Completable fuzzy metric spaces

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Abstract

In the context of fuzzy metrics in the sense of George and Veeramani, we introduce the concept of stratified fuzzy metric. Many well-known fuzzy metrics are stratified. We prove that stratified strong fuzzy metric spaces $(X, M, *)$ are completable, under the assumption that $*$ is integral (positive). In particular, stratified fuzzy ultrametric spaces are completable.

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1. Introduction

The notion of fuzzy metric space has been presented from different points of view. One of them, which we deal in this paper, was introduced by George and Veeramani in [1, 3], where the authors studied this notion topologically and showed that every fuzzy metric M on a set X generates a topology τ_M on X, which is metrizable. Moreover, it was proved later in [2, 10], that the class of topological spaces which are *fuzzy metrizable* agrees with the class of metrizable spaces. So, many results on classical metric spaces have been

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stated in the fuzzy metric setting [10, 13, 17]. Nevertheless, we can find some differences between classical metrics and fuzzy metrics, for instance in Fixed Point Theory, which has developed a wide range of research in these last years (see [16, 18, 20, 21, 22]).

Another topic which differs with the classical theory is the fuzzy metric completion. Indeed, Gregori and Romaguera [11] proved that there exist fuzzy metric spaces which are not completable. Later, the same authors gave a characterization of those fuzzy metric spaces that are completable. We reformulate that characterization in the following theorem, for better observing the advances of the theory on completion of fuzzy metric spaces.

Theorem 1.1. *[12] A fuzzy metric space $(X, M, *)$ is completable if and only if for each pair of Cauchy sequences $\{a_n\}$ and $\{b_n\}$ in X the following three conditions are fulfilled:*

(c1) The assignment $t \to \lim_n M(a_n, b_n, t)$ for each $t > 0$ is a continuous function on $]0, \infty[$, provided with the usual topology of $\mathbb{R}$.
(c2) Each pair of point-equivalent Cauchy sequences is equivalent, i.e., $\lim_n M(a_n, b_n, s) = 1$ for some $s > 0$ implies $\lim_n M(a_n, b_n, t) = 1$ for all $t > 0$.
(c3) $\lim_n M(a_n, b_n, t) > 0$ for all $t > 0$.

Since then, to find large classes of completable fuzzy metric spaces turned an interesting question. Recently, it has been proved [6] that conditions $(c1) - (c3)$ constitute an independent axiomatic system, i.e., two of such conditions do not imply the third one.

The first non-completable fuzzy metric space which appears in the literature was Example 4.2, which fulfils $(c1) - (c2)$ but not $(c3)$. The second one was Example 3.4 (b), which fulfils $(c1)$ and $(c3)$ but not $(c2)$. In [6] a non-completable fuzzy metric space fulfilling $(c2) - (c3)$ but not $(c1)$ was given (Example 4.1).

For getting condition $(c3)$ in a fuzzy metric space $(X, M, *)$ it suffices (see Lemma 3.7) that $*$ be integral (Definition 1). Under this assumption a strong (non-Archimedean) fuzzy metric (Definition 2.5) is completable if and only if $(c2)$ is satisfied ([6], Theorem 4.7). This result is a consequence of the fact that condition $(c1)$ is always satisfied in a strong fuzzy metric space ([7], Theorem 35). Up we know no approaches on the condition $(c2)$ have been made until now.

The aim of this paper is to find a large class of fuzzy metric spaces satisfying condition $(c2)$. So, we introduce the concept of stratified fuzzy metric space (Definition 3.2). We show that the class of stratified fuzzy metric spaces is a large class that contains many well-known fuzzy metric spaces. We also provide examples of non-stratified fuzzy metric spaces. The main result of this paper (Theorem 3.5) proves that condition $(c2)$ is satisfied by stratified fuzzy metric spaces. As a consequence, under the assumption that $*$ is integral, we can assert that a stratified fuzzy metric space is completable if and only if it satisfies condition $(c1)$ (Corollary 3.8), and then stratified strong fuzzy metric spaces are completable (Theorem 3.9).

Appropriate examples illustrate our results and, in particular, that the converse of Theorem 3.9 is not true and also that we cannot remove any condition in this theorem.

The structure of the paper is as follows. After the preliminaries section, in Section 3 we introduce the concept of stratified fuzzy metric space and prove our results related to the completability of this class of fuzzy metric spaces. In Section 4 we provide appropriate examples to illustrate the theory and for supporting our conclusions.

2. Preliminaries

In this section, we present some results related to fuzzy metric spaces, introduced by George and Veeramani in [1], and their completiability. We begin recalling the the concept of t-norm, which plays an important role in the definition of this concept.

Definition 2.1. A t-norm (as it is used today) is a binary operator T on the unit interval $[0,1]$, i.e., a function $T:[0,1]^2 \to [0,1]$, such that for all $x,y,z \in [0,1]$ the following four axioms are satisfied:

(T1) $T(x,y) = T(y,x)$; (Commutativity)

(T2) $T(x,T(y,z)) = T(T(x,y),z)$; (Associativity)

(T3) $T(x,y) \geq T(x,z)$; (Monotonicity)

(T4) $T(x,1) = x$. (Boundary Condition)

An interesting class of t-norms, especially to the study that we will carry out in this paper is the class of integral or positive t-norms which satisfy the next condition:

$$x * y > 0 \text{ whenever } x, y \in]0, 1]. \tag{1}$$

Notice that the continuous t-norm minimum, denoted by $\wedge$, and the usual product are integral. The Lukasievicz continuous t-norm given by $x\mathfrak{L}y = \max\{x + y - 1, 0\}$ is not integral.

After recalling the concept of t-norm, we are able to present the concept of fuzzy metric space, which we deal in this paper, since it is used to define the transitivity.

Definition 2.2. (George and Veeramani [1]) A fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a (non-empty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times]0, \infty[$ satisfying the following conditions, for all $x, y, z \in X,\ s, t > 0$:

(GV1) $M(x, y, t) > 0$

(GV2) $M(x, y, t) = 1$ if and only if $x = y$

(GV3) $M(x, y, t) = M(y, x, t)$

(GV4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$

(GV5) $M(x, y, \_) :]0, \infty[\to]0, 1]$ is continuous.

If $(X, M, *)$ is a fuzzy metric space, we will say that $(M, *)$ is a *fuzzy metric* on X. If no confusion can arise, we simply write M. Notice that if $(M, *)$ is a fuzzy metric on X and $\diamond$ is a continuous t-norm such that $\diamond \leq *$, then $(M, \diamond)$ is a fuzzy metric on X.

Remark 2.3. *It is well-known that the function* $M(x, y, \_) : \mathbb{R}^+ \to]0, 1]$ *is non-decreasing for all* $x, y \in X$*. From now on, it will be denoted as* M_{xy}*.*

Two interesting classes of fuzzy metric spaces are presented in the next definitions:

Definition 2.4. (Gregori and Romaguera [12]) A fuzzy metric M on X is said to be *stationary* if M does not depend on t, i.e., if for each $x, y \in X$, the function $M_{x,y}(t) = M(x, y, t)$ is constant. In this case we write $M(x, y)$ instead of $M(x, y, t)$.

Definition 2.5. (Gregori et al. [7], Istrăţescu [14]) A fuzzy metric space $(X, M, *)$ is said to be *strong* (non-Archimedean) if for all $x, y, z \in X$ and all $t > 0$ satisfies

$$M(x, z, t) \geq M(x, y, t) * M(y, z, t).$$

A strong fuzzy metric for the minimum t-norm is called a *fuzzy ultrametric*.

George and Veeramani proved in [1] that every fuzzy metric M on X generates a topology τ_M on X which has as a base the family of open sets of the form $\{B_M(x, \epsilon, t) : x \in X, 0 < \epsilon < 1, t > 0\}$, where $B_M(x, \epsilon, t) = \{y \in X : M(x, y, t) > 1 - \epsilon\}$ for all $x \in X$, $\epsilon \in]0, 1[$ and $t > 0$.

Let (X, d) be a metric space and let M_d a fuzzy set on $X \times X \times]0, \infty[$ defined by

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}$$

Then $(X, M_d, \cdot)$ is a fuzzy metric space, [1], and M_d is called the *standard fuzzy metric* induced by d. It is easy to check and a well-known fact that the topology on X deduced from d agrees with τ_{M_d}.

Furthermore, in [1] the authors provided the next characterization of convergent sequences.

Proposition 2.6. *Let $(X, M, *)$ be a fuzzy metric space. A sequence $\{x_n\}$ in X converges to x if and only if $\lim_n M(x_n, x, t) = 1$, for all $t > 0$.*

Attending to the last characterization, it was introduced in a natural way in [1], the next concept of Cauchy sequence and the corresponding concept of complete fuzzy metric spaces that we recall below.

Definition 2.7. A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is said to be *M-Cauchy*, or simply Cauchy, if for each $\epsilon \in]0, 1[$ and each $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \epsilon$ for all $n, m \geq n_0$ or, equivalently, $\lim\limits_{n,m} M(x_n, x_m, t) = 1$ for all $t > 0$. X is said to be *complete* if every Cauchy sequence in X is convergent with respect to τ_M. In such a case M is also said to be complete.

Remark 2.8. *Notice that there exist other notions of Cauchyness in the literature (see for example [9]).*

In the light of the preceding concepts, we are able to recall the concept of completion of a fuzzy metric space that introduced Gregori and Romaguera in [11].

Definition 2.9. Let $(X, M, *)$ and $(Y, N, \diamond)$ be two fuzzy metric spaces. A mapping f from X to Y is said to be an *isometry* if for each $x, y \in X$ and $t > 0$, $M(x, y, t) = N(f(x), f(y), t)$ and, in this case, if f is a bijection, X and Y are called *isometric*. A *fuzzy metric completion* of (X, M) is a complete fuzzy metric space (X^*, M^*) such that (X, M) is isometric to a dense subspace of X^*. X is said to be *completable* if it admits a fuzzy metric completion.

Finally, we recall a concept introduced by the aforementioned authors in [12], which they used to provide a characterization of completable fuzzy metric spaces that we also show following the other one.

Definition 2.10. Let $X, M, *)$ be a fuzzy metric space. Then a pair $\{a_n\}$ and $\{b_n\}$ of Cauchy sequences in X is called:

(a) point-equivalent if there exists s_0 such that $\lim_n M(a_n, b_n, s_0) = 1$.

(b) equivalent if $\lim_n M(a_n, b_n, t) = 1$ for all $t > 0$.

Theorem 2.11. *(Gregori and Romaguera [12]) A fuzzy metric space $(X, M, *)$ is completable if and only if it satisfies the following conditions:*

(C1) Given Cauchy sequences $\{a_n\}$ and $\{b_n\}$ in X, then $t \to \lim_n M(a_n, b_n, t)$ is a continuous function on $]0, +\infty[$ with values in $]0, 1]$.

(C2) Each pair of point-equivalent Cauchy sequences is equivalent.

3. Stratified fuzzy metric spaces

We start this section with the following proposition.

Proposition 3.1. *Let $(X, M*)$ be a fuzzy metric space. The following are equivalent:*

(i) $M(a, b, s) = M(a', b', s)$ implies $M(a, b, t) = M(a', b', t)$ for all $t > 0$.

(ii) $M(a, b, s) < M(a', b', s)$ implies $M(a, b, t) < M(a', b', t)$ for all $t > 0$.

Proof. $(i) \Rightarrow (ii)$ Suppose that there exists $s > 0$ such that $M(a,b,s) < M(a',b',s)$. If $M(a,b,t_0) = M(a',b',t_0)$ for some $t_0 > 0$ then $M(a,b,t) = M(a',b',t)$ for all $t > 0$, a contradiction. Then $M(a,b,t) \neq M(a',b',t)$ for all $t > 0$.

Suppose now that there exists $t_1 > 0$ such that $M(a,b,t_1) > M(a',b',t_1)$. So, since the functions $M_{ab}(t)$ and $M_{a'b'}(t)$ are continuous, then the function $M_{ab} - M_{a'b'}$ is also continuous. Now, we have that $(M_{ab} - M_{a'b'})(s) < 0$ and $(M_{ab} - M_{a'b'})(t_1) > 0$ and so there exists $t_2 \in]s, t_1[$, or $t_2 \in]t_1, s[$, such that $(M_{ab} - M_{a'b'})(t_2) = 0$, i.e. $M(a,b,t_2) = M(a',b',t_2)$ and by (i) we have that $M(a,b,t) = M(a',b',t)$ for all $t > 0$, a contradiction.

$(ii) \Rightarrow (i)$ It is obvious. □

Definition 3.2. Let $(X, M, *)$ be a fuzzy metric space. We will say that $(X, M, *)$ is a stratified fuzzy metric space if it satisfies one of the previous conditions in Proposition 3.1. In this case, we say that $(M, *)$ (or simply M) is a stratified fuzzy metric on X.

Observe that in a stratified fuzzy metric space, if the graphs of the functions M_{xy} and $M_{x'y'}$ intersect, then $M_{xy} = M_{x'y'}$. Next we will show some examples of stratified fuzzy metrics.

Example 3.3. (a) Stationary fuzzy metrics are stratified.

(b) Any fuzzy metric space $(X, M, *)$ where M is defined by means of a classical metric d on X, i.e. in which in the expression of the fuzzy mettric appears "explicitly" a classical distance ([8], Examples 4-6), is stratified. In particular, the standard fuzzy metric M_d and $M(x,y,t) = e^{\frac{-d(x,y)}{t}}$ introduced in [1] are stratified.

(c) Let $(M, *)$ be a stationary fuzzy metric on a set X where $*$ is integral and let $\varphi :]0, +\infty[\to]0, 1]$ be a non-decreasing continuous function. Then the pair $(N, *)$ where N is defined by

$$N(x,y,t) = \begin{cases} 1 & \text{if} \quad x = y \\ M(x,y,t) * \varphi(t) & \text{if} \quad x \neq y \end{cases}$$

is a stratified fuzzy metric on X.

Next example shows three non-stratified fuzzy metrics.

Example 3.4. (a) Consider the fuzzy metric space $(X, M, *)$ where $X =]0, +\infty[$, $M(x, y, t) = \frac{\min\{x,y\}+t}{\max\{x,y\}+t}$ and $*$ the usual product [23]. We have that $M(1, 2, 1) = \frac{1+1}{2+1} = \frac{2}{3}$ and $M(3, 5, 1) = \frac{3+1}{5+1} = \frac{2}{3}$. On the other hand, $M(1, 2, 2) = \frac{1+2}{2+2} = \frac{3}{4}$ and $M(3, 5, 2) = \frac{3+2}{5+2} = \frac{5}{7} \left(\neq \frac{3}{4}\right)$. Hence M is not stratified.

(b) Let $\{x_n\}$ and $\{y_n\}$ be two strictly increasing sequences of positive real numbers, which converge to 1 with respect to the Euclidean metric, with $A \cap B = \emptyset$, where $A = \{x_n : n \in \mathbb{N}\}$ and $B = \{y_n : n \in \mathbb{N}\}$. Put $X = A \cup B$ and define M on X by

$M(x_n, x_n, t) = M(y_n, y_n, t) = 1$ for all $n \in \mathbb{N},\ t > 0,$

$M(x_n, x_m, t) = x_n \wedge x_m$ for all $n, m \in \mathbb{N}$ with $n \neq m,\ t > 0,$

$M(y_n, y_m, t) = y_n \wedge y_m$ for all $n, m \in \mathbb{N}$ with $n \neq m,\ t > 0,$

$M(x_n, y_m, t) = M(y_m, x_n, t) = x_n \wedge y_m$ for all $n, m \in \mathbb{N},\ t \geq 1,$

$M(x_n, y_m, t) = M(y_m, x_n, t) = x_n \wedge y_m \wedge t$ for all $n, m \in \mathbb{N},\ 0 < t < 1.$

In [12] it is proved that $(X, M, \wedge)$ is a fuzzy metric space. Now, let $x_1 < y_1 < x_2$, we have that $M(x_1, x_2, 1) = x_1 \wedge x_2 = x_1$ and $M(x_1, y_1, 1) = x_1 \wedge y_1 = x_1$. Let $t < x_1$ then $M(x_1, x_2, t) = x_1 \wedge x_2 = x_1$ and $M(x_1, y_1, t) = x_1 \wedge y_1 \wedge t = t < x_1$. Hence M is not stratified.

(c) Let $\{x_n\} \subset]0, 1[$ be a strictly increasing sequence convergent to 1 respect to the usual topology of $\mathbb{R}$. Put $X = \{x_n\} \cup \{1\}$. Define the function M given by

$$M(x, x, t) = 1 \text{ for each } x \in X,\ t > 0$$

$$M(x_n, x_m, t) = \min\{x_n, x_m\} \text{ for all } m, n \in \mathbb{N},\ t > 0$$

$$M(x_n, 1, t) = M(1, x_n, t) = \min\{x_n, t\} \text{ for all } n \in \mathbb{N},\ t > 0$$

In [15] it is proved that $(X, M, \wedge)$ is a fuzzy metric space. Now, $M(\frac{1}{2}, 1, \frac{1}{4}) = M(\frac{1}{3}, 1, \frac{1}{4}) = \frac{1}{4}$. Nevertheless, $M(\frac{1}{2}, 1, 1) = \frac{1}{2}$ and $M(\frac{1}{3}, 1, 1) = \frac{1}{3}$ and, in consequence, $(M, \wedge)$ is not stratified.

Theorem 3.5. *Let $(M,*)$ be a stratified fuzzy metric on X. Let $\{a_n\}$ and $\{b_n\}$ be two Cauchy sequences in X satisfying $\lim_n M(a_n,b_n,s)=1$ for some $s>0$. Then $\lim_n M(a_n,b_n,t)=1$ for all $t>0$, that is, each pair of point-equivalent Cauchy sequences in X is equivalent.*

Proof. Let $\{a_n\}$ and $\{b_n\}$ be two Cauchy sequences. If both are constant from a certain stage $n_1\in\mathbb{N}$ then, clearly, $a_n=b_n$ for all $n\geq n_1$ and the result is obvious.

Suppose, without loss of generality, that $\{a_n\}$ is not constant. Consider the subsequences $\{x_n\}$ and $\{y_n\}$ of $\{a_n\}$ given by $x_n=a_{2n-1}$ and $y_n=a_{2n}$, $n\geq 1$. Clearly $\{x_n\}$ and $\{y_n\}$ are Cauchy and it is easy to see that

$$\lim_n M(x_n,y_n,t)=1 \text{ for all } t>0. \tag{2}$$

Now, since $\lim_n M(a_n,b_n,s)=1$ we can find $n_1=\min\{m\in\mathbb{N}: M(a_m,b_m,s)\geq M(x_1,y_1,s)\}$ and also $n_2=\min\{m\in\mathbb{N}\text{ with } m>n_1 : M(a_m,b_m,s)\geq M(x_2,y_2,s)\}$. In this way, we can define by induction a subsequence $\{M(a_{n_i},b_{n_i},s\}_i$ of $\{M(a_n,b_n,s)\}_n$ which satisfies $M(a_{n_i},b_{n_i},s)\geq M(x_i,y_i,s)$ for $i\geq 1$. Since M is stratified and condition (2) is satisfied then $\lim_i M(a_{n_i},b_{n_i},t)\geq \lim_n M(x_i,y_i,t)=1$ for all $t>0$.

Let $t>0$ and $\varepsilon\in]0,1[$. Choose $\delta\in]0,1[$ such that $\delta*\delta*\delta>\varepsilon$. We have that $M(a_n,b_n,t)\geq M(a_n,a_{n_i},\frac{t}{3})*M(a_{n_i},b_{n_i},\frac{t}{3})*M(b_{n_i},b_n,\frac{t}{3})$ and since $\{a_n\}$ and $\{b_n\}$ are Cauchy, there exists $p\in\mathbb{N}$ such that $M(a_n,a_{n_i},\frac{t}{3})>\delta$ and $M(b_{n_i},b_n,\frac{t}{3})>\delta$ for all $n,n_i\geq p$. Since $\lim_i M(a_{n_i},b_{n_i},\frac{t}{3})=1$, there exists $q\in\mathbb{N}$ such that $M(a_{n_i},b_{n_i},\frac{t}{3})>\delta$ for all $i\geq q$. Choose $n_0=\max\{p,q\}$. We have that $M(a_n,b_n,t)>\delta*\delta*\delta>\varepsilon$ for all $n\geq n_0$ and, in consequence, $\lim_n M(a_n,b_n,t)=1$. $\square$

Remark 3.6. *(i) The converse of Theorem 3.5 is not true. Indeed, it is easy to verify that in the fuzzy metric space $(X,M,*)$ of Example 3.4 (a) every pair of point-equivalent Cauchy sequences is equivalent and, as it has been shown, $(X,M,*)$ is not stratified.*

(ii) We cannot replace stratified by strong in Theorem 3.5. Indeed, it is easy to verify that $(X,M,*)$ of Example 3.4 (b) is strong. Now, it was shown in [12] the existence of a pair of point-equivalent Cauchy sequences in X which are not equivalent.

Lemma 3.7. *(Gregori et. al. [7]) Let $(X, M, *)$ be a fuzzy metric space and suppose $*$ integral. Let $\{a_n\}$ and $\{b_n\}$ be Cauchy sequences in X and let $t > 0$. If $\{M(a_n, b_n, t)\}$ converges (in the usual topology of $\mathbb{R}$) to c, then $c > 0$.*

As a consequence of Theorem 3.5 and Lemma 3.7, we obtain the following corollary.

Corollary 3.8. *Under the assumption that $*$ is integral, a stratified fuzzy metric space $(X, M, *)$ is completable if and only if $(c1)$ is satisfied.*

The following theorem gives a large class of completable fuzzy metric spaces.

Theorem 3.9. *Let $(M, *)$ be a stratified strong fuzzy metric on X and suppose that $*$ is integral. Then $(X, M, *)$ is completable.*

Proof. We will see that Theorem 1.1 is accomplished. Let $\{a_n\}$ and $\{b_n\}$ be two Cauchy sequences in X. From [6], Theorem 21, the assignment $t \to \lim_n M(a_n, b_n, t)$ is a continuous function on $]0, +\infty[$, equipped with the usual topology on $\mathbb{R}$, since M is strong, and so $(c1)$ is satisfied. By Lemma 3.7 we have that $\lim_n M(a_n, b_n, t) > 0$ and so $(c3)$ is fulfilled. Finally, by Theorem 3.5, condition $(c2)$ is satisfied, and thus $(X, M, *)$ is completable.□

Corollary 3.10. *Let $(M, \wedge)$ be a stratified fuzzy ultrametric on X. Then $(X, M, \wedge)$ is completable.*

Remark 3.11. *Notice that, in general, a fuzzy ultrametric space is not completable (see Example 4.3).*

4. Examples

In this section we give appropriate examples that illustrate the results obtained in the previous section. The following three examples prove that we cannot remove any condition in Theorem 3.9

Example 4.1. *A non-completable stratified fuzzy metric space with integral t-norm.* Let d be the usual metric on $\mathbb{R}$ restricted to $X =]0, 1]$ and consider the standard fuzzy metric M_d induced by d. Define the function

$$M(x,y,t) = \begin{cases} M_d(x,y,t) & 0 < t \leq d(x,y) \\ M_d(x,y,2t) \cdot \frac{t-d(x,y)}{1-d(x,y)} + M_d(x,y,t) \cdot \frac{1-t}{1-d(x,y)} & d(x,y) < t \leq 1 \\ M_d(x,y,2t) & t > 1 \end{cases}$$

In [4] it is proved that $(M,\cdot)$ is neither a completable nor a strong fuzzy metric on X. As we have observed, the t-norm $\cdot$ is integral. Next we will see that $(M,\cdot)$ is a stratified fuzzy metric on X.

Note that if $d(x,y)=d(x',y')$ then $M(x,y,t)=M(x',y',t)$ for all $t>0$.

Suppose that $M(x,y,t_0)=M(x',y',t_0)$ for some $t_0>0$. We will see that $M(x,y,t)=M(x',y',t)$ for all $t>0$. To prove it we distinguish four cases:

1) If $0<t_0\leq d(x,y$ and $0<t_0\leq d(x',y')$, then we have that $M(x,y,t_0)=\frac{t_0}{t_0+d(x,y)}=\frac{t_0}{t_0+d(x',y')}=M(x',y',t_0)$ and so $d(x,y)=d(x',y')$. Therefore, attending to the above observation, we have $M(x,y,t)=M(x',y',t)$ for all $t>0$.

2) If $t_0>1$ then $M(x,y,t_0)=\frac{2\,t_0}{2\,t_0+d(x,y)}=\frac{2\,t_0}{2\,t_0+d(x',y')}$ and so $d(x,y)=d(x',y')$. Thus, as before, $M(x,y,t)=M(x',y',t)$ for all $t>0$.

3) Suppose that $d(x,y)<t_0\leq 1$ and $d(x',y')<t_0\leq 1$. Obviously, if $d(x,y)=d(x',y')$, then $M(x,y,t)=M(x',y',t)$ for all $t>0$.

Suppose, without loss of generality, that $d(x,y)<d(x',y')$. Then $M_d(x,y,t)>M_d(x',y',t)$ for all $t>0$. Besides $M_d(x,y,2t)\geq M_d(x,y,t)$ for all $x,y\in]0,1]$. Now,

$$\begin{aligned}
&M(x,y,t_0)=M_d(x,y,2\,t_0)\cdot\tfrac{t_0-d(x,y)}{1-d(x,y)}+M_d(x,y,t_0)\cdot\tfrac{1-t_0}{1-d(x,y)}=\\
&=M_d(x,y,2\,t_0)\cdot\tfrac{t_0-d(x',y')}{1-d(x',y')}+M_d(x,y,t_0)\cdot\tfrac{1-t_0}{1-d(x',y')}+M_d(x,y,2\,t_0)\cdot\left(\tfrac{t_0-d(x,y)}{1-d(x,y)}-\tfrac{t_0-d(x',y')}{1-d(x',y')}\right)+\\
&+M_d(x,y,t_0)\cdot\left(\tfrac{1-t_0}{1-d(x,y)}-\tfrac{1-t_0}{1-d(x',y')}\right)>\\
&>M_d(x',y',2\,t_0)\cdot\tfrac{t_0-d(x',y')}{1-d(x',y')}+M_d(x',y',t_0)\cdot\tfrac{1-t_0}{1-d(x',y')}+\\
&+M_d(x,y,t_0)\cdot\left(\tfrac{t_0-d(x,y)}{1-d(x,y)}-\tfrac{t_0-d(x',y')}{1-d(x',y')}+\tfrac{1-t_0}{1-d(x,y)}-\tfrac{1-t_0}{1-d(x',y')}\right)=\\
&=M_d(x',y',2\,t_0)\cdot\tfrac{t_0-d(x',y')}{1-d(x',y')}+M_d(x',y',t_0)\cdot\tfrac{1-t_0}{1-d(x',y')}=M(x',y',t_0),
\end{aligned}$$

a contradiction.

4) Finally, we will see that the case $d(x,y)<t_0$ and $0<t_0\leq d(x',y')$ is not possible. Indeed, in such a case $d(x,y)<d(x',y')$ and, by our initial assumption, we have that

$$M(x',y',t_0)=\frac{t_0}{t_0+d(x',y')}=\frac{2\,t_0}{2\,t_0+d(x,y)}\cdot\frac{t_0-d(x,y)}{1-d(x,y)}+\frac{t_0}{t_0+d(x,y)}\cdot\frac{1-t_0}{1-d(x,y)}=M(x,y,t_0).$$

In this case it is easy to verify that $M(x,y,t_0)\geq\frac{t_0}{t_0+d(x,y)}$ and then we have that $M(x,y,t_0)\geq\frac{t_0}{t_0+d(x,y)}>\frac{t_0}{t_0+d(x',y')}=M(x',y',t_0)$, a contradiction.

Example 4.2. *A non-completable stratified strong fuzzy metric space [11].* Let $\{x_n\}_{n\geq 3}$ and $\{y_n\}_{n\geq 3}$ be two sequences of distinct points such that $A\cap B=\emptyset$, where $A=\{x_n:n\geq 3\}$ and $B=\{y_n:n\geq 3\}$.

Put $X = A \cup B$. Define a real valued function M on $X \times X \times]0, \infty[$ as follows:

$$\begin{aligned} M(x_n, x_m, t) &= M(y_n, y_m, t) = 1 - \left[\frac{1}{n \wedge m} - \frac{1}{n \vee m}\right], \\ M(x_n, y_m, t) &= M(y_m, x_n, t) = \frac{1}{n} + \frac{1}{m}, \end{aligned}$$

for all $n, m \geq 3$.

In [11] it is proved that $(X, M, \mathfrak{L})$ is a non-completable fuzzy metric space.

It is obvious that $(M, \mathfrak{L})$ is a stratified strong fuzzy metric on X, since it is stationary. Now, notice that the continuous t-norm $\mathfrak{L}$ is not integral.

Example 4.3. *A non-completable strong fuzzy metric space with an integral t-norm [12].* Consider the fuzzy metric space $(X, M, \wedge)$ given in Example 3.4. It is easy to verify that $(M, \wedge)$ is strong and, clearly, $\wedge$ is integral. In [12] it is proved that $(M, \wedge)$ is a non-completable fuzzy metric on X. Here we have shown that $(M, \wedge)$ is not a stratified fuzzy metric. Observe that $(X, M, \wedge)$ is a fuzzy ultrametric space.

Remark 4.4. *Under the assumption that the continuous t-norm is integral, the converse of Theorem 3.9 is not true. Indeed, the fuzzy metric space of Example 3.4 (a) is strong and completable but it is not stratified. On the other hand, if d is a metric, which is not ultrametric on X, then $(X, M_d, \wedge)$ is stratified and completable [5] but it is not strong [19].*

Remark 4.5. *The authors in [6] have proved that the assignment $t \to \lim_n M(a_n, b_n, t)$, where $\{a_n\}$ and $\{b_n\}$ are Cauchy sequences in a strong fuzzy metric space, is a continuous function. This assertion is not true, in general, for stratified fuzzy metrics (indeed, in [4] the authors have found two Cauchy sequences in the stratified fuzzy metric space of Example 4.1 for which the mentioned assignment is not a continuous function). Nevertheless, the following question remains open.*

Open question. Let $(X, M, *)$ be a stratified fuzzy metric space and let $\{a_n\}$ and $\{b_n\}$ be two Cauchy sequences in X. Does it exist $\lim_n M(a_n, b_n, t)$ for all $t > 0$?

Notice that this question has affirmative answer for a strong fuzzy metric space [7].

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