polyhedral* partition; these polyhedral partitions are the ones appearing in the seminal literature (Johansson, Rantzer, and Arzen 1999); non-polyhedral partitions with circular boundaries are considered in (Gonzalez, Bernal, and Marquez 2014). Polyhedral partitions of the state space have the form $\sigma_k(x) := \bar{E}_k \bar{x} \succeq 0$, where $\bar{E}_k = \begin{bmatrix} E_k & e_k \end{bmatrix}$, $x \in \Omega_k$, $k \in \{1, 2, \ldots, q\}$. A systematic procedure for their construction is described in (Johansson, Rantzer, and Arzen 1999; Hedlund and Johansson 1999). Note that if $e_k = 0$ the inequality $E_k x \succeq 0$ defines a polyhedral cone with its vertex at the origin.

For each region Ω_k, all constraints can be joined in a vector of functions $\sigma_k(\cdot) := \begin{bmatrix} \sigma_1^k(\cdot) & \ldots & \sigma_{n_k}^k(\cdot) \end{bmatrix}^T$; thus, we could define $\Omega_k = \{x : \sigma_k(x) \succeq 0\}$, where "$\succeq 0$" stands for element-wise "greater than 0".

5.2.3 Lyapunov-based domain of attraction estimation for PWATS

Classical estimates of the domain of attraction of the origin resort to well-known invariant set ideas such as Lyapunov level sets (Khalil 2002). The Lyapunov level-set concept can be generalised including prior estimates of the DA. In particular, the following result will be later exploited:

Theorem 5.2.1 ((Pitarch, Sala, and C.V. Ariño 2014))**.** *Consider two sets A, B, such that $B \subset A$. If A is invariant and there exist $\gamma > 0$ and $V(x)$, bounded in A, such that $\dot{V}(x) < -\gamma$ for all $x \in (A - B)$, where $A - B := \{x | x \in A, x \notin B\}$, then all trajectories starting in A enter B in finite time.*

LMIs in stability analysis of TS systems usually resort to expressions of the form $A_i^T P + PA_i < 0$. Let us review some already-known stability results for PWATS systems.

[2] In this work, as in (Johansson, Rantzer, and Arzen 1999), upper indexes of matrix expressions such as k in A_h^k are *not* powers, but only for indexation purposes.

Defining an augmented state and augmented matrices:

$$\bar{x} := \begin{bmatrix} x \\ 1 \end{bmatrix}, \ \bar{A}_i^k := \begin{bmatrix} A_i^k & b_i^k \\ 0 & 0 \end{bmatrix}, \ i \in \{1, 2, \dots, r\}, \ k \in K_1. \tag{5.18}$$

The PWATS stability analysis in (Johansson, Rantzer, and Arzen 1999) can be straightforwardly applied if Ω_k conform to a polyhedral partition of the operating region in the state space.

To see this, consider PWQLFs of the form

$$V(x) := \bar{x}^T \bar{P}_k \bar{x}, \quad x \in \Omega_k, \tag{5.19}$$

so that $V(0) = 0$, with continuity of the Lyapunov function across the boundaries, i.e., $V_k(x(t)) = V_l(x(t))$, $\forall x(t) \in (\Omega_k \cap \Omega_l)$, guaranteed by parameterising $\bar{P}_k$ as

$$\bar{P}_k := \bar{F}_k^T T \bar{F}_k, \tag{5.20}$$

where T is a symmetric matrix of adequate dimensions, $\bar{F}_k = \begin{bmatrix} F_k & f_k \end{bmatrix}$ with $f_k = 0$ for $k \in K_0$, satisfying $\bar{F}_k \bar{x} = \bar{F}_l \bar{x}$ for $x \in (\Omega_k \cap \Omega_l)$, $k, l \in \{1, 2, \dots, q\}$. Partition information can be systematically incorporated into the analysis via the S-procedure (Boyd et al. 1994). Notation $\mathbf{I}_\gamma := \text{blkdiag}(\gamma I, 0)$, and $\mathbf{0}_\gamma := \text{diag}(0, 0, \dots, 0, \gamma)$ will be later used. "blkdiag$(\cdot)$" stands for a square block-diagonal matrix in which the diagonal elements are the matrices in the argument. Thus, the following slight generalisations of (Johansson, Rantzer, and Arzen 1999; Gonzalez and Bernal 2016) are given:

Theorem 5.2.2. *If there exist symmetric matrices T, $U_k \succeq 0$, and $W_{ki} \succeq 0$ such that, for a given small $\gamma > 0$, the LMIs*

$$\begin{gathered} \bar{P}_k - \bar{E}_k^T U_k \bar{E}_k \geq \mathbf{I}_\gamma \\ \left(\bar{A}_i^k\right)^T \bar{P}_k + \bar{P}_k \bar{A}_i^k + \bar{E}_k^T W_{ki} \bar{E}_k \leq -\Phi_\gamma^k \end{gathered} \tag{5.21}$$

hold for $i \in \{1, 2, \dots, r\}$, being $\Phi_\gamma^k = \mathbf{I}_\gamma$ if $k \in K_0$, and $\Phi_\gamma^k = \mathbf{0}_\gamma$ if $k \in K_1$, then $x(t)$ tends to zero exponentially for every continuous differentiable piecewise trajectory in $\Omega = \bigcup_{k=1}^q \Omega_k$ satisfying the model equations (5.17) with initial conditions $x_0 \in V_\beta$, where $V_\beta := \{x : V(x) < \beta\}$ is any level set of the piecewise $V(x)$ defined in (5.19) such that $V_\beta \subset \Omega$.

Proof outline. First condition proves $V(x) > \gamma x^T x$ in region Ω_k, and second one proves $\dot{V}(x) \leq -\gamma x^T x$ in regions Ω_k, $k \in K_0$, and $\dot{V}(x) \leq -\gamma$ in regions Ω_k, $k \in K_1$. □

Remark 5.2.1. *From* (5.19)*, in regions containing the origin ($k \in K_0$), $V(x)$ is a standard quadratic form without constant or linear terms. As quadratic forms are positive in cones, only the set of conditions with $\bar{E}_k = \begin{bmatrix} E_k & 0 \end{bmatrix}$ are relevant if $k \in K_0$. In the original reference (Johansson, Rantzer, and Arzen 1999), conditions* (5.21) *were separated in two groups according to $k \in K_0$ or $k \in K_1$; however, such separation is implicitly considered in Φ_γ above. In fact, in a region where $e_k = 0$ and the model is given by TS representation $\bar{A}_i^k = blkdiag(A_i^k, 0)$, LMIs* (5.21) *would entail the Lyapunov function to be forcedly homogeneous quadratic if $V(0) = 0$ were enforced. Due to this reason, such separation between K_0 and K_1 will be no longer pursued in this work.*

Theorem 5.2.2 has been extended to the case of non-polyhedral partitions with circular boundaries in the conference paper (Gonzalez, Bernal, and Marquez 2014). For brevity, it will not be discussed here as it will be a particular case of the proposal in this work.

5.2.4 Farkas Lemma and Positivstellensatz

The above-reviewed prior results can be understood as proving positiveness of quadratic functions in regions with affine/quadratic boundaries; they are instances of the Positivstellensatz argumentation (Jarvis-Wloszek et al. 2005, Theorem 1), which in the quadratic-only case amount to the S-procedure (Boyd et al. 1994), and in the affine-only case are a version of Farkas lemma (Jönsson 2001). Computationally, conditions are posed as linear programming (affine case), LMIs (quadratic case) or generic sum-of-squares constraints (Jarvis-Wloszek et al. 2005). However, the latter exacerbates the computational cost, so it is intentionally left out of the scope of this thesis.

Decision variables U_k and W_{ki} are generically known as *multipliers*. In general, the above multiplier-based conditions are only sufficient for emptiness of semialgebraic sets or for sign-definiteness of some polynomial functions of the state in particular regions[3].

However, there are a few well-known situations in which *exact* results can be asserted with few computational resources. These situations are: the S-procedure with a single quadratic constraint, and the Farkas Lemma for affine constraints (in linear programming setups). The latter can be stated as:

[3]More general conditions may be obtained by transforming the multipliers into polynomials of arbitrary degree; however, as pointed out at the introduction, it is at the expense of a heavy computational cost (Pitarch, Sala, and C.V. Ariño 2014).

LEMMA 5.2.2 ((Jönsson 2001)). *Consider an affine function $V(x) = p^T x + \delta$, where $p \in \mathbb{R}^{n\times 1}$ and $\delta \in \mathbb{R}$, and a polyhedral region $\Omega := \{\sigma(x) \succeq 0\}$ being $\sigma(x) := \begin{bmatrix} E & e \end{bmatrix} \bar{x}$, where $E \in \mathbb{R}^{N\times n}$ and $e \in \mathbb{R}^{N\times 1}$. Let $\sigma_l(x)$ be the l-th element of vector $\sigma(x)$. Then, the following expressions are equivalent:*

a) $V(x) = p^T x + \delta \geq 0$ *for all* $x \in \Omega$

b) There exist $\tau_l \geq 0$, $l \in \{1, 2, \ldots, N\}$ *such that*

$$V(x) - \sum_{l=1}^{N} \tau_l \sigma_l(x) \geq 0, \quad \forall x \in \mathbb{R}^n \tag{5.22}$$

Corollary 5.2.2.1. *Under the same settings, the following expressions are equivalent:*

a) $V(x) = p^T x + \delta = 0$ *for all* $x \in \Omega$, *and* $\Omega \neq \emptyset$.

b) There exist arbitrary τ_l, $l \in \{1, 2, \ldots, N\}$ *such that*

$$V(x) - \sum_{l=1}^{N} \tau_l \sigma_l(x) = 0, \quad \forall x \in \mathbb{R}^n \tag{5.23}$$

Proof. The result can be proved considering $V(x) = 0$ as $V(x) \geq 0$, $-V(x) \geq 0$, and applying twice the above lemma, i.e., for $V(x) = 0$, $\forall x \in \Omega$, there exist $c_1 \geq 0$ and $c_2 \geq 0$ such that:

$$V(x) - \sum_{l=1}^{N} \tau_l' \sigma_l(x) = c_1 \geq 0, \; -V(x) - \sum_{l=1}^{N} \tau_l^* \sigma_l(x) = c_2 \geq 0$$

where linearity of V forces c_1 and c_2 being constants. Adding, we would have:

$$-\sum_{l=1}^{N} (\tau_l' + \tau_l^*)\sigma_l = c_1 + c_2$$

but, if we assume the region Ω is not empty, the above cannot happen unless $c_1 = c_2 = 0$ (standard Positivstellensatz). Now, subtracting and dividing by 2, we obtain:

$$V - \sum_{l=1}^{N} \frac{1}{2}(\tau_l' - \tau_l^*)\sigma_l = 0$$

so $\tau_l = 0.5(\tau_l' - \tau_l^*)$. □

In the next sections, earlier results will be generalised using the ideas in Sections 5.2.4 and 5.2.3; asymptotical exactness of the proposed approach will be established via universal-approximation argumentations.

5.3 Main Results

Let us consider a connected modelling region Ω partitioned into q subregions Ω_k with disjoint interiors where each region is defined[4] as:

$$\Omega_k = \left\{x : E_k\bar{x} \succeq 0,\ \bar{x}^T Q_{lk}\bar{x} \geq 0, l \in \{1, 2, \ldots, \ell_k\}\right\} \tag{5.24}$$

where $\bar{x}$ is obtained from x using (5.18). The j-th affine constraint, corresponding to the j-th row of E_k will be denoted as E_{jk}[5]. The "faces" of Ω_k will be defined by changing just one of the affine or quadratic inequalities to equality.

If $Q_{lk} = 0$, or, equivalently, $\ell_k = 0$, the partition will be said to be *polyhedral*. Given that the regions have disjoint interior by assumption, the intersection of two regions Ω_k and Ω_l must be a subset of a face in each of them. The region Ω_k will have a number of vertices located at the intersection of n faces.

5.3.1 Continuity in the Piecewise Lyapunov Function

Continuity of the piecewise Lyapunov function was enforced via (5.20) in prior works. A more flexible alternative will be proposed next. Consider a non-empty set

$$\mathcal{X} := \{\bar{x} : E\bar{x} = 0, \bar{x}^T Q_1\bar{x} = 0, \bar{x}^T Q_2\bar{x} = 0, \ldots, \bar{x}^T Q_{\bar{\ell}}\bar{x} = 0\},$$

such that $\Omega_k \cap \Omega_m \subset \mathcal{X}$, for some k, m.

LEMMA 5.3.1. *The piecewise quadratic function*

$$V(x) = \begin{cases} \bar{x}^T\bar{P}_k\bar{x} \text{ for } x \in \Omega_k, \\ \bar{x}^T\bar{P}_m\bar{x} \text{ for } x \in \Omega_m, \end{cases}$$

is continuous in the "face" $\Omega_k \cap \Omega_m$ if, given $\mathcal{X}$ in the above form such that $\Omega_k \cap \Omega_m \subset \mathcal{X}$, there exists an arbitrary multiplier matrix U and arbitrary scalars

[4] For notational simplicity, denoting constraints associated to regions containing the origin with E_k, and those where $0 \notin \Omega_k$ with $\bar{E}_k$ (established in (Johansson, Rantzer, and Arzen 1999)), will no longer be used. All matrices in (5.24) will be assumed to apply on the extended state $\bar{x}$. In this way cluttering all matrices with barred notation is avoided while leaving $\bar{E}$ available for future definitions.

[5] Following notation in (Johansson, Rantzer, and Arzen 1999), indexes will be stacked together in order to avoid long expressions; system matrices will use upper and lower ones.

τ_j *such that:*

$$\bar{P}_k - \bar{P}_m + UE + E^T U + \sum_{j=1}^{\bar{\ell}} \tau_j Q_j = 0 \tag{5.25}$$

Proof. Since $0 = \bar{x}^T(\bar{P}_k - \bar{P}_m + UE + E^T U + \sum_{j=1}^{\bar{\ell}} \tau_j Q_j)\bar{x} = \bar{x}^T(\bar{P}_k - \bar{P}_m)\bar{x}$ for $\bar{x} \in \mathcal{X}$, then the result is trivial. □

In this way, matrices F and decision variables T parameterising the sought Lyapunov functions, used in prior literature, are not needed in this proposal, giving more clarity and flexibility, in exchange for additional multipliers.

Remark 5.3.1. *Note that, from analytical prolongation (or Taylor series), if two functions coincide on an infinitesimal fragment of a face (i.e., a small lower-dimensional affine or quadratic region), they do on all prolongations. This is the reason of considering the above set $\mathcal{X}$ which disregards inequalities in $\Omega_k \cap \Omega_m$ (for instance, with $\Omega_1 = \{9 - x^T x \geq 0, x^T x - 1 \geq 0, x_2 \geq 0\}$, $\Omega_2 = \{1 - x^T x \geq 0, x_2 \geq 0\}$, we would have that $\Omega_1 \cap \Omega_2 = \{1 - x^T x = 0, x_2 \geq 0\}$, and $\mathcal{X} = \{1 - x^T x = 0\}$; adding a multiplier associated to constraint $x_2 \geq 0$ would be useless).*

5.3.2 Extension of piecewise quadratic stability analysis

In Theorem 5.2.2, taken from (Johansson, Rantzer, and Arzen 1999), only multipliers U_k in $E_k^T U_k E_k$ (and W_{ki}, with the same role) appeared to enforce local positiveness (negativeness) of the Lyapunov function (and its derivative).

However, we can state a more general condition.

Lemma 5.3.2. *Consider the set*

$$X := \left\{ x \in \mathbb{R}^n : \begin{array}{c} E\bar{x} \succeq 0 \\ \bar{x}^T Q_l \bar{x} \geq 0, l \in \{1, 2, \ldots, \ell\} \\ R\bar{x} = 0 \\ \bar{x}^T \mathcal{Q}_j \bar{x} = 0, j \in \{1, 2 \ldots, \bar{\ell}\} \end{array} \right\}$$

Consider, too, a quadratic polynomial $\bar{x}^T \Xi \bar{x}$. Then, $\bar{x}^T \Xi \bar{x} \geq 0$ for all $x \in X$ if there exist arbitrary scalars ξ_j, $j \in \{1, 2, \ldots, \bar{\ell}\}$, arbitrary matrix Z, positive scalars τ_l, $l \in \{1, 2, \ldots, \ell\}$, and element-wise positive matrix U such that the following matrix inequality holds:

$$-\Xi + \sum_{l=1}^{\ell} \tau_l Q_l + \bar{E}^T U \bar{E} + \sum_{j=1}^{\bar{\ell}} \xi_j \mathcal{Q}_j + Z^T R + R^T Z \leq 0 \tag{5.26}$$

where[6]

$$\bar{E} := \begin{bmatrix} [0\ 0\ \cdots\ 1] \\ E \end{bmatrix}. \tag{5.27}$$

Proof. Indeed, for any $x \in X$, we have

$$\sum_{l=1}^{\ell} \tau_l Q_l + \bar{E}^T U \bar{E} + \sum_{j=1}^{\bar{\ell}} \xi_j \mathcal{Q}_j + Z^T R + R^T Z \geq 0.$$

Hence, if (5.26) holds, it proves that $-\bar{x}^T \Xi \bar{x} \leq 0$ in X, i.e., $\bar{x}^T \Xi \bar{x} \geq 0$. □

Corollary 5.3.0.1. *Letting* $\Xi = diag(0, 0, ..., 0, -1)$*, if there exists the above-mentioned multipliers then* X *is empty.*

Proof. Indeed, we proved $0 \geq 1$ on X so forcefully X should be empty. □

Corollary 5.3.0.2. *If* $\bar{x}^T \Xi \bar{x}$ *is a degree-1 polynomial, and* X *is a full-dimensional polyhedron* ($Q_l = \mathcal{Q}_j = 0$, $R = 0$)*, then conditions in Lemma 5.3.2 are necessary and sufficient.*

Proof. It can be shown that the choice of multipliers encompasses those in Farkas lemma, i.e., the multipliers τ_l in (5.22) from Lemma 5.2.2. Details omitted for brevity. □

Remark 5.3.2. *The fact that the last element of* $\bar{x}$ *is equal to 1, as well as the seemingly "trivial" addition of* $1 \geq 0$ *in the construction of* $\bar{E}$*, introduces additional multipliers, which were not considered in prior literature; this enables the above generalisation and exactness in the affine case (Corollary 5.3.0.2). Without* $\bar{E}$*,* (5.26) *cannot be written as* (5.22) *in the polyhedral case* ($Q_l = Q_j = 0$)*. Apart, combined affine/quadratic boundaries are considered, as well as equalities which do not appear in* (5.24)*, but will be relevant when geometric conditions are pursued.*

Consider now a PWATS model (5.17) defined over a quadratic/polyhedral partition of a region Ω with sets $\Omega_k = \{x : \sigma_j^k(x) \geq 0, j \in \{1, 2, \ldots, n_k\}\}$, $k \in \{1, 2, \ldots, q\}$ defined as (5.24), i.e. being each of the constraints $\sigma_j^k(\cdot)$ either affine or quadratic.

The following definition will single out constraints which take part in the shape of the overall modelling region $\Omega = \cup_k \Omega_k$ defining its outer boundary:

[6] Recall $\bar{E}$ carrying the meaning in (Johansson, Rantzer, and Arzen 1999) is henceforth no longer in use.

Definition 5.3.1. *The face generated by constraint $\sigma_j^k(\cdot)$ will be denoted as:*

$$\mathcal{F}_j^k := \{x : \sigma_j^k(x) = 0\} \cap \Omega_k \tag{5.28}$$

Such face (and the constraint σ_j^k itself) is called "outer" if:

$$\mathcal{F}_j^k \not\subset \bigcup_{l \neq k} \Omega_l \tag{5.29}$$

An illustration of the meaning of the above definition appears on Figure 5.3, where outer faces are labelled with $\mathcal{F}_{out}^k$, $k = \{1, 2, 3, 4\}$.

Obviously, the boundary $\partial\Omega$, fulfills $\partial\Omega \subset \bigcup_{\mathcal{F}_j^k \text{ is outer}} \mathcal{F}_j^k$.

Let us denote as $\partial\Omega^{\downarrow}$ as the set of points in the boundary of Ω such that system trajectories which contain them "enter" Ω, i.e., in formal terms:

$$\partial\Omega^{\downarrow} := \{x \in \partial\Omega : \exists\omega > 0 \; s.t. \; \phi(\epsilon, x) \in \Omega \; \forall 0 < \epsilon < \omega\}$$

Let us denote as $\partial\Omega^{\uparrow}$ the complementary of $\partial\Omega^{\downarrow}$ in $\partial\Omega$, i.e., the points in the boundary of Ω such that trajectories do not immediately enter the interior of Ω.

For later use, we will denote the set of all outer constraints as:

$$\mathcal{I}_k := \{j : \sigma_j^k(\cdot) \text{ is outer}\}$$

Given an arbitrary point $x \in \partial\Omega_k$, let us denote as $\Gamma_k(x)$ the set of outer constraints in Ω_k which are active at x, i.e., the ones associated to the outer faces x belongs to:

$$\Gamma_k(x) := \{j \in \mathcal{I}_k : \sigma_j^k(x) = 0\}$$

Proposition 5.3.1. *Given $x \in \partial\Omega_k \cap \partial\Omega$, if $\dot{\sigma}_j^k(x) > 0$ for all $j \in \Gamma_k(x)$, then $x \in \partial\Omega^{\downarrow}$.*

Proof. First, note that, for the active constraints $\sigma_k^k(x) = 0$, $\dot{\sigma}_j^k(x) > 0$ entails $\sigma(\phi(\epsilon, x)) > 0$ for all ϵ such that $0 < \epsilon < \omega$ for small enough ω. Given that $\sigma(x) > 0$ for inactive constraints, then for small enough ω, $\sigma(\phi(\epsilon, x)) > 0$ will still hold for such constraints for all $0 < \epsilon < \omega$. Hence, no other constraint will be active and all active ones will render inactive: $\phi(\epsilon, x)$ will belong to the interior of Ω. □

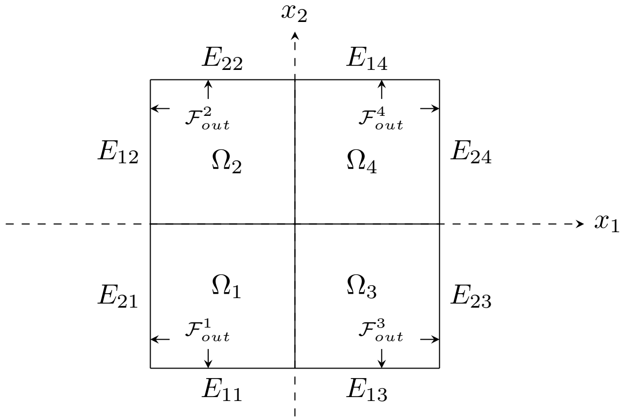

Figure 5.3: Bounding hyperplanes $\mathcal{F}^k_{out}$ delimiting Ω.

Consider, given x and the constraints indexed in $\Gamma_k(x)$, that a particular active constraint is either affine $\sigma^k_j(x) = E_{jk}\bar{x}$, being E_{jk} a row vector, or quadratic $\sigma^k_j(x) = \bar{x}^T Q_{jk}\bar{x}$, being Q_{jk} a matrix of adequate size.

Corollary 5.3.0.3. *Given $x \in \partial\Omega_k \cap \partial\Omega$, if, for all $i \in \{1, 2, \dots r\}$, for all $j \in \Gamma_k(x)$ either:*

- $E_{jk}\bar{A}^k_i\bar{x} > 0$, *if $\sigma^k_j(\cdot)$ is affine, or*
- $\bar{x}^T \left(Q_{jk}\bar{A}^k_i + (\bar{A}^k_i)^T Q^T_{jk}\right)\bar{x} > 0$ *if $\sigma^k_j(\cdot)$ is quadratic,*

then $x \in \partial\Omega^{\downarrow}$.

Proof. The conditions on the vertices of the PWATS model are sufficient to ensure that conditions in Proposition 5.3.1 hold, as $\dot{\bar{x}}$ belongs to the convex hull of the vertex derivative estimates $\bar{A}^k_i\bar{x}$. □

Now, we are in conditions to state the main result of the chapter.

Theorem 5.3.1. *Consider a nonlinear system* (5.1), *and a PWATS model* (5.17) *of it, defined over a partition of a compact region* Ω *with sets* Ω_k, $k \in \{1,2,\dots,q\}$ *defined as in* (5.24). *Consider, too, a collection of ellipsoids* $\mathcal{E}_s^k = \{x : \bar{x}^T \bar{G}_{ks}\bar{x} > 0\}$ *for* $s \in \{1,2,\dots,\bar{s}_k\}$, *such that* $\mathcal{E}_s^k \cap \Omega_k$ *belongs to the DA of* $x=0$ *for the nonlinear system* (5.1), *and a second collection of ellipsoids* $\hat{\mathcal{E}}_{js}^k = \{x : \bar{x}^T \hat{G}_{kjs}\bar{x} > 0\}$, $s \in \{1,2,\dots,\hat{s}_{kj}\}$, *associated to each face* $\mathcal{F}_j^k$ *such that* $\hat{\mathcal{E}}_s^k \cap \mathcal{F}_j^k$, *too, belongs to the DA of* $x=0$. *Then, if there exist symmetric matrices* $\bar{P}_k$ *satisfying the continuity conditions*[7]

$$\bar{x}^T \bar{P}_k \bar{x} = \bar{x}^T \bar{P}_m \bar{x},\ \forall x \in (\Omega_k \cap \Omega_m)\,, \tag{5.30}$$

symmetric matrices $U_{ki}^1 \succeq 0$, $U_{kji}^2 \succeq 0$, *arbitrary row vectors* Z_{jk}, *positive scalars* τ_{kl}^1, τ_{ks}^2, τ_{kl}^3, τ_{ks}^4, τ_{kji}^5, τ_{kjs}^6, $i \in \{1,2,\dots,r\}$, *and arbitrary scalars* τ_{kj}^7, $j \in \mathcal{I}_k$, $m \in \{1,2,\dots,q\}$, *yielding a feasible solution for the following inequalities, given* $\gamma > 0$, *first:*

$$\bar{P}_k \bar{A}_i^k + \left(\bar{A}_i^k\right)^T \bar{P}_k + \bar{E}_k^T U_{ki}^1 \bar{E}_k + \sum_{l=1}^{\ell_k} \tau_{kl}^1 Q_{lk} - \sum_{s=1}^{\bar{s}_k} \tau_{ks}^2 \bar{G}_{sk} \leq -\Phi_\gamma^k; \tag{5.31}$$

being $\Phi_\gamma^k = \mathbf{I}_\gamma$ *if* $k \in K_0$, *and* $\Phi_\gamma^k = \mathbf{0}_\gamma$ *if* $k \in K_1$;

and, second, either, if $\sigma_j^k = E_{jk}\bar{x}$ *(affine constraints):*

$$Z_{jk}^T E_{jk} + (*) + \bar{P}_k - \mathbf{E}_{kji}^T U_{kji}^2 \mathbf{E}_{kji} - \sum_{l=1}^{\ell_k} \tau_{kl}^3 Q_{lk} + \sum_{s=1}^{\bar{s}_k} \tau_{ks}^4 \bar{G}_{sk} + \sum_{s=1}^{\hat{s}_{kj}} \tau_{kjs}^6 \hat{G}_{kjs} \geq 0, \tag{5.32}$$

where

$$\mathbf{E}_{kij} = \begin{bmatrix} \bar{E}_k \\ -E_{jk}\bar{A}_i^k \end{bmatrix}; \tag{5.33}$$

or, if $\sigma_j^k = \bar{x}^T Q_{jk}\bar{x}$ *(quadratic constraints):*

$$\tau_{kj}^7 Q_{jk} + \bar{P}_k - \bar{E}_k^T U_{kji}^2 \bar{E}_k + \tau_{kji}^5 \left(Q_{jk}\bar{A}_i^k + \left(\bar{A}_i^k\right)^T Q_{jk}^T \right) - \sum_{l=1}^{\ell_k} \tau_{kl}^3 Q_{lk} + \sum_{s=1}^{\bar{s}_k} \tau_{ks}^4 \bar{G}_{sk} + \sum_{s=1}^{\hat{s}_{kj}} \tau_{kjs}^6 \hat{G}_{kjs} \geq 0, \tag{5.34}$$

then, $\{x : \bar{x}^T \bar{P}_k \bar{x} < 0\} \cap \Omega_k$ *belongs to the DA of* $x=0$ *for every* k, *for the nonlinear system under study.*

[7] Which can be enforced via LMI conditions (5.25) on all shared faces.

Proof. Consider the regions $\hat{\mathcal{E}}_k := \bigcup_{j=1}^{n_k} \left(\bigcup_{s=1}^{\hat{s}_k} \hat{\mathcal{E}}_{js}^k \cap \mathcal{F}_j^k \right)$ and $\mathcal{E}_k := \Omega_k \cap \bigcup_{s=1}^{\bar{s}_k} \mathcal{E}_s^k$. Consider, too, the regions $\mathcal{E} := \bigcup_{k=1}^q \mathcal{E}_k$ and $\hat{\mathcal{E}} := \bigcup_{k=1}^q \hat{\mathcal{E}}_k$. Then, by assumption, each $\mathcal{E}_k$, $\hat{\mathcal{E}}_k$, and, evidently, the whole $\hat{\mathcal{E}}$, and $\mathcal{E}$ belong to the DA of the origin.

Using the argumentations in Lemma 5.3.2 with $\Xi = \dot{V}_k(x) + \gamma\|x\|^2$ and $V_k(x) := \bar{x}^T \bar{P}_k \bar{x}$, we can state that (5.31) ensures that the time derivative of $V_k(x)$ is strictly negative for nonzero x (lower or equal than $-\gamma\|x\|^2$), in $\Omega_k - \mathcal{E}_k$, because such set is given by:

$$\Omega_k - \mathcal{E}_k = \{x : E_k\bar{x} \succeq 0,\ \bar{x}^T Q_{lk}\bar{x} \geq 0,\ \bar{x}^T \bar{G}_{sk}\bar{x} \leq 0\},$$

for $l \in \{1, 2, \ldots, \ell_k\}$ and $s \in \{1, 2, \ldots, \bar{s}_k\}$, so suitable multipliers $U_{ki}^1 \succeq 0$, $\tau_{kl}^1 \geq 0$, $\tau_{ks}^2 \geq 0$ are introduced.

Let us discuss now inequality (5.32). In this case, we want to show that the level set $\{V_k(x) < 0\} \cap (\Omega_k - \mathcal{E}_k - \hat{\mathcal{E}}_k)$ does not intersect $\partial\Omega^\uparrow$, as $\partial\Omega^\uparrow$ is the subset of $\partial\Omega_k$ where the trajectories of the system do not immediately enter Ω.

In order to show that, we will combine Corollary 5.3.0.3 with Lemma 5.3.2, posing the conditions of $\bar{P} \geq 0$ for all x in the set $\partial\Omega^\uparrow \cap (\Omega_k - \mathcal{E}_k - \hat{\mathcal{E}}_k)$.

As $\partial\Omega^\uparrow \subset \bigcup_{j \in \mathcal{I}_k} \{x : \dot{\sigma}_j^k(x) \leq 0\}$ we can assert that, if the following assertion holds for all $j \in \mathcal{I}_k$:

$$\bar{P} \geq 0 \ \forall x \in \Sigma_j^k \tag{5.35}$$

where $\Sigma_j^k := \{x : \dot{\sigma}_j^k(x) \leq 0\} \cap (\Omega_k - \mathcal{E}_k - \hat{\mathcal{E}}_k)$, then $\bar{P} \geq 0$ on $\partial\Omega^\uparrow \cap (\Omega_k - \mathcal{E}_k - \hat{\mathcal{E}}_k)$.

Now, we replace Σ_j^k by the larger (shape-independent) set on which at least one of the vertices of the PWATS model proves $\dot{\sigma}_j^k(x) \leq 0$, as discussed on Corollary 5.3.0.3. Then, application of Lemma 5.3.2 for each of the outer constraints in (5.35) and model vertices yields conditions (5.32) if the constraint in consideration is affine, and (5.34) if it were quadratic.

Now, by considering all regions we have:

1. a continuous piecewise quadratic function $V(x)$, defined as $V_k(x) = \bar{x}^T \bar{P}_k \bar{x}$ in Ω_k;

2. $V(x)$ is non-increasing, i.e., for a sufficiently small ϵ, $V(x(t+\epsilon)) \leq V(x(t))$; actually $V(x(t+\epsilon)) < V(x(t))$ if $x(t) \neq 0$. Indeed, along the trajectories of the nonlinear system (5.1), $\dot{V} \leq 0$ if $x(t)$ is in the interior of any Ω_k; if $x(t)$

is in the boundary of several regions, we can ensure that:

$$D^+V(t) := \lim_{\epsilon\to 0^+} \frac{V(x(t+\epsilon)) - V(x(t))}{\epsilon} \leq \max_{k \text{ s.t. } x(t)\in\Omega_k} \dot{V}_k \leq 0 \tag{5.36}$$

3. $V(x)$ has a level-zero set $V_0 := \{V(x) < 0\}$ that verifies

$$V_0 \cap \left(\partial\Omega^{\uparrow} \cap (\Omega_k - \mathcal{E}_k - \hat{\mathcal{E}}_k)\right) = \emptyset.$$

Denoting $\boldsymbol{\mathcal{E}} := \mathcal{E} \cup \hat{\mathcal{E}}$, Let us define the following sets:

$$\mathcal{V} := \{V(x) < 0\} \cap \Omega, \quad \mathcal{W} := \mathcal{V} - \boldsymbol{\mathcal{E}}, \tag{5.37}$$

$$\mathcal{W}_{\neg\boldsymbol{\varepsilon}} := \{x \in \mathcal{W} : \phi(t,x) \notin \boldsymbol{\mathcal{E}}\ \forall t \geq 0\}, \mathcal{W}_{\boldsymbol{\varepsilon}} := \mathcal{W} - \mathcal{W}_{\neg\boldsymbol{\varepsilon}}. \tag{5.38}$$

With the above definition, $\mathcal{W}$ is the set of points who have not (yet) been proven to belong to the DA. Such set is partitioned in two: $\mathcal{W}_{\neg\boldsymbol{\varepsilon}}$, i.e., the set of points of $\mathcal{W}$ which do not enter $\boldsymbol{\mathcal{E}}$ in finite time, and $\mathcal{W}_{\boldsymbol{\varepsilon}}$.

Now, note that when starting in $\mathcal{W}$, it is impossible to abandon $\mathcal{W}$ without entering $\boldsymbol{\mathcal{E}}$, due to:

- As $V(x)$ is non-increasing in time in $\mathcal{W}$, the boundary $V(x) = 0$ will never be reached.
- As $\dot{\sigma}_j^k(x) > 0$ for all x lying both in the outer faces and in $V(x) < 0$ (proven due to the third of the above-enumerated conditions), trajectories cannot exit Ω through such outer faces.

Thus, all points in $\mathcal{W}$ either enter $\boldsymbol{\mathcal{E}}$ in finite time or remain indefinitely in $\mathcal{W}$. As the latter points are, by definition, those in $\mathcal{W}_{\neg\boldsymbol{\varepsilon}}$, forcedly $\mathcal{W}_{\boldsymbol{\varepsilon}}$ is the set of points who *do* enter $\boldsymbol{\mathcal{E}}$ in finite time.

Obviously, all $x \in \mathcal{W}_{\boldsymbol{\varepsilon}}$ belong to the DA of the origin, because they enter $\boldsymbol{\mathcal{E}}$ in finite time without leaving Ω, so they converge to the origin later on.

Let us prove that all $x \in \mathcal{W}_{\neg\boldsymbol{\varepsilon}}$ belong, too, to the DA of the origin. Indeed, $\mathcal{W}_{\neg\boldsymbol{\varepsilon}}$ is invariant, because trajectories always remain inside it in future time: they do not enter $\boldsymbol{\mathcal{E}}$ and, due to the above reasons, they do not exit $\mathcal{V}$, and they do not enter $\mathcal{W}_{\boldsymbol{\varepsilon}}$ because in such a case they would eventually enter $\boldsymbol{\mathcal{E}}$, which cannot happen by definition.

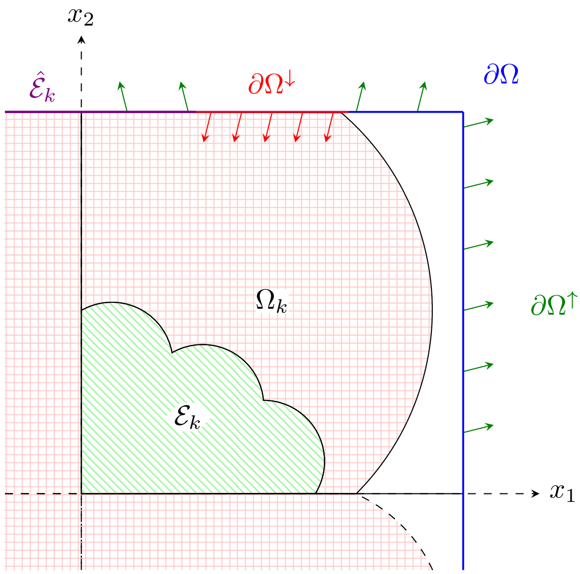

Figure 5.4: Subsets Ω_k, $\mathcal{E}_k$, $\hat{\mathcal{E}}_k$, $\partial\Omega$, and $\partial\Omega^{\downarrow}$.

As $V(x)$ is continuous, piecewise polynomial, it is bounded on $\mathcal{W}_{\neg\boldsymbol{\varepsilon}}$, i.e., there exist

$$V_{\min} := \inf_{x \in \mathcal{W}_{\neg\boldsymbol{\varepsilon}}} V(x), \quad V_{\max} := \sup_{x \in \mathcal{W}_{\neg\boldsymbol{\varepsilon}}} V(x).$$

Given any $x \in \mathcal{W}_{\neg\boldsymbol{\varepsilon}}$, as $V(\phi(t,x))$ is nonincreasing and bounded from below at all times, there must exist a limit $a := \lim_{t\to\infty} V(\phi(t,x))$, so, as a consequence $\lim_{t\to\infty} D^+V(\phi(t,x)) = 0$. As $\dot{V}_k(x) \leq -\gamma$ in regions Ω_k not containing the origin, and $\dot{V}_k(x) \leq -\gamma\|x\|^2$ if the region contains the origin, the only point in which such situation ($D^+V = \max_{k \text{ s.t. } x(t)\in\Omega_k} \dot{V}_k = 0$) can happen is the origin. So, all initial conditions $x \in \mathcal{W}_{\neg\boldsymbol{\varepsilon}}$ tend to the origin, i.e., belong to the DA of the origin[8]. Given that both $\mathcal{W}_{\boldsymbol{\varepsilon}}$ and $\mathcal{W}_{\neg\boldsymbol{\varepsilon}}$ belong to the DA of the origin, so does their union $\mathcal{W}$. □

Remark 5.3.3. *Theorem 5.3.1 requires a prior estimate of the DA of the origin $\mathcal{E}$. In order to apply the above result to prove stability of a PWATS model without such "initialisation" (to get results with the same a priori assumptions as usual literature), the theorem should be modified by setting $\bar{G}_{ks} = 0$, thus initialising the ellipsoids $\mathcal{E}_s^k$ to* empty *sets (equivalently, forgetting about the terms with G in the LMIs, letting $\bar{s}_k = 0$). The result is as follows.*

[8]Note that, if $0 \in \boldsymbol{\mathcal{E}}$, forcefully $\mathcal{W}_{\neg\boldsymbol{\varepsilon}} = \emptyset$; this is in accordance with Theorem 5.2.1.

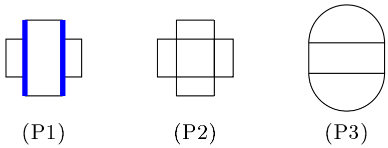

Figure 5.5: Example partitions: (P1) is not a honeycomb; (P2,P3) are.

Corollary 5.3.1.1. *A PWATS model* (5.17)*, defined over a partition of a region* Ω *with sets* Ω_k*,* $k \in \{1, 2, \ldots, q\}$ *defined as in* (5.24)*, is locally stable if there exist decision variables fulfilling Theorem 5.3.1 with* $\bar{s}_k = 0$ *and* $\hat{s}_{kj}$*, such that the set* $\mathcal{V}$ *in* (5.37) *is not empty.*

Proof. Indeed, applying the prior theorem, $\{x : \bar{x}^T \bar{P} \bar{x} < 0\} \cap \Omega$ belongs to the DA of $x = 0$ and, by assumptions in the corollary statement, it is not empty. In this particular situation, contrarily to footnote 8, the set $\mathcal{W}_{\boldsymbol{\varepsilon}}$ would be empty, and $\mathcal{W} = \mathcal{V} = \mathcal{W}_{\neg\boldsymbol{\varepsilon}}$, actually containing the origin, deduced with an identical argumentation to the one in the theorem's proof for this particular case $\boldsymbol{\mathcal{E}} = \emptyset$. □

Note that non-emptiness of $\mathcal{V}$ can be enforced in the LMI conditions with some geometric conditions. This is the objective of next subsection.

In order to avoid conservatism, we will assume that the chosen partition conforms a *honeycomb* (Coexeter 1973), defined as a partition where vertices of the regions are common to neighboring ones (a region Ω_j will be understood to be neighboring to Ω_k if $\Omega_j \cap \Omega_k \neq \emptyset$, $\text{int}(\Omega_j) \cap \text{int}(\Omega_k) = \emptyset$; vertices will be the points formed by intersection of n faces).

For instance, Figure 5.5 shows a partition (P1) which does not fulfill the honeycomb assumption, and a pair of another ones which do. The reason of such assumption is that the faces of the central region in partition (P1) (marked as a thick blue line) are outer, so the theorem would preclude a level set including the subset of the face where trajectories enter the neighboring regions, which is clearly undesired. The second partition (P2) is a honeycomb and such issue does not appear. Partition (P3) is, too, a honeycomb with quadratic boundaries.

5.3.3 Geometric optimisation

In order for the theorem to be useful, some additions enforcing how to obtain the "largest" estimate of the domain of attraction should be added, for instance, maximising the size of some prefixed-shape set which can be fit inside the obtained DA estimate (via maximisation of scaling factors).

Consider a prefixed-shape region in the form:

$$\hat{\Omega} := \{x : \tilde{E}\bar{x} \succeq 0, \bar{x}^T\tilde{Q}_1\bar{x} \geq 0, \ldots, \bar{x}^T\tilde{Q}_{\tilde{q}}\bar{x} \geq 0\}$$

where some affine inequalities (rows of $\tilde{E}$) and $\tilde{q}$ quadratic ones hold. Let us define the geometric transformation below:

$$\bar{x}_\lambda := \begin{bmatrix} x_c + \lambda^{-1}(x - x_c) \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda^{-1}I & x_c - \lambda^{-1}x_c \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} = \Lambda\bar{x},$$

being λ a "scaling factor" and x_c a "scaling centroid", both parameters assumed known. The scaled region $\tilde{\Omega}$ by factor λ around x_c is defined as:

$$\tilde{\Omega}(\lambda) := \{x : \tilde{E}\Lambda\bar{x} \succeq 0, \bar{x}^T\Lambda^T\tilde{Q}_1\Lambda\bar{x} \geq 0, \ldots, \bar{x}^T\Lambda^T\tilde{Q}_{\tilde{q}}\Lambda\bar{x} \geq 0\}. \tag{5.39}$$

Note that setting $x_c = 0$ reduces the scaling to the standard scaling around the origin.

Theorem 5.3.2. *Consider a PWATS model* (5.17) *defined over a partition of a region Ω with sets Ω_k, $k \in \{1, 2, \ldots, q\}$ defined as in* (5.24)*. Consider, too, a collection of ellipsoids $\mathcal{E}_s^k = \{x : \bar{x}^T\bar{G}_{sk}\bar{x} > 0\}$ for $s \in \{1, 2, \ldots, \bar{s}_k\}$, such that $\mathcal{E}_s^k \cap \Omega_k$ belongs to the DA of $x = 0$ for the nonlinear system* (5.1)*, and a second collection of ellipsoids $\hat{\mathcal{E}}_{js}^k = \{x : \bar{x}^T\hat{G}_{kjs}\bar{x} > 0\}$, $s \in \{1, 2, \ldots, \hat{s}_{kj}\}$, associated to each face $\mathcal{F}_j^k$ such that $\hat{\mathcal{E}}_s^k \cap \mathcal{F}_j^k$, too, belongs to the DA of $x = 0$. Then, if there exist symmetric matrices $\bar{P}_k$, $U_{ki}^1 \succeq 0$, $U_{kji}^2 \succeq 0$, $U_k^3 \succeq 0$, $U_k^4 \succeq 0$, arbitrary column vectors Z_{jk}, arbitrary scalars τ_{kj}^7, and positive scalars τ_{kl}^1, τ_{ks}^2, τ_{kl}^3, τ_{ks}^4, τ_{kji}^5, τ_{kjs}^6, $\tau_{k'l}^8$, $\tau_{k'm}^9$, $\tau_{k's}^{10}$, $i \in \{1, 2, \ldots, r\}$, $j \in \mathcal{I}_k$, $m \in \{1, 2, \ldots, \hat{q}\}$, yielding a feasible solution for the inequalities* (5.30)*, either* (5.31) *or* (5.32)*,* (5.34)*, and, for a given k', and $\gamma > 0$:*

$$\bar{P}_{k'} + \bar{E}_{k'}^T U_{k'}^3 \bar{E}_{k'} + \Lambda^T \tilde{E}_{k'}^T U_{k'}^4 \tilde{E}_{k'} \Lambda + \sum_{l=1}^{\ell_{k'}} \tau_{k'l}^8 Q_{lk'} + \sum_{m=1}^{\hat{q}} \tau_{k'm}^9 \Lambda^T \tilde{Q}_m \Lambda - \sum_{s=1}^{\bar{s}_{k'}} \tau_{k's}^{10} \bar{G}_{sk'} \leq -\mathbf{0}_\gamma, \tag{5.40}$$

then, the region $\hat{\Omega}(\lambda) \cap \Omega_{k'}$ belongs to the DA of $x = 0$.

Proof. In this case, we want to show that $\hat{\Omega}(\lambda)$ belongs to the domain of attraction of $x = 0$, by showing that it is included in the subset of the DA proven in Theorem 5.3.1, where constraints for the level set $\mathcal{V}$ for being part of the DA are enforced ((5.30), (5.31), (5.32), (5.34)).

We want to enforce that the region $\mathcal{E}_{k'} \cup \{\bar{x}^T \bar{P}_{k'} \bar{x} < 0\} \cap \Omega_k$ contains $\hat{\Omega}(\lambda) \cap \Omega_{k'}$. We will do that by proving that $\bar{x}^T \bar{P}_{k'} \bar{x} \leq -\gamma$ in $(\Omega_{k'} - \mathcal{E}_{k'}) \cap \hat{\Omega}(\lambda)$. Indeed, if that holds, all points of $\hat{\Omega}(\lambda) \cap \Omega_{k'}$ either lie in $\mathcal{E}_{k'}$ or in $\{\bar{x}^T \bar{P}_{k'} \bar{x} < 0\} \cap \Omega_k$, both belonging to the DA of the origin.

Thus, conditions for inclusion of $\bar{x}^T \bar{P}_{k'} \bar{x} \leq -\gamma$ in the required set are written as (5.40) by using the S-procedure argumentation and positive multipliers $\tau^8_{k'm}$ associated to the quadratic constraints in $\hat{\Omega}(\lambda)$, $U^4_{k'}$ associated to the linear inequalities in $\hat{\Omega}(\lambda)$, $U^3_{k'}$ and $\tau^7_{k'l}$ associated to the corresponding region Ω_k, and positive constants $\tau^9_{k's}$ associated to ellipsoids $\mathcal{E}^k_s$. □

Note that $\hat{\mathcal{E}}$ has not been used in conditions (5.40); indeed, such $\hat{\mathcal{E}}$ is formed by fragments of outer faces with no volume, but $\hat{\Omega}(\lambda) \cap \Omega_{k'}$ will have nonzero volume except in degenerate cases, so behaviour at the faces is irrelevant for the level sets of $\bar{P}_{k'}$ in Ω_k.

Remark 5.3.4. *The above theorem can be extended to forcing shape constraints in several regions, by repeating* (5.40) *for different k' in a selected set (or even all of them). The fixed-shape conditions above can be particularised to spherical regions, polytopes (boxes), or intersections thereof, extending analogous geometrical conditions in LMI setups for classical (non-affine) TS systems (Boyd et al. 1994; Tanaka and H. Wang 2001).*

Remark 5.3.5. *Theorem 5.3.2 provides only feasibility conditions. Trivially, they can be converted to optimisation ones on the centroid/size "shape" parameters (x_c, λ). If only one of them is to be optimised (either scale or translation), such optimisation setups can be cast as bisection problems and, in some particular cases as GEVP ones or even LMI ones in Lyapunov and shape parameters. Such developments are transcriptions to the affine case of well-studied geometric problems*[9] *and are omitted for brevity, leaving details to particular examples later.*

The following corollary shows that our result extends prior literature.

[9]For instance, the smallest or largest circle inside an ellipsoid, the largest ellipsoid inside a polytope, etc. in (Boyd et al. 1994).

Corollary 5.3.2.1. *In the polyhedral partition case, if LMIs in Theorem 5.2.2 are feasible, and Ω contains a neighborhood of the origin, then conditions on Corollary 5.3.1.1 hold for some non-empty domain of attraction.*

Proof. Suppose that a feasible solution $\{P_k^{Joh},\, U_k^{Joh},\, W_{ki}^{Joh}\}$ for (5.21) has been obtained, i.e.:

$$\begin{gathered}\bar{P}_k^{Joh} - \bar{E}_k^T U_k^{Joh} \bar{E}_k \geq \mathbf{I}_\gamma, \\ \left(\bar{A}_i^k\right)^T \bar{P}_k^{Joh} + \bar{P}_k^{Joh} \bar{A}_i^k + \bar{E}_k^T W_{ki}^{Joh} \bar{E}_k \leq -\Phi_\gamma^k.\end{gathered} \tag{5.41}$$

We will prove that there exist some $\beta > 0$ such that V_β in Theorem 5.2.2 belongs to the DA of the origin, provable with Theorem 5.3.1. As the level set considered in the latter theorem is in the form $\{\bar{x}^T \bar{P}_k \bar{x} < 0\}$, whereas the condition $x^T \bar{P}_k^{Joh} \bar{x} \geq I_\gamma$ in Theorem 5.2.2 would need level sets in the form $\{\bar{x}^T \bar{P}_k^{Joh} \bar{x} < \beta\}$, we will consider $\bar{P}_k = \bar{P}_k^{Joh} - \mathbf{0}_\beta$, without loss of generality, for some β. In this way, $\{\bar{x}^T \bar{P}_k^{Joh} \bar{x} < \beta\} \equiv \{\bar{x}^T \bar{P}_k \bar{x} < 0\}$.

Consider inequality (5.31). As partition is polyhedral then $\ell_k = 0$ and if the prior estimates of the DA are empty, then $\bar{s}_k = 0$ and $\hat{s}_{jk} = 0$. Furthermore if only the rows E_k are considered from $\bar{E}_k$, the result is the second LMI in (5.41), with the notational changes in footnote 4. As subtracting a constant from the Lyapunov function does not influence its derivative (algebraically, it can be proved from the fact that the last row of $\bar{A}_i^k$ is zero), Johansson's multipliers W_{ki}^{Joh} would render (5.31) feasible (padded with zeros to conform the larger size of $\bar{E}_k$).

Consider now that the first inequality in (5.41) holds. Then, we will prove that there exists $\beta > 0$ and arbitrary row-vector multipliers Z_{jk} such that

$$Z_{jk}^T E_{jk} + (*) + (\bar{P}_k^{Joh} - \mathbf{0}_\beta) - E_k^T U_k^{Joh} E_k \geq 0, \tag{5.42}$$

where the above expression has been obtained from (5.32) removing the absent elements Q_{lk}, $\bar{G}_{sk}$, $\hat{G}_{kjs}$, and also setting the multiplier for the term $E_{jk}\bar{A}_i^k$ in U_{kji}^2 equal to zero (hence, the original multiplier U_{kji}^2 no longer depends on i, j), setting the remaining terms equal to the corresponding ones in U_k^{Joh}.

Indeed, consider the problem of finding E_{jk} such that the following expression is feasible for all outer constraints E_{jk}:

$$Z_{jk}^T E_{jk} + (*) + \operatorname{blkdiag}(\gamma I, -\beta) \geq 0, \tag{5.43}$$

The above problem is feasible if the circle $\gamma x^T x \leq \beta$ is inside Ω. So, if there exists a circle around the origin which is contained in Ω, true by assumption, a feasible solution for (5.43) exists. Now, adding the first matrix inequality of (5.41) and

(5.43) results in (5.42), proving that (5.32) was feasible in Theorem 5.3.1 with the choice of multipliers in (5.42).

In summary, the above argumentation proves that if (5.41) are feasible, so they are (5.31) and (5.32). Continuity is also enforced in Johansson's result, so we proved that Theorem 5.3.1 is feasible in all cases (5.41) is, for suitable Ω. ☐

Our proposal, apart from giving the same (or better) solutions as Theorem 5.2.2 in an identical setting, improving over (Johansson, Rantzer, and Arzen 1999; Gonzalez and Bernal 2016), applies to regions with quadratic boundaries, it is less conservative (due to $\bar{E}$, and to the fact that the level set can get "out" of Ω) and, last, Ω can even *not* contain the origin as long as a fraction of it is proven (elsewhere) to belong to the DA of the origin.

Example 5.3.1. *(Pitarch, Sala, and C.V. Ariño 2014) Consider the following nonlinear system:*

$$\dot{x}_1 = 0.5x_2 - 3x_1, \quad \dot{x}_2 = (3\sin x_1 - 2)\, x_2 \tag{5.44}$$

where the state is assumed to lie in the compact set $\Omega = \{x : |x_i| \leq 1.2, i = 1,2\}$*. Consider a partition of the compact set* Ω *in* $q = 16$ *subsets, as it is shown in Figure 5.6. An initial estimation of the DA was obtained using a quadratic Lyapunov function and a standard 2-rule TS model resulting from choosing* $\rho(x) = 3\sin(x_1)x_2$*, computed in a smaller modelling region* $\Omega_{TS} = \{x : |x_i| \leq 0.72, i = 1,2\}$*. The resulting largest level set in* Ω_{TS} *is given by* $\mathcal{V}_Q = \{x : x^T\bar{P}_Q x < 0\}$*, with:*

$$\bar{P}_Q = \begin{bmatrix} 1.9104 & -0.2365 & 0 \\ -0.2365 & 1.9104 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Such level set is depicted in red in the referred figure.

Now, a PWATS model has been generated with the same choice of $\rho(x)$ *applying the optimisation setup discussed in (Gonzalez, Sala, Bernal, and Robles 2015). Theorem 5.3.2 was applied in order to find the largest circle* $\tilde{\Omega}(\lambda) = \{x : -\lambda^{-1}x^Tx+1 \geq 0\}$ *inside the proven domain of attraction, minimising* λ^{-1} *by bisection, stating conditions* (5.40) *for all the regions. The knowledge that the red region already belonged to the DA has been exploited in the LMI conditions. In Figure 5.6, the larger resulting level-set* $\mathcal{V}$ *is shown in green. The level set intersects with the frontier of* Ω*, as the theorem allows for it; the only regions out of it are the top and bottom right white zones.*

For comparison, a estimation of the DA using classical Theorem 5.2.2 for the same PWATS model is shown in blue. In this case, level sets from earlier results cannot exit Ω*.*

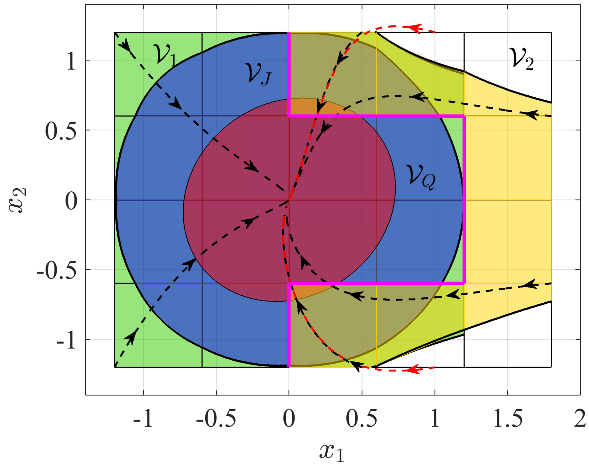

Figure 5.6: Estimation of the DA for example 5.3.1: quadratic TS case (region in red, (Tanaka and H. Wang 2001)), Thm. 5.2.2 (region in blue, (Gonzalez and Bernal 2016)), and Thm. 5.3.2 (region in green). Yellow region also depicts the result of a second execution of Theorem 5.3.2 only on the squares at the right of the magenta boundary, seeding it with the prior green region.

Last, the 8 squares containing the yellow regions in the figure are used in a new estimation of the DA with a partition which does not *contain the origin but contains as initial DA estimates both the prior green piecewise-ellipsoidal fragments conforming $\mathcal{E}$ and the magenta lines conforming $\hat{\mathcal{E}}$. With the same geometric objective, the referred yellow region can be proved to belong to the domain of attraction*[10]*. Some simulated trajectories show that, indeed, the DA estimate is correct.*

5.3.4 Iterative Enlargement of the Domain of Attraction

The basic idea in this section is proving a large DA estimate by modifying Ω as the region proved with Theorem 5.3.2 grows larger, removing "empty" regions (in order to be less conservative at next iteration), and adding new neighboring regions around the ones that contain any points in the proven DA, i.e. around those in which there exists an ellipsoid $\mathcal{E}_s^k$ such that $\mathcal{E}_s^k \cap \Omega_k \neq \emptyset$. In order to carry out such operation, the following result will be used:

[10] Actually, as complete faces are in the DA, instead of being considered in $\hat{\mathcal{E}}_k$, they can be equivalently removed from the set of outer faces, details omitted for brevity.

LEMMA 5.3.3. *Consider a region Ω_k defined as in* (5.24) *and a collection of ellipsoids $\mathcal{E}_s^k = \{x : \bar{x}^T \bar{G}_{ks} \bar{x} > 0\}$ for $s \in \{1, 2, \dots, \bar{s}_k\}$. Then, the two assertions below are true:*

a) if $\exists \tau_s^1 \geq 0, \tau_l^2 \geq 0$, $U = U^T \succeq 0$ such that

$$\mathbf{0}_1 - \sum_{l=1}^{\bar{s}_k} \tau_s^1 \bar{G}_{ks} + \hat{E}_k^T U \hat{E}_k + \sum_{l=1}^{\ell_k} \tau_l^2 Q_{lk} \leq 0,$$

then $\Omega_k \subset \cup_s^{\bar{s}_k} \mathcal{E}_s^k$.

b) if $\exists \tau_s^1 \geq 0, \tau_{l,s}^2 \geq 0$, $U_s = U_s^T \succeq 0$ such that, for all s,

$$\mathbf{0}_1 + \tau_s^1 \bar{G}_{ks} + \hat{E}_k^T U_s \hat{E}_k + \sum_{l=1}^{\ell_k} \tau_{l,s}^2 Q_{lk} \leq 0,$$

then $\Omega_k \cap \cup_s^{\bar{s}_k} \mathcal{E}_s^k = \emptyset$.

Proof. The first condition a) proves that $\Omega_k \cap \left(\cap_{s=1}^{\bar{s}_k} \{x : \bar{x}^T \bar{G}_{ks} \bar{x} \leq 0\}\right)$ is empty from Corollary 5.3.0.1, and therefore $\Omega_k \subset \cup_s^{\bar{s}_k} \mathcal{E}_s^k$, because $\cap_{s=1}^{\bar{s}_k} \{x : \bar{x}^T \bar{G}_{ks} \bar{x} \leq 0\}$ is the set of $\bar{x}$ lying *outside* the union of the ellipsoids $\mathcal{E}_s^k$.

The second condition b) proves that $\Omega_k \cap \mathcal{E}_s^k = \emptyset$ for every s, from Corollary 5.3.0.1, and, hence, so it is $\Omega_k \cap \cup_s^{\bar{s}_k} \mathcal{E}_s^k = \emptyset$. □

If the ellipsoids are those in Theorem 5.3.1, Lemma 5.3.3 ensures that regions fulfilling the first LMI have been totally proven to belong to the DA, and regions fulfilling the second set of LMIs (one for each s) have no point in them proven to belong to the DA. The former ones will be labelled as "full" and the latter ones, as "empty".

Algorithm

Based on Theorem 5.3.1 and the discussed idea above, Algorithm 1 on top of next page is proposed, initialising on a prior feasible solution and iteratively improving the DA estimate by suitably modifying the partition (adding, removing and dividing regions). Some remarks are presented below detailing the ideas in some of its steps.

Algorithm 1. Start from a compact set $\Omega^{[0]}$ defined by a list of sets from a associated partition Ω_k, $k \in \{1, 2, \ldots, q\}$. Consider a previous estimate of the DA, see Remark 5.3.6, as a list of sets in the form $\mathcal{E}_k^{[0]} = \{x : \bar{x}^T \bar{G}_k^0 \bar{x} > 0\} \cap \Omega_k$. Set $c = 1$ and perform the following steps:

1. Test Lemma 5.3.3 for each region $\Omega_k \in \Omega^{[c-1]}$.
   (a) If a) is feasible, set full$(k) = 1$ else full$(k) = 0$.
   (b) If b) is feasible, set empty$(k) = 1$ else empty$(k) = 0$.
2. Generate the list of sets for a new partition $\Omega^{[c]}$, as follows:
   (a) If empty$(k) = 1$, then reject Ω_k, do not add it to $\Omega^{[c]}$;
   (b) Else, add Ω_k to the list $\Omega^{[c]}$, and enlarge the region of study adding to $\Omega^{[c]}$ a neighbouring region Ω', see Remark 5.3.7.
   (c) if full$(k) = 1$, Ω_k can be taken out, if so wished, from $\Omega^{[c]}$, if the steps in Remark 5.3.8 are taken.
3. Obtain a new PWATS model from the new region.
4. Obtain a PWQLF from Theorem 5.3.2 under some chosen geometric performance maximisation, see Remark 5.3.9.
5. If Theorem 5.3.2 is feasible, then add $\{x : \bar{x}^T \bar{P}_k^c \bar{x} < 0\} \cap \Omega_k$ to the list of sets conforming the current DA estimate, and set $c = c + 1$.
6. If Theorem 5.3.2 is not feasible, then subdivide some of the regions where empty$(k) = 0$ and full$(k) = 0$. See Remark 5.3.10.
7. Check a suitable termination criteria (see Remark 5.3.11), and if it not satisfied, go to Step 1.

Remark 5.3.6. *[*Initialization*] The algorithm will be initialised with any piecewise partition of an initial compact set $\Omega^{[0]}$ where a PWQLF has been obtained via a feasible solution of any LMI in literature, for instance:*

- *a single region with a TS model, as done in Example 5.3.1,*
- *a feasible piecewise-quadratic DA estimate from Johansson's Theorem 5.2.2 or, better,*
- *a solution from Corollary 5.3.1.1 (with some geometric optimisation, Theorem 5.3.2) with initial empty DA estimate, proved to be more general than Theorem 5.2.2.*

Remark 5.3.7. *[*Neighbouring region generation*] Depending on the geometry of the chosen partition (simplicial, parallelotopic, etc.), generating these new neighbouring regions might require different code implementations; in later examples, a particular hyper-cube-based setting will be explained, based on the fact that a space-filling tessellation is possible with congruent copies of any parallelotope.*

Remark 5.3.8. *[*Removing fully covered regions*] If* full(k)*=1, as the whole region is proved to belong to the DA of the origin, such a region can be actually removed from* $\Omega^{[c]}$ *in step 2 of Algorithm 1; in order to keep this information, the faces of neighboring regions can be "marked" to belong to the DA via suitable set up of ellipsoids* $\hat{\mathcal{E}}_k$*.*

Remark 5.3.9. *[*Geometric optimisation goal*] In general, there are no LMI conditions to maximise the volume of a piecewise estimation of the DA. An indirect way to achieve this goal is to maximise the radius of a sphere centered at the origin (Gonzalez, Sala, Bernal, and Robles 2015), but it may be inadequate for nonconvex regions. An alternative to the sphere-based maximisation is trying to maximise in a region the scaling* (5.39) *of a degenerate ellipsoid (with very small axis length in all directions but a random one) with a random center point.*

Remark 5.3.10. *[*Finer partition granularity*] As expected, there are several ways of dividing regions as to apply the algorithm above; in later examples in this work, the regions have been split into* 2^n *equal smaller parallelotopes. Obviously, other implementations may be conceivable, such as generating a random splitting direction for some regions.*

Remark 5.3.11. *[*Termination*] There might be different options to be used as termination criteria: (a) some geometric goal reached, or slow progress of it, (b) number of regions or computation time at step 4 above a predefined limit.*

Comparative analysis with other DA analysis proposals

In (Gonzalez and Bernal 2016), an algorithm to get progressively better estimates of the DA was given. Nevertheless, in contrast with Algorithm 1 above, the proposal in (Gonzalez and Bernal 2016) (a) is unable to establish asymptotical exactness (see next section); (b) it includes no geometrical optimisation conditions, thus stopping when any arbitrary piecewise Lyapunov set which fits the DA is found; (c) it is computationally over-demanding since at each step the whole region is reconsidered in the new partition. All these issues make the prior algorithm provide worse numerical results than the one here presented (see example below).

Example 5.3.2. *Consider the following nonlinear system*

$$\dot{x}_1 = -x_2, \tag{5.45}$$

$$\dot{x}_2 = x_1 - x_2 + x_2 x_1^2. \tag{5.46}$$

The system has one equilibrium point at the origin and one unstable limit cycle, which implies the DA is bounded by the latter. In order to obtain the largest

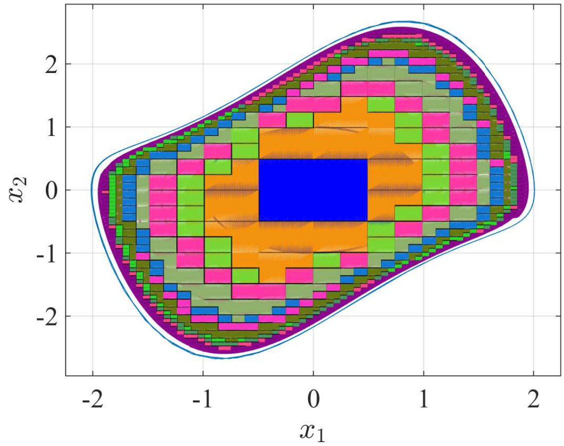

Figure 5.7: Estimation of the DA for Example 5.3.2.

possible estimate of the DA, Algorithm 1 comes at hand. We started it with the region $\Omega^{[0]} = \{x \in \mathbb{R}^2 : |x_i| \leq 0.99\}$, $i \in \{1, 2\}$, *on which a quadratic Lyapunov function has been used as an initial estimate of the DA.*

Figure 5.7 plots the limit cycle (the outermost blue closed curve was obtained with backwards-in-time simulation) and compares it with different estimates of the DA obtained by the iterations of Algorithm 1. The figure shows, in different colors, the estimate of DA for each iteration of Algorithm 1. Note that, in this example, the chosen geometry partition is based on a square tessellation, and we maximised the radius of a sphere center at the origin as the geometric optimisation goal. A colored square means that the entire region belongs to the DA. The different sizes of the regions are caused by the splitting into smaller squares at step 6 of the Algorithm. The region proven to belong to the DA is the union of all colored regions.

Figure 5.8 shows the DA estimate in Figure 5.7 as a red line, very close to the actual exact limit cycle (black line). For comparison, it also shows the result applying the approach in (Gonzalez and Bernal 2016) with a blue closed solid line. The approach in (Gonzalez and Bernal 2016) does not incorporate the geometric border conditions neither previous estimates, reaching a high computational cost with slow progress, obtaining inferior results. Both algorithms were stopped when 4 GB of memory were exhausted in the computations.

As the algorithm progresses, it gets progressively closer to the actual domain of attraction of the origin (the open set inside the limit circle). However, as the

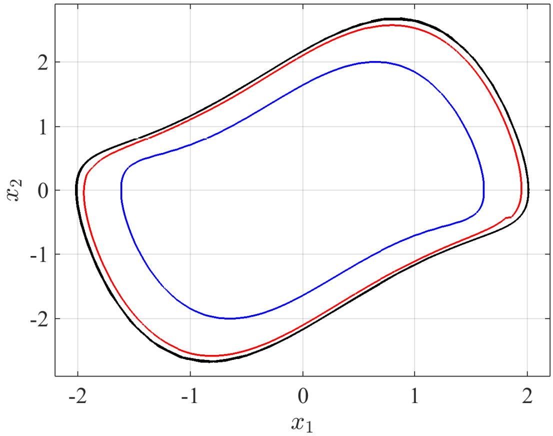

Figure 5.8: Estimation of the DA for Example 5.3.2 (Black: exact Limit Cycle; red: proposal here; blue: estimate in (Gonzalez and Bernal 2016)).

boundary of the limit cycle is not quadratic, we would, in theory, need an infinite amount of piecewise-quadratic fragments to approximate it, this is why the number of regions ends up increasing greatly.

Next section analysis in depth the algorithm behaviour when the number of regions increases: it can be proved that, under some assumptions, as the partitions get finer, the accuracy of the DA estimate improves, reaching *asymptotical exactness* i.e., limited only by finite computational resources in DA estimation (disturbances and controller design induce other limitations as more complex/BMI problems arise, out of the scope of this work).

5.4 Asymptotical exactness

In this section, Farkas Lemma (here recalled as Lemma 5.2.2) will allow to prove asymptotical exactness of the above algorithm: with enough computational resources, the algorithm is non-conservative in the precise sense to be discussed next.

Indeed, Theorems 5.3.1 and 5.3.2, obviously, apply to the particular case in which the Lyapunov function has the form

$$V_k(x) = \bar{x}^T \begin{bmatrix} 0 & 0 & \cdots & 0.5p_k^1 \\ 0 & 0 & \cdots & 0.5p_k^2 \\ \vdots & \vdots & \ddots & \vdots \\ 0.5p_k^1 & 0.5p_k^2 & \cdots & p_k^{n+1} \end{bmatrix} \bar{x} = \bar{x}^T \bar{P}_k \bar{x} \tag{5.47}$$

These Lyapunov functions are piecewise-affine, as $V_k(x) = \sum_{i=1}^{n} p_k^i x_i + p_k^{n+1}$, short-handed to PWALF. In this way, piecewise-polyhedral level sets could be proven to belong to the DA of the origin.

The key fact about the use of the above class of functions is that, due to Lemma 5.2.2, the proposed conditions in Theorem 5.3.1 are necessary and sufficient in the sense that, if conditions in the referred theorem with the above Lyapunov function structure (5.47) are not feasible then *there is no* PWALF for the set partition fulfilling the needed Lyapunov condition[11] with a *single* affine expression for the PWALF in each Ω_k. So, forcedly, the partition must be changed, because no other theorem would find a PWALF on it if Theorem 5.3.1 does not work.

The above idea, jointly with universal-approximation capabilities of PWALF and PWATS models as regions get smaller, allow to prove the following key result, which states that if there exists any smooth Lyapunov function proving that a particular point x^* belongs to the DA of the origin, a PWALF will also prove that x^* belongs to such DA for a fine enough partition.

LEMMA 5.4.1. *For any $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, there exist a fine enough partition of a compact set Ω such that a PWALF in the form (5.47), $V_{PW}(x) := V_k(x)$ for $x \in \Omega_k$, approximates any function V of class $\mathscr{C}^2$ and its gradient as follows, for all $x \in \Omega$:*

$$\|V_{PW}(x) - V(x)\| \le \varepsilon_1, \tag{5.48}$$

$$\|\nabla V_{PW}(x) - \nabla V(x)\| \le \varepsilon_2. \tag{5.49}$$

[11] Contrarily, in the quadratic case, such a Lyapunov function might exist but might be only provable to be so with higher-degree Positivstellensatz multipliers, requiring a Sum-of-Squares version of the theorems; anyway, there are also positive polynomials which are not SOS (Jarvis-Wloszek et al. 2005) so these conservatism sources cannot be removed in general, except in the above-referred affine case.

Proof. First, note that the gradient of a PWALF is a piecewise-constant function[12]. If a function $V(x)$ is of class $\mathscr{C}^2$, then its partial derivative ∇V is of class $\mathscr{C}^1$, meaning that ∇V is bounded in Ω and can be approximated by a piecewise constant function ∇V_{PW} to any arbitrary error ε_3, as piecewise constant functions are universal function approximators, as long as the partition is fine enough, so there exists $\psi(x)$ such that $\|\psi(x)\| \leq \varepsilon_3$ for all $x \in \Omega$ and $\nabla V(x) = \nabla V_{PW}(x) + \psi(x)$.

Integrating the gradient, we get:

$$V(x) = \int_0^1 \nabla V(\lambda x)^T x \, d\lambda = \int_0^1 (\nabla V_{PW}(\lambda x) + \psi(\lambda x))^T x d\lambda \tag{5.50}$$

where $\psi(\lambda x)$ is the approximation error, which verifies $\|\psi(\lambda x)\| \leq \varepsilon_3$. Hence,

$$V(x) = \int_0^1 \nabla V_{PW}(\lambda x)^T x \, d\lambda + \int_0^1 \psi(\lambda x)^T x \, d\lambda \tag{5.51}$$

so we can assert:

$$\|V(x) - V_{PW}(x)\| \leq \int_0^1 \|\psi(\lambda x)\| \cdot \|x\| \, d\lambda \leq \varepsilon_3 \|x\| \tag{5.52}$$

Choosing ε_3 such that $\varepsilon_1 \geq \max_{x \in \Omega} \varepsilon_3 \|x\|$, and $\varepsilon_3 \leq \varepsilon_2$, we can prove (5.48) and (5.49). As a result, we can approximate *both* ∇V and V as closely as desired by increasing the partition granularity. □

LEMMA 5.4.2. *For any $\varepsilon > 0$, there exist a fine enough partition of a compact set Ω such that, given a continuous function $f(x)$, a PWATS model can be obtained fulfilling:*

$$\begin{aligned} &\left\| \left(A_i^k x + b_i^k\right) - f(x) \right\| \leq \varepsilon, \\ &\forall i \in \{1, 2, \ldots, r\}, \forall x \in \Omega_k \end{aligned} \tag{5.53}$$

Proof. Consider the 2-rule PWATS model given by $A_i^k := 0$, $b_1^k := \min_{x \in \Omega_k} f(x)$, $b_2^k := \max_{x \in \Omega_k} f(x)$, where maximum and minimum have been considered to be computed element-wise (b_1^k and b_2^k are vectors) on a compact set Ω_k. As $f(x)$ is continuous, by assumption, there exists a fine enough partition such that $\|b_i^k - f(x)\| \leq \|b_2^k - b_1^l\| \leq \varepsilon$ for any arbitrary choice of ε. □

[12] Understanding the gradient at faces common to several regions to be defined as the average of the different piecewise gradients. As such faces are zero-measure sets, such formal definition will not have any influence in the integral-based results in the remaining of the proof.

Now, we can state the key result of this chapter, proving that we can be at least as good as any conceivable algorithm based on Lyapunov level-sets.

Theorem 5.4.1. *Let $x = 0$ be an asymptotically stable equilibrium point for the nonlinear system*

$$\dot{x} = f(x) \tag{5.54}$$

where $f : \Omega \to \mathbb{R}^n$ is locally Lipschitz, $\Omega \subset \mathcal{D}$ is compact. Assume that a (possibly small) polyhedron $\mathcal{B}$ containing the origin has been proved to belong to the DA, and define a compact set $\Theta := \Omega - int(\mathcal{B})$. If there exists a function $V : \Theta \to \mathbb{R}$, and $\varepsilon > 0$ such that:

1. *$V(x)$ is of class $\mathscr{C}^2$ in an open set including Ω.*
2. *$\dot{V}(x) = \dfrac{\partial V}{\partial x} \cdot f(x) \leq -\varepsilon$, for all $x \in \Theta$.*
3. *There exists a level set in the form $V_{\alpha_2} := \{x : V(x) \leq \alpha_2\}$, for some $\alpha_2 > 0$ such that $V_{\alpha_2} \subset \Omega$.*

Then, there exist a fine enough partition of Θ such that any PWATS model fulfilling conditions in Lemma 5.4.2 allows finding a PWQLF ($V_{PW}(x)$) which fulfills conditions in Theorem 5.3.1, and a level set of the PWQLF allowing to prove that any point in the interior of V_{α_2} belongs to the DA of the origin.

Proof. By Lemma 5.4.1, there exists a fine enough partition such that there exists a PWA function fulfilling: $\nabla V_{PW}(x) + \psi(x) = \nabla V(x)$, $\|\psi(x)\| \leq \varepsilon_3$, and, by Lemma 5.4.2, that for all vertices, for all regions there exists $\phi_i^k(x)$ such that $A_i^k x + b_i^k + \phi_i^k = f(x)$, $\|\phi_i^k(x)\| \leq \varepsilon_4$, for any $\varepsilon_4 > 0$. Then, we can state, denoting $f_i^k(x) := A_i^k x + b_i^k$, by continuity of $f(x)$ that there exists $\hat{f} := \max_{x\in\Omega} \|f(x)\|$, and by continuity of ∇V, that there exists $\hat{V} := \max_{x\in\Omega} \|\nabla V(x)\|$. Now, we have:

$$\begin{aligned}\nabla V_{PW}(x) f_i^k(x) &= (\nabla V - \psi(x))(f(x) - \phi_i^k(x)) \\ &= \nabla V \cdot f(x) - \psi(x) \cdot f(x) - \nabla V \cdot \phi_i^k(x) + \psi(x)\phi_i^k(x) \\ &\leq -\varepsilon + \varepsilon_2 \cdot \hat{f} + \varepsilon_4 \cdot \hat{V} + \varepsilon_4\varepsilon_2.\end{aligned}$$

So, for any $0 < \gamma' < \varepsilon$, a suitable choice of small enough ε_2 and ε_4 can prove that there exists a fine enough partition so that:

$$\nabla V_{PW}(x) f_i^k(x) \leq -\gamma'. \tag{5.55}$$

Now, from Farkas Lemma, the existence of the multipliers U_{ki}^1 in (5.31) in the affine case ($l_k = 0$, $\bar{s}_k = 0$) are a *necessary and sufficient* condition for (5.55) to hold, as the region Ω_k does not contain the origin by assumption. Regarding the multiplier-based continuity conditions (5.25), Corollary 5.2.2.1 ensures that they are also necessary and sufficient for the PWA case.

Last, regarding geometric conditions (level set), any point in the interior of V_{α_2} is in the (closed) level set α_1 for some $\alpha_1 < \alpha_2$.

Consider now $\varepsilon_1 < 0.5(\alpha_2 - \alpha_1)$. Then, select any choice of α such that $\alpha_1 + \varepsilon_1 < \alpha < \alpha_2 - \varepsilon_1$. In this way, given the above ε_1, there exists a fine enough partition so that (5.48) holds; hence, the level set of V_{PW}, denoted as $\tilde{V}_{PW}(\alpha) := \{x : V_{PW}(x) \leq \alpha\}$, fulfills

$$V_{\alpha_1} \subset \tilde{V}_{PW}(\alpha) \subset V_{\alpha_2} \tag{5.56}$$

because all $x \in V_{\alpha_1}$ will belong to the level set of V_{PW} given by $\tilde{V}_{PW}(\alpha_1 + \varepsilon_1)$, and also, all elements of the level set $\tilde{V}_{PW}(\alpha_2 - \varepsilon_1)$ will be included in V_{α_2}.

If a fine enough partition is chosen such that both (5.55) and (5.56) hold, we have found a PWALF fulfilling the required derivative conditions and including in a level set any desired point in the interior of the level set of the “true” Lyapunov function. If we consider that piecewise-affine Lyapunov functions are a particular case of piecewise-quadratic ones, the theorem is proved. □

Remark 5.4.1. *Note that, by Theorem 5.2.1, all trajectories of the nonlinear system inside the level set V_{α_2} will enter $\mathcal{B}$, because forcedly $V_{\alpha_2} \cap \mathcal{B} \neq \emptyset$, as the trajectories should abandon V_{α_2} in at most α_2/ε time units, and they cannot abandon Ω if they start in the interior of V_{α_2}. For any of such interior initial conditions, a PWQLF proving that it belongs to the DA of the origin can be found because of the same argumentations.*

Example 5.4.1. *As a last example, for the sake of comparison, consider the system in (Y. Chen et al. 2015, Example 3):*

$$\dot{x}_1 = -x_1 + x_1^2 + x_1^3 + x_1^2 x_2 - x_1 x_2^2 + x_2, \quad \dot{x}_2 = -\sin x_1 - x_2,$$

altogether with a PWATS model of it, (Gonzalez, Sala, Bernal, and Robles 2015), as an input to Algorithm 1. Figure 5.9 shows the DA estimate in the referred work (obtained via BMIs and SOS tools) with a red closed solid line whereas our estimate is shown with a green-coloured area. Clearly, our proposal reaches much better estimations than (Y. Chen et al. 2015), as expected due to the asymptotical exactness; however, region size needs to be decreased as the border of the “true” domain of attraction is approached, as discussed in earlier examples.

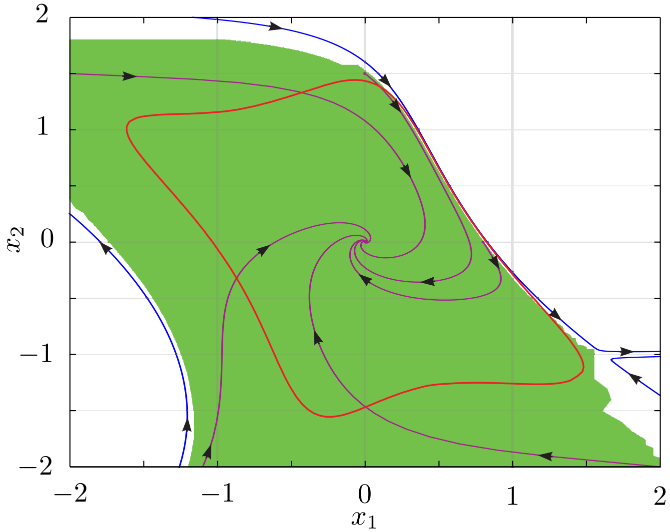

Figure 5.9: Estimation of the DA for Example 5.4.1. (Red: estimate in (Y. Chen et al. 2015); Green: proposal here; magenta: some trajectories inside the DA; blue: some trajectories outside DA.

Remark 5.4.2. *With prefixed regions, our proposal renders LMI conditions (even linear programming ones, in some cases) so the computational cost is basically identical to prior PWATS literature (increasing just a small amount due to the handful of extra multipliers proposed here). However, the actual DA of nonlinear systems is, in general,* not *piecewise quadratic, so the* exact *domain of attraction cannot be obtained with finite computational resources with our approach*[13]*.: as the required estimation accuracy increases, the number of regions must increase (with decreasing size). Hence, Theorem 5.4.1 can only prove that finite computational resources are needed to find a particular point in the* interior *of the "true" DA.*

5.5 Conclusion

In this chapter, an iterative linear matrix inequality methodology has been presented for estimation of the domain of attraction of a nonlinear model. The proposal, based on a systematic exploitation of geometrical and stability facts via piecewise affine Takagi-Sugeno models and piecewise Lyapunov functions, has been

[13] In fact, neither with any alternative conceivable approach: it is well known that nonlinear differential equations rarely admit explicit solutions (or DA expressions) in closed form, requiring numerical simulation (Slotine and W. Li 1991)

shown to outperform the most relevant works on the subject. Estimates of the domain of attraction have been increased by "emptying" previously proven regions and extending the modelling region in "promising" neighboring areas. Moreover, based on universal-approximation properties of TS models, it has been proved that the estimate of the domain of attraction approaches the level set of any existing $\mathscr{C}^2$ Lyapunov function of the original nonlinear system, as the partition where the piecewise TS model is obtained gets finer (smaller regions): the proposed procedures are asymptotically exact.

Chapter 6

Parameter-dependent Lyapunov function

This chapter is concerned with nonquadratic conditions for stabilization of continuous-time nonlinear systems via exact Takagi-Sugeno models and generalized parameter-dependent Lyapunov function. The approach hereby proposed feeds back the time derivatives of the membership functions through a multi-index control law that cancels out the terms responsible of former a priori local conditions. Thus, a nonquadratic controller design in the form of linear matrix inequalities is achieved; it does not require bounds on the time derivatives nor any extra parameters. The examples included are shown to outperform former approaches.

The contents of this chapter appeared in the journal article:

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6.1 Introduction

Among the variety of nonlinear control techniques, those based on exact convex representations have progressively gained the attention of the control community, due to their combination of mathematical formality and numerical applicability (T.M. Guerra, Sala, and Tanaka 2015). The simplest of these representations is the Takagi-Sugeno (TS) model (Takagi and Sugeno 1985), originally appeared in the fuzzy context for practical engineering problems (L. Wang 1997), later cast as a rewriting of nonlinearities into convex forms within a compact subset of the state space, a methodology referred to as the sector nonlinearity approach (Ohtake, Tanaka, and H. Wang 2001; Taniguchi, Tanaka, and H. Wang 2001). The convex structure allows the direct Lyapunov method to be applied (Tanaka and H. Wang 2001), which usually leads to conditions in the form of linear matrix inequalities (LMIs): these exhibit numerical advantages because they are efficiently solved via convex optimization techniques (Boyd et al. 1994), which are already implemented in commercially available software (Gahinet et al. 1995; Sturm 1999). Moreover, due to their exactness, conclusions drawn on the TS model of a nonlinear one, are directly valid for the latter (Z. Lendek, T.M Guerra, et al. 2010).

As in many other areas of control theory, quadratic Lyapunov functions $V = x^T(t)Px(t)$ were originally used because of their simplicity: results thus obtained remained sufficient, this is to say, with a certain degree of conservativeness (H. Wang, Tanaka, and Griffin 1996). Therefore, larger classes of Lyapunov functions that include the quadratic one as a particular case were tried: piecewise (Johansson, Rantzer, and Arzen 1999; Campos et al. 2013), line-integral (Rhee and Won 2006; Marquez, T.M. Guerra, et al. 2013), and parameter-dependent (also known as nonquadratic or fuzzy) (Blanco, Perruqueti, and Borne 2001). The latter class replaces the common positive-definite matrix P by a convex sum of positive-definite matrices P_i, weighted by the membership functions (MFs) in the TS model (those that capture the system nonlinearities and hold the convex sum property). While results in the discrete-time case made an impressive progress (T.M. Guerra and Vermeiren 2004; T.M. Guerra, Kruszewski, and Bernal 2009; Ding 2010; Z. Lendek, T.M. Guerra, and Lauber 2015), the use of parameter-dependent Lyapunov functions (PDLFs) in the continuous-time case was restrained.

The reason behind the stagnation of the nonquadratic continuous-time framework has been the appearance of the time derivatives of the MFs when a PDLF is involved (Tanaka, Hori, Taniguchi, et al. 2001): these derivatives cannot be directly cast as convex expressions and, when controller design is under consideration, they lead to algebraic loops, making it difficult to obtain LMI expressions. A way out of these issues has been found in the introduction of artificial a-priori bounds on the time derivatives of the MFs (D. Lee and D. Kim 2014) or in the LMI-imposed bounds on partial derivatives (Pan et al. 2012): both solutions are local.

Contribution: This chapter is concerned with nonquadratic controller design of nonlinear systems via exact TS representations, based on which a multi-index control law is proposed that feeds back the time derivatives of the MFs. In contrast with former approaches, it does not require a priori bounds on the derivatives (Tanaka, Hori, and H. Wang 2003; L. Mozelli, Palhares, and Avellar 2009; D. Lee and D. Kim 2014) nor in their partial form (T.M. Guerra, Bernal, et al. 2012; Pan et al. 2012); this is achieved via a suitable control law instead of restrictive path-independent conditions (Rhee and Won 2006). The proposal employs: (a) a generalized parameter-dependent Lyapunov function (GPDLF) (Bernal and T. M. Guerra 2010) along with a tensor-product notation in order to fully exploit Polya-like relaxations, which are asymptotically sufficient and necessary (Sala and Ariño 2007); (b) a generalized multi-index control law that cancels out the terms that cause a priori locality in the Lyapunov analysis; moreover, the resulting conditions are purely LMI. A preliminary version of this work has appeared in (Aguiar, Márquez, and Bernal 2015). Some conditions for regularity of the possible algebraic loops arising from derivative-feedback are proposed, as well as a robust-observer based implementation (following (Levant 1998)) for environments with bounded disturbances or modelling errors.

The contents in this chapter are now described. Section II introduces a multi-index notation for exact TS models and GPDLFs: the issues raised by former nonquadratic schemes are discussed in order to naturally lead the reader to the problem statement. In Section III a generalized multi-index control law that employs the time derivatives of the MFs is proposed: it is shown that, thanks to the control law structure, these derivatives can be directly obtained from the closed-loop model. Section IV provides examples on how the proposed methodology improves both the feasibility set of former approaches as well as the quality of solutions. This report concludes in Section V where final remarks and future work are discussed.

6.2 Preliminaries

A well-established procedure for convex rewriting of nonlinear systems within a compact set $\mathcal{C} \supset \{0\}$ of the state space, called the sector nonlinearity methodology (Taniguchi, Tanaka, and H. Wang 2001), is available; it considers nonlinear models of the form

$$\dot{x}(t) = f(z(x))x(t) + g(z(x))u(t), \tag{6.1}$$

with $f(\cdot)$ and $g(\cdot)$ being nonlinear vector functions of the state $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ being the input vector, and $z(x) \in \mathbb{R}^p$ the premise vector:

$$z(x) = \begin{bmatrix} z_1(x) & z_2(x) & \cdots & z_p(x) \end{bmatrix}^T,$$

which collects nonlinearities $z_j(\cdot)$, $j \in \{1, 2, \ldots, p\}$ in (6.1), which are assumed to be continuous in $\mathcal{C}$ and chosen in such a way so that $f(z)$ and $g(z)$ are multilinear in z.

Since, by continuity and compactness, the premise vector $z(x)$ is bounded, assume $z_j(\cdot) \in \left[\underline{z}_j, \overline{z}_j\right]$, $j \in \{1, 2, \ldots, p\}$ in $\mathcal{C}$. By defining the following *weighting functions* (WFs):

$$w_0^j(\cdot) = \frac{\overline{z}_j - z_j(\cdot)}{\overline{z}_j - \underline{z}_j}, \; w_1^j(\cdot) = 1 - w_0^j(\cdot), \; j \in \{1, 2, \ldots, p\},$$

each premise variable is written as $z_j(x) = w_0^j \underline{z}_j + w_1^j \overline{z}_j$, with $0 \leq w_i^j \leq 1$, $w_0^j + w_1^j = 1$. Thus, grouping all of them leads to a TS model with p nested convex sums:

$$\begin{aligned} \dot{x}(t) &= A_w x(t) + B_w u(t), \qquad (6.2)\\ A_w &= \sum_{i_1=0}^{1} \sum_{i_2=0}^{1} \cdots \sum_{i_p=0}^{1} w_{i_1}^1 w_{i_2}^2 \cdots w_{i_p}^p A_{(i_1, i_2, \ldots, i_p)},\\ B_w &= \sum_{i_1=0}^{1} \sum_{i_2=0}^{1} \cdots \sum_{i_p=0}^{1} w_{i_1}^1 w_{i_2}^2 \cdots w_{i_p}^p B_{(i_1, i_2, \ldots, i_p)},\\ A_{(i_1, i_2, \ldots, i_p)} &= f(z(x))|_{w_{i_1}^1 = w_{i_2}^2 = \cdots = w_{i_p}^p = 1},\\ B_{(i_1, i_2, \ldots, i_p)} &= g(z(x))|_{w_{i_1}^1 = w_{i_2}^2 = \cdots = w_{i_p}^p = 1}, \end{aligned}$$

with $A_{(i_1, i_2, \ldots, i_p)} \in \mathbb{R}^{n \times n}$, $B_{(i_1, i_2, \ldots, i_p)} \in \mathbb{R}^{n \times m}$, $i_j \in \{0, 1\}$, $j \in \{1, 2, \ldots, p\}$. This sort of notation for TS models corresponds to the tensor-product approach (C. Ariño and Sala 2007; Campos et al. 2013).

The following adaptation of the standard multi-index notation will be used (Sala and Ariño 2007), being $\mathbf{a}$ and $\mathbf{b}$ p-dimensional multi-indexes (p-tuples):

$$\begin{aligned} \mathbf{w}_0^{\mathbf{a}} &= \left(w_0^1\right)^{a_1} \left(w_0^2\right)^{a_2} \cdots \left(w_0^p\right)^{a_p} \text{ with } \mathbf{a} = (a_1, a_2, \ldots, a_p), \; a_j \in \{0 \cup \mathbb{N}\},\\ \mathbf{w}_1^{\mathbf{b}} &= \left(w_1^1\right)^{b_1} \left(w_1^2\right)^{b_2} \cdots \left(w_1^p\right)^{b_p} \text{ with } \mathbf{b} = (b_1, b_2, \ldots, b_p), \; b_j \in \{0 \cup \mathbb{N}\}, \end{aligned}$$

from which it is clear that nested convex sums A_w and B_w in (6.2) can be compactly rewritten as:

$$A_w = \sum_{\mathbf{j}_0+\mathbf{i}_0=\mathbf{1}} \mathbf{w}_0^{\mathbf{j}_0}\mathbf{w}_1^{\mathbf{i}_0} A_{\mathbf{i}_0},\ B_w = \sum_{\mathbf{j}_0+\mathbf{i}_0=\mathbf{1}} \mathbf{w}_0^{\mathbf{j}_0}\mathbf{w}_1^{\mathbf{i}_0} B_{\mathbf{i}_0},$$

with $\mathbf{j}_0 + \mathbf{i}_0 = \left(j_0^1 + i_0^1, j_0^2 + i_0^2, \ldots, j_0^p + i_0^p,\right)$ being the element-wise sum of p-tuples, $\mathbf{1} = \underbrace{(1, 1, \ldots, 1)}_{p \text{ ones}}$.

The previous notation will be key for the following developments and the reason behind the appearance of multi-indexes **a** and **b** will be the possibility of using a higher number of convex sums to relax the results[1]. Traditionally, 2^p composite MFs of the form $h_i = \Pi_{j=1}^p w_{i_j}^j$ have been often used; in contrast, this chapter has privileged the use of the so called WFs $w_{i_j}^j$ due to the fact that (a) they lead to better relaxations with a fewer number of LMIs due to the tensor-product structure (C. Ariño and Sala 2007; Campos et al. 2013), and (b) only p time derivatives $\dot{w}_{i_j}^j$ will be required, instead of 2^p which would be the case if composite functions h_i are used (Aguiar, Márquez, and Bernal 2015).

Consider a GPDLF candidate of the form

$$V(x) = x^T P_{\mathbf{w}}^{-1} x \tag{6.3}$$

where $P_{\mathbf{w}}$ is a convex summation with tensor-product structure, as follows: for a given degree vector $\mathbf{c} = (c_1, c_2, \ldots, c_p)$, $c_j \in \mathbb{N}$, where c_i is the degree of $V(x)$ in (w_0^i, w_1^i), $P_{\mathbf{w}}$ is defined as:

$$\begin{aligned}
P_{\mathbf{w}} &= \left(\sum_{i_1^1=0}^{1} w_{i_1^1}^1 \cdots \sum_{i_1^{c_1}=0}^{1} w_{i_1^{c_1}}^1\right)\left(\sum_{i_2^1=0}^{1} w_{i_2^1}^2 \cdots \sum_{i_2^{c_2}=0}^{1} w_{i_2^{c_2}}^2\right) \cdots \left(\sum_{i_p^1=0}^{1} w_{i_p^1}^p \cdots \sum_{i_p^{c_p}=0}^{1} w_{i_p^{c_p}}^p\right) P_{b_1 b_2 \cdots b_p} \\
&= \sum_{a_1+b_1=c_1} \left(w_0^1\right)^{a_1}\left(w_1^1\right)^{b_1} \sum_{a_2+b_2=c_2} \left(w_0^2\right)^{a_2}\left(w_1^2\right)^{b_2} \cdots \\
&\qquad \cdots \sum_{a_p+b_p=c_p} (w_0^p)^{a_p}(w_1^p)^{b_p} \binom{c_1}{b_1}\binom{c_2}{b_2} \cdots \binom{c_p}{b_p} P_{b_1 b_2 \cdots b_p} \\
&= \sum_{\mathbf{a}+\mathbf{b}=\mathbf{c}} \mathbf{w}_0^{\mathbf{a}} \mathbf{w}_1^{\mathbf{b}} \binom{\mathbf{c}}{\mathbf{b}} P_{\mathbf{b}},
\end{aligned} \tag{6.4}$$

[1] Note that $f(z(x))$ can be a polynomial of $z(x)$. For instance, if $z_1(x) = \sin x$ and $z_2(x) = \cos x$, then $f(x) = \sin^2(x) + 3\cos(x)\sin(x) = z_1^2 + 3z_1z_2$, which corresponds to $\mathbf{j}_0 + \mathbf{i}_0 = (2, 1)$.

with $P_{\mathbf{b}} = P_{\mathbf{b}}^T > 0$, $b_j = i_j^1 + i_j^2 + \cdots + i_j^{c_j}$, $j \in \{1, 2, \ldots, p\}$, $\binom{c_j}{b_j} = \frac{c_j!}{b_j!\,(c_j - b_j)!}$, and $\binom{\mathbf{c}}{\mathbf{b}} = \prod_{j=1}^{p} \binom{c_j}{b_j}$. Notice that each $P_{\mathbf{b}}$ is a single variable grouping all the terms that share the same membership monomial $\mathbf{w}_0^{\mathbf{a}}\mathbf{w}_1^{\mathbf{b}}$. The addition of the combinatorial number will be convenient later on, as

$$\sum_{\mathbf{a}+\mathbf{b}=\mathbf{c}} \mathbf{w}_0^{\mathbf{a}}\mathbf{w}_1^{\mathbf{b}} \binom{\mathbf{c}}{\mathbf{b}} = \left(\sum_{i_1=0}^{1} w_{i_1}^1\right)^{c_1} \cdots \left(\sum_{i_p=0}^{1} w_{i_p}^1\right)^{c_p} = 1.$$

Example: In order to illustrate the notation just introduced, consider a TS model with $p = 2$ nonlinearities. By the sector nonlinearity methodology described above, only functions w_0^1, $w_1^1 = 1 - w_0^1$, w_0^2, and $w_1^2 = 1 - w_0^2$ arise. Therefore, if $\mathbf{c} = (c_1, c_2) = (1, 1)$ then

$$\begin{aligned}
P_{\mathbf{w}} &= \sum_{i_1^1=0}^{1} w_{i_1^1}^1 \sum_{i_2^1=0}^{1} w_{i_2^1}^2 P_{b_1 b_2} \\
&= w_0^1 w_0^2 P_{00} + w_0^1 w_1^2 P_{01} + w_1^1 w_0^2 P_{10} + w_1^1 w_1^2 P_{11} \\
&= \sum_{a_1+b_1=1} \left(w_0^1\right)^{a_1} \left(w_1^1\right)^{b_1} \sum_{a_2+b_2=1} \left(w_0^2\right)^{a_2} \left(w_1^2\right)^{b_2} P_{\mathbf{b}},
\end{aligned}$$

with $b_1 = i_1^1$, $b_2 = i_2^1$, and $\mathbf{b} = (b_1, b_2)$.

If $\mathbf{c} = (c_1, c_2) = (2, 2)$ then

$$\begin{aligned}
P_{\mathbf{w}} &= \sum_{i_1^1=0}^{1} \sum_{i_1^2=0}^{1} w_{i_1^1}^1 w_{i_1^2}^1 \sum_{i_2^1=0}^{1} \sum_{i_2^2=0}^{1} w_{i_2^1}^2 w_{i_2^2}^2 P_{b_1 b_2} \\
&= \left(w_0^1\right)^2 \left(w_0^2\right)^2 P_{00} + \left(w_0^1\right)^2 w_0^2 w_1^2 2P_{01} + \left(w_0^1\right)^2 \left(w_1^2\right)^2 P_{02} \\
&\quad + w_0^1 w_1^1 \left(w_0^2\right)^2 2P_{10} + w_0^1 w_1^1 w_0^2 w_1^2 4P_{11} + w_0^1 w_1^1 \left(w_1^2\right)^2 2P_{12} \\
&\quad + \left(w_1^1\right)^2 \left(w_0^2\right)^2 P_{20} + \left(w_1^1\right)^2 w_0^2 w_1^2 2P_{21} + \left(w_1^1\right)^2 \left(w_1^2\right)^2 P_{22} \\
&= \sum_{a_1+b_1=2} \left(w_0^1\right)^{a_1} \left(w_1^1\right)^{b_1} \sum_{a_2+b_2=2} \left(w_0^2\right)^{a_2} \left(w_1^2\right)^{b_2} \binom{c_1}{b_1} \binom{c_2}{b_2} P_{\mathbf{b}},
\end{aligned}$$

with $b_1 = i_1^1 + i_1^2$, $b_2 = i_2^1 + i_2^2$, $\mathbf{b} = (b_1, b_2)$.

This form includes Lyapunov functions previously appeared in non-quadratic schemes; for instance, those in (Blanco, Perruqueti, and Borne 2001; T.M. Guerra and Vermeiren 2004; Tanaka, Hori, and H. Wang 2003) have the form

$$V(x) = x^T(t)P_h^{-1}x(t) : \; P_h = \sum_{i=1}^{r} h_i(z(t))P_i, \tag{6.5}$$

with $P_i = P_i^T > 0$, $i \in \{1, 2, \ldots, r\}$. But, since every $h_i = \Pi_{j=1}^{p} w_{i_j}^{j}$, it is clear that the latter is equivalent to (6.3) with $\mathbf{c} = \mathbf{1}$.

Generalizations of the sort appeared in (D. Lee and D. Kim 2014; Bernal and T. M. Guerra 2010), which use multiple convex sums on MFs h_i, are also described by the GFLF presented above, since

$$V(x) \;=\; x^T(t)P_{\mathbf{h}}^{-1}x(t) \;:\; P_{\mathbf{h}} \;=\; \sum_{i_1=1}^{r}\sum_{i_2=1}^{r}\cdots\sum_{i_q=1}^{r} h_{i_1}h_{i_2}\cdots h_{i_q}P_{i_1 i_2 \cdots i_q}, \tag{6.6}$$

with $P_{i_1 i_2 \cdots i_q} = P_{i_1 i_2 \cdots i_q}^T > 0$, $i_j \in \{1, 2, \ldots, r\}$. Expressions like (6.6), which is a homogenous polynomial in h_i of degree q, can be trivially transformed in a tensor-product expression by replacing h_i as the product of p 2-rule individual weighting functions and reordering the factors. The resulting degree vector is $\mathbf{c} = \underbrace{(q, q, \ldots, q)}_{p \text{ q's}}$. Details are omitted for brevity.

The following notation will be used in the sequel:

$$\dot{\Upsilon}_{\mathbf{w}} = \tfrac{d}{dt}\left(\Upsilon_{\mathbf{w}}\right), \quad \dot{\Upsilon}_{\mathbf{w}}^{-1} = \tfrac{d}{dt}\left(\Upsilon_{\mathbf{w}}^{-1}\right),$$
$$\begin{bmatrix} A & (*) \\ Y & B \end{bmatrix} = \begin{bmatrix} A & Y^T \\ Y & B \end{bmatrix}, \quad A + (*) = A + A^T.$$

Arguments will be omitted when convenient.

The Lyapunov function (6.5) has been usually combined with a control law $u(t) = F_h P_h^{-1}$, where $F_i \in \mathbb{R}^{m\times n}$, $i \in \{1, 2, \ldots, r\}$ are gains to be determined. Analogously, when Lyapunov function (6.6) is used, a generalization of the previous control law is used, i.e., $u(t) = F_{\mathbf{h}}P_{\mathbf{h}}^{-1}$, with $\mathbf{h}$ standing for multi-indexes associated with nested convex sums. Naturally, these control laws can be generalized as the ones in the tensor-product form below:

$$u(t) = F_{\mathbf{w}}P_{\mathbf{w}}^{-1}x(t), \quad F_{\mathbf{w}} = \sum_{\mathbf{a}+\mathbf{b}=\mathbf{c}} \mathbf{w}_0^{\mathbf{a}}\mathbf{w}_1^{\mathbf{b}} \binom{\mathbf{c}}{\mathbf{b}} F_{\mathbf{b}},$$

with $F_{\mathbf{b}} \in \mathbb{R}^{m\times n}$ grouping all the terms that share the same membership monomial $\mathbf{w}_0^{\mathbf{a}}\mathbf{w}_1^{\mathbf{b}}$, and $P_{\mathbf{w}}$ as in (6.3), both for a given $\mathbf{c} = (c_1, c_2, \ldots, c_p)$, $c_j \in \mathbb{N}$.

These control schemes lead to *local* stability conditions because a priori bounds (D. Lee and D. Kim 2014) or LMI-imposed ones (T.M. Guerra, Bernal, et al. 2012) on the time derivatives of the membership functions have to be employed. The reason behind locality is the fact that the term $\dot{P}_{\mathbf{w}}$ appears when investigating the stability of the closed-loop model $\dot{x}(t) = \left(A_w + B_w F_{\mathbf{w}} P_{\mathbf{w}}^{-1}\right) x(t)$, since the time-derivative of the corresponding Lyapunov function (6.3) is:

$$\begin{aligned}\dot{V} &= \dot{x}^T P_{\mathbf{w}}^{-1} x + x^T P_{\mathbf{w}}^{-1} \dot{x} + x^T \dot{P}_{\mathbf{w}}^{-1} x \\ &= x^T P_{\mathbf{w}}^{-1} \left(A_w + B_w F_{\mathbf{w}} P_{\mathbf{w}}^{-1}\right) x + (*) + x^T \dot{P}_{\mathbf{w}}^{-1} x < 0 \\ &\Leftarrow A_w P_{\mathbf{w}} + B_w F_{\mathbf{w}} + (*) + P_{\mathbf{w}} \dot{P}_{\mathbf{w}}^{-1} P_{\mathbf{w}} \\ &= A_w P_{\mathbf{w}} + B_w F_{\mathbf{w}} + (*) - \dot{P}_{\mathbf{w}} < 0.\end{aligned}$$

It turns out that the term $\dot{P}_{\mathbf{w}}$ is hard to cast as a convex sum without conservative steps (Tanaka, Hori, and H. Wang 2003; Rhee and Won 2006; T.M. Guerra, Bernal, et al. 2012; D. Lee and D. Kim 2014) and cannot be therefore associated with the rest of convex expressions in order to obtain LMIs.

Problem statement: The objective of this work is providing suitable elements in the feedback controller able to cancel out the effect of the time derivative of the WFs in $\dot{P}_{\mathbf{w}}$, in order to achieve stability up to the modeling area $\mathcal{C}$, assuming absence of control saturation.

6.3 Main Results

Consider the TS model (6.2) altogether with the multi-index control law with derivative feedback given by:

$$u(t) = \left(F_{\mathbf{w}} + \dot{G}_{\mathbf{w}}\right) P_{\mathbf{w}}^{-1} x(t), \tag{6.7}$$

with $P_{\mathbf{w}}$ as in (6.4), $F_{\mathbf{w}}$ and $\dot{G}_{\mathbf{w}}$ defined as follows[2]

$$\begin{aligned}F_{\mathbf{w}} &= \sum_{\mathbf{a}+\mathbf{b}=\mathbf{c}} \mathbf{w}_0^{\mathbf{a}} \mathbf{w}_1^{\mathbf{b}} \binom{\mathbf{c}}{\mathbf{b}} F_{\mathbf{b}}, \\ \dot{G}_{\mathbf{w}} &= \sum_{\mathbf{a}+\mathbf{b}=\mathbf{c}} \frac{d}{dt} \left(\mathbf{w}_0^{\mathbf{a}} \mathbf{w}_1^{\mathbf{b}}\right) \binom{\mathbf{c}}{\mathbf{b}} G_{\mathbf{b}},\end{aligned}$$

[2]For simplicity, it has been assumed that the number of nested convex sums in $F_{\mathbf{w}}$ is the same as that of the Lyapunov function (6.3), i.e., "**c**", but of course it can be chosen independently as a new index "**d**" with straightforward modifications. For $\dot{G}_{\mathbf{w}}$, such an adaptation will be more involved.

with $F_{\mathbf{b}}, G_{\mathbf{b}} \in \mathbb{R}^{m\times n}$, and $P_{\mathbf{b}} \in \mathbb{R}^{n\times n}$, being matrices to be found, all of them sharing a given degree vector $\mathbf{c} = (c_1, c_2, \ldots, c_p)$, $c_j \in \mathbb{N}$ as in (6.3) and (6.4).

In the next subsection, stability of the corresponding closed-loop TS model

$$\dot{x}(t) = \left(A_w + B_w F_{\mathbf{w}} P_{\mathbf{w}}^{-1} + B_w \dot{G}_{\mathbf{w}} P_{\mathbf{w}}^{-1}\right) x(t) \tag{6.8}$$

is analyzed via the GFLF candidate (6.3). It will be assumed that the time derivatives $\dot{w}_0^l$, $l \in \{1, 2, \ldots, p\}$ in (6.7) are well defined and available, so they will appear in the control law. Computation of these derivatives will be discussed in more detail in subsection 6.3.2.

6.3.1 Lyapunov analysis

Lyapunov analysis of the previous system involves the time derivatives $\dot{P}_{\mathbf{w}}$ and $\dot{G}_{\mathbf{w}}$; they come from the time derivative of nested convex sums such as $P_{\mathbf{w}}$ in (6.4) and have to be therefore analyzed under the same notation searching for (a) maximum relaxation (algebraic association of similar terms) and (b) a way to cancel out the effects of $\dot{P}_{\mathbf{w}}$ through the terms in $\dot{G}_{\mathbf{w}}$. We begin by noticing that

$$\begin{aligned}
\frac{d}{dt}\left(\mathbf{w}_0^{\mathbf{a}}\right) &= \frac{d}{dt}\left(\left(w_0^1\right)^{a_1} \cdots \left(w_0^p\right)^{a_p}\right) = \sum_{l=1}^{p} a_l \dot{w}_0^l (w_0^l)^{-1} \mathbf{w}_0^{\mathbf{a}},\\
\frac{d}{dt}\left(\mathbf{w}_1^{\mathbf{b}}\right) &= \frac{d}{dt}\left(\left(w_1^1\right)^{b_1} \cdots \left(w_1^p\right)^{b_p}\right) = \sum_{l=1}^{p} b_l \dot{w}_1^l (w_1^l)^{-1} \mathbf{w}_1^{\mathbf{b}},
\end{aligned}$$

which, considering the identity $\dot{w}_1^l = -\dot{w}_0^l$, leads to

$$\begin{aligned}
\frac{d}{dt}\left(\mathbf{w}_0^{\mathbf{a}}\mathbf{w}_1^{\mathbf{b}}\right) &= \mathbf{w}_1^{\mathbf{b}}\dot{\mathbf{w}}_0^{\mathbf{a}} + \mathbf{w}_0^{\mathbf{a}}\dot{\mathbf{w}}_1^{\mathbf{b}}\\
&= \mathbf{w}_1^{\mathbf{b}} \sum_{l=1}^{p} a_l \dot{w}_0^l \left(w_0^l\right)^{-1} \mathbf{w}_0^{\mathbf{a}} - \mathbf{w}_0^{\mathbf{a}} \sum_{l=1}^{p} b_l \dot{w}_0^l \left(w_1^l\right)^{-1} \mathbf{w}_1^{\mathbf{b}}\\
&= \sum_{l=1}^{p} \dot{w}_0^l \mathbf{w}_0^{\mathbf{a}} \mathbf{w}_1^{\mathbf{b}} \left(a_l w_1^l - b_l w_0^l\right) \left(w_0^l w_1^l\right)^{-1}.
\end{aligned} \tag{6.9}$$

But, since $w_0^l + w_1^l = 1$, we have that

$$\begin{aligned}
a_l w_1^l - b_l w_0^l &= \left(a_l w_1^l - b_l w_0^l\right)\left(w_0^l + w_1^l\right)\\
&= -w_0^l w_0^l b_l - w_0^l w_1^l b_l + w_1^l w_0^l a_l + w_1^l w_1^l a_l\\
&= \sum_{d+e=2} \left(w_0^l\right)^d \left(w_1^l\right)^e \left(\operatorname{sgn}(e) a_l - \operatorname{sgn}(d) b_l\right),
\end{aligned}$$

where $d, e \in \{0 \cup \mathbb{N}\}$ and $\text{sgn}(\cdot)$ stands for the sign function with $\text{sgn}(0) = 0$. Therefore, $\frac{d}{dt}\left(\mathbf{w}_0^{\mathbf{a}}\mathbf{w}_1^{\mathbf{b}}\right)$ in (6.9) can be rewritten as

$$\sum_{l=1}^{p} \dot{w}_0^l \mathbf{w}_0^{\mathbf{a}}\mathbf{w}_1^{\mathbf{b}} \sum_{d+e=2} \left(w_0^l\right)^{d-1}\left(w_1^l\right)^{e-1}(\text{sgn}(e)a_l - \text{sgn}(d)b_l),$$

which can be now substituted in $\dot{P}_{\mathbf{w}}$ in order to get

$$\begin{aligned}
\dot{P}_{\mathbf{w}} &= \sum_{\mathbf{a}+\mathbf{b}=\mathbf{c}} \left(\dot{\mathbf{w}}_0^{\mathbf{a}}\mathbf{w}_1^{\mathbf{b}} + \mathbf{w}_0^{\mathbf{a}}\dot{\mathbf{w}}_1^{\mathbf{b}}\right)\binom{\mathbf{c}}{\mathbf{b}} P_{\mathbf{b}} \\
&= \sum_{\mathbf{a}+\mathbf{b}=\mathbf{c}} \sum_{l=1}^{p} \dot{w}_0^l \mathbf{w}_0^{\mathbf{a}}\mathbf{w}_1^{\mathbf{b}} \sum_{d+e=2} \left(w_0^l\right)^{d-1}\left(w_1^l\right)^{e-1}(\text{sgn}(e)a_l - \text{sgn}(d)b_l)\binom{\mathbf{c}}{\mathbf{b}} P_{\mathbf{b}} \\
&= \sum_{l=1}^{p} \dot{w}_0^l \sum_{\bar{\mathbf{a}}+\bar{\mathbf{b}}=\mathbf{c}^{l+2}} \mathbf{w}_0^{\bar{\mathbf{a}}^{l-}}\mathbf{w}_1^{\bar{\mathbf{b}}^{l-}} \sum_{(a_l,b_l,d,e)\in\mathcal{P}(\bar{a}_l,\bar{b}_l,c_l)} (\text{sgn}(e)a_l - \text{sgn}(d)b_l)\binom{\mathbf{c}}{\bar{\mathbf{b}}^l} P_{\bar{\mathbf{b}}^l},
\end{aligned} \tag{6.10}$$

under the following definitions:

$$\begin{aligned}
\bar{\mathbf{a}}^{l-} &= (\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_l - 1, \ldots, \bar{a}_p), \ \bar{a}_j \in \{0 \cup \mathbb{N}\}, \\
\bar{\mathbf{b}}^{l-} &= \left(\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_l - 1, \ldots, \bar{b}_p\right), \\
\bar{\mathbf{b}}^{l} &= \left(\bar{b}_1, \bar{b}_2, \ldots, b_l, \ldots, \bar{b}_p\right), \bar{b}_j \in \{0 \cup \mathbb{N}\}, \\
\mathbf{c}^{l+2} &= (c_1, c_2, \ldots, c_l + 2, \ldots, c_p), \ c_j \in \mathbb{N} \\
\mathcal{P}(\bar{a}_l, \bar{b}_l, c_l) &= \left\{ (a_l, b_l, d, e) : \begin{array}{l} a_l + d = \bar{a}_l, \\ b_l + e = \bar{b}_l, \\ a_l + b_l = c_l, \\ d + e = 2 \end{array} \right\}.
\end{aligned}$$

Similarly to (6.10), $\dot{G}_{\mathbf{w}}$ in (6.7) can be expressed as:

$$\dot{G}_{\mathbf{w}} = \sum_{l=1}^{p} \dot{w}_0^l \sum_{\bar{\mathbf{a}}+\bar{\mathbf{b}}=\mathbf{c}^{l+2}} \mathbf{w}_0^{\bar{\mathbf{a}}^{l-}}\mathbf{w}_1^{\bar{\mathbf{b}}^{l-}} \sum_{(a_l,b_l,d,e)\in\mathcal{P}(\bar{a}_l,\bar{b}_l,c_l)} (\text{sgn}(e)a_l - \text{sgn}(d)b_l)\binom{\mathbf{c}}{\bar{\mathbf{b}}^l} G_{\bar{\mathbf{b}}^l} \tag{6.11}$$

Theorem 6.3.1. *The origin $x = 0$ of the nonlinear system (6.1) under the control law (6.7) is asymptotically stable for any trajectory starting in the outermost Lyapunov level within the modeling area $\mathcal{C}$ where (6.2) is a valid TS model of the system and (6.3) an associated valid GFLF, if there exist matrices $F_{\mathbf{b}}, G_{\mathbf{b}}$, and $P_{\mathbf{b}} \in \mathbb{R}^{n\times n}$, $P_{\mathbf{b}} = P_{\mathbf{b}}^T > 0$, all of them sharing a given degree vector $\mathbf{c} = (c_1, c_2, \ldots, c_p)$, $c_j \in \mathbb{N}$ as in (6.4), such that the following conditions hold* *for all* $b_j \leq c_j$, $\bar{b}_j \leq c_j + 1$, $\tilde{\mathbf{a}} + \tilde{\mathbf{b}} = (\mathbf{c} + \mathbf{1})^{l+2}$*:*

$$\sum_{i_0^j + b_j = \bar{b}_j} \binom{\mathbf{c}}{\mathbf{b}} \left(A_{\mathbf{i}_0} P_{\mathbf{b}} + B_{\mathbf{i}_0} F_{\mathbf{b}} + (*) \right) < 0, \tag{6.12}$$

$$\sum_{\left(\bar{\mathbf{a}}, \bar{\mathbf{b}}, \mathbf{j}_0, \mathbf{i}_0\right) \in \mathcal{Q}(\tilde{\mathbf{a}}, \tilde{\mathbf{b}}, \mathbf{c}^{l+2})} \sum_{(a_l, b_l, d, e) \in \mathcal{P}(\bar{a}_l, \bar{b}_l, c_l)} (sgn(e) a_l - sgn(d) b_l) \binom{\mathbf{c}}{\bar{\mathbf{b}}^l} \left(B_{\mathbf{i}_0} G_{\bar{\mathbf{b}}^l} + (*) - P_{\bar{\mathbf{b}}^l} \right) = 0, \tag{6.13}$$

with

$$\mathcal{Q}(\tilde{\mathbf{a}}, \tilde{\mathbf{b}}, \mathbf{c}^{l+2}) = \left\{ (\bar{\mathbf{a}}, \bar{\mathbf{b}}, \mathbf{j}_0, \mathbf{i}_0) : \begin{array}{l} \bar{\mathbf{a}} + \mathbf{j}_0 = \tilde{\mathbf{a}}, \\ \bar{\mathbf{b}} + \mathbf{i}_0 = \tilde{\mathbf{b}}, \\ \bar{\mathbf{a}} + \bar{\mathbf{b}} = \mathbf{c}^{l+2}, \\ \mathbf{j}_0 + \mathbf{i}_0 = \mathbf{1} \end{array} \right\}.$$

Proof. Condition $P_{\mathbf{b}} = P_{\mathbf{b}}^T > 0$ guarantees (6.3) is a valid Lyapunov function candidate. Taking into account the closed-loop system in (6.8), the time derivative of $V(x)$ is:

$$\begin{aligned} \dot{V} &= \dot{x}^T P_{\mathbf{w}}^{-1} x + x^T P_{\mathbf{w}}^{-1} \dot{x} + x^T \dot{P}_{\mathbf{w}}^{-1} x \\ &= x^T P_{\mathbf{w}}^{-1} \left(A_w + B_w F_{\mathbf{w}} P_{\mathbf{w}}^{-1} + B_w \dot{G}_{\mathbf{w}} P_{\mathbf{w}}^{-1} \right) x + (*) + x^T \dot{P}_{\mathbf{w}}^{-1} x < 0 \\ &\Leftarrow A_w P_{\mathbf{w}} + B_w F_{\mathbf{w}} + B_w \dot{G}_{\mathbf{w}} + (*) - \dot{P}_{\mathbf{w}} < 0, \end{aligned}$$

which can be guaranteed if

$$A_w P_{\mathbf{w}} + B_w F_{\mathbf{w}} + (*) < 0, \tag{6.14}$$

$$B_w \dot{G}_{\mathbf{w}} + (*) - \dot{P}_{\mathbf{w}} = 0. \tag{6.15}$$

Condition (6.14) can be rewritten as:

$$\begin{aligned} & A_w P_{\mathbf{w}} + B_w F_{\mathbf{w}} + (*) \\ &= \sum_{\mathbf{j}_0 + \mathbf{i}_0 = \mathbf{1}} \mathbf{w}_0^{\mathbf{j}_0} \mathbf{w}_1^{\mathbf{i}_0} A_{\mathbf{i}_0} \sum_{\mathbf{a} + \mathbf{b} = \mathbf{c}} \mathbf{w}_0^{\mathbf{a}} \mathbf{w}_1^{\mathbf{b}} \binom{\mathbf{c}}{\mathbf{b}} P_{\mathbf{b}} \\ &\quad + \sum_{\mathbf{j}_0 + \mathbf{i}_0 = \mathbf{1}} \mathbf{w}_0^{\mathbf{j}_0} \mathbf{w}_1^{\mathbf{i}_0} B_{\mathbf{i}_0} \sum_{\mathbf{a} + \mathbf{b} = \mathbf{c}} \mathbf{w}_0^{\mathbf{a}} \mathbf{w}_1^{\mathbf{b}} \binom{\mathbf{c}}{\mathbf{b}} F_{\mathbf{b}} + (*) \\ &= \sum_{\bar{\mathbf{a}} + \bar{\mathbf{b}} = \mathbf{c} + \mathbf{1}} \mathbf{w}_0^{\bar{\mathbf{a}}} \mathbf{w}_1^{\bar{\mathbf{b}}} \sum_{i_0^j + b_j = \bar{b}_j} \binom{\mathbf{c}}{\mathbf{b}} \left(A_{\mathbf{i}_0} P_{\mathbf{b}} + B_{\mathbf{i}_0} F_{\mathbf{b}} + (*) \right) < 0. \end{aligned}$$

Thus, LMIs (6.12) guarantee the previous inequality, i.e., (6.14). Condition (6.15) is developed as follows:

$$
\begin{aligned}
B_w\dot{G}_{\mathbf{w}} + (*) - \dot{P}_{\mathbf{w}} &= \sum_{\mathbf{j}_0+\mathbf{i}_0=\mathbf{1}} \mathbf{w}_0^{\mathbf{j}_0}\mathbf{w}_1^{\mathbf{i}_0} B_{\mathbf{i}_0} \sum_{l=1}^{p} \dot{w}_0^l \sum_{\bar{\mathbf{a}}+\bar{\mathbf{b}}=\mathbf{c}^{l+2}} \mathbf{w}_0^{\bar{\mathbf{a}}^{l-}} \mathbf{w}_1^{\bar{\mathbf{b}}^{l-}} \sum_{(a_l,b_l,d,e)\in\mathcal{P}(\bar{a}_l,\bar{b}_l,c_l)} (\text{sgn}(e)a_l - \text{sgn}(d)b_l) \\
&\quad \times \binom{\mathbf{c}}{\bar{\mathbf{b}}^l} G_{\bar{\mathbf{b}}^l} + (*) - \sum_{l=1}^{p} \dot{w}_0^l \sum_{\bar{\mathbf{a}}+\bar{\mathbf{b}}=\mathbf{c}^{l+2}} \mathbf{w}_0^{\bar{\mathbf{a}}^{l-}} \mathbf{w}_1^{\bar{\mathbf{b}}^{l-}} \sum_{(a_l,b_l,d,e)\in\mathcal{P}(\bar{a}_l,\bar{b}_l,c_l)} (\text{sgn}(e)a_l - \text{sgn}(d)b_l) \binom{\mathbf{c}}{\bar{\mathbf{b}}^l} P_{\bar{\mathbf{b}}^l} \\
&= \sum_{\mathbf{j}_0+\mathbf{i}_0=\mathbf{1}} \mathbf{w}_0^{\mathbf{j}_0}\mathbf{w}_1^{\mathbf{i}_0} \sum_{l=1}^{p} \dot{w}_0^l \sum_{\bar{\mathbf{a}}+\bar{\mathbf{b}}=\mathbf{c}^{l+2}} \mathbf{w}_0^{\bar{\mathbf{a}}^{l-}} \mathbf{w}_1^{\bar{\mathbf{b}}^{l-}} \sum_{(a_l,b_l,d,e)\in\mathcal{P}(\bar{a}_l,\bar{b}_l,c_l)} (\text{sgn}(e)a_l - \text{sgn}(d)b_l) \\
&\quad \times \binom{\mathbf{c}}{\bar{\mathbf{b}}^l} (B_{\mathbf{i}_0} G_{\bar{\mathbf{b}}^l} + (*) - P_{\bar{\mathbf{b}}^l}) \\
&= \sum_{l=1}^{p} \dot{w}_0^l \sum_{\tilde{\mathbf{a}}+\tilde{\mathbf{b}}=(\mathbf{c}+\mathbf{1})^{l+2}} \mathbf{w}_0^{\tilde{\mathbf{a}}^{l-}} \mathbf{w}_1^{\tilde{\mathbf{b}}^{l-}} \sum_{(\bar{\mathbf{a}},\bar{\mathbf{b}},\mathbf{j}_0,\mathbf{i}_0)\in\mathcal{Q}(\tilde{\mathbf{a}},\tilde{\mathbf{b}},\mathbf{c}^{l+2})} \sum_{(a_l,b_l,d,e)\in\mathcal{P}(\bar{a}_l,\bar{b}_l,c_l)} (\text{sgn}(e)a_l - \text{sgn}(d)b_l) \\
&\quad \times \binom{\mathbf{c}}{\bar{\mathbf{b}}^l} (B_{\mathbf{i}_0} G_{\bar{\mathbf{b}}^l} + (*) - P_{\bar{\mathbf{b}}^l})
\end{aligned} \tag{6.16}
$$

with

$$
\mathcal{Q}(\tilde{\mathbf{a}}, \tilde{\mathbf{b}}, \mathbf{c}^{l+2}) = \left\{ (\bar{\mathbf{a}}, \bar{\mathbf{b}}, \mathbf{j}_0, \mathbf{i}_0) : \bar{\mathbf{a}} + \mathbf{j}_0 = \tilde{\mathbf{a}}, \bar{\mathbf{b}} + \mathbf{i}_0 = \tilde{\mathbf{b}}, \bar{\mathbf{a}} + \bar{\mathbf{b}} = \mathbf{c}^{l+2}, \mathbf{j}_0 + \mathbf{i}_0 = \mathbf{1} \right\}.
$$

Thus, equation (6.16) can be seen as a sum of p terms, each of them multiplied by its corresponding $\dot{w}_0^l$. Clearly, if the matrix equalities (6.13) hold, the corresponding term in the p-term sum (6.16) is zero, thus concluding the proof. ☐

Remark 6.3.1. *In order to reduce conservatism, apart from increasing the complexity of* $\mathbf{c}$ *in conditions* (6.12) *and* (6.13)*, relaxations of convex summations as those in (Tuan et al. 2001) or (Peaucelle et al. 2000) may be used to slightly increase the number of decision variables striking a reasonable tradeoff between accuracy and computational resource requirements (see, for instance, the related work (Aguiar, Márquez, and Bernal 2015)). These ideas have been used quite a few times in numerical examples in TS literature, but they will be intentionally left out as computational efficiency issues are out of the scope of this thesis.*

Example: Consider a 2-rule TS system $\dot{x} = w_0(A_0x + B_0u) + w_1(A_1x + B_1u)$, and a GFLF $P_{\mathbf{w}} = w_0^2P_0 + 2w_0w_1P_1 + w_1^2P_2$. Then, condition (6.12) would amount to enumerating the degree-3 monomials:

$$(w_0)^3 : A_0P_0 + B_0F_0 + (*) < 0$$

$$
\begin{aligned}
(w_0)^2 w_1 &: 2(A_0P_1 + B_0F_1) + (A_1P_0 + B_1F_0) + (*) < 0 \\
w_0(w_1)^2 &: 2(A_1P_1 + B_1F_1) + (A_0P_2 + B_0F_1) + (*) < 0 \\
(w_1)^3 &: A_1P_2 + B_1F_2 + (*) < 0
\end{aligned}
$$

and condition (6.13) would amount to:

$$
\begin{aligned}
\dot{w}_0(w_0)^3 &: 2B_0G_0 - 2B_0G_1 + (*) + 2P_1 - 2P_0 = 0 \\
\dot{w}_0(w_0)^2 w_1 &: 2B_0G_0 + 2B_1G_0 - 2B_0G_2 - 2B_1G_1) + (*) + 2P_1 + 2P_2 - 4P_0 = 0 \\
\dot{w}_0 w_0 (w_1)^2 &: 2B_0G_1 + 2B_1G_0 - 2B_0G_2 - 2B_1G_2 + (*) + 4P_2 - 2P_1 - 2P_0 = 0 \\
\dot{w}_0(w_1)^3 &: 2B_1G_1 - 2B_1G_2 + (*) + 2P_2 - 2P_1 = 0
\end{aligned}
$$

Should the degrees of $P_{\mathbf{w}}$, $F_{\mathbf{w}}$, and $G_{\mathbf{w}}$ increase, conditions would be more relaxed. The theorem statement provides the expression for such general case, contemplating, too, the general power-of-two tensor-product case in the problem statement.

Now, consider a 4-rule TS system $\dot{x} = w_0^1 w_0^2 (A_{00}x + B_{00}u) + w_0^1 w_1^2 (A_{01}x + B_{01}u) + w_0^2 w_1^1 (A_{10}x + B_{10}u) + w_1^1 w_1^2 (A_{11}x + B_{11}u)$. Then, some of the conditions in 6.13 corresponding to $\dot{w}_0^1$ are:

$$
\begin{aligned}
\left(w_0^1\right)^3 \left(w_0^2\right)^2 &: 2B_{00}G_{00} - 2B_{00}G_{10} + (*) + 2P_{10} - 2P_{00} = 0 \\
\left(w_0^1\right)^3 w_0^2 w_1^2 &: 2B_{00}G_{01} + 2B_{01}G_{00} - 2B_{00}G_{11} - 2B_{01}G_{10} + (*) \\
&\quad + 2P_{10} - 2P_{01} - 2P_{00} + 2P_{11} = 0 \\
\left(w_0^1\right)^3 \left(w_1^2\right)^2 &: 2B_{01}G_{01} - 2B_{01}G_{11} + (*) + 2P_{11} - 2P_{01} = 0 \\
\left(w_0^1\right)^2 w_1^1 \left(w_0^2\right)^2 &: 2B_{00}G_{00} + 2B_{10}G_{00} - 2B_{00}G_{20} - 2B_{10}G_{10} + (*) \\
&\quad + 2P_{10} - 4P_{00} + 2P_{20} = 0 \\
&\vdots \\
\left(w_1^1\right)^3 \left(w_1^2\right)^2 &: 2B_{11}G_{11} - 2B_{11}G_{21} + (*) + 2P_{21} - 2P_{11} = 0
\end{aligned}
$$

6.3.2 Computation of Time Derivatives of the WFs

Once the previous theorem finds a feasible solution, feeding back the time derivatives of the WFs $\dot{w}_0^l$, $l \in \{1, 2, \ldots, p\}$ needs a way of obtaining it from measurements. Several situations arise:

Undisturbed-case, model-based approach

In the undisturbed case, if memberships' arguments have relative degree greater or equal to one with respect to the input, then state measurement is enough, i.e., if z is such that

$$0 = \frac{\partial \dot{z}_l}{\partial u} = \frac{\partial}{\partial u}\left(\frac{\partial z_l}{\partial x}\left(A_w x + B_w u\right)\right) \tag{6.17}$$

i.e., $(\partial z_l/\partial x)B_w = 0$, then the terms

$$\dot{w}_0^l = \left(\frac{\partial w_0^l}{\partial z_l}\right)^T \dot{z}_l, \quad l \in \{1, 2, \ldots, p\},$$

will all depend exclusively on the states. This fact was used in prior literature, such as (Tanaka, Hori, and H. Wang 2003), in order to feed back such derivatives to the control law without extra measurements, though they were remodelled as convex expressions in order to obtain LMI conditions. However, if relative degree of z_l with respect to the input is zero, i.e., $\dot{z}_l$ explicitly depends on u so (6.17) does not hold, then an algebraic loop appears: the control depends on the derivatives of z_l but, these derivatives depend on the control. Thus, such loop must be algebraically solved at each sample. As x is measurable, such step can be easily done, as follows.

First, note that the time derivatives of the WFs can be directly solved from the p equations below:

$$\dot{w}_0^i = \left(\frac{\partial w_0^i}{\partial x}\right)^T \left(A_w + B_w F_{\mathbf{w}} P_{\mathbf{w}}^{-1} + B_w \dot{G}_{\mathbf{w}} P_{\mathbf{w}}^{-1}\right) x, \tag{6.18}$$

for $i \in \{1, 2, \ldots, p\}$, where $\dot{G}_{\mathbf{w}}$ must be substituted by (6.11). For a given measured x, the above results in a linear system of equations to be solved at each sample. Indeed, defining $\tilde{G}_l$, $l \in \{1, 2, \ldots, p\}$, as

$$\tilde{G}_l = \sum_{\bar{\mathbf{a}}+\bar{\mathbf{b}}=\mathbf{c}^{l+2}} \mathbf{w}_0^{\bar{\mathbf{a}}^{l-}} \mathbf{w}_1^{\bar{\mathbf{b}}^{l-}} \sum_{(a_l,b_l,d,e)\in\mathcal{P}(\bar{a}_l,\bar{b}_l,c_l)} (\mathrm{sgn}(e)a_l - \mathrm{sgn}(d)b_l) \begin{pmatrix} \mathbf{c} \\ \bar{\mathbf{b}}^l \end{pmatrix} G_{\bar{\mathbf{b}}^l},$$

the term $\dot{G}_{\mathbf{w}}$ in (6.11) can be written as $\dot{G}_{\mathbf{w}} = \sum_{l=1}^{p} \dot{w}_0^l \tilde{G}_l$, from which, each of the p equations (6.18) becomes:

$$\begin{aligned} \dot{w}_0^i\left(1 - \left(\frac{\partial w_0^i}{\partial x}\right)^T B_w \tilde{G}_i P_{\mathbf{w}}^{-1} x\right) - \sum_{l=1, l\neq i}^{p} \dot{w}_0^l \left(\frac{\partial w_0^i}{\partial x}\right)^T B_w \tilde{G}_l P_{\mathbf{w}}^{-1} x \\ = \left(\frac{\partial w_0^i}{\partial x}\right)^T \left(A_w + B_w F_{\mathbf{w}} P_{\mathbf{w}}^{-1}\right) x, \ i \in \{1, 2, \ldots, p\}. \end{aligned} \tag{6.19}$$

These p equations can be grouped as:

$$\underbrace{\begin{bmatrix} 1-W_{11} & -W_{12} & \cdots & -W_{1p} \\ -W_{21} & 1-W_{22} & \cdots & -W_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ -W_{p1} & -W_{p2} & \cdots & 1-W_{pp} \end{bmatrix}}_{I-W} \underbrace{\begin{bmatrix} \dot{w}_0^1 \\ \dot{w}_0^2 \\ \vdots \\ \dot{w}_0^p \end{bmatrix}}_{\dot{w}} = \underbrace{\begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_p \end{bmatrix}}_{X}, \tag{6.20}$$

where expressions W_{il} (elements of a block-matrix W) and X_i (elements of a block-matrix X), $i, l \in \{1, 2, \ldots, p\}$, are given by

$$\begin{aligned} W_{il} &= \left(\frac{\partial w_0^i}{\partial x}\right)^T B_w \tilde{G}_l P_{\mathbf{w}}^{-1} x \\ X_i &= \left(\frac{\partial w_0^i}{\partial x}\right)^T \left(A_w + B_w F_{\mathbf{w}} P_{\mathbf{w}}^{-1}\right) x \end{aligned}$$

Therefore, $\dot{w} = (I - W)^{-1} X$ is a vector whose entries are the desired time derivatives of the WFs, where both W and X are functions of state x.

Remark 6.3.2. *In general, solving nonlinear algebraic loops during on-line operation would require iterative approaches (Kelley 1995) without a guarantee of termination time: such approach would pose severe drawbacks regarding real-time controller implementation. However, given the explicit expressions in TS form, the computational cost of the proposed derivatives is small and predictable (non-iterative): it requires computing the weighting functions, carrying out the summations, evaluating some gradients and inverting a small* $p \times p$ *matrix.*

Regularity conditions

Though the inclusion of the term $\dot{G}_{\mathbf{w}}$ was key in solving the algebraic loop, obtaining the time derivatives of the WFs depends on whether the inverse of matrix $I - W$ exists or not, i.e., there might be points of $x(t)$ where $I - W$ is singular. To guarantee regularity of $I - W$, (conservative) LMI conditions can be imposed based on small-gain argumentations.

From the knowledge of the explicit expressions of WF's, a bound on the maximum singular value of

$$\mathbf{J}_{w_0}(x) := \left[\left(\frac{\partial w_0^1}{\partial x}\right)^T \;\; \cdots \;\; \left(\frac{\partial w_0^p}{\partial x}\right)^T\right]^T \tag{6.21}$$

will be assumed, in the form $\bar{\sigma}(\mathbf{J}_{w_0}(x)) \leq \kappa$ for all x in a circular region of a prefixed radius ρ. Then, we can assert:

Theorem 6.3.2. *The largest spherical region where $I - W$ is regular inside $\mathcal{C}$ has radius greater or equal to ρ if, given bound κ, there exist matrices $M_{\mathbf{i}_1\mathbf{i}_2\cdots\mathbf{i}_p} \in \mathbb{R}^{p\cdot n\times(p-1)\cdot n}$ for a given $\mathbf{c} = (c_1, c_2, \ldots, c_p)$, $c_j \in \mathbb{N}$ as in (6.4), such that the following optimization problem is feasible:*

$\max \rho$ *subject to*

$$P_{\mathbf{w}} \geq \rho^2 I \tag{6.22}$$

$$\begin{bmatrix} M_{\mathbf{w}}\mathcal{I} + (*) + \frac{1}{p\kappa^2}\bar{P}_{\mathbf{w}} & (*) \\ \begin{bmatrix} H^1_{\mathbf{w}} & H^2_{\mathbf{w}} & \ldots & H^p_{\mathbf{w}} \end{bmatrix} & I \end{bmatrix} \geq 0, \tag{6.23}$$

with $\bar{P}_{\mathbf{w}} = P_{\mathbf{w}} \otimes$ *block-diag* $\underbrace{\begin{bmatrix} I & I & \ldots & I \end{bmatrix}}_{p\ times}$, $H^l_{\mathbf{w}} = B_w\tilde{G}_l$, $l \in \{1, 2, \ldots, p\}$, *and*

$$\mathcal{I} = \begin{bmatrix} I & -I & 0 & \cdots & 0 & 0 \\ 0 & I & -I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I & -I \end{bmatrix}.$$

Proof. We will prove that inequalities (6.22) and (6.23), along with the LMI objective, maximize the radius of the quadratically invariant sphere contained in $\mathcal{C}$, while keeping $\dot{w}_0$ bounded and allowing to W to be regular inside the sphere. Let us verify first inequality (6.22). We are looking to:

$$\left\{\frac{x^Tx}{\rho^2} < 1\right\} \subset \left\{V(x) = x^TP^{-1}_{\mathbf{w}}x < 1\right\}, \tag{6.24}$$

where ρ is the radius of the largest sphere inside $V(x) < 1$. Hence, expressing the initial ellipsoid and the sphere on this way, the S-procedure comes at hand to obtain:

$$\frac{x^Tx}{\rho^2} - x^TP^{-1}_{\mathbf{w}}x > 0 \Leftrightarrow \frac{I}{\rho^2} - P^{-1}_{\mathbf{w}} \Leftrightarrow P_{\mathbf{w}} \geq \rho^2 I,$$

where condition (6.22) guarantees that $P_{\mathbf{w}} \geq \rho^2 I$.

Let us now discuss the second condition (6.23). In this case, we want to guarantee regularity of $I - W$ inside the Lyapunov level set at the right-hand side of (6.24). Following (6.20), the matrix $I - W$ will be invertible if W has a maximum singular value lower than one. Rewriting W as:

$$W = \mathbf{J}_{w_0}(x)\left[B_w\tilde{G}_1P^{-1}_{\mathbf{w}}x, B_w\tilde{G}_2P^{-1}_{\mathbf{w}}x, \ldots, B_w\tilde{G}_pP^{-1}_{\mathbf{w}}x\right]$$

where $\mathbf{J}_{w_0}(x)$ is the Jacobian matrix of the vector w_0, see (6.21), and since $\mathbf{J}_{w_0}(x)$ is known, we can bound its worst-case gain, by assumption, as $\bar{\sigma}(\mathbf{J}_{w_0}(x)) \leq \kappa$. Note that, with $\eta = P_{\mathbf{w}}^{-1}x$, the level set $x^T P_{\mathbf{w}}^{-1} x \leq 1$ is $\eta^T P_{\mathbf{w}} \eta \leq 1$. Extracting $P_{\mathbf{w}}^{-1}x$ as common factor, we obtain:

$$W = \mathbf{J}_{w_0}(x)[H_{\mathbf{w}}^1 \; H_{\mathbf{w}}^2 \dots H_{\mathbf{w}}^p]\bar{\eta} = \mathbf{J}_{w_0}(x) H_{\mathbf{w}} \bar{\eta}$$

where $\bar{\eta} = [\eta^T \;\; \eta^T \;\; \dots \;\; \eta^T]^T$.

A sufficient condition $I - W$ being invertible is the small-gain one $\bar{\sigma}(W) < 1$. From the assumption $\bar{\sigma}(\mathbf{J}_{w_0}(x)) \leq \kappa$, such small-gain condition holds if:

$$\bar{\eta}^T H_{\mathbf{w}}^T H_{\mathbf{w}} \bar{\eta} < \frac{1}{\kappa^2}. \tag{6.25}$$

So, (6.25) should hold in $V(x) \leq 1$, i.e., in $\eta^T P_{\mathbf{w}} \eta < 1$. From $\eta^T P_{\mathbf{w}} \eta < 1$, we have $\bar{\eta}^T \bar{P}_{\mathbf{w}} \bar{\eta} < p$; then, along with the S-procedure, Finsler's lemma, and $\mathcal{I}\bar{\eta} = 0$, we obtain the following inequalities

$$M_{\mathbf{w}}\mathcal{I} + (*) + \frac{1}{p\kappa^2}\bar{P}_{\mathbf{w}} - H_{\mathbf{w}}^T H_{\mathbf{w}} \geq 0 \Leftrightarrow$$

$$\begin{bmatrix} M_{\mathbf{w}}\mathcal{I} + (*) + \frac{1}{p\kappa^2}\bar{P}_{\mathbf{w}} & H_{\mathbf{w}}^T \\ H_{\mathbf{w}} & I \end{bmatrix} \geq 0.$$

Clearly, if condition (6.23) holds, the last inequality is greater or equal to zero, thus concluding the proof. □

LMIs can be obtained from (6.22) and (6.23) by simply dropping off the WFs (since they hold the convex sum property), and testing inequalities for each vertex model (polynomial coefficient), as usual, replacing $\mathbf{w}$ by the each multi-index $\mathbf{b}$, $\mathbf{b} \leq \mathbf{c}$.

Robust differentiators (disturbed/modelling error cases)

In the presence of disturbances or modelling errors, the algebraic solution of (6.18) or, equivalently (6.20) would give a "biased" estimate of the true membership derivatives, as the closed-loop equation in the "real" controlled system is not the one at the right-hand side of (6.18). This, hence, would introduce an additional error in the controller implementation.

To address this problem, a proposal based on an s-th order Levant's robust differentiator, $s \geq 1$, can be employed (Levant 1998; Levant 2003); it ensures finite-time convergence to $\dot{w}_0^j$, $j \in \{1, 2, \ldots, p\}$ in the undisturbed case; its structure is:

$$
\begin{aligned}
\dot{v}_j^0 &= -\lambda_j^0 \left| v_j^0 - w_0^j \right|^{\frac{s}{s+1}} \operatorname{sign}\left(v_j^0 - w_0^j \right) + v_j^1 \\
\dot{v}_j^1 &= -\lambda_j^1 \left| v_j^1 - v_j^0 \right|^{\frac{s-1}{s}} \operatorname{sign}\left(v_j^1 - v_j^0 \right) + v_j^2 \\
&\vdots \\
\dot{v}_j^{s-1} &= -\lambda_j^{s-1} \left| v_j^{s-1} - v_j^{s-2} \right|^{\frac{1}{2}} \operatorname{sign}\left(v_j^{s-1} - v_j^{s-2} \right) + v_j^s \\
\dot{v}_j^s &= -\lambda_j^s \operatorname{sign}\left(v_j^s - v_j^{s-1} \right),
\end{aligned}
\tag{6.26}
$$

where $\lambda_j^0 > 0$ and $\lambda_j^i > L_j$, $i \in \{1, 2, \ldots, s\}$ are tuning parameters with $L_j > 0$ being a Lipschitz constant for $\dot{w}_0^j$.

Should the parameters be properly chosen, $\dot{v}_j^0 = \dot{w}_0^j$ after a finite time of a transient process in the absence of input noises. In (Aguiar, Márquez, and Bernal 2015), 1st-order Levant's robust differentiators were employed to provide estimates of the WFs derivatives; this solution can still be used.

An important property of the above differentiators is that, in the disturbed case, any sth-order Levant's robust differentiator, $s \geq 1$, can be employed to estimate the required time derivatives in finite-time with an accuracy of ϵ^{-2s}, where ϵ is the maximal (possibly unknown) measurement-noise magnitude, which implies that convergence time as well as accuracy can be improved as the differentiator order s goes higher. Once the order s is fixed, the differentiator performance only improves with the sampling step reduction (Levant 2003), even in the presence of exploding signals and feedback setups (Levant and Livne 2012). The above differentiators are employed in many real-time applications of contemporary sliding mode control (Shtessel et al. 2013), and its computational cost is just the one of integrating an s-th order ODE which, of course, depends on sampling rate and desired accuracy; most applications just use $s = 2$ whose computing cost per sample is negligible.

Note that finite-time convergence to zero error allows proving a separation-like stability result with the observer in the undisturbed case: indeed, as a TS system cannot have finite escape time[3], forcedly state trajectories will converge to zero from bounded initial conditions once the observer has converged, if they do not leave the region where $(I - W)$ is regular during such transient.

[3]TS systems are, trivially, globally Lipschitz with Lipschitz constant $\max_w \bar{\sigma}(A_w)$ so their solution is defined for every positive time.

6.4 Examples

Example 6.4.1. *Consider the 2nd-order 2-rule TS model in (D. Lee and D. Kim 2014), $\dot{x}(t) = A_w x(t) + B_w u(t)$, with $x(t)$ and $u(t)$ as the state and input vector, respectively, WFs $w_0 = 0.5(1 + \sin x_1)$ and $w_1 = 1 - w_0$, and system matrices A_w and B_w being convex sums of the following matrices:*

$$A_0 = \begin{bmatrix} 4 & -4 \\ -1 & -2 \end{bmatrix}, \; A_1 = \begin{bmatrix} -2 & -4 \\ 20 & -2 \end{bmatrix}, \; B_0 = \begin{bmatrix} 1 \\ 10 \end{bmatrix}, \; B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Note that the WFs hold the convex sum property everywhere in the state space $\mathbb{R}^2$.

Quadratic stabilization of this model is not possible (D. Lee, J. Park, and Joo 2012). Thus, four approaches will be tested:

1. *Conditions in (Aguiar, Márquez, and Bernal 2015) along with Theorem 6.3.1 with $\mathbf{c} = 2$. Time derivatives of the WFs are obtained from a 1st-order Levant's robust differentiator*[4].
2. *Theorem 6.3.1 with $\mathbf{c} = 2$ where the time derivatives of the WFs are also obtained from a 1st-order differentiator.*
3. *Theorem 6.3.1 with $\mathbf{c} = 2$ where the time derivatives of the WFs are algebraically solved from* (6.20).
4. *Theorem 6.3.1 and Theorem 6.3.2 with $\mathbf{c} = 2$.*

For the second and third cases, the matrices in the Lyapunov function as well as the set of gains for the control law are the same, since they come from the same LMI conditions. The first ones are:

$$P_0 = \begin{bmatrix} 0.3811 & 0.5957 \\ 0.5957 & 1.5697 \end{bmatrix}, \; P_1 = \begin{bmatrix} 0.5889 & 0.2732 \\ 0.2732 & 2.1096 \end{bmatrix}, \; P_2 = \begin{bmatrix} 0.2077 & 0.0675 \\ 0.0675 & 1.0033 \end{bmatrix}.$$

Notice that not all of them are obliged to be positive definite.

The controller gains are the following:

$$F_0 = \begin{bmatrix} 0.5107 & 0.3237 \end{bmatrix}, \; F_1 = \begin{bmatrix} 0.9204 & -0.5108 \end{bmatrix}, \; F_2 = \begin{bmatrix} 0.1375 & 0.0840 \end{bmatrix},$$

[4] The preliminary version (Aguiar, Márquez, and Bernal 2015) considered inequality constraints in (6.16) depending on the sign of the derivatives. Although that idea was considered of interest at the time of writing (Aguiar, Márquez, and Bernal 2015), subsequent analysis showed that there was no loss in generality considering just equality when global cancellation was pursued. Anyway, the inequality-based version of (6.16) might be worthwhile in local/saturated control extensions to the ideas presented here, which will be pursued in further research.

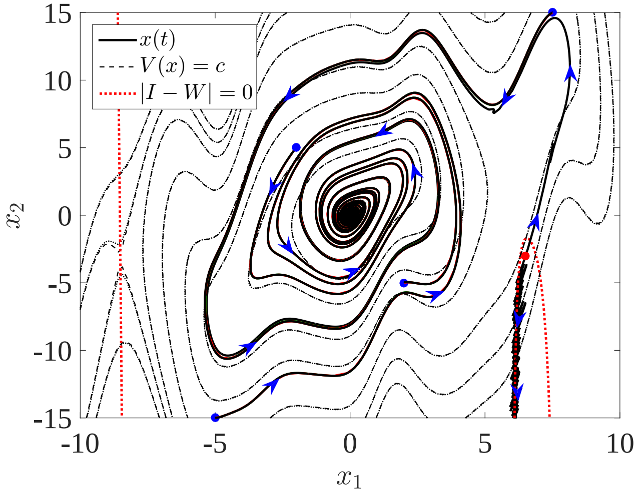

Figure 6.1: Lyapunov level sets (dashed lines), system trajectories (solid lines), and $\det(I - W) = 0$ (dotted lines) for Example 6.4.1.

$$G_0 = \begin{bmatrix} 0.0175 & 0.0502 \end{bmatrix}, \; G_1 = \begin{bmatrix} -0.0517 & 0.0488 \end{bmatrix}, \; G_2 = \begin{bmatrix} -0.0692 & -0.0013 \end{bmatrix}.$$

The resulting Lyapunov level sets are shown in Fig. 6.1 with dashed lines; the solid ones are the system trajectories. Clearly, the system has been stabilized, but how far can this be guaranteed? Dotted lines correspond to the places where $\det(I - W) = 0$*: obviously, trajectories crossing these lines whose control laws get the time derivatives of the MFs from* (6.20)*, may diverge. This is the case of the divergent trajectory beginning at* $x_1(0) = 6.5$*,* $x_2(0) = -3$*. On the other hand, if the time derivatives of the MFs are obtained from the Levant's robust differentiator, a stable trajectory starting at the same point is obtained. This behaviour outside the guaranteed regularity region is left for further research. Inside the regularity region, there is no substantial difference between the observer-based simulations and the algebraic-solution ones.*

The similarity of results among 1) those obtained with the enhanced version of the switching control in (Aguiar, Márquez, and Bernal 2015), 2) conditions in Theorem 6.3.1 with the time derivatives of the WFs coming from a 1st-order Levant's robust differentiator, and 3) those in Theorem 6.3.1 whose time derivatives are algebraically solved, is explained by the fact that gains $G^1_{\mathbf{i}_1}$ *and* $G^2_{\mathbf{i}_1}$ *in (Aguiar, Márquez, and Bernal 2015) tend to be the same, i.e., a* single *set of gains* $G_{\mathbf{b}}$ *along with the multi-index nature of the controller design, is enough to guarantee* equality (6.13).

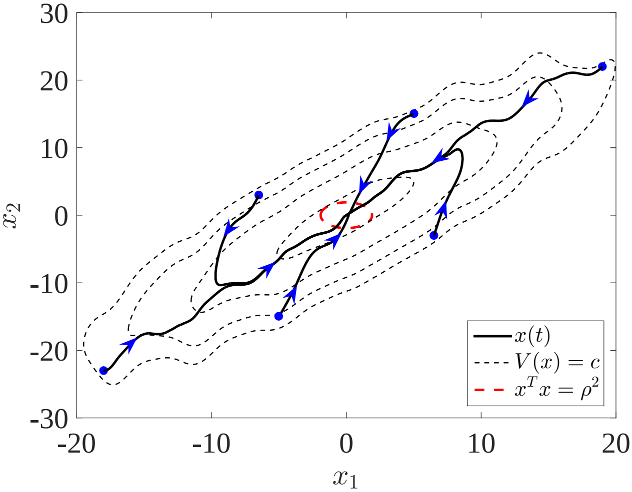

Figure 6.2: Lyapunov level sets (dashed lines), system trajectories (solid lines), and maximum guaranteed circle with $\det(I - W) \neq 0$ (dotted lines in red) for Example 6.4.1

Now, consider the case 4) where LMIs in Theorem 6.3.2 are tested along with those in Theorem 6.3.1. We can guarantee the existence of $(I - W)^{-1}$ in a circle of radius $\rho = 1.9096$ which is shown in dashed lines at the center of Fig. 6.2. This is certainly a conservative estimate as can be easily proved by plotting $\det(I - W) = 0$, which is far beyond the limits of this figure. Lyapunov sets are shown also in dashed lines, while trajectories are shown in solid lines. It is important to notice that the local estimations of the domain of attraction in (D. Lee and D. Kim 2014) are all subsets of those hereby provided. Note also that, as the stabilising controller is not unique, the geometry of the level sets obtained in case 4) is quite different from that in cases 1)-3).

Example 6.4.2. *The 2-nd-order 2-rule TS model $\dot{x}(t) = A_w x(t) + B_w u(t)$ in (Pan et al. 2012) has WFs w_0, w_1 as in example 6.4.1, and system matrices*

$$A_0 = \begin{bmatrix} a & -5 \\ 1 & 2 \end{bmatrix}, \; A_1 = \begin{bmatrix} 2 & -10 \\ 2 & 0 \end{bmatrix}, \; B_0 = \begin{bmatrix} b \\ 2 \end{bmatrix}, \; B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

with parameters $a \in [-20, 11]$ and $b \in [0, 25]$. Within these ranges, it is tested under the quadratic case (Tanaka and H. Wang 2001), conditions in (Rhee and Won 2006), and those of Theorem 6.3.1 with $\mathbf{c} = 2$ and $\mathbf{c} = 3$. Clearly, the proposed approach overcomes the feasibility set of former approaches, as shown in Fig. 6.3; increasing the Polya degree $\mathbf{c}$ from 2 to 3 achieves a handful of additional feasible points. The feasibility set reported in (Pan et al. 2012), though

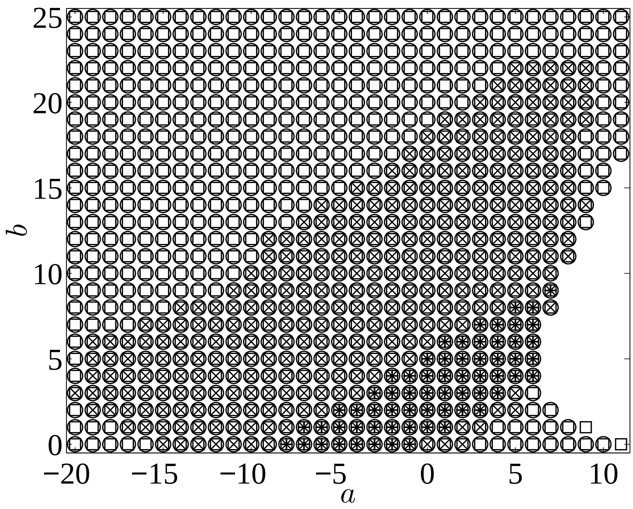

Figure 6.3: Feasibility sets for Example 6.4.2: (+) for quadratic (Tanaka and H. Wang 2001); (×) for line-integral (Rhee and Won 2006); (○) for Th. 6.3.1, $\mathbf{c} = 2$; (□) for Th. 6.3.1, $\mathbf{c} = 3$.

not included because of its use of a priori bounds, is smaller than those obtained with the proposed approach.

It is important to underline the fact that the proposed improvements are compatible with further relaxations such as those based on matrix transformations, which may improve numerical efficiency.

In order to illustrate the quality of a particular solution, consider the case $a = 11$, $b = 0$, *which has no solution in the quadratic framework (Tanaka and H. Wang 2001) nor in the line-integral approach of (Rhee and Won 2006) nor with Th. 6.3.1 with* $\mathbf{c} < 3$. *For* $\mathbf{c} = 3$ *a controller has been found: due to this number of sum relaxations, 8 triplets of matrices* P_{b_1}, F_{b_1}, *and* G_{b_1} *were found; they are omitted for brevity.*

The time evolution of the control signal $u(t)$, *the states* $x(t)$, *and the Lyapunov function* $V(t)$ *is shown in Fig. 6.4, all of them corresponding to a simulation of the system under the initial condition* $x(0) = \begin{bmatrix} 1 & -2 \end{bmatrix}^T$. *Clearly, the control task has been achieved as expected: states are driven to zero and* $V(t)$ *is indeed a Lyapunov function.*

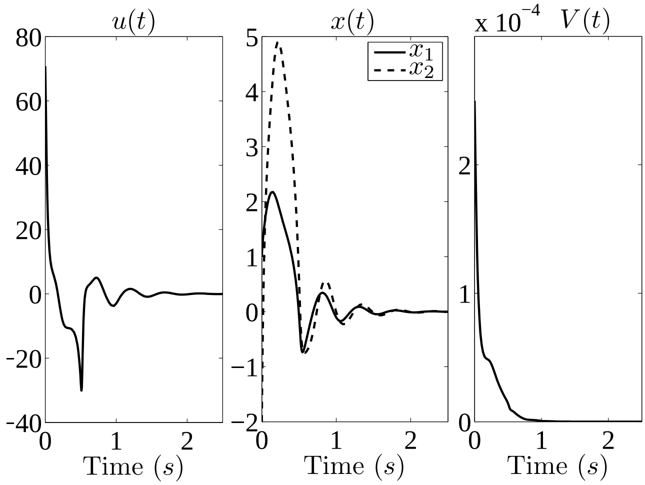

Figure 6.4: From left to right: time evolution of the control signal $u(t)$, states $x(t)$, and Lyapunov function $V(x(t))$ for Example 6.4.2.

EXAMPLE 6.4.3. *Consider the 4-rule, 2-nd order TS model* $\dot{x}(t) = A_w x(t) + B_w u(t)$, *with*

$$A_{00} = \begin{bmatrix} 1.59 & -7.29 \\ 0.01 & 0 \end{bmatrix}, \; A_{01} = \begin{bmatrix} 0.02 & -4.64 \\ 0.35 & 0.21 \end{bmatrix},$$

$$A_{10} = \begin{bmatrix} 0 & -4.33 \\ 0 & 0.05 \end{bmatrix}, \; A_{11} = \begin{bmatrix} 0.89 & -5.29 \\ 0.1 & 0 \end{bmatrix},$$

$$B_{00} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; B_{01} = \begin{bmatrix} 8 \\ 0 \end{bmatrix}, \; B_{10} = \begin{bmatrix} 6 \\ -1 \end{bmatrix}, \; B_{11} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

and WFs $w_{01} = 0.5\,(1 + \sin x_1)$, $w_{02} = 0.25\left(4 - x_2^2\right)$, $w_{11} = 1 - w_{01}$, *and* $w_{12} = 1 - w_{02}$, *within the compact set* $\mathcal{C} = \{(x_1, x_2) : x_1 \in \mathbb{R}, x_2 \in [-2, 2]\}$. *This example is a 4-rule extension of a system shown in (Sala and Ariño 2007; Fang et al. 2006; Marquez, T.M Guerra, et al. 2016); it produces a feasible solution of conditions in Theorem 6.3.1.*

In this example, Polya relaxations can prove that the system has a quadratic LF. If we seek to optimize the guaranteed radius in our approach, the found radius can be increased to arbitrarily large values, and the LMIs find, of course, the quadratic solution. This shows that there is no loss in the presented proposal with respect to quadratically-feasible solutions.

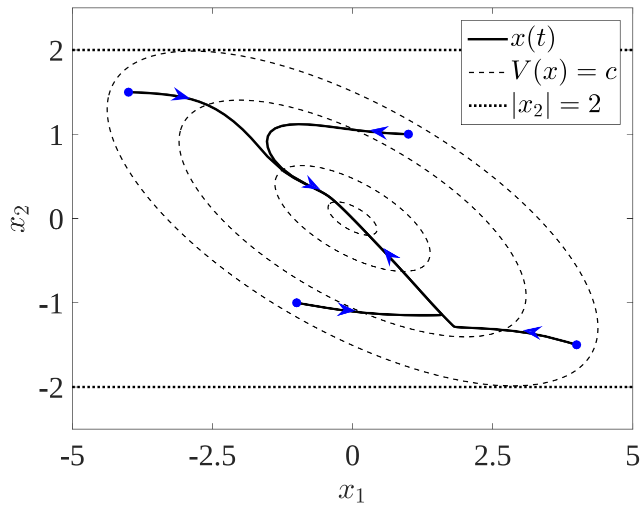

Figure 6.5: Lyapunov sets (dashed lines), modelling border (dotted lines), and trajectories (solid lines) for Example 6.4.3. As conditions were quadratically feasible, such solution is also obtained by the here presented approach.

6.5 Conclusion

A novel solution for nonquadratic stabilization of continuous-time nonlinear systems via exact Takagi-Sugeno models and generalized parameter-dependent Lyapunov functions has been presented. The main contribution of this work has been a multi-index control law that cancels out the terms that cause a priori locality of former approaches, by using the time derivatives of the membership functions obtained from the closed loop expression of the system. The resulting LMI conditions have outperformed well-known examples taken from the literature on the subject. Levant's robust observer-based implementations are suggested in applications where noise or modelling error is present.

As for future work, it is worth exploring how to overcome the current limitations on the regularity of $(I - W)^{-1}$, by finding either less conservative LMIs guaranteeing it or new control schemes which naturally avoid such terms. Global nonquadratic stabilization seems possible if a suitable combination of such improvements and input saturation is explored.

Chapter 7

Polynomial-Integral Lyapunov Function

In this chapter, a new integral Lyapunov Function is presented, which generalises the line-integral Lyapunov function in Rhee and Won 2006 for stability analysis of continuous-time nonlinear models expressed as fuzzy systems. The referred result applied only to Takagi-Sugeno representations, and required memberships to be a tensor-product of functions of a single state; these are generalised here so that membership arguments can be arbitrary polynomials of the state variables; in this way, systems for which earlier results cannot be applied are now covered. Both the modelling and the integral terms appearing in the Lyapunov functions are generalised to a fuzzy polynomial case. Illustrative examples show the advantage of the proposed method against previous literature, even in the TS case.

The contents of this chapter appeared in the journal article:

- T. Gonzalez, A. Sala, and M. Bernal (2018). "A Generalised Integral Polynomial Lyapunov Function for Nonlinear Systems". In: *Fuzzy Sets and Systems*. In press.

7.1 Introduction

Stability analysis of nonlinear systems has benefited in the last twenty years from a representation as a combination of linear models, denoted as Takagi-Sugeno (TS) (Takagi and Sugeno 1985) or quasi-LPV (J. S. Shamma and Cloutier 1992) representations. Obtaining such models via the sector nonlinearity approach (Taniguchi, Tanaka, and H. Wang 2001) allows to exactly rewrite a nonlinear system as a convex sum of linear models within a compact set of the state space (modeling region), the nonlinearities being captured in so-called membership functions (MFs) which are in general state-dependent and hold the convex sum property (Tanaka and H. Wang 2001). Later on, in (Sala and C. Ariño 2009; Chesi 2009), via the Taylor-series approach, the sector nonlinearity idea was extended to polynomial fuzzy models: this representation expresses non-polynomial nonlinearities as an equivalent convex sum of polynomial consequents, blended together by MFs.

When a TS model is available, stability analysis and controller design are usually performed via the direct Lyapunov method, which usually leads to conditions in the form of linear matrix inequalities (LMIs) (Tanaka and H. Wang 2001). LMI conditions are highly appreciated as their feasibility can be decided via convex optimization techniques (Boyd et al. 1994). Different classes of Lyapunov functions have been used to overcome the conservatism of the common quadratic one, first proposed in (Tanaka and Sugeno 1990): piecewise (Johansson, Rantzer, and Arzen 1999; Gonzalez, Sala, and Bernal 2017), parameter-dependent (also unspecifically known as “non-quadratic” or “fuzzy”) (T.M. Guerra and Vermeiren 2004; T.M. Guerra and Bernal 2012), and fuzzy line-integral (LI) (Rhee and Won 2006). Other recent proposals, intentionally left out of this quest, are based on polyhedron manipulations and set-invariance considerations (C. Ariño, Sala, et al. 2017); these proposals avoid the need of fixing a structure of a Lyapunov function and, importantly, are asymptotically exact (under some conditions) for the TS case; however, they cannot be extended to the fuzzy-polynomial setup below.

In (F. Wu and Prajna 2005; Tanaka, Yoshida, et al. 2007a), the quadratic LMI/TS framework was extended to the sum-of-squares (SOS) approach (Prajna, Papachristodoulou, Seiler, et al. 2004; Prajna, Papachristodoulou, Seiler, et al. 2005), which use polynomial Lyapunov functions for stability analysis of nonlinear systems in fuzzy-polynomial form, posing SOS conditions which are actually reducible to LMIs. Later on, a fuzzy polynomial Lyapunov function was employed to generalize results for fuzzy polynomial models (Bernal, Sala, et al. 2011). In that work, the time-derivative of the MFs is *a priori* bounded by polynomials of the state, thus obtaining a fuzzy polynomial model of the time derivative of the MFs. As a last option on these issues, (Y. Chen et al. 2015) presented a piecewise Lyapunov function defined by the minimum or maximum of polynomials.

The widely-cited work (Rhee and Won 2006) proposed an interesting fuzzy line-integral Lyapunov function, presenting LMI stability conditions which are global and avoided the time derivative of the MFs. The goal of this chapter is generalising the fuzzy LI approach in the above-referred work to the polynomial case: it turns out that path independency conditions for line integrals are automatically verified if the integral can be expressed as a sum of single-variable terms. Let us, next, discuss in detail the motivation behind our proposal.

In (Rhee and Won 2006), a Lyapunov function with integral terms was pursued. However, since such Lyapunov function depended on necessary path-independence conditions, the approach was only applicable to a specific class of TS models where the MFs are a tensor-product expression (C. Ariño and Sala 2007) of at most n nonlinear components where each of them depends exactly on one state variable. For this class of models, only the “diagonal” terms of the Lyapunov function were actually using fuzzy summations and, moreover, if the MFs depend on multiple variables and cannot be factorised, e.g., $w_i(x_1 + x_2) \neq \alpha(x_1)\beta(x_2)$, the approach in (Rhee and Won 2006) cannot be directly applied.

In order to generalize the class of TS model on where the LI approach can be applied, the LF in (Rhee and Won 2006) is expressed as a sum of single-variable integrals, as above mentioned. Resorting to such parametrisation, path-independence conditions are automatically fulfilled. This was the idea behind a preliminary result presented in (Gonzalez, Sala, Bernal, and Robles 2017), introducing a larger class of path-independent line-integral Lyapunov functions whenever the MFs depended on an arbitrary set of *linear* functions of the system states. Other refinements on the work of (Rhee and Won 2006) can be found in (Marquez, T.M. Guerra, et al. 2013; Marquez, T.M. Guerra, et al. 2014); they exploit a relaxation from a determinant formula which applies only to second-order TS systems, but do not correspond to the point of view hereby adopted (pursuing results applicable to higher-order systems).

Motivated by the ideas above, this chapter presents a Polynomial Lyapunov function including integral terms, for the stability analysis of a class of nonlinear models so the results in (Rhee and Won 2006; Gonzalez, Sala, Bernal, and Robles 2017) are a particular case. The results in this manuscript apply to nonlinear systems that can be expressed in terms of single-variable non-polynomial nonlinearities with a polynomial argument.

The chapter is organized as follows: section 7.2 presents the classical sector non-linearity approach to obtain TS models, previous results about the line-integral Lyapunov approach, and a review on the standard polynomial fuzzy framework; section 7.3 develops the main result, where a new Polynomial+Integral Lyapunov function is built; section 7.4 gives some examples to illustrate the effectiveness of the proposed approach; finally, discussion, concluding remarks and ideas for future work are given in sections 7.5 and 7.6.

7.2 Preliminaries and problem statement

7.2.1 Takagi-Sugeno models

Consider a nonlinear system:

$$\dot{x}(t) = h(x(t)), \tag{7.1}$$

with $x \in \mathbb{R}^n$ being the state vector, and $x = 0$ being an equilibrium point, i.e., $h(0) = 0$. Let us assume that $h(\cdot)$ can be expressed in the form:

$$\dot{x}(t) = \tilde{h}(\eta(x), x), \tag{7.2}$$

where $\tilde{h}(\cdot)$ is linear in $x(t)$ and multiaffine in $\eta(x) \in \mathbb{R}^q$, where

$$\eta(x) = [\eta_1(x)\ \eta_2(x)\ \cdots\ \eta_q(x)]^T$$

is a set of continuous functions which collects all nonlinearities present in $h(\cdot)$ in (7.1). Then, the above model can be written as (Robles et al. 2017):

$$\dot{x}(t) = \tilde{f}(\eta(x))x(t), \tag{7.3}$$

with $\tilde{f}(\cdot) : \mathbb{R}^q \mapsto \mathbb{R}^n$ being a multiaffine function in its arguments.

A well-established procedure for convex rewriting of such nonlinear systems within a compact set $\Omega \supset \{0\}$ of the state space, called the sector nonlinearity methodology (Taniguchi, Tanaka, and H. Wang 2001), is available. Let us outline the main ideas of it in order to introduce notation which will be used in later developments in the chapter.

Since, by continuity and compactness, the components of vector $\eta(x)$ are bounded in Ω, assume $\eta_j(x) \in \left[\underline{\eta}_j, \overline{\eta}_j\right]$, $j \in \{1, 2, \ldots, q\}$ in Ω. By defining the following *weighting functions* (WFs):

$$w_0^j(x) := \frac{\overline{\eta}_j - \eta_j(x)}{\overline{\eta}_j - \underline{\eta}_j},\ w_1^j(x) = 1 - w_0^j(x),\ j \in \{1, 2, \ldots, q\}, \tag{7.4}$$

each nonlinearity is written as $\eta_j(x) = w_0^j(x)\underline{\eta}_j + w_1^j(x)\overline{\eta}_j$, with $0 \leq w_i^j \leq 1$, $w_0^j + w_1^j = 1$. On the sequel, dependence of w_i^j on the state x will be omitted for notational brevity if clear from the context.

As $\tilde{f}$ is multiaffine, straightforward manipulations lead to a TS model with q nested convex sums:

$$\dot{x}(t) = A_w x(t), \tag{7.5}$$

$$A_w := \sum_{i_1=0}^{1} \sum_{i_2=0}^{1} \cdots \sum_{i_q=0}^{1} w_{i_1}^1 w_{i_2}^2 \cdots w_{i_q}^q A_{(i_1,i_2,\ldots,i_q)}, \tag{7.6}$$

$$A_{(i_1,i_2,\ldots,i_q)} = \tilde{f}(\eta(x))|_{w_{i_1}^1 = w_{i_2}^2 = \cdots = w_{i_q}^q = 1},$$

with $A_{(i_1,i_2,\ldots,i_q)} \in \mathbb{R}^{n\times n}$, $i_j \in \{0,1\}$, $j \in \{1,2,\ldots,q\}$. This sort of notation for TS models corresponds to the tensor-product modelling approach (C. Ariño and Sala 2007; Campos et al. 2013). The reader is referred to these works for further details on the above fuzzy modelling steps, which routinely appear in systems with several nonlinearities.

Example 7.2.1. *Consider the following nonlinear system:*

$$\dot{x} = \begin{bmatrix} -a - b(1+\cos\rho_2) + 0.2\cos\rho_3 & -3 + \cos\rho_3 - 5\sin\rho_1 \\ -0.5a + 0.2b(1+\cos\rho_3) & -4.6 + \cos\rho_3 + \sin\rho_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where $\eta_1(x) := \sin(\rho_1(x))$ *with* $\rho_1(x) := x_2$, $\eta_2(x) := \cos(\rho_2(x))$ *with* $\rho_2(x) := 2x_2^2 - x_1x_2$, *and* $\eta_3(x) := \cos(\rho_3(x))$ *with* $\rho_3(x) := x_2 - 4x_1^2$. *If we independently model* $\sin(\rho_1)$, $\cos(\rho_2)$, *and* $\cos(\rho_3)$ *in the previous system via standard sector nonlinearity, we get the following tensor-product TS model with* 2^3 *vertices:*

$$\dot{x} = \sum_{i_1=0}^{1} \sum_{i_2=0}^{1} \sum_{i_3=0}^{1} w_{i_1}^1(\rho_1) w_{i_2}^2(\rho_2) w_{i_3}^3(\rho_3) A_{(i_1,i_2,i_3)} x, \tag{7.7}$$

with

$$A_{000} = \begin{bmatrix} -a-0.2 & 1 \\ -0.5a & -6.6 \end{bmatrix}, \; A_{001} = \begin{bmatrix} 0.2-a & 3 \\ 0.4b-0.5a & -4.6 \end{bmatrix},$$

$$A_{010} = \begin{bmatrix} -a-2b-0.2 & 1 \\ -0.5a & -6.6 \end{bmatrix}, \; A_{011} = \begin{bmatrix} 0.2-2b-a & 3 \\ 0.4b-0.5a & -4.6 \end{bmatrix},$$

$$A_{100} = \begin{bmatrix} -a-0.2 & -9 \\ -0.5a & -4.6 \end{bmatrix}, \; A_{101} = \begin{bmatrix} 0.2.a & -7 \\ 0.4b-0.5a & -2.6 \end{bmatrix},$$

$$A_{110} = \begin{bmatrix} -a-2b-0.2 & -9 \\ -0.5a & -4.6 \end{bmatrix}, \quad A_{111} = \begin{bmatrix} 0.2-2b-a & -7 \\ 0.4b-0.5a & -2.6 \end{bmatrix},$$

and WFs $w_0^1(\rho_1) = 0.5(1 - \sin(\rho_1))$, $w_1^1(\rho_1) = 1 - w_0^1(\rho_1)$, $w_0^2(\rho_2) = 0.5(1 - \cos(\rho_2))$, $w_1^2(\rho_2) = 1 - w_0^2(\rho_2)$, $w_0^3(\rho_3) = 0.5(1 - \cos(\rho_3))$, $w_1^3(\rho_3) = 1 - w_0^3(\rho_3)$.

In order to get a more compact notation, the multi-index shorthand notation from, for instance, (Tognetti, R.C.L.F. Oliveira, and P.L.D. Peres 2011; Gonzalez, Bernal, Sala, et al. 2017), will be used with $\mathbf{a} := (a_1, a_2, \ldots, a_q)$, $a_j \in \{0 \cup \mathbb{N}\}$, and $\mathbf{b} := (b_1, b_2, \ldots, b_q)$, $b_j \in \{0 \cup \mathbb{N}\}$, q-dimensional multi-indices (q-tuples):

$$\mathbf{w}_0^{\mathbf{a}} := \left(w_0^1\right)^{a_1} \left(w_0^2\right)^{a_2} \cdots \left(w_0^q\right)^{a_q}, \quad \mathbf{w}_1^{\mathbf{b}} := \left(w_1^1\right)^{b_1} \left(w_1^2\right)^{b_2} \cdots \left(w_1^q\right)^{b_q}.$$

Then, the nested convex sum A_w in (7.6) can be equivalently rewritten as:

$$A_w = \sum_{\mathbf{j}+\mathbf{i}=\mathbf{1}} \mathbf{w}_0^{\mathbf{j}} \mathbf{w}_1^{\mathbf{i}} A_{\mathbf{i}}, \tag{7.8}$$

with $\mathbf{j}+\mathbf{i} := (j_1 + i_1, j_2 + i_2, \ldots, j_q + i_q,)$ being the element-wise sum of q-tuples, and $\mathbf{1} := (1, 1, \ldots, 1)$. For instance, in the above example $q = 3$.

Well-known conditions for quadratic stability, with Lyapunov function $V(x) = x^T P x$, of the above model are (Tanaka and H. Wang 2001):

$$P > 0, \quad PA_{\mathbf{i}} + A_{\mathbf{i}}^T P \leq 0.$$

However, these conditions are known to be conservative. Other options, called non-quadratic LF, have appeared in literature (see, for instance, (Blanco, Perruqueti, and Borne 2001; Tanaka, Hori, and H. Wang 2003; T.M. Guerra and Vermeiren 2004)), in which the LF is in the form:

$$V(x) = x^T P_w x = x^T \left(\sum_{\mathbf{j}+\mathbf{i}=\mathbf{1}} \mathbf{w}_0^{\mathbf{j}} \mathbf{w}_1^{\mathbf{i}} P_{\mathbf{i}} \right) x. \tag{7.9}$$

However, as $\dot{V}(x)$ depends on $\dot{P}_w$, time-derivative bounds on the WFs are needed (Tanaka, Hori, and H. Wang 2003; L. Mozelli, Palhares, and Avellar 2009) or, via chain-rule argumentations, bounds on the partial derivatives of them (T.M. Guerra and Bernal 2009; Bernal and T. M. Guerra 2010). In some cases, a cancellation-based controller design approach can be crafted to avoid the WF derivative bounds (Gonzalez, Bernal, Sala, et al. 2017) in the resulting closed-loop expressions.

7.2.2 Line-integral fuzzy Lyapunov Functions in prior literature

Consider the particular case of a model (7.2) with $\eta_j(x)$ depending only on $x_j(t)$ and $q \leq n$. Then, from (7.4), each $w_k^j(\cdot)$ only depends on x_j. On the sequel, given $f: \mathbb{R}^n \mapsto \mathbb{R}^h$ notation ∇f denotes the Jacobian matrix of size $h \times n$.

In the work (Rhee and Won 2006), based on line-integral considerations, the following line-integral fuzzy Lyapunov function was proposed:

$$V(x) = \int_{\Gamma(0,x)} f(\psi)\, d\psi \tag{7.10}$$

where $\Gamma(0,x)$ was any one-dimensional path betwen 0 and x, $\psi \in \mathbb{R}^n$ is a dummy vector for the integral argument, and $d\psi \in \mathbb{R}^n$ is an infinitesimal displacement vector along the path, and $f(\psi)$ was given by:

$$f(\psi) = \psi^T P + \begin{bmatrix} \sum\limits_{j_1+i_1=1} (w_0^1)^{j_1} (w_1^1)^{i_1} s_{i_1}^1 \psi_1 \\ \sum\limits_{j_2+i_2=1} (w_0^2)^{j_2} (w_1^2)^{i_2} s_{i_2}^2 \psi_2 \\ \vdots \\ \sum\limits_{j_n+i_n=1} (w_0^n)^{j_n} (w_1^n)^{i_n} s_{i_n}^n \psi_n \end{bmatrix}^T \tag{7.11}$$

being P a constant, symmetrical matrix with null diagonal in the said reference.

Expression (7.10) was proved path-independent (proving that $\partial f_i/\partial\psi_j = p_{ij} = p_{ji} = \partial f_j/\partial\psi_i$), so the integral is identical for any Γ. Choosing the particular path formed by the 1-dimensional segments going from $(0,\dots,0)$ to $(x_1, 0, \dots, 0)$, then to $(x_1, x_2, 0, \dots)$, then to $(x_1, x_2, x_3, 0, \dots)$ and so on until $(x_1, \dots, x_n)$ is reached[1], the integral (7.10) results in the actual explicit expression:

$$V(x) = x^T P x + \sum_{k=1}^{n} \int_0^{x_k} \sum_{j_k+i_k=1} \left(w_0^k(\psi)\right)^{j_k} \left(w_1^k(\psi)\right)^{i_k} s_{i_k}^k \psi\, d\psi, \tag{7.12}$$

where $\psi \in \mathbb{R}$ is now a *one*-dimensional dummy variable and P is a matrix (without loss of generality, with null diagonal). Conversely, its gradient $\nabla V(x)$ is $f(x)$ being $f(\cdot)$ defined in (7.11). Such a fact can be proven from path-independence considerations, as originally done in (Rhee and Won 2006), or, alternatively, by explicitly carrying out the straightforward differentiation of (7.12).

Then, a reformulation of the main result of (Rhee and Won 2006), adapted to our notation, is the following theorem:

[1] or, evidently, any other path, if desired.

Theorem 7.2.1. *The system* (7.5) *with* $w_j^k(x_k(t))$, $k \in \{1, 2, \dots, n\}$ *is asymptotically stable if the following conditions hold:*

$$x^T P x + \sum_{k=1}^{n} (s_{j_k}^k - \varepsilon) x_k^2 \geq 0 \tag{7.13}$$

$$-\sum_{\mathbf{k}_0+\mathbf{k}_1=\mathbf{2}} \mathbf{w}_0^{\mathbf{k}_0} \mathbf{w}_1^{\mathbf{k}_1} \sum_{\mathbf{i}+\mathbf{j}=\mathbf{k}_1,\ \mathbf{i}\leq\mathbf{1},\ \mathbf{j}\leq\mathbf{1}} x^T \left(\bar{P}_{\mathbf{j}} A_{\mathbf{i}} + A_{\mathbf{i}}^T \bar{P}_{\mathbf{j}} + \varepsilon I\right) x \geq 0. \tag{7.14}$$

where $\bar{P}_{\mathbf{j}} = P + diag\left(s_{j_1}^1, s_{j_2}^2, ..., s_{j_n}^n\right)$, $j_k \in \mathbb{B}$, $\mathbb{B} = 0, 1$ *and* ε *is a small positive constant.*

The reader is referred to the cited references for further details and proofs of the above-presented results. Trivially, by removing x^T and x and using Polya relaxations (C. Ariño and Sala 2007), the above scalar inequalities get converted into standard LMIs:

$$\bar{P}_{\mathbf{k}} - \varepsilon I \geq 0 \qquad \forall\, \mathbf{k} \leq \mathbf{1} \tag{7.15}$$

$$\sum_{\mathbf{i}+\mathbf{j}=\mathbf{k},\ \mathbf{i}\leq\mathbf{1},\ \mathbf{j}\leq\mathbf{1}} \bar{P}_{\mathbf{j}} A_{\mathbf{i}} + A_{\mathbf{i}}^T \bar{P}_{\mathbf{j}} + \varepsilon I \leq 0 \qquad \forall\, \mathbf{k} \leq \mathbf{2}. \tag{7.16}$$

7.2.3 Polynomial fuzzy models

Consider now a more general case where $\tilde{h}$ in (7.2) is a polynomial in nonlinearities η, say, of degree c_k in η_k for $k \in \{1, 2, \dots, q\}$. Then, the model can be expressed as the multi-dimensional TS one below where memberships have degree greater than 1 in the summations:

$$\dot{x}(t) = \sum_{\mathbf{j}+\mathbf{i}=\mathbf{c}} n_{\mathbf{i}}^{\mathbf{c}} \mathbf{w}_0^{\mathbf{j}} \mathbf{w}_1^{\mathbf{i}} A_{\mathbf{i}} x(t), \tag{7.17}$$

being $\mathbf{c} := (c_1, c_2, \dots, c_q)$ a degree vector where c_k, $k = \{1, 2, \dots, q\}$ is the degree each nonlinearity η has in the polynomial $\tilde{h}$ in (7.2), and $n_{\mathbf{i}}^{\mathbf{c}} := \prod_{k=1}^{q} \frac{c_k!}{i_k!(c_k-i_k)!}$. The combinatorial number $n_{\mathbf{i}}^{\mathbf{c}}$ is the number of similar terms sharing a specific combination $\mathbf{w}_0^{\mathbf{j}} \mathbf{w}_1^{\mathbf{i}}$, which allows writing $\sum_{\mathbf{j}+\mathbf{i}=\mathbf{c}} n_{\mathbf{i}}^{\mathbf{c}} \mathbf{w}_0^{\mathbf{j}} \mathbf{w}_1^{\mathbf{i}} = 1$, a property that proves to be useful in the quest for less conservative conditions derived from convex sums (Sala and Ariño 2007). Note that the previously-considered tensor-product TS case in (7.2) is the particular case of $\mathbf{c} = (1, \dots, 1)$.

Actually, if $\tilde{h}$ were a polynomial in both η and x, then a so-called fuzzy-polynomial model[2] would have been obtained in the form:

$$\dot{x}(t) = \sum_{\mathbf{j}+\mathbf{i}=\mathbf{c}} n_{\mathbf{i}}^{\mathbf{c}} \mathbf{w}_0^{\mathbf{j}} \mathbf{w}_1^{\mathbf{i}} F_{\mathbf{i}}(x(t)) := F_w(x(t)), \tag{7.18}$$

where $F_{\mathbf{i}}(x(t))$ are vertex polynomial models (Sala and C. Ariño 2009). These general fuzzy polynomial models will, thus, be the subject of inquiry in the sequel.

The sum-of-squares (SOS) paradigm is widely used to prove stability of the above models. Indeed, a polynomial $p(x)$ is SOS (to be denoted by $p(x) \in \Sigma_x$) if it can be decomposed as $\zeta^T(x)\Gamma\zeta(x)$ where $\zeta(x)$ is a vector of monomials and the so-called Gram-matrix Γ is a positive semi-definite matrix, $\Gamma \geq 0$. Obviously, all SOS polynomials are non-negative, although the converse is not true (Chesi 2007).

Theorem 7.2.2 ((Sala and C. Ariño 2009; Tanaka, Yoshida, et al. 2009; Prajna, Papachristodoulou, Seiler, et al. 2005))**.** *The polynomial fuzzy model* (7.18) *is asymptotically stable if a polynomial Lyapunov function $V(x) = P(x)$ can be found verifying*

$$P(x) - \varepsilon(x) \in \Sigma_x, \tag{7.19}$$

$$-\nabla P(x) F_{\mathbf{i}}(x) - \varepsilon(x) \in \Sigma_x, \quad \forall\, \mathbf{i} \leq \mathbf{c}, \tag{7.20}$$

where $\varepsilon(x)$ is a radially unbounded positive polynomial.

For a high-enough degree of $P(x)$ and $F_{\mathbf{i}}(x)$ if the nonlinear system admits a smooth Lyapunov function, the polynomial approach will eventually succeed, up to the gap of positive polynomials which are not SOS (Chesi 2007), if sufficient computational resources were available.

Fuzzy-polynomial Lyapunov functions

In (Bernal, Sala, et al. 2011), a fuzzy-polynomial LF was proposed $P_w(x)$, improving over Theorem 7.2.2 due to its larger representation capabilities. However, there was the need of explicitly bounding $\frac{\partial w}{\partial x}$ by, for instance, other polynomials of the state (the authors proposed carrying out a fuzzy-polynomial model of the mentioned partial derivatives). This is an extension of the idea of bounding the value or $\dot{w}$ in (Tanaka, Hori, and H. Wang 2003) or bounding the gradient of the membership functions in (Bernal and T. M. Guerra 2010). Notwithstanding, as the goal of this work is enhancing the integral terms in Lyapunov functions, no

[2] As discussed in (Sala and C. Ariño 2009), if h in (7.1) is of class $\mathcal{C}^p$, a Taylor-series argumentation can prove the existence of such a fuzzy-polynomial model of degree p.

further discussion of gradient/time-derivative bounding will be considered in the sequel (actually, combination of approaches is possible, see discussion in Section 7.5).

7.2.4 Problem statement

The objective of this chapter is generalising the LI Lyapunov function proposal in (Rhee and Won 2006; Gonzalez, Sala, Bernal, and Robles 2017) to a class of fuzzy-polynomial models in the form (7.18). Specifically, we will assume that the nonlinear model, written as expression (7.2), has the particular form:

$$\dot{x}(t) = \tilde{h}(\eta_1(\rho_1(x)), \dots, \eta_q(\rho_q(x)), x) \tag{7.21}$$

where $\tilde{h}$ is a polynomial in its arguments (η, x), with each $\eta_j : \mathbb{R} \mapsto \mathbb{R}$ being a real function of one variable, and being $\rho_j : \mathbb{R}^n \mapsto \mathbb{R}$ the argument to η_j; furthermore, $\rho_j(x)$ which will be assumed to be a polynomial in the state. Then, sector-nonlinearity modelling of η_j allows building membership functions in (7.18) which depend on $\rho_j(x)$:

$$w_0^j(\rho_j) = \frac{\overline{\eta}_j - \eta_j(\rho_j)}{\overline{\eta}_j - \underline{\eta}_j}, \; w_1^j(\rho_j) = 1 - w_0^j(\rho_j), \; j \in \{1, 2, \dots, q\}. \tag{7.22}$$

Thus, in the case under study, we will consider $w_i^j : \mathbb{R} \mapsto \mathbb{R}$, having the polynomial ρ_i as argument, instead of the "generic" dependence $w_i^j(x)$ considered in the original expression (7.4). Actually, it can be easily shown that the cases in (Rhee and Won 2006; Gonzalez, Sala, Bernal, and Robles 2017) are a particular case of the above setup, details left to the reader. For instance, in (Rhee and Won 2006), condition $\rho_i \equiv x_i$ was needed, as well as $q \leq n$. These assumptions are no longer needed in the present work, as discussed below.

The main goal of this chapter is generalising ρ_i to arbitrary polynomials, and to also consider the case in which the number of nonlinearities q can be larger than the system's order n. Given that polynomials appear, the generalisation of (Rhee and Won 2006) to the polynomial case (from LMI to SOS) comes as a side result but, importantly, advantages of the ideas here proposed can be achieved even in an LMI-only setup, as discussed in our conference paper (Gonzalez, Sala, Bernal, and Robles 2017). Hence, the LMIs in the cited works will be a particular case of our SOS approach.

Note that we do not need to model the gradient of the memberships because of the integral nature of the LF (following the main idea in the seminal work (Rhee and Won 2006)), thus obtaining simpler conditions than (Bernal, Sala, et al.

2011) (which require such gradient model), but more powerful than standard SOS conditions (Theorem 7.2.2), due to the incorporation of $w(\cdot)$ in the LI Lyapunov function.

Example 7.2.1 (Continued). *Considering the model in* (7.7)*, the approach in (Rhee and Won 2006) cannot be "directly" applied to the above model using all three weighting functions: Theorem 7.2.1 can be applied by considering only fuzziness in the WFs* $w_j^1(\cdot)$ *in the Lyapunov function* (7.12)*, because it is the only one which depends on exactly a single state variable. Thus, Theorem 7.2.1 can consider the following integral form for* $V(x)$*:*

$$V(x) = x^T P x + \int_0^{x_2} \sum_{i+j=1} \left(w_0^1(\psi)\right)^i \left(w_1^1(\psi)\right)^j s_{j_1}^1 \psi \, d\psi. \tag{7.23}$$

The above example motivates the need for extending the Lyapunov function with further integral terms depending on w_0^2, w_1^2, w_0^3 and w_1^3, to be dealt with in our proposals in next section. Note that a fuzzy-polynomial model (7.18) may be obtained for the model in example 7.2.1, if so wished; anyway, as the goal of this chapter is comparing the flexibility of the more general Lyapunov function proposals, we intentionally restrain ourselves to just the TS model (7.7) in the later numerical computations over the nonlinear system in this example, in order to suitably compare with prior literature; such further improvements from more general polynomial modelling are left to the reader.

7.3 Main Result

Let us first consider a generic integral expression, motivated by (7.12), in the form:

$$\bar{V}(\mu, \lambda) := P(\mu, \lambda) + \sum_{k=1}^{q} \int_0^{\lambda_k} \pi_w^{[k]}(\mu, \psi) \, d\psi \tag{7.24}$$

where $\lambda \in \mathbb{R}^q$, $\mu \in \mathbb{R}^s$, for some s to be later specified, are symbolic arguments (which will be later on replaced by state-dependent expressions), $P(\mu, \lambda)$ is an arbitrary polynomial function (depending on some decision variables), $\psi \in \mathbb{R}$ is a uni-dimensional dummy integral variable, and $\pi_w^{[k]}(\mu, \psi) : \mathbb{R}^{s+1} \mapsto \mathbb{R}$ are given by the fuzzy summations

$$\pi_w^{[k]}(\mu, \psi) = \sum_{\ell_k + l_k = d_k} n_{l_k}^{d_k} (w_0^k(\psi))^{\ell_k} (w_1^k(\psi))^{l_k} s_{l_k}^k(\mu, \psi) \tag{7.25}$$

where d_k is a Polya complexity parameter (Sala and Ariño 2007), $n_{l_k}^{d_k}$ is a combinatorial number $n_{l_k}^{d_k} = \dfrac{d_k!}{l_k!(d_k - l_k)!}$, and $s_{l_k}^k(\mu, \psi)$ is a polynomial parameterised, too, with some decision variables. As before, the Polya complexity parameters will be arranged into a "Polya degree vector", to be denoted as $\mathbf{d} := (d_1, \ldots, d_q) \in \mathbb{N}^q$.

In order to be used as a Lyapunov function, the gradient of $\bar{V}$ needs to be computed. Instead of line-integral argumentations, we will use explicit differentiation, as justified earlier on. Thus, the components of the gradient of $\bar{V}$ are given by:

$$\frac{\partial \bar{V}}{\partial \mu_i} = \frac{\partial P}{\partial \mu_i}(\mu, \lambda) + \sum_{k=1}^{q} \int_0^{\lambda_k} \frac{\partial \pi_w^{[k]}}{\partial \mu_i}(\mu, \psi)\, d\psi \tag{7.26}$$

and

$$\frac{\partial \bar{V}}{\partial \lambda_k} = \frac{\partial P}{\partial \lambda_k}(\mu, \lambda) + \pi_w^{[k]}(\mu, \lambda_k) \tag{7.27}$$

The above structure (7.24) will be used to build Lyapunov functions in Section 7.3.1, once relevant positiveness conditions formulated below do hold.

Theorem 7.3.1. *If $P(\mu, \lambda) \in \Sigma_{\mu,\lambda}$ and $s_{l_k}^k(\mu, \psi)\psi \in \Sigma_{\mu,\psi}$, for all $0 \le l_k \le d_k$, then $\bar{V}(\mu, \lambda) \ge 0$.*

Proof. Condition $s_{l_k}^k(\mu, \psi)\psi \in \Sigma_{\mu,\psi}$ implies that $s_{l_k}^k$ has the same sign as ψ. As $\pi_w^{[k]}$ is a sum of $s_{l_k}^k$ multiplied by positive coefficients, we can assert that $\pi_w^{[k]}(\mu, \psi)\psi \ge 0$ and, for any $\tau > 0$, we have $\pi_w^{[k]}(\mu, \psi)\psi/\tau \ge 0$. Hence,

$$\int_0^{\lambda_k} \pi_w^{[k]}(\mu, \psi)\, d\psi = \lim_{h \to 0^+} \int_h^1 \pi_w^{[k]}(\mu, \tau\lambda_k)\lambda_k\, d\tau \ge 0$$

where the rightmost integral comes from the change $\psi = \tau\lambda_k$, hence τ should range from zero to 1, and the last inequality comes from the fact that $\pi_w^{[k]}(\mu, \tau\lambda_k)\lambda_k = \pi_w^{[k]}(\mu, \psi)\psi/\tau \ge 0$. Note that the limit in the above expression exists from continuity of $\pi_w^{[k]}$. Therefore, $\bar{V}$ is expressed as the sum of two non-negative quantities if conditions in the theorem statement hold. $\square$

The above theorem can be made less conservative, introducing some additional decision variables (non-fuzzy polynomials $\underline{s}_k$) which "link" the non-integral and integral parts, as follows:

Theorem 7.3.2. *If there exist polynomials $\underline{s}_k(\mu,\psi)$, for $k \in \{1,2,\ldots,q\}$, such that*

$$\left(s^k_{l_k}(\mu,\psi) - \underline{s}_k(\mu,\psi)\right)\psi \in \Sigma_{\mu,\psi} \tag{7.28}$$

and

$$V_1(\mu,\lambda) := P(\mu,\lambda) + \sum_{k=1}^{q} \int_0^{\lambda_k} \underline{s}_k(\mu,\psi)\, d\psi \in \Sigma_{\mu,\lambda} \tag{7.29}$$

then $\bar{V}(\mu,\lambda) \geq 0$ for all μ, λ.

Proof. We can express:

$$\bar{V}(\mu,\lambda) = V_1(\mu,\lambda) + V_2(\mu,\lambda) \tag{7.30}$$

where V_1 is the polynomial defined in (7.29) and

$$V_2(\mu,\lambda) := \sum_{k=1}^{q} \int_0^{\lambda_k} \left(\pi^{[k]}_w(\psi) - \underline{s}_k(\mu,\psi)\right) d\psi \tag{7.31}$$

and Theorem 7.3.1 can now be applied changing the original $\bar{V}(\cdot)$ by $V_1(\cdot)$, and changing $s^k_{l_k}$ in the referred theorem for $s^k_{l_k} - \underline{s}_k$, as stated in (7.28). □

Next section will apply the above results to building Lyapunov functions. In order to avoid integral terms in the gradient of V, the restriction $s^k_{l_k}(\mu,\psi)$ being only dependent on ψ will be enforced in the sequel, i.e., we will only consider $s^k_{l_k}(\psi)$.

7.3.1 Stability

Consider now a Lyapunov function, using the structure (7.24), defined as:

$$V(x) := \bar{V}(Ex,\rho(x)) = P(Ex,\rho(x)) + \sum_{k=1}^{q} \int_0^{\rho_k} \pi^{[k]}_w(\psi)\, d\psi \tag{7.32}$$

where Ex selects only the components of the state which do not *explicitly* appear in $\rho(x)$ (thus, avoiding repeated arguments): for instance, in the original setting in (Rhee and Won 2006), E would be zero as $\rho(x) \equiv x$; in the 2nd-order system in Example 7.2.1, we would set $E := (1\ 0)$, so $Ex = x_1$ because $\rho_1(x) = x_2$.

Using positiveness results in Theorem 7.3.2 and adding derivative-related decrescence conditions allows to state the main result below:

Theorem 7.3.3. *Consider a polynomial fuzzy model* (7.18), *with degree vector* $\mathbf{c}$, *with the membership function structure* $w_j^k(\rho_k(x))$, *arising from* (7.21) *and* (7.22). *Consider, too, a given degree vector* $\mathbf{d} = (d_1, d_2, \ldots, d_q)$, *see* (7.25), *and the Lyapunov function structure* (7.32) *and an arbitrary radially unbounded polynomial* $\varepsilon(x)$, *such that* $\varepsilon(0) = 0$ *and* $\varepsilon(x) > 0$ *elsewhere. Then, the origin* $x(t) = 0$ *of such system is asymptotically stable if there exist polynomial functions* $P(Ex, \rho(x))$, $s_{l_k}^k(\psi)$, *and* $\underline{s}_k(\psi)$, *such that the following SOS conditions hold for all* $0 \le l_k \le d_k$, $0 \le b_j \le e_j$, $j, k \in \{1, 2, \ldots, q\}$, $\mathbf{e} = (c_1 + d_1, c_2 + d_2, \ldots, c_q + d_q)$*:*

$$(s_{l_k}^k(\psi) - \underline{s}_k(\psi))\psi \in \Sigma_\psi, \tag{7.33}$$

$$P(Ex, \rho(x)) + \sum_{k=1}^{q} \int_0^{\rho_k} \underline{s}_k(\psi)\, d\psi - \varepsilon(x) \in \Sigma_x, \tag{7.34}$$

$$-\sum_{l_j+i_j=b_j} n_{\mathbf{l}}^{\mathbf{d}} n_{\mathbf{i}}^{\mathbf{c}} \left(\nabla P(Ex, \rho(x)) \begin{bmatrix} E & \nabla\rho \end{bmatrix} + \sum_{k=1}^{q} s_{l_k}^k(\rho_k) \nabla\rho_k \right) F_{\mathbf{i}}(x) - \varepsilon(x) \in \Sigma_x. \tag{7.35}$$

Proof. Conditions (7.33) and (7.34) are the translation[3] to the current notation of conditions (7.28) and (7.29). Thus, application of Theorem 7.3.2 ensures that $V(x)$ in (7.32) fulfills $V(x) \ge \varepsilon(x)$.

Now, the derivative of the Lyapunov function can be expressed as:

$$\dot{\bar{V}}(\mu, \lambda) = \left(\frac{\partial \bar{V}}{\partial \mu} \frac{\partial \mu}{\partial x} + \frac{\partial \bar{V}}{\partial \lambda} \frac{\partial \lambda}{\partial x} \right) \cdot \dot{x}(t)$$

so, with the choice of arguments to $\bar{V}(\cdot)$ being $\mu := Ex$ and $\lambda := \rho(x)$, we have that the time derivative above (corresponding to the time derivative of (7.32)) becomes:

$$\dot{V}(x) = \nabla P(Ex, \rho(x)) \begin{bmatrix} E & \nabla\rho \end{bmatrix} \dot{x}(t) + [\pi_w^{[1]}(\rho_1(x)) \;\cdots\; \pi_w^{[q]}(\rho_q(x))] \nabla\rho(x)\dot{x}(t) \le 0.$$

Replacing $\dot{x}(t)$ by its model (7.18), and $\pi_w^{[k]}$ by its definition (7.25), we get:

$$\sum_{\mathbf{j}+\mathbf{i}=\mathbf{c}} n_{\mathbf{i}}^{\mathbf{c}} \mathbf{w}_0^{\mathbf{j}} \mathbf{w}_1^{\mathbf{i}} \nabla P(Ex, \rho(x)) [E \;\nabla\rho] F_{\mathbf{i}}(x)$$
$$+ \sum_{k=1}^{q} \sum_{\ell_k + l_k = d_k} (w_0^k(\rho_k))^{\ell_k} (w_1^k(\rho_k))^{l_k} n_{l_k}^{d_k} s_{l_k}^k(\rho_k) \nabla\rho_k \sum_{\mathbf{j}+\mathbf{i}=\mathbf{c}} n_{\mathbf{i}}^{\mathbf{c}} \mathbf{w}_0^{\mathbf{j}} \mathbf{w}_1^{\mathbf{i}} F_{\mathbf{i}}(x) \le 0,$$

[3]Actually, note that (7.29) poses SOS conditions on two variables (μ, λ) so applicability of Theorem 7.3.2 would hold even if the explicit relationship between these variables were unknown. However, as ρ_i are known polynomials in (7.32), substitution of these polynomials by their explicit expressions renders an easier SOS problem only in variables x in (7.34).

which is equivalent to the homogeneous summation of degree vector $\mathbf{e} := (c_1 + d_1, c_2 + d_2, \ldots, c_q + d_q)$ below:

$$\sum_{\mathbf{a}+\mathbf{b}=\mathbf{e}} \mathbf{w}_0^{\mathbf{a}} \mathbf{w}_1^{\mathbf{b}} \sum_{l_j+i_j=b_j} n_{\mathbf{l}}^{\mathbf{d}} n_{\mathbf{i}}^{\mathbf{c}} \Bigg(\nabla P(Ex, \rho(x))[E \ \nabla\rho] F_{\mathbf{i}}(x) + \left(\sum_{k=1}^{q} s_{l_k}^k (\rho_k) \nabla \rho_k \right) F_{\mathbf{i}}(x) \Bigg) + \varepsilon(x) \leq 0, \qquad (7.36)$$

Carrying out fuzzy-summation manipulations as to isolate each of the summation coefficients, we get the sufficient condition (7.35), which guarantees $\dot{V}(x) < 0$, thus concluding the proof. □

Note that Polya relaxations of the fuzzy summations (7.36) may be carried out to further reduce conservatism, but details on them are omitted for brevity.

In the particular case where $\rho_k(x)$ is an arbitrary linear function of the state $x(t)$, i.e., $\rho_k(x) = l_1^k x_1(t) + l_2^k x_2(t) + \cdots + l_n^k x_n(t) = L^{[k]} x(t)$, $\forall k \in \{1, 2, \ldots, q\}$, if the Lyapunov function is also chosen to be quadratic, then Theorem 7.3.3 reduces to the stability conditions in (Gonzalez, Sala, Bernal, and Robles 2017, Thm. 4), as stated next:

Corollary 7.3.3.1. *The origin $x(t) = 0$ of the TS model (7.17) with the membership function structure $w_j^k(\rho_k(x))$ and $\rho_k(x)$ being an arbitrary linear function of the state $x(t)$, i.e., $\rho_k(x) = l_1^k x_1(t) + l_2^k x_2(t) + \cdots + l_n^k x_n(t) = L^{[k]} x(t)$, $\forall k \in \{1, 2, \ldots, q\}$, is asymptotically stable if the following conditions hold:*

$$x^T P x + \sum_{k=1}^{q} s_{j_k}^k x^T \left(L^{[k]} \right)^T L^{[k]} x(t) - \varepsilon x^T x \geq 0, \forall j_k \in \{0, 1\} \qquad (7.37)$$

$$- \sum_{\mathbf{k}_0+\mathbf{k}_1=\mathbf{2}} \mathbf{w}_0^{\mathbf{k}_0} \mathbf{w}_1^{\mathbf{k}_1} \sum_{\mathbf{i}+\mathbf{j}=\mathbf{k}_1, \ \mathbf{i}\leq\mathbf{1}, \ \mathbf{j}\leq\mathbf{1}} x^T \left(\bar{P}_{\mathbf{j}} A_{\mathbf{i}} + A_{\mathbf{i}}^T \bar{P}_{\mathbf{j}} + \varepsilon I \right) x \geq 0, \qquad (7.38)$$

where $\bar{P}_{\mathbf{j}} = P + diag\left(s_{j_1}^1, s_{j_2}^2, ..., s_{j_n}^n\right)$, being $P = P^T \in \mathbb{R}^{n \times n}$ with null diagonal, and ε is a small positive constant. Obviously the above quadratic SOS conditions can be, trivially, considered to be an LMI[4].

Proof. Considering the Lyapunov function candidate (7.32) with

$$P(Ex, \rho(x)) = x^T P x,$$

[4]See conditions (11) and (12) in (Gonzalez, Sala, Bernal, and Robles 2017).

P defined as above with null diagonal, and

$$\pi_w^{[k]}(\psi) = 2 \sum_{\ell_k + l_k = 1} \left(w_0^k(\psi)\right)^{\ell_k} \left(w_1^k(\psi)\right)^{l_k} s_{l_k}^k \psi.$$

Rewriting (7.32), we get the following function:

$$V(x) = x^T P x + \sum_{k=1}^{q} \int_0^{L^{[k]} x_k} \pi_w^{[k]}(\psi)\, d\psi. \tag{7.39}$$

By Theorem 7.3.2, expression (7.39) is positive if there exists $\underline{s}_k$ such that:

$$p_{l_k}^k \psi^2 - \underline{s}_k \psi^2 \in \Sigma_\psi$$

$$x^T P x + \sum_{k=1}^{q} \int_0^{L^{[k]} x_k} \underline{s}_k \psi \, d\psi = x^T P x + \sum_{k=1}^{q} \underline{s}_k x^T \left(L^{[k]}\right)^T L^{[k]} x(t) - \varepsilon x^T x \in \Sigma_x.$$

Since P is null diagonal, setting $\underline{s}_k = \min(s_0^k, s_1^k)$, then (7.37) implies the previous condition.

The following condition on the time derivative of the Lyapunov function (7.39)

$$\dot{V}(x) = x^T \left(P A_w + A_w^T P\right) x + 2 \sum_{k=1}^{q} \pi_w^{[k]}(x) A_w x < 0,$$

is equivalent to that in (7.38) as can be seen performing similar steps as those in proof of Theorem 7.3.3. □

7.4 Examples

Example 7.2.1 (Continued). *The motivating example considering the model in* (7.7) *will be now numerically solved with the proposed results, and compared with alternative prior approaches. In particular, stability of the system* (7.7) *will be studied for different values of constant parameters* $a \in [10, 13]$ *and* $b \in [50, 60]$.

First, recall that the results in (Rhee and Won 2006), i.e., Theorem 7.2.1 can be applied only with the Lyapunov function (7.23), *with integral terms only depending on* x_2, *as previously discussed on page 149.*

However, our proposal in Theorem 7.3.3 can consider all three nonlinearities. If we apply Theorem 7.3.3 with $\varepsilon(x) = 10^{-4} x^T x$ *and the following polynomial Lyapunov*

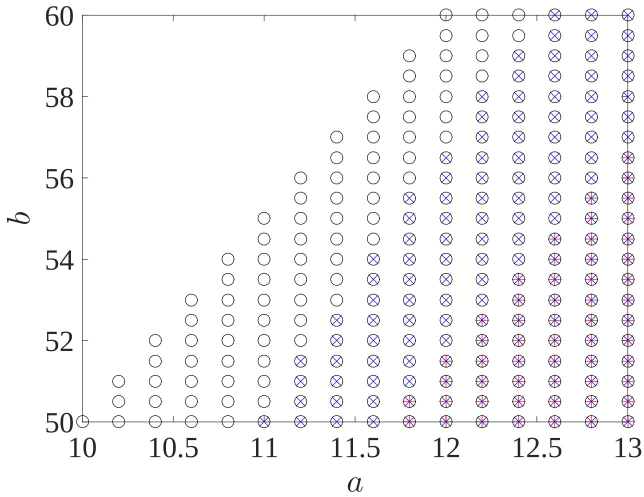

Figure 7.1: Feasibility sets for Example 7.2.1: (∘) for Theorem 7.3.3; (×) for Theorem 7.2.1; (+) for Quadratic Lyapunov function.

function with integral terms

$$V(x) = p_1 x_1^2 + p_2 x_1 x_2 + p_3 x_2^2 + \sum_{k=1}^{3} \int_0^{\rho_k} (w_0^k(\psi) s_0^k \psi + w_1^k(\psi) s_1^k \psi)\, d\psi$$

where p_1, p_2, p_3, and s_j^k are decision variables, the obtained feasible set of solutions is marked with (∘) in Figure 7.1, within the ranges of a and b above mentioned. For the sake the comparison, in Figure 7.1 the feasible set of solutions obtained if the classical quadratic approach $V = x^T P x$ is applied is marked with a (+); last, (×) points out the feasible set of solutions obtained if the approach in (Rhee and Won 2006) is applied considering only the WFs $w_j^1(\cdot)$ in the Lyapunov function (7.23) with, too $\varepsilon = 10^{-4}$. As expected, (Rhee and Won 2006) improves over the plain quadratic case, but our new approach produces the largest feasible set of solutions[5] *due to the two additional integral terms apart from the one in (7.23).*

Example 7.4.2. *In this example, we will compare our proposal with a "standard" sum-of-squares approach (recalled here as Theorem 7.2.2), i.e., with a polynomial non-fuzzy Lyapunov function (without integral terms). In order to carry out such*

[5] Note that, although this example has detailed the developments for polynomial arguments to ρ, similar improvements occur even if the arguments of ρ were just linear functions, as discussed earlier in this work (Corollary 7.3.3.1, taken from our conference paper (Gonzalez, Sala, Bernal, and Robles 2017)).

a comparison, we will consider the following nonlinear model:

$$\dot{x}_1 = x_2 \tag{7.40}$$
$$\dot{x}_2 = -2x_1 - x_2 - 0.5\,\kappa\, x_1\left(1+\sin\left(\rho_1(x)\right)\right), \tag{7.41}$$

where $\rho_1(x) = -4x_2 - 5x_2x_1 + x_1^2 - 2x_2^2$ *and* κ *is a non-negative parameter, so the objective is finding the largest possible* κ *such that several sets of SOS conditions (corresponding to different LF proposals) render feasible, to compare them. Applying the sector nonlinearity approach to* $\sin(\rho_1(x))$*, we obtained the following TS model:*

$$\dot{x} = \sum_{i_1=0}^{1} w_{i_1}^1(\rho_1) A_{i_1} x,$$

where

$$A_0 = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 1 \\ -2-\kappa & -1 \end{bmatrix},$$

and $w_0^1(\rho_1) = 0.5(1-\sin(\rho_1))$, $w_1^1(\rho_1) = 1 - w_0^1(\rho_1)$.

Note that, as ρ_1 *is neither a state nor a linear function of the state variables, integral LF terms from the proposals in (Rhee and Won 2006) or (Gonzalez, Sala, Bernal, and Robles 2017) cannot be applied.*

Following the approach in this chapter, if Theorem 7.3.3 is applied with $\mathbf{d} = (1)$, $\varepsilon(x) = 10^{-4}\left(x_1^2 + x_2^2\right)$*, and the following Polynomial Line-Integral Lyapunov function, which incorporates degree-4 monomials:*

$$V(x) = p_1x_1^2 + p_2x_1x_2 + p_3x_2^2 + p_4x_1^3 + p_5x_1^2x_2 + p_6x_1x_2^2 + p_7x_2^3 + p_8x_1^4$$
$$+p_9x_1^3x_2 + p_{10}x_1^2x_2^2 + p_{11}x_1x_2^3 + p_{12}x_2^4 + \int_0^{\rho_1}\left(w_0^1(\psi)s_0^1\psi + w_1^1(\psi)s_1^1\psi\right)d\psi$$

such that $\left(s_j^1\psi - \underline{s}_1\psi\right)\psi \in \Sigma_\psi$, $j \in \{0,1\}$ *where* p_i, $i \in \{1,2,\ldots,12\}$, $\underline{s}_1$, s_1^1, *and* s_0^1 *are decision variables, our approach can guarantee stability for* $\kappa = 6.5046$*. The resulting Lyapunov function for* $\kappa = 6.5046$ *is*

$$V(x) = 266.1084x_1^4 + 91.8725x_1^3x_2 + 54.6768x_1^3 + 116.435x_1^2x_2^2 + 10.7059x_1^2x_2$$
$$+76.9601x_1^2 + 11.1967x_1x_2^3 - 38.9486x_1x_2^2 + 12.3850x_1x_2 + 5.6442x_2^4$$
$$-20.2489x_2^3 - 3.8261x_2^2 + \int_0^{\rho_1}\left(w_0^1(\psi)3.2041\psi + w_1^1(\psi)1.9146\psi\right)d\psi,$$

with $\underline{s}_k = 1.391$*. In Figure 7.2 some level sets of* $V(x)$ *and some system trajectories are shown for illustration purposes.*

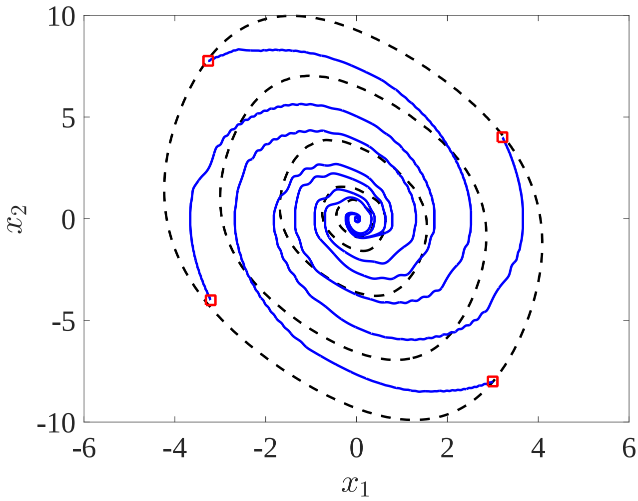

Figure 7.2: Lyapunov sets (dashed lines) and some trajectories (solid lines) for Example 2.

For the sake of comparison, Table 7.1 presents the maximum value of the parameter κ keeping conditions in Theorem 7.3.3 feasible[6] *for several degrees of the classical polynomial LF component $P(Ex, \rho_1(x))$ (left column) and the integral ones (middle column) with $\mathbf{d} = (2)$. Thus, the standard SOS approach corresponds to the rows where $\deg(s^k_{l_k}(\psi))$ is empty (labelled with a dash). For instance, a 4th-degree non-integral term plus a degree 1 integral term achieves better results than a non-integral LF of degree 12. From the numerical figures in the table, either increasing the non-integral polynomial degree or that of the integral term seem to improve results, however the incorporation of integral terms seems very effective with significantly less decision variables than the high-degree non-integral options, while achieving better performance.*

For information, the used solver in the numerical examples in this chapter was Mosek 7.1 (E. D. Andersen and K. D. Andersen 2000), under the programming language YALMIP 20150919 (Löofberg 2004), and running on Matlab R2015a with default tolerances.

[6] The function $\varepsilon(x)$ was chosen, following (Papachristodoulou and Prajna 2005), as:$\varepsilon(x) = \sum_{i=1}^{n}\sum_{j=1}^{d}\epsilon_{ij}x_i^{2j}$ where d is the degree of $P(Ex, \rho(x))$ and the ϵ's satisfy $\sum_{j=1}^{d}\epsilon_{ij} > \gamma, \quad \forall i \in \{1, 2, \ldots, n\}$ with γ a positive number (1×10^{-4}), and $\epsilon_{ij} \geq 0$ for all i and j.

Table 7.1: Maximum κ for polynomial line-integral LF with Theorem 7.3.3, and standard SOS Theorem 7.2.2.

$\deg(P(Ex, \rho(x)))$	$\deg(s^k_{l_k}(\psi))$	κ	Decision variables in $V(x)$	Average Solver time (s)
2	-	3.8284	3	0.1660
4	-	5.7393	12	0.1740
8	-	6.3981	42	0.1960
12	-	6.6537	88	0.3440
4	1	7.0880	16	0.1840
4	3	7.1018	24	0.1960
8	1	7.2990	46	0.1940
8	3	7.6879	54	0.2160
12	1	8.3010	92	0.4760
12	3	8.9234	100	0.4920

7.5 Discussion

In this section, once the results and example have been presented, a brief discussion on the advantages provided and room for further enhancements will be provided next.

Regarding the chosen nonlinear model for the examples, note that they have been intentionally written as TS models in order to compare with prior literature, but other polynomial models for the same nonlinear systems may be amenable to our proposal (such as the Taylor-series approach (Sala and C. Ariño 2009)), details left to the reader.

Also, for simplicity, global bounds on the nonlinearities have been considered (they are trigonometric functions). Nevertheless, the approach would equally work on compact modelling regions where suitable bounds for x and ρ would be available. Obviously, the advantages of non-quadratic/fuzzy-LF-SOS approaches would vanish for very small modelling regions, as the resulting model would equal the linearisation (in a TS case) or the truncated Taylor series (in the generic polynomial setup). Nevertheless, comparison of results with different sizes of modelling region has not been considered of interest, for brevity.

Apart from the concrete example, in a generic case, our approach has advantages if the nonlinearities can be expressed as a single-variable real function composed with a polynomial one; in this case, the polynomial nature of the arguments to nonlinearities is duly exploited. It would not apply to, for instance to $\rho(x) = \cos(e^{x_1} - \arctan x_2^2)$.

Note, too, that further relaxation of the result would be obtained by combining it with a variety of approaches in fuzzy control literature, which relax conservatism based on other ideas unrelated to our integral Lyapunov function proposal:

1. Increasing the degree of the polynomial term of the Lyapunov function $P(\cdot,\cdot)$ in (7.32).
2. Get a less conservative model via increasing the degree of the polynomial consequents, (Sala and C. Ariño 2009).
3. Use a standard "fuzzy"-polynomial Lyapunov function in the non-integral part of (7.32) replacing $P(Ex, \rho)$ by $P(Ex, \rho, \mathbf{w})$ with expressions similar to (7.9), incorporating information on the time-derivatives or the gradient of the memberships (Bernal, Sala, et al. 2011).
4. Use other results depending on membership shape. For instance, in Example 7.2.1, based on the actual nonlinearities, we could assert expressions such as $\gamma(w) := (w_0^1)^2 - 0.5w_0^2w_0^3 - 1 \leq 0$ or/and $\gamma(w) := w_0^2w_0^3 - \rho_2(x)^2\rho_3(x)^2 \leq 0$, a restriction that can be included via a suitable Positivstellensatz multiplier $R(x, \rho)\gamma(w)$ in the SOS conditions (C. Ariño and Sala 2007; Lam 2012).

7.6 Conclusion

This chapter presents a general SOS condition for the stability analysis of a class of nonlinear models via a polynomial Lyapunov function with integral terms which has been suitably parameterised. Compared to prior literature, two improvements are presented: first, the generalisation to a polynomial case of earlier LMI line-integral results; second, the new approach allows the line-integral approach to be applied to a larger class of TS models, where their WFs arguments can be arbitrary sets of polynomial functions of the system states, instead of only each of the states being the argument to a single WF considered in (Rhee and Won 2006). Unfortunately, as in the original reference, controller design problems cannot be cast as convex optimisation ones.

Chapter 8

Conclusions

In this thesis, solutions to some drawbacks in the TS/LMI/SOS-framework for analysis and control of nonlinear systems were proposed; namely, problems arising from the use of different classes of Lyapunov functions were addressed: handling of exact representations of nonlinear systems (affine Takagi-Sugeno models) allowing the inclusion of geometrical restrictions for piecewise analysis, a solution to the algebraic loops appearing when parameter-dependent Lyapunov functions are employed for control purposes, and enlarging of the class of systems that can be treated with line-integral Lyapunov functions.

A summary of the thesis contributions addressing the aforementioned problems follows:

- *The use of piecewise Lyapunov functions for the estimation of the domain of attraction of nonlinear systems.*

  The approach presented on chapter 5 allows obtaining asymptotically exact estimation of the DA of nonlinear systems. The algorithm therein presented is based on getting finer piecewise TS model and taking into account previously proven regions and “promising” neighboring areas, all within the LMI framework.

Nonetheless, the procedure has its own limitations. One of them, is the fact that the actual DA of a nonlinear system is, in general, not piecewise quadratic, so the exact domain of attraction cannot be obtained with finite computational resources. Hence, our proposal can only prove that finite computational resources are needed to find if a particular point in the state space belong to the interior of the "true" DA. Additionally, as the required estimation accuracy increases, the number of regions must accordingly increase (with decreasing size). Very small sizes would need heavy memory and processing requirements and an accurate handling of tolerances and numerical precision issues to obtain meaningful results.

- *A multi-index control law for stabilisation of nonlinear systems that feeds back the time derivatives of the membership functions.*

  In chapter 6, a new generalised PDLF is proposed along with a generalised multi-index control law that cancels out the terms that cause *a priori* locality in the Lyapunov analysis; moreover, the resulting conditions are purely LMI. For this sake, the control law uses the time derivative of the MFs obtained from the closed-loop expression of the system. The examples show that results in previous literature on the subject has been outperformed.

  Due to the inclusion of the time derivative of the MFs, a possible algebraic loops may arise. Thus, some additional LMI conditions are proposed to guarantee regularity of the control. Nevertheless, these LMI conditions are conservative and could lead to locality in the control law.

- *The use of the polynomial Lyapunov functions with integral terms is generalised for a larger class of nonlinear models.*

  This thesis shows a new polynomial Lyapunov function with integral terms that generalise works in prior literature for cases on which the later cannot be directly applied. It also goes beyond the TS framework including the polynomial one: it turns out that path independency conditions for line integrals are automatically verified if the integral is expressed as a sum of single-variable terms.

A word on future work

The main advantage of the convex approach is the fact that there exists a systematic methodology to model a smooth nonlinear system as a convex-linear one (sector-nonlinearity approach) or as a convex polynomial one (Taylor series approach, being the sector-nonlinearity generalisation), and then, via convex optimisation techniques, search for Lyapunov functions and controllers. For stability analysis this search can be conducted in terms of LMIs or SOS; several approaches are even asymptotically exact, though computationally demanding.

This is not, in general, the case of controller synthesis. As soon as the quadratic framework is left, a number of problems arise besides the loss of necessity of results; among them, the emergence of conditions not amenable to LMIs or SOS problems is a major obstacle. Usually, further assumptions and change of variables are performed on these problems to yield a convex formulation (see, for instance, the proposal in chapter 6), but of course these solutions lack generality. If, as many authors did in the past, conditions are left as BMI problems, the spirit of the whole methodology is lost, as BMIs cannot be optimally solved. A growing misunderstanding of this point seems to be motivated by the availability of hit-and-miss BMI solvers.

More specifically, since the power of piecewise methodologies lies in incorporating local information about a number of regions of interest where the system is supposed to operate, controller design cannot simultaneously preserve this information and modify the system trajectories through control; thus, the BMI nature of the problem. Mathematically, since most of the LMIs involving synthesis involve an inversion of the Lyapunov function, such transformation implies a change of variables which destroys the convex formulation of local restrictions.

In the case of parameter-dependent Lyapunov functions, well-posedness of the problem is an issue as results in this thesis show: a matrix inversion prevents controller design from being global. A non-conservative solution to ensure regularity of the control law is BMI, which obliged us to adopt an LMI conservative reduction of the problem. It is clear that feeding back the time derivatives of the MFs is both related with descriptor forms (as the left-hand time derivative of the state gets enriched by right-hand terms) and dynamical controllers (as the time derivative of the MFs might be subsumed in a chain of integrators). In the TS context, this might translate into non-affine in control systems, i.e., models that include x and u in their MFs.

As shown in this thesis, line-integral Lyapunov functions had no need to be path-independent if properly defined. While enriching the solution set for stability purposes, controller design shares the BMI nature of the problems above. In this case, the reason lies on the imposed structure of the Lyapunov function that does not allow the inversion already discussed.

Publications authored or co-authored

Listed below are all the publications obtained to date. In particular, publications [6,8,9,14,16] are the contributions of this thesis. The rest of contributions arose from work by the author in collaboration with other researchers on related nonlinear-TS issues.

Journal papers

1. B. Aguiar, T. Gonzalez, and M. Bernal (2015). "Comments on "Robust Stability and Stabilization of Fractional-Order Interval Systems With the Fractional Order: The Case"". In: *IEEE Transactions on Automatic Control* 60.2, pp. 582–583.

2. M. Ramirez, R. Villafuerte, T. Gonzalez, and M. Bernal (2015). "Exponential Estimates Of A Class Of Time-Delay Nonlinear Systems With Convex Representations". In: *International Journal of Applied Mathematics and Computer Science* 25.4, pp. 815–826.

3. T. Gonzalez, R. Marquez, M. Bernal, and T.M. Guerra (2016). "Nonquadratic Controller and Observer Design for Continuous TS Models: A Discrete-Inspired Solution". In: *International Journal of Fuzzy Systems* 18.1, pp. 1–14.

4. T. Gonzalez and M. Bernal (2016). “Progressively better estimates of the domain of attraction for nonlinear systems via piecewise Takagi-Sugeno models: Stability and stabilization issues”. In: *Fuzzy Sets and Systems* 297, pp. 73–95

5. B. Aguiar, T. Gonzalez, and M. Bernal (2016). “A Way to Exploit the Fractional Stability Domain for Robust Chaos Suppression and Synchronization via LMIs”. In: *IEEE Transactions on Automatic Control* 61.10, pp. 2796–2807.

6. T. Gonzalez, M. Bernal, A. Sala, and B. Aguiar (2017). “Cancellation-Based Nonquadratic Controller Design for Nonlinear Systems via Takagi-Sugeno Models”. In: *IEEE Transactions on Cybernetics* 47.9, pp. 2628–2638.

7. R. Robles, A. Sala, M. Bernal, and T. Gonzalez (2017). “Subspace-Based Takagi-Sugeno Modeling for Improved LMI Performance”. In: *IEEE Transactions on Fuzzy Systems* 25.4, pp. 754–767.

8. T. Gonzalez, A. Sala, and M. Bernal (2017). “Piecewise-Takagi-Sugeno asymptotically exact estimation of the domain of attraction of nonlinear systems”. In: *Journal of the Franklin Institute* 354.3, pp. 1514–1541.

9. T. Gonzalez, A. Sala, and M. Bernal (2018). “A Generalised Integral Polynomial Lyapunov Function for Nonlinear Systems”. In: *Fuzzy Sets and Systems*. In press.

Conference papers

10. T. Gonzalez, P. Rivera, and M. Bernal (2012). “Nonlinear control for plants with partial information via Takagi-Sugeno models: An application on the twin rotor MIMO system”. In: *Proceedings of the 9th International Conference on Electrical Engineering, Computing Science, and Automatic Control*. Mexico City, Mexico, pp. 1–6.

11. B. Aguiar, S. Angulo, T. Gonzalez, and M. Bernal (2013). “Synchronization of a class of chaotic systems via Takagi-Sugeno representations”. In: *Proceedings of the 2013 IEEE International Conference on Fuzzy Systems*. Hyderabad, India, pp. 1–6.

12. T. Gonzalez, M. Bernal, and R. Marquez (2014). “Stability analysis of nonlinear models via exact piecewise Takagi-Sugeno models”. In: *Proceedings of the 19th IFAC World Congress*. Cape Town, South Africa, pp. 1–6.

13. R. Robles, A. Sala, M. Bernal, and T. Gonzalez (2015). “Choosing a Takagi-Sugeno model for improved performance”. In: *Proceedings of the 2015 IEEE International Conference on Fuzzy Systems*. Istanbul, Turkey.

14. T. Gonzalez, A. Sala, M. Bernal, and R. Robles (2015). “Invariant Sets of Nonlinear Models via Piecewise Affine Takagi-Sugeno Model”. In: *Proceedings of the 2015 IEEE International Conference on Fuzzy Systems*. Istanbul, Turkey.

15. R. Robles, A. Sala, M. Bernal, and T. Gonzalez (2016). “Optimal-Performance Takagi-Sugeno Models via the LMI Null Space”. In: *IFAC-PapersOnLine* 49.5, pp. 13–18.

16. T. Gonzalez, A. Sala, M. Bernal, and R. Robles (2017). “A generalisation of Line-Integral Lyapunov Function for Takagi-Sugeno systems”. In: *Proceedings of the 20th World Congress of the International Federation of Automatic Control*. Toulouse, France.

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