INTEGRAL CLOSURE AND BOUNDS FOR QUOTIENTS OF MULTIPLICITIES OF MONOMIAL IDEALS

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Abstract. Given a pair of monomial ideals I and J of finite colength of the ring of analytic function germs $(\mathbb{C}^n, 0) \to \mathbb{C}$, we prove that some power of I admits a reduction formed by homogeneous polynomials with respect to the Newton filtration induced by J if and only if the quotient of multiplicities $e(I)/e(J)$ attains a suitable upper bound expressed in terms of the Newton polyhedra of I and J. We also explore other connections between mixed multiplicities, Newton filtrations and the integral closure of ideals.

1. Introduction

Let us denote by $\mathcal{O}_n$ the ring of complex analytic function germs $f : (\mathbb{C}^n, 0) \to \mathbb{C}$. Let $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ be a complex analytic map. We say that g is *finite* when $g^{-1}(0) = \{0\}$; in this case, we refer to the number $e(g) = \dim_{\mathbb{C}} \mathcal{O}_n/I(g)$ as the *multiplicity of* g, where $I(g)$ denotes the ideal of $\mathcal{O}_n$ generated by the components of g (see [1, §5], [8, §2] or [9, §2] for several characterizations of this number). More generally, if I is any ideal of $\mathcal{O}_n$ of finite colength, then the multiplicity of I, in the sense of Hilbert-Samuel, is denoted by $e(I)$ (see [10, 12, 23]). We recall that, when I admits a generating system formed by n elements, then $e(I) = \dim_{\mathbb{C}} \mathcal{O}_n/I$. It is well-known that, if we fix a vector $w = (w_1, \dots, w_n) \in \mathbb{Z}^n_{\geqslant 1}$ and g is semi-weighted homogeneous with respect to w, then $e(g)$ can be expressed as

$$e(g) = \frac{d_1 \cdots d_n}{w_1 \cdots w_n}$$

where d_i is the degree of g_i with respect to w, for all $i = 1, \dots, n$ (see for instance [1, §12.3] or [8, §10.3]). This result was generalized in [7] by replacing the weighted homogeneous filtration induced by w by the Newton filtration induced by a given Newton polyhedron of $\mathbb{R}^n_{\geqslant 0}$ (see Theorem 4.2). That is, let $\Gamma_+ \subseteq \mathbb{R}^n_{\geqslant 0}$ be a Newton polyhedron such that $\Gamma_+ \neq \mathbb{R}^n_{\geqslant 0}$ and Γ_+ intersects each coordinate axis. Let Γ be the union of all compact faces of Γ_+ and let ν_Γ be the Newton filtration induced by Γ_+ (see Section 4 for details). If $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is any finite analytic map, then

$$e(g) \geqslant \frac{d_1 \cdots d_n}{M_\Gamma^n} n! \, \mathrm{V}_n \left(\mathbb{R}^n_{\geqslant 0} \smallsetminus \Gamma_+ \right), \tag{1}$$

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where $d_i = \nu_\Gamma(g_i)$, for all $i = 1, \dots, n$, V_n denotes the n-dimensional volume and M_Γ is the value of ν_Γ over the monomials whose exponent belongs to Γ. The maps $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ for which equality holds in (1) are called *non-degenerate on* Γ_+. This class of maps is characterized in [7, Theorem 3.3].

If K is a monomial ideal of $\mathcal{O}_n$ of finite colength, then we recall that the multiplicity of K is expressed as $e(K) = n!\,\mathrm{V}_n(\mathbb{R}^n_{\geqslant 0} \smallsetminus \Gamma_+(K))$, where $\Gamma_+(K)$ denotes the Newton polyhedron of K (see for instance [21, 22]). Therefore, relation (1) also shows a lower bound for the quotient $e(g)/e(J)$, where J is the integrally closed monomial ideal such that $\Gamma_+ = \Gamma_+(J)$. We also refer to non-degenerate maps on Γ_+ as *J-non-degenerate maps*. We show that equality holds in (1) if and only if there exists some integers $a_1, \dots, a_n, d \in \mathbb{Z}_{\geqslant 1}$ such that $\overline{\langle g_1^{a_1}, \dots, g_n^{a_n} \rangle} = \overline{J^d}$, where the bar denotes integral closure.

Moreover, if I is a monomial ideal of $\mathcal{O}_n$ of finite colength, then we use the respective Newton polyhedra of I and J to define an increasing sequence of positive rational numbers $a_{1,J}(I), \dots, a_{n,J}(I)$ that leads to an upper bound for the quotient $e(I)/e(J)$, that is,

$$\frac{e(I)}{e(J)} \leqslant \frac{a_{1,J}(I) \cdots a_{n,J}(I)}{M_J^n}, \tag{2}$$

where M_J is a positive integer defined in terms of the Newton filtration of J (see Section 4). We prove that equality holds in (2) if and only if there exists some $s \geqslant 1$ such that $\overline{I^s} = \overline{\langle g_1, \dots, g_n \rangle}$, for some map $(g_1, \dots, g_n) : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ which is J-non-degenerate. This result appears in Theorem 5.5. The proof of this result is preceded by a characterization of the notion of J-non-degeneracy of n-tuples of monomial ideals (see Theorem 4.10 and Definition 4.3), which in turn depends on previous combinatorial results proven in Section 3. Let us remark that, by interchanging the roles of I and J in (2) we automatically obtain a lower bound for $e(I)/e(J)$ (see Corollary 5.7).

The motivation of our work in this article arises from our previous work [4], where we characterized when the integral closure of a given monomial ideal of $\mathcal{O}_n$ of finite colength is equal to the integral closure of the ideal generated by n homogeneous polynomials. In turn, [4] was motivated by the results of Hickel in [11].

The article is organized as follows. In Section 2 we recall some definitions and results related with mixed multiplicities, joint reductions of families of ideals and Newton polyhedra that we will need in the article. Let $I_1, \dots, I_n$ be n monomial ideals of $\mathcal{O}_n$. Due to its importance in subsequent sections, we recall in Theorem 2.3 the result of Rees and Sally (see [15, Theorem 1.6] and [12, §17.3]) about the existence of joint reductions of $(I_1, \dots, I_n)$, in the sense of Rees [14].

Section 3 is devoted to showing a combinatorial characterization of the finiteness of $\sigma(I_1, \dots, I_n)$ (see Theorem 3.2), where $\sigma(I_1, \dots, I_n)$ denotes what we call the Rees' mixed multiplicity of $I_1, \dots, I_n$ (see (5)) and $I_1, \dots, I_n$ are monomial ideals of $\mathcal{O}_n$. This result will be fundamental in the proofs of some results of Section 4.

The objective of Section 4 is to show a combinatorial characterization of those pairs formed by an n-tuple $(I_1, \dots, I_n)$ of monomial ideals of $\mathcal{O}_n$ such that $\sigma(I_1, \dots, I_n) < \infty$ and a monomial ideal J of $\mathcal{O}_n$ of finite colength for which $(I_1,\dots,I_n)$ is J-non-degenerate (see Definition 4.3), which is a generalization of the notion of J-non-degenerate map. This is given in Theorem 4.10.

In Section 5 we prove the existence of what we call *central maps* with respect to a given pair of monomial ideals of $\mathcal{O}_n$ of finite colength (see Theorem 5.3 and Corollary 5.4). The existence of central maps supports the proof of the upper bound mentioned in (2) and the characterization of the equality in (2) (see Theorem 5.5). We remark that in Corollary 5.7(c) we show a characterization of the equality in (2) that is expressed only in terms of the respective Newton filtrations induced by I and J.

2. Preliminary concepts

This section is devoted to recalling some definitions and fundamental facts that we will use along the paper.

2.1. **Mixed multiplicities and joint reductions**

Along this section we suppose that $(R,\mathbf{m})$ is a Noetherian local ring with infinite residue field $\mathbf{k}=R/\mathbf{m}$ and of dimension n. We recall some concepts and results from [2, 3, 5]. If I is an ideal of R, then we denote by $\overline{I}$ the integral closure of I (see [10, 12, 23]).

Let $I_1,\dots,I_n$ be ideals of R of finite colength. We denote by $e(I_1,\dots,I_n)$ the mixed multiplicity of $I_1,\dots,I_n$ defined by Teissier and Risler in [19, §2] (see also [12, Section 17.4] or [16, Section 2.5]). We recall briefly the definition of $e(I_1,\dots,I_n)$. Let us consider the function $H:\mathbb{Z}^n_{\geqslant 0}\to\mathbb{Z}_{\geqslant 0}$ given by

$$H(r_1,\dots,r_n)=\ell\left(\frac{R}{I_1^{r_1}\cdots I_n^{r_n}}\right), \tag{3}$$

for all $(r_1,\dots,r_n)\in\mathbb{Z}^n_{\geqslant 0}$, where $\ell(M)$ denotes the length of a given R-module M. It is proven in [19, §2] that there exists a polynomial $P(x_1,\dots,x_n)\in\mathbb{Q}[x_1,\dots,x_n]$ of degree n such that

$$H(r_1,\dots,r_n)=P(r_1,\dots,r_n),$$

for all sufficiently large $r_1,\dots,r_n\in\mathbb{Z}_{\geqslant 0}$. Moreover, the coefficient of the monomial $x_1\cdots x_n$ in $P(x_1,\dots,x_n)$ is a positive integer. This integer is called the *mixed multiplicity* of $I_1,\dots,I_n$ and is denoted by $e(I_1,\dots,I_n)$.

We remark that if $I_1,\dots,I_n$ are all equal to a given ideal I of finite colength of R, then $e(I_1,\dots,I_n)=e(I)$, where $e(I)$ denotes the Samuel multiplicity of I. We refer to [12, §17.4] or [18] for fundamental results concerning mixed multiplicities of ideals.

Moreover Rees showed in [14] that the mixed multiplicity $e(I_1,\dots,I_n)$ can be computed in terms of Samuel multiplicities via the following formula:

$$e(I_1,\dots,I_n)=\frac{1}{n!}\sum_{\substack{\mathsf{J}\subseteq\{1,\dots,n\}\\ \mathsf{J}\neq\emptyset}}(-1)^{n-|\mathsf{J}|}e\Bigg(\prod_{j\in\mathsf{J}}I_j\Bigg),$$

where we denote by $|X|$ the cardinal of a given finite set X.

Given two ideals I and J of R of finite colength and an integer $i \in \{1,\dots,n\}$, we define

$$e_i(I,J) = e(I,\dots,I,J,\dots,J), \tag{4}$$

where I is repeated i times and J is repeated $n-i$ times.

Let $I_1,\dots,I_n$ be proper ideals of R (not necessarily of finite colength). In [2] we studied the following number:

$$\sigma(I_1,\dots,I_n) = \sup_{r\in\mathbb{Z}_{\geqslant 0}} e(I_1+\mathbf{m}^r,\dots,I_n+\mathbf{m}^r). \tag{5}$$

When the set of integers $\{e(I_1+\mathbf{m}^r,\dots,I_n+\mathbf{m}^r) : r\in\mathbb{Z}_{\geqslant 0}\}$ is bounded, then we refer to $\sigma(I_1,\dots,I_n)$ as the *Rees' mixed multiplicity of* $I_1,\dots,I_n$. Obviously, if I_i has finite colength, for all $i=1,\dots,n$, then $\sigma(I_1,\dots,I_n)=e(I_1,\dots,I_n)$.

In Proposition 2.2 we recall a result from [2] that interprets $\sigma(I_1,\dots,I_n)$ as a multiplicity in the usual sense. First we need to introduce a preliminary concept.

Definition 2.1. Let $I_1,\dots,I_r$ be proper ideals of R. Let $a_{i1},\dots,a_{is_i}$ be a minimal generating system of I_i, where $s_i\in\mathbb{Z}_{\geqslant 1}$, for all $i=1,\dots,r$. Let $s=s_1+\cdots+s_r$. We say that a property holds for *sufficiently general* elements $(g_1,\dots,g_r)\in I_1\oplus\cdots\oplus I_r$ if there exists a non-empty Zariski-open set U in $\mathbf{k}^s$ verifying that if

(a) $g_i=\sum_j u_{ij}a_{ij}$, where $u_{ij}\in R$, for all $j=1,\dots,s_i$, $i=1,\dots,r$, and
(b) the image of $(u_{11},\dots,u_{1s_1},\dots,u_{r1},\dots,u_{rs_r})$ in $\mathbf{k}^s$ belongs to U,

then the said property holds for $(g_1,\dots,g_r)$.

Proposition 2.2. [2, 2.9] *Let $I_1,\dots,I_n$ be proper ideals of R. Then $\sigma(I_1,\dots,I_n)<\infty$ if and only if there exist elements $g_i\in I_i$, for $i=1,\dots,n$, such that $\langle g_1,\dots,g_n\rangle$ has finite colength. If $\sigma(I_1,\dots,I_n)<\infty$, then $\sigma(I_1,\dots,I_n)=e(g_1,\dots,g_n)$ for sufficiently general elements $(g_1,\dots,g_n)\in I_1\oplus\cdots\oplus I_n$.*

Let I and J be ideals of R such that $J\subseteq I$. We recall that an ideal J is called a *reduction* of I if there exists some $r\in\mathbb{Z}_{\geqslant 0}$ such that $I^{r+1}=JI^r$. It is well known that J is a reduction of I if and only if $\overline{I}=\overline{J}$ (see for instance [12, Corollary 1.2.5]). In turn, if we assume that the ideals I and J have finite colength, $J\subseteq I$ and R is quasi-unmixed, then the celebrated Rees' multiplicity theorem says that the equality $\overline{I}=\overline{J}$ holds if and only if $e(I)=e(J)$ (see [10, p. 147] or [12, p. 222]).

Let $I_1,\dots,I_n$ be ideals of R. Let $g_1,\dots,g_n\in R$ such that $g_i\in I_i$, for all $i=1,\dots,n$. Then $(g_1,\dots,g_n)$ is called a *joint reduction of* $(I_1,\dots,I_n)$ when $g_1I_2\cdots I_n+\cdots+g_nI_1\cdots I_{n-1}$ is a reduction of $I_1\cdots I_n$. By the relation between reductions and integral closure mentioned before, this condition is equivalent to saying that

$$\overline{g_1I_2\cdots I_n+\cdots+g_nI_1\cdots I_{n-1}}=\overline{I_1\cdots I_n}. \tag{6}$$

Let us fix a family $I_1,\dots,I_p$ of proper ideals of R. We recall that $\dim(R)=n$. In [14, Theorem 1.3], D. Rees showed that there exists a family of elements $\{x_{ij} : i=1,\dots,p, j=1,\dots n\}\subseteq R$ such that $x_{i1},\dots,x_{in}\in I_i$, for all $i=1,\dots,p$, and if $y_j=x_{1j}\cdots x_{pj}$, for all $j = 1, \dots, n$, then the ideal $\langle y_1, \dots, y_n \rangle$ is a reduction of the product ideal $I_1 \cdots I_p$. We remark that p is not assumed to be equal to n in this result. Any set of elements x_{ij} satisfying the above properties is called a *complete reduction of* $(I_1, \dots, I_p)$ (see [14, p. 402] or [12, Definition 17.1.3]).

Now let us suppose that $p = n$. In [14, Theorem 1.4] D. Rees easily proved that, if $\{x_{ij} : i, j = 1, \dots, n\}$ is a complete reduction of $(I_1, \dots, I_n)$, then $x_{1\sigma(1)}, \dots, x_{n\sigma(n)}$ is a joint reduction of $(I_1, \dots, I_n)$, for any permutation $i \mapsto \sigma(i)$ of $\{1, \dots, n\}$. In [15, Theorem 1.6] Rees and Sally proved that joint reductions of sets of n ideals exist, we next recall this result (see also [12, §17.3]).

Theorem 2.3. *Let $I_1, \dots, I_n$ be ideals of R. Then $(g_1, \dots, g_n)$ is a joint reduction of $(I_1, \dots, I_n)$, for sufficiently general elements $(g_1, \dots, g_n) \in I_1 \oplus \cdots \oplus I_n$.*

2.2. Newton polyhedra of ideals and non-degeneracy conditions

Let us fix a coordinate system $x_1, \dots, x_n$ in $\mathbb{C}^n$. If $k = (k_1, \dots, k_n) \in \mathbb{Z}^n_{\geqslant 0}$, then we denote the monomial $x_1^{k_1} \cdots x_n^{k_n}$ by x^k. Given a proper ideal J of $\mathcal{O}_n$, we will say that J is *monomial* when J admits a generating system formed by monomials.

If $h \in \mathcal{O}_n$ and $h = \sum_k a_k x^k$ is the Taylor expansion of h around the origin, then the *support* of h, denoted by $\mathrm{supp}(h)$ is defined as the set $\{k \in \mathbb{Z}^n_{\geqslant 0} : a_k \neq 0\}$. Given a subset $\Delta \subseteq \mathbb{R}^n_{\geqslant 0}$, we denote by h_Δ the sum of those terms $a_k x^k$ such that $k \in \Delta \cap \mathrm{supp}(h)$. If $\Delta \cap \mathrm{supp}(h) = \emptyset$, then we set $h_\Delta = 0$. If I is an ideal of $\mathcal{O}_n$, then the *support* of I is defined as $\mathrm{supp}(I) = \cup_{g \in I} \mathrm{supp}(g)$.

If $A \subseteq \mathbb{Z}^n_{\geqslant 0}$, $A \neq \emptyset$, then we define the *Newton polyhedron determined by* A, denoted by $\Gamma_+(A)$, as the convex hull in $\mathbb{R}^n$ of the set $\{k + v : k \in A,\, v \in \mathbb{R}^n_{\geqslant 0}\}$. A subset $\Gamma_+ \subseteq \mathbb{R}^n_{\geqslant 0}$ is called a *Newton polyhedron* when $\Gamma_+ = \Gamma_+(A)$, for some $A \subseteq \mathbb{Z}^n_{\geqslant 0}$.

Let us fix a Newton polyhedron $\Gamma_+ \subseteq \mathbb{R}^n_{\geqslant 0}$. If $v \in \mathbb{R}^n_{\geqslant 0}$, then we define

$$\ell(v, \Gamma_+) = \min \big\{ \langle v, k \rangle : k \in \Gamma_+ \big\}$$
$$\Delta(v, \Gamma_+) = \big\{ k \in \Gamma_+ : \langle v, k \rangle = \ell(v, \Gamma_+) \big\},$$

where $\langle \, , \rangle$ stands for the standard scalar product in $\mathbb{R}^n$. A *face* of Γ_+ is any set Δ of the form $\Delta = \Delta(v, \Gamma_+)$, for some $v \in \mathbb{R}^n_{\geqslant 0} \smallsetminus \{0\}$; in this case we say that Δ is *supported* by v. Given a face Δ of Γ_+, we observe that Δ is compact if and only if it is supported by a vector $v \in \mathbb{R}^n_{>0}$. The *dimension* of Δ is defined as the minimum of the dimensions of the affine subspaces of $\mathbb{R}^n$ containing Δ. We denote by $C(\Delta)$ the cone formed by all half-lines emanating from the origin and passing through some point of Δ. Let us denote by $\mathcal{R}_\Delta$ the subring of $\mathcal{O}_n$ formed by the functions $h \in \mathcal{O}_n$ such that $\mathrm{supp}(h) \subseteq C(\Delta)$.

The faces of dimension 0 and the faces of dimension $n-1$ of Γ_+ are known, respectively, as the *vertices* and the *facets* of Γ_+. We denote by $\mathbf{v}(\Gamma_+)$ the set of vertices of Γ_+. The union of all compact faces of Γ_+ will be denoted by Γ. We will refer to Γ as the *Newton boundary of* Γ_+. We remark that Γ_+ is univocally determined by the set Γ, since $\Gamma_+ = \Gamma + \mathbb{R}^n_{\geqslant 0}$. We denote by Γ_- the union of all segments joining the origin and some point of Γ.

If $h \in \mathcal{O}_n$, then the *Newton polyhedron of* h is defined as $\Gamma_+(h) = \Gamma_+(\mathrm{supp}(h))$. Moreover, if I is an ideal of $\mathcal{O}_n$, then the *Newton polyhedron of* I is defined as $\Gamma_+(I) = \Gamma_+(\mathrm{supp}(I))$. We recall that $\Gamma_+(I) = \Gamma_+(\overline{I})$ (see for instance [7, Lemma 2.3]).

If $\{g_1, \dots, g_r\}$ is a generating set of I, then it is straightforward to see that $\Gamma_+(I)$ equals the convex hull of $\Gamma_+(g_1) \cup \dots \cup \Gamma_+(g_r)$. We denote the Newton boundary of $\Gamma_+(I)$ by $\Gamma(I)$ and the union of all segments joining the origin with some point of $\Gamma(I)$ by $\Gamma_-(I)$.

Let I be a proper ideal of $\mathcal{O}_n$ and let $g_1, \dots, g_s$ be a generating system of I. We recall that I is called *Newton non-degenerate* (see [2, 7, 17]) when

$$\big\{x \in \mathbb{C}^n : (g_1)_\Delta(x) = \dots = (g_s)_\Delta(x) = 0\big\} \subseteq \big\{x \in \mathbb{C}^n : x_1 \cdots x_n = 0\big\},$$

as set germs at $0 \in \mathbb{C}^n$, for each compact face Δ of $\Gamma_+(I)$. It was proven by Saia [17] that an ideal I of $\mathcal{O}_n$ is Newton non-degenerate if and only if the integral closure of I is equal to the ideal generated by those monomials x^k such that $k \in \Gamma_+(I)$ (see also [7, Corollary 2.6] or [2, Proposition 3.6]).

As a consequence of [2, Proposition 3.6] we have that, if I is a monomial ideal of $\mathcal{O}_n$ and $J \subseteq I$, then J is a reduction of I if and only if J is Newton non-degenerate and $\Gamma_+(I) = \Gamma_+(J)$.

We remark that, if $f \in \mathcal{O}_n$, then the condition of Newton non-degeneracy of the ideal $\langle x_1 \frac{\partial f}{\partial x_1}, \dots, x_n \frac{\partial f}{\partial x_n} \rangle$ allows to obtain a lot of information about the topology of f by means of $\Gamma_+(f)$ (see [13] and [24]).

3. The Rees' mixed multiplicity of a family of monomial ideals

Given a non-empty subset $\mathtt{L} \subseteq \{1, \dots, n\}$, if $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, then we define $\mathbb{K}^n_{\mathtt{L}} = \{x \in \mathbb{K}^n : x_i = 0, \text{ for all } i \notin \mathtt{L}\}$. Let $r = |\mathtt{L}|$ and let us write $\mathtt{L} = \{j_1, \dots, j_r\}$, for some integers $1 \leqslant j_1 < \dots < j_r \leqslant n$. If $S \subseteq \mathbb{K}^n$, then we denote by $S^{\mathtt{L}}$ the intersection $S \cap \mathbb{K}^n_{\mathtt{L}}$. Let us define

$$\mathbf{H} = \big\{x \in \mathbb{C}^n : x_1 \cdots x_n = 0\big\}$$
$$\mathbf{H}_{\mathtt{L}} = \big\{(x_{j_1}, \dots, x_{j_r}) \in \mathbb{C}^r : x_{j_1} \cdots x_{j_r} = 0\big\}.$$

Hence, if $|\mathtt{L}| = 1$, then $\mathbf{H}_{\mathtt{L}} = \{0\} \subseteq \mathbb{C}$.

If we fix a non-empty subset $\mathtt{L} \subseteq \{1, \dots, n\}$, then $h^{\mathtt{L}}$ will denote the sum of all terms $a_k x^k$, such that $k \in \mathrm{supp}(h)^{\mathtt{L}}$. Then $\mathrm{supp}(h^{\mathtt{L}}) = \mathrm{supp}(h)^{\mathtt{L}}$. Let $\mathcal{O}_{n,\mathtt{L}}$ denote the subring of $\mathcal{O}_n$ formed by all elements $h \in \mathcal{O}_n$ with $\mathrm{supp}(h) \subseteq \mathbb{R}^n_{\mathtt{L}}$. Let us remark that the map $\pi_{\mathtt{L}} : \mathcal{O}_n \to \mathcal{O}_{n,\mathtt{L}}$ given by $\pi_{\mathtt{L}}(h) = h^{\mathtt{L}}$, for all $h \in \mathcal{O}_n$, is a ring morphism. In order to simplify the notation, if I is an ideal of $\mathcal{O}_n$, then we also denote the ideal $\pi_{\mathtt{L}}(I)$ by $I^{\mathtt{L}}$.

If $I = \langle g_1, \dots, g_r \rangle$ is an ideal of $\mathcal{O}_n$, then we denote by $\mathbf{V}(I)$, or by $\mathbf{V}(g_1, \dots, g_r)$, the zero set germ of I at 0. Let $\mathbf{m}_n$ denote the maximal ideal of $\mathcal{O}_n$.

Proposition 3.1. *Let I be a proper ideal of $\mathcal{O}_n$. Then I has finite colength if and only if $\mathbf{V}(I^{\mathtt{L}}) \subseteq \mathbf{H}_{\mathtt{L}}$, for all $\mathtt{L} \subseteq \{1, \dots, n\}$, $\mathtt{L} \neq \emptyset$.*

Proof. Let us prove first the *only if* part. Let us fix a subset $\mathtt{L} \subseteq \{1,\dots,n\}$, $\mathtt{L} \neq \emptyset$. Since I has finite colength, there exists some integer $r \geqslant 1$ such that $\mathbf{m}^r \subseteq I$. If we apply $\pi_{\mathtt{L}}$ to both sides of this inclusion, we conclude that the ideal $I^{\mathtt{L}}$ has also finite colength in $\mathcal{O}_{n,\mathtt{L}}$. In particular, $\mathbf{V}(I^{\mathtt{L}}) = \{0\} \subseteq \mathbf{H}_{\mathtt{L}}$.

Let us prove the *if* part. Let $g_1,\dots,g_r$ be any generating system of I and let $\mathtt{L} \subseteq \{1,\dots,n\}$, $\mathtt{L} \neq \emptyset$. The condition $\mathbf{V}(I^{\mathtt{L}}) \subseteq \mathbf{H}_{\mathtt{L}}$ implies that $I^{\mathtt{L}} \neq 0$; otherwise $\mathbf{V}(I^{\mathtt{L}}) = \mathbb{C}^n_{\mathtt{L}}$. Therefore $\{i : g_i^{\mathtt{L}} \neq 0\} \neq \emptyset$. Then we have that

$$\mathbf{V}(I) \cap \mathbb{C}^n_{\mathtt{L}} = \mathbf{V}(g_1,\dots,g_r) \cap \mathbb{C}^n_{\mathtt{L}} = \mathbf{V}(g_1^{\mathtt{L}},\dots,g_r^{\mathtt{L}}) = \mathbf{V}(I^{\mathtt{L}}).$$

By the same reason, we have

$$\mathbf{V}(I^{\mathtt{L}'}) \cap \mathbb{C}^n_{\mathtt{L}} = \mathbf{V}(I^{\mathtt{L}}) \tag{7}$$

for all $\mathtt{L}' \subseteq \{1,\dots,n\}$ containing $\mathtt{L}$.

Let $\mathtt{L} \subseteq \{1,\dots,n\}$ such that $|\mathtt{L}| = 1$. By hypothesis we have $\mathbf{V}(I^{\mathtt{L}}) \subseteq \mathbf{H}_{\mathtt{L}} = \{0\} \subseteq \mathbb{C}$. Since $I \subseteq \mathbf{m}_n$, we have $I^{\mathtt{L}} \subseteq \mathbf{m}_n^{\mathtt{L}}$ and then $\{0\} \subseteq \mathbf{V}(I^{\mathtt{L}})$. Thus $\mathbf{V}(I^{\mathtt{L}}) = \{0\}$.

Let $j \in \{1,\dots,n-1\}$ and let us suppose that $\mathbf{V}(I^{\mathtt{L}}) = \{0\}$, for all $\mathtt{L} \subseteq \{1,\dots,n\}$ such that $|\mathtt{L}| \leqslant j$. Let us fix a subset $\mathtt{L}' \subseteq \{1,\dots,n\}$ such that $|\mathtt{L}'| = j+1$. The condition $\mathbf{V}(I^{\mathtt{L}'}) \subseteq \mathbf{H}_{\mathtt{L}'}$ implies that

$$\mathbf{V}(I^{\mathtt{L}'}) = \bigcup_{\substack{\mathtt{L} \subseteq \{1,\dots,n\} \\ |\mathtt{L}|=j,\, \mathtt{L} \subseteq \mathtt{L}'}} \left(\mathbf{V}(I) \cap \mathbb{C}^n_{\mathtt{L}}\right) = \bigcup_{\substack{\mathtt{L} \subseteq \{1,\dots,n\} \\ |\mathtt{L}|=j,\, \mathtt{L} \subseteq \mathtt{L}'}} \mathbf{V}(I^{\mathtt{L}})$$

where the first equality comes from (7). Then $\mathbf{V}(I^{\mathtt{L}'}) = \{0\}$. By finite induction on $|\mathtt{L}|$ we deduce that $\mathbf{V}(I) = \mathbf{V}(I^{\{1,\dots,n\}}) = \{0\}$.

□

Theorem 3.2. *Let $I_1,\dots,I_n$ be monomial ideals of $\mathcal{O}_n$. Then the following conditions are equivalent.*

(a) $\sigma(I_1,\dots,I_n) < \infty$.
(b) *For each $\mathtt{L} \subseteq \{1,\dots,n\}$, $\mathtt{L} \neq \emptyset$, we have $|\{i : I_i^{\mathtt{L}} \neq 0\}| \geqslant |\mathtt{L}|$.*

Proof. Let us prove (a) ⇒ (b). If $\sigma(I_1,\dots,I_n) < \infty$, then there exist some $(g_1,\dots,g_n) \in I_1 \oplus \cdots \oplus I_n$ such that, if I denotes the ideal generated by $g_1,\dots,g_n$, then I has finite colength in $\mathcal{O}_n$ and $e(I) = \sigma(I_1,\dots,I_n)$, by Proposition 2.2. In particular, there exists some $r \in \mathbb{Z}_{\geqslant 1}$ such that $\mathbf{m}_n^r \subseteq I$. Given a subset $\mathtt{L} \subseteq \{1,\dots,n\}$, $\mathtt{L} \neq \emptyset$, if we apply $\pi_{\mathtt{L}}$ to both sides of the inclusion $\mathbf{m}_n^r \subseteq I$, then we obtain that the ideal $I^{\mathtt{L}}$ has also finite colength. This implies that the set $\{g_1^{\mathtt{L}},\dots,g_n^{\mathtt{L}}\}$ contains at least $|\mathtt{L}|$ non-zero elements, since $\dim \mathcal{O}_{n,\mathtt{L}} = |\mathtt{L}|$. Then, as $\mathrm{supp}(g_i^{\mathtt{L}}) \subseteq I_i^{\mathtt{L}}$, for all $i = 1,\dots,n$, condition (b) holds.

Let us prove (b) ⇒ (a). By Proposition 2.2, it suffices to see that there exists an element $(g_1,\dots,g_n) \in I_1 \oplus \cdots \oplus I_n$ such that the ideal generated by $g_1,\dots,g_n$ in $\mathcal{O}_n$ has finite colength.

Let us fix a subset $\mathtt{L} \subseteq \{1,\dots,n\}$, $\mathtt{L} \neq \emptyset$. By hypothesis we can choose a set of indexes $B_{\mathtt{L}} \subseteq \{1,\dots,n\}$ such that $|B_{\mathtt{L}}| = |\mathtt{L}|$ and $I_j^{\mathtt{L}} \neq 0$, for all $j \in B_{\mathtt{L}}$. Let us write $B_{\mathtt{L}}$ as $B_{\mathtt{L}} = \{j_{\mathtt{L},1},\dots,j_{\mathtt{L},|\mathtt{L}|}\} \subseteq \{1,\dots,n\}$, for some integers $1 \leqslant j_{\mathtt{L},1} < \cdots < j_{\mathtt{L},|\mathtt{L}|} \leqslant n$.

Let G_i be a fixed minimal generating system of I_i formed by monomials, for all $i = 1,\dots,n$, and let $\mathbf{G} = (G_1,\dots,G_n)$. Let us denote by $G_i^{\mathtt{L}}$ be the set of monomials of G_i whose support belongs to $\mathbb{R}^n_{\mathtt{L}}$. Hence $G_i^{\mathtt{L}}$ is a generating system of $I_i^{\mathtt{L}}$, for all $i = 1,\dots,n$. Let $\mathbf{G}^{\mathtt{L}} = (G^{\mathtt{L}}_{j_{\mathtt{L},1}},\dots,G^{\mathtt{L}}_{j_{\mathtt{L},|\mathtt{L}|}})$. Let us identify the set of $\mathbf{G}^{\mathtt{L}}$-maps with $\mathbb{C}^{N_{\mathtt{L}}}$, where $N_{\mathtt{L}} = |G^{\mathtt{L}}_{j_{\mathtt{L},1}}| + \cdots + |G^{\mathtt{L}}_{j_{\mathtt{L},|\mathtt{L}|}}|$.

Since $\dim \mathcal{O}_{n,\mathtt{L}} = |\mathtt{L}|$, we can apply Theorem 2.3 to the $|\mathtt{L}|$-tuple of ideals $(I^{\mathtt{L}}_{j_{\mathtt{L},1}},\dots,I^{\mathtt{L}}_{j_{\mathtt{L},|\mathtt{L}|}})$. Thus, there exist a Zariski-open subset $U_{\mathtt{L}} \subseteq \mathbb{C}^{N_{\mathtt{L}}}$, such that any $\mathbf{G}^{\mathtt{L}}$-map with coefficients in $U_{\mathtt{L}}$ is a joint reduction of $(I^{\mathtt{L}}_{j_{\mathtt{L},1}},\dots,I^{\mathtt{L}}_{j_{\mathtt{L},|\mathtt{L}|}})$. Hence there exists a $\mathbf{G}$-map $(g_1,\dots,g_n) \in I_1 \oplus \cdots \oplus I_n$ such that the set of coefficients of $(g^{\mathtt{L}}_{j_{\mathtt{L},1}},\dots,g^{\mathtt{L}}_{j_{\mathtt{L},|\mathtt{L}|}})$ belongs to $U_{\mathtt{L}}$, for all non-empty $\mathtt{L} \subseteq \{1,\dots,n\}$.

Let us fix again a subset $\mathtt{L} \subseteq \{1,\dots,n\}$, $\mathtt{L} \neq \emptyset$. Since $(g^{\mathtt{L}}_{j_{\mathtt{L},1}},\dots,g^{\mathtt{L}}_{j_{\mathtt{L},|\mathtt{L}|}})$ is a joint reduction of $(I^{\mathtt{L}}_{j_{\mathtt{L},1}},\dots,I^{\mathtt{L}}_{j_{\mathtt{L},|\mathtt{L}|}})$, by (6), we obtain that

$$\overline{g^{\mathtt{L}}_{j_{\mathtt{L},1}} I^{\mathtt{L}}_{j_{\mathtt{L},2}} \cdots I^{\mathtt{L}}_{j_{\mathtt{L},|\mathtt{L}|}} + \cdots + g^{\mathtt{L}}_{j_{\mathtt{L},|\mathtt{L}|}} I^{\mathtt{L}}_{j_{\mathtt{L},1}} \cdots I^{\mathtt{L}}_{j_{\mathtt{L},|\mathtt{L}|-1}}} = \overline{I^{\mathtt{L}}_{j_{\mathtt{L},1}} \cdots I^{\mathtt{L}}_{j_{\mathtt{L},|\mathtt{L}|}}}.$$

Hence $\overline{I^{\mathtt{L}}_{j_{\mathtt{L},1}} \cdots I^{\mathtt{L}}_{j_{\mathtt{L},|\mathtt{L}|}}} \subseteq \overline{\langle g^{\mathtt{L}}_{j_{\mathtt{L},1}},\dots,g^{\mathtt{L}}_{j_{\mathtt{L},|\mathtt{L}|}} \rangle}$, which in turn implies that

$$\mathbf{V}(g_1^{\mathtt{L}},\dots,g_n^{\mathtt{L}}) \subseteq \mathbf{V}(g^{\mathtt{L}}_{j_{\mathtt{L},1}},\dots,g^{\mathtt{L}}_{j_{\mathtt{L},|\mathtt{L}|}}) \subseteq \mathbf{V}(I^{\mathtt{L}}_{j_{\mathtt{L},1}} \cdots I^{\mathtt{L}}_{j_{\mathtt{L},|\mathtt{L}|}}) \subseteq \mathbf{H}_{\mathtt{L}},$$

where the last inclusion follows from the fact that $I_1,\dots,I_n$ are monomial ideals. Then we have deduced that $\mathbf{V}(g_1^{\mathtt{L}},\dots,g_n^{\mathtt{L}}) \subseteq \mathbf{H}_{\mathtt{L}}$, for all non-empty subsets $\mathtt{L} \subseteq \{1,\dots,n\}$. In particular, the ideal $\langle g_1,\dots,g_n\rangle$ has finite colength in $\mathcal{O}_n$, by Proposition 3.1, and the result follows. □

4. Characterization of J-non-degeneracy of sequences of ideals

This section is devoted to characterize those sequences of ideals whose Rees' mixed multiplicity attains a lower bound formulated in terms of a fixed Newton filtration.

4.1. **The Newton filtration and the computation of multiplicities**

Let $\Gamma_+ \subseteq \mathbb{R}^n$ be a Newton polyhedron. We say that Γ_+ is *convenient* when $\Gamma_+ \neq \mathbb{R}^n_{\geqslant 0}$ and Γ_+ intersects each coordinate axis. If Γ_+ is convenient, then Γ_- is equal to the closure of $\mathbb{R}^n_{\geqslant 0} \smallsetminus \Gamma_+$ (in the usual Euclidean sense). We observe that, if I is an ideal of $\mathcal{O}_n$ of finite colength, then $\Gamma_+(I)$ is convenient.

Let $v \in \mathbb{Z}^n_{\geqslant 0}$, $v \neq 0$, we say that v is *primitive* when the non-zero coordinates of v are mutually prime integer numbers. Then any facet of Γ_+ is supported by a unique primitive vector of $\mathbb{Z}^n_{\geqslant 0}$. Let us denote by $\mathcal{F}(\Gamma_+)$ the set of primitive vectors of $\mathbb{Z}^n_{\geqslant 0}$ supporting some facet of Γ_+ and let $\mathcal{F}_c(\Gamma_+) = \mathcal{F}(\Gamma_+) \cap \mathbb{Z}^n_{\geqslant 1}$. Let us remark that, if Γ_+ is convenient, then $\mathcal{F}(\Gamma_+) = \mathcal{F}_c(\Gamma_+) \cup \{e_1,\dots,e_n\}$, where $e_1,\dots,e_n$ is the canonical basis of $\mathbb{R}^n$.

Let us fix a convenient Newton polyhedron $\Gamma_+ \subseteq \mathbb{R}^n_{\geqslant 0}$. Let us write $\mathcal{F}_c(\Gamma_+) = \{v^1,\dots,v^r\}$. Then $\ell(v^i,\Gamma_+) > 0$, for all $i = 1,\dots,r$. We denote by M_Γ the least common multiple of the set of integers $\{\ell(v^1, \Gamma_+), \dots, \ell(v^r, \Gamma_+)\}$. We define the *filtrating map* associated to Γ_+ as the map $\phi_\Gamma : \mathbb{R}^n_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ given by

$$\phi_\Gamma(k) = \min\left\{ \frac{M_\Gamma}{\ell(v^i, \Gamma_+)} \langle k, v^i\rangle : i = 1, \dots, r \right\}, \quad \text{for all } k \in \mathbb{R}^n_{\geqslant 0}.$$

We observe that $\phi_\Gamma(\mathbb{Z}^n_{\geqslant 0}) \subseteq \mathbb{Z}^n_{\geqslant 0}$, $\phi_\Gamma(k) = M_\Gamma$, for all $k \in \Gamma$, and the map ϕ_Γ is linear on each cone $C(\Delta)$, where Δ is any compact face of Γ_+. As mentioned in [13, p. 10], given $a, b \in \mathbb{Z}^n_{\geqslant 0}$, it is easy to prove that $\phi_\Gamma(a+b) \geqslant \phi_\Gamma(a) + \phi_\Gamma(b)$. Moreover, it is a straightforward exercise to see that equality holds if and only if there exists some compact face Δ of Γ_+ such that $a, b \in C(\Delta)$.

Let us define the map $\nu_\Gamma : \mathcal{O}_n \to \mathbb{R}_{\geqslant 0} \cup \{+\infty\}$ by $\nu_\Gamma(h) = \min\{\phi_\Gamma(k) : k \in \operatorname{supp}(h)\}$, for all $h \in \mathcal{O}_n$, $h \neq 0$; we set $\nu_\Gamma(0) = +\infty$. We refer to ν_Γ as the *Newton filtration* induced by Γ_+ (see also [7, 13]).

By abuse of notation, if $A \subseteq \mathbb{R}^n_{\geqslant 0}$ denotes a non-empty closed set, then we define $\nu_\Gamma(A) = \min\{\phi_\Gamma(k) : k \in A\}$. Hence $\nu_\Gamma(h) = \nu_\Gamma(\operatorname{supp}(h))$, for all $h \in \mathcal{O}_n$, $h \neq 0$. If I is a non-zero ideal of $\mathcal{O}_n$, then we also define $\nu_\Gamma(I) = \nu_\Gamma(\operatorname{supp}(I))$. Given a proper ideal J of $\mathcal{O}_n$ of finite colength, then we denote by ϕ_J the filtrating map associated to $\Gamma_+(J)$ and the integer $M_{\Gamma(J)}$ by M_J. We will also write ν_J instead of $\nu_{\Gamma(J)}$ and we will also refer to ν_J as the *Newton filtration induced by* J.

Definition 4.1. Let us fix a convenient Newton polyhedron $\Gamma_+ \subseteq \mathbb{R}^n_{\geqslant 0}$. If $h \in \mathcal{O}_n$ and $h = \sum_k a_k x^k$ is the Taylor expansion of h around the origin, then we define the *principal part of* h *with respect to* Γ_+, denoted by $\mathrm{p}_\Gamma(h)$, as the polynomial obtained as the sum of all terms $a_k x^k$ such that $\nu_\Gamma(h) = \nu_\Gamma(x^k)$. If Δ is a compact face of Γ_+, we define the *principal part of* h *over* Δ, denoted by $\mathrm{p}_{\Gamma,\Delta}(h)$, as the sum of all terms $a_k x^k$ such that $k \in C(\Delta)$ and $\nu_\Gamma(h) = \nu_\Gamma(x^k)$. If no such terms exist, then we set $\mathrm{p}_{\Gamma,\Delta}(h) = 0$. When there is no risk of confusion, then we denote $\mathrm{p}_{\Gamma,\Delta}(h)$ simply by $\mathrm{p}_\Delta(h)$.

Let us fix a monomial ideal J of $\mathcal{O}_n$ of finite colength. Let Δ be a compact face of $\Gamma_+(J)$. If $h \in \mathcal{O}_n$, then we denote the polynomials $\mathrm{p}_{\Gamma(J)}(h)$ and $\mathrm{p}_{\Gamma(J),\Delta}(h)$ by $\mathrm{p}_J(h)$ and $\mathrm{p}_{J,\Delta}(h)$, respectively. If I is an arbitrary ideal of $\mathcal{O}_n$, then we define the ideals:

$$\mathrm{p}_{J,\Delta}(I) = \big\langle\, \mathrm{p}_{J,\Delta}(h) : h \in I,\ \nu_J(h) = \nu_J(I) \big\rangle \tag{8}$$

$$I_{C(\Delta)} = \big\langle h_{C(\Delta)} : h \in I \big\rangle. \tag{9}$$

Given a polynomial $h \in \mathbb{C}[x_1, \dots, x_n]$, we say that h is *J-homogeneous* when $\nu_J(x^k) = \nu_J(h)$, for all $k \in \operatorname{supp}(h)$. If $H = (h_1, \dots, h_p) : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is a polynomial map, then we say that H is *J-homogeneous* when h_i is J-homogeneous, for all $i = 1, \dots, p$.

We denote the n-dimensional volume of a compact set $K \subseteq \mathbb{R}^n$ by $\mathrm{V}_n(K)$. Joining [7, Theorem 3.3] and [5, Proposition 3.5, Corollary 3.8], we have the following result.

Theorem 4.2. *Let* $g = (g_1, \dots, g_n) : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ *be an analytic map germ such that* $g^{-1}(0) = \{0\}$. *Let* $\Gamma_+ \subseteq \mathbb{R}^n_{\geqslant 0}$ *be a convenient Newton polyhedron, let* $d_i = \nu_\Gamma(g_i)$, *for all*

$i = 1, \dots, n$. *Then*

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle g_1, \dots, g_n \rangle} \geqslant \frac{d_1 \cdots d_n}{M_\Gamma^n} n! \, \mathrm{V}_n(\Gamma_-) \tag{10}$$

and the following conditions are equivalent:

(a) *equality holds in* (10)*;*
(b) *for each compact facet* Δ *of* Γ_+*, the ideal of* $\mathcal{R}_\Delta$ *generated by* $\mathrm{p}_\Delta(g_1), \dots, \mathrm{p}_\Delta(g_n)$ *has finite colength in* $\mathcal{R}_\Delta$*;*
(c) *the set germ at* 0 *of common zeros of* $\mathrm{p}_\Delta(g_1), \dots, \mathrm{p}_\Delta(g_n)$ *is contained in* $\{x \in \mathbb{C}^n : x_1 \cdots x_n = 0\}$*, for all compact faces* Δ *of* Γ_+*.*

If I is a non-zero ideal of $\mathcal{O}_n$, then we define the *order of* I, denoted by $\mathrm{ord}(I)$, as $\mathrm{ord}(I) = \max\{r \in \mathbb{Z}_{\geqslant 0} : I \subseteq \mathbf{m}_n^r\}$. We set $\mathrm{ord}(0) = \infty$.

It is a well-known fact that if J denotes a monomial ideal of finite colength of $\mathcal{O}_n$, then $e(J) = n! \, \mathrm{V}_n(\Gamma_-(J))$ (see for instance [22, p. 239]). Let $I_1, \dots, I_n$ be ideals of $\mathcal{O}_n$ such that $\sigma(I_1, \dots, I_n) < \infty$. By Proposition 2.2 we have that $\sigma(I_1, \dots, I_n) = e(g_1, \dots, g_n)$, for some $(g_1, \dots, g_n) \in I_1 \oplus \cdots \oplus I_n$. In particular $\nu_J(I_i) \leqslant \nu_J(g_i)$, for all $i = 1, \dots, n$. Hence relation (10) implies that

$$\sigma(I_1, \dots, I_n) \geqslant \frac{\nu_J(I_1) \cdots \nu_J(I_n)}{M_J^n} e(J). \tag{11}$$

Definition 4.3. Under the conditions of the above paragraph, we say that the n-tuple of ideals $(I_1, \dots, I_n)$ is *J-non-degenerate* when equality holds in (11).

We will denote the term of the right hand side of (11) by $A_J(I_1, \dots, I_n)$. We remark that $A_{\mathbf{m}_n}(I_1, \dots, I_n) = \mathrm{ord}(I_1) \cdots \mathrm{ord}(I_n)$. If $h \in \mathcal{O}_n$, $h \neq 0$, then we will write $A_J(h, I_2, \dots, I_n)$ instead of $A_J(\langle h \rangle, I_2, \dots, I_n)$ and we accordingly extend this notation when any other ideal appearing in $A_J(I_1, \dots, I_n)$ is principal. In particular, if $g_1, \dots, g_n$ are non-zero elements of $\mathcal{O}_n$, then we write $A_J(g_1, \dots, g_n)$ instead of $A_J(\langle g_1 \rangle, \dots, \langle g_n \rangle)$.

Let $g = (g_1, \dots, g_n) : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ be an analytic map germ. We denote by $I(g)$ the ideal of $\mathcal{O}_n$ generated by the components of g. We say that g is *J-non-degenerate* when the n-tuple of ideals $(\langle g_1 \rangle, \dots, \langle g_n \rangle)$ is J-non-degenerate.

Proposition 4.4. *Let* J *be a monomial ideal of* $\mathcal{O}_n$ *of finite colength and let* $I_1, \dots, I_n$ *be ideals of* $\mathcal{O}_n$ *such that* $\sigma(I_1, \dots, I_n) < \infty$*. Then* $(I_1, \dots, I_n)$ *is J-non-degenerate if and only if there exist* $a_1, \dots, a_n, d \in \mathbb{Z}_{\geqslant 1}$ *such that* $\sigma(I_1^{a_1}, \dots, I_n^{a_n}) = e(J^d)$ *and* $\nu_J(I_1^{a_1}) = \cdots = \nu_J(I_n^{a_n}) = dM_J$*.*

Proof. Let $M = M_J$. Let us see the *only if* part. So, let us assume that

$$\sigma(I_1, \dots, I_n) = \frac{d_1 \cdots d_n}{M^n} e(J),$$

where $d_i = \nu_J(I_i)$, for all $i = 1, \dots, n$. Let us define $d = d_1 \cdots d_n$ and $a_i = \frac{d}{d_i}M$, for all $i = 1, \dots, n$. Clearly we have $\nu_J(I_i^{a_i}) = a_i \nu_J(I_i) = \frac{d}{d_i} M d_i = dM$, for all $i = 1, \dots, n$, and

$$\sigma(I_1^{a_1}, \dots, I_n^{a_n}) = a_1 \cdots a_n \sigma(I_1, \dots, I_n) = \frac{d^n}{d} M^n \frac{d_1 \cdots d_n}{M^n} e(J) = d^n e(J) = e(J^d).$$

Let us see the *if* part. Let $a_1, \dots, a_n, d \in \mathbb{Z}_{\geqslant 1}$ such that $\sigma(I_1^{a_1}, \dots, I_n^{a_n}) = e(J^d)$ and $\nu_J(I_1^{a_1}) = \cdots = \nu_J(I_n^{a_n}) = dM_J$. In particular $\nu_J(I_i) = \frac{d}{a_i}M$, for all $i = 1, \dots, n$.

The equality $\sigma(I_1^{a_1}, \dots, I_n^{a_n}) = e(J^d)$ is equivalent to saying that $a_1 \cdots a_n \sigma(I_1, \dots, I_n) = d^n e(J)$. Therefore

$$\sigma(I_1, \dots, I_n) = \frac{d^n}{a_1 \cdots a_n} M^n \frac{1}{M^n} e(J) = \frac{\nu_J(I_1) \cdots \nu_J(I_n)}{M^n} e(J).$$

Hence the result follows. □

As we will see in Theorem 4.10, the J-non-degeneracy of sequences $(I_1, \dots, I_n)$ of monomial ideals admits a combinatorial characterization.

Proposition 4.5. *Let J be a monomial ideal of $\mathcal{O}_n$ of finite colength and let $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ be a complex analytic map such that $g^{-1}(0) = \{0\}$. Let $a_1, \dots, a_n, d \in \mathbb{Z}_{\geqslant 1}$. Then the following conditions are equivalent.*

(a) $e(g_1^{a_1}, \dots, g_n^{a_n}) = e(J^d)$ *and* $\nu_J(g_1^{a_1}) = \cdots = \nu_J(g_n^{a_n}) = dM_J$.
(b) $\overline{\langle g_1^{a_1}, \dots, g_n^{a_n} \rangle} = \overline{J^d}$.

Proof. Let $M = M_J$. Let us suppose that (a) holds. The condition $\nu_J(g_1^{a_1}) = \cdots = \nu_J(g_n^{a_n}) = dM$ implies that $g_i^{a_i} \in \overline{J^d}$, for all $i = 1, \dots, n$. Then relation (b) follows, by the Rees' multiplicity theorem [12, p. 222].

Let us now assume that (b) holds. Then the relation $e(g_1^{a_1}, \dots, g_n^{a_n}) = e(J^d)$ follows automatically, since the Samuel multiplicity is invariant by integral closures.

Since J^d is a monomial ideal of finite colength, there exists at least one compact face Δ of $\Gamma_+(J^d)$ of dimension $n-1$. Let $\mathcal{R}_\Delta$ be the subring of $\mathcal{O}_n$ given by all $h \in \mathcal{O}_n$ whose support is contained in $C(\Delta)$. The equality (b) implies that the ideal $\langle g_1^{a_1}, \dots, g_n^{a_n} \rangle$ is a reduction of $\overline{J^d}$, which is to say that $\langle g_1^{a_1}, \dots, g_n^{a_n} \rangle$ is Newton non-degenerate and $\Gamma_+(g_1^{a_1}, \dots, g_n^{a_n}) = \Gamma_+(J^d)$ (see [7, Theorem 2.11] for the characterization of reductions of monomial ideals). In particular, the ideal of $\mathcal{R}_\Delta$ generated by $(g_1^{a_1})_\Delta, \dots, (g_n^{a_n})_\Delta$ has finite colength in $\mathcal{R}_\Delta$. Hence $(g_i^{a_i})_\Delta \neq 0$, for all $i = 1, \dots, n$, since $\dim \mathcal{R}_\Delta = n$. In particular, this says that $\nu_J(g_i^{a_i}) = dM$, for all $i = 1, \dots, n$, and then (a) follows. □

For the sake of completeness we add the following consequence of Propositions 4.4 and 4.5.

Corollary 4.6. *Let J be a monomial ideal of $\mathcal{O}_n$ of finite colength and let $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ be a complex analytic map such that $g^{-1}(0) = \{0\}$. Then g is J-non-degenerate if and only if there exist $a_1, \dots, a_n, d \in \mathbb{Z}_{\geqslant 1}$ such that $\overline{\langle g_1^{a_1}, \dots, g_n^{a_n} \rangle} = \overline{J^d}$.*

Proof. By Proposition 4.4, the map g is J-non-degenerate if and only if there exist some $a_1, \dots, a_n, d \in \mathbb{Z}_{\geqslant 1}$ such that $\sigma(\langle g_1^{a_1}\rangle, \dots, \langle g_n^{a_n}\rangle) = e(J^d)$ and $\nu_J(g_1^{a_1}) = \cdots = \nu_J(g_n^{a_n}) = dM_J$. Since $\sigma(\langle g_1^{a_1}\rangle, \dots, \langle g_n^{a_n}\rangle) = e(g_1^{a_1}, \dots, g_n^{a_n})$ (see Proposition 2.2), then the result follows from Proposition 4.5. □

Corollary 4.6 motivates the following definition in the context of an arbitrary Noetherian local ring.

Definition 4.7. Let $(R, \mathbf{m})$ be a Noetherian local ring and let J be a proper ideal of $\mathcal{O}_n$. Let $g_1, \dots, g_n$ be a sequence of non-zero elements of R. We say that the n-tuple $(g_1, \dots, g_n)$ is *J-non-degenerate* if and only if there exists some $a_1, \dots, a_n, d \in \mathbb{Z}_{\geqslant 1}$ such that $\overline{\langle g_1^{a_1}, \dots, g_n^{a_n}\rangle} = \overline{J^d}$.

It is interesting to remark that if $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is J-non-degenerate, where J is a monomial ideal of $\mathcal{O}_n$ of finite colength, then the sequence of mixed multiplicities $e_i(I(g), J)$, $i = 1, \dots, n$, is determined by the Newton filtration induced by J.

Proposition 4.8. *Let J be a monomial ideal of $\mathcal{O}_n$ of finite colength and let $g = (g_1, \dots, g_n) : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ be a J-non-degenerate map. Let $d_i = \nu_J(g_i)$, for all $i = 1, \dots, n$. Let us suppose that $d_1 \leqslant \cdots \leqslant d_n$. Then*

$$e_i(I(g), J) = \frac{d_1 \cdots d_i}{M^i} e(J).$$

Proof. Let $I = I(g)$. By the Theorem of existence of joint reductions (see [18, p. 4] or [12, p. 336]), there exist a sufficiently general element $(f_1, \dots, f_i, f_{i+1}, \dots, f_n)$ of $I \oplus \cdots \oplus I \oplus J \oplus \cdots \oplus J$ such that

$$e_i(I, J) = e(I, \dots, I, J, \dots, J) = e(f_1, \dots, f_i, f_{i+1}, \dots, f_n). \tag{12}$$

Since f_j is a generic linear combination of $g_1, \dots, g_n$, for all $j = 1, \dots, i$, we observe that

$$\langle f_1, \dots, f_i\rangle = \Big\langle g_1 + \sum_{\ell=i+1}^{n} \alpha_{1\ell} g_\ell, \dots, g_i + \sum_{\ell=i+1}^{n} \alpha_{i\ell} g_\ell \Big\rangle, \tag{13}$$

for some constants $\alpha_{j\ell} \in \mathbb{C}$, $j = 1, \dots, i$, $\ell = i+1, \dots, n$. Let $h_j = g_j + \sum_{\ell=i+1}^{n} \alpha_{j\ell} g_\ell$, for all $j = 1, \dots, i$.

By appropriately taking the coefficients $\alpha_{j\ell}$, by virtue of Theorem 4.2, we obtain that $(h_1, \dots, h_i, f_{i+1}, \dots, f_n)$ is J-non-degenerate with $\nu_J(h_j) = \nu_J(g_j) = d_j$, for all $j = 1, \dots, i$, and $\nu_J(f_j) = M$, for all $j = i+1, \dots, n$. Hence, by (12) and (13) we obtain that

$$e_i(I, J) = e(h_1, \dots, h_i, f_{i+1}, \dots, f_n) = \frac{d_1 \cdots d_i M^{n-i}}{M^n} e(J) = \frac{d_1 \cdots d_i}{M^i} e(J).$$

□

Remark 4.9. Under the hypothesis of Proposition 4.8, we observe that the condition

$$e_0(I(g), J) = e_1(I(g), J) = \cdots = e_n(I(g), J) \tag{14}$$

is equivalent to saying that $d_1 = \cdots = d_n = M$, which in turn is equivalent to the condition $\overline{\langle g_1, \dots, g_n \rangle} = \overline{J}$, by Proposition 4.5. Then we recovered a particular case of the result of Teissier characterizing the equality $\overline{I} = \overline{J}$ when no inclusion relation between I and J is assumed (see [20, Théorème 4.2, p. 341]).

4.2. **Characterization of J-non-degeneracy of sequences of monomial ideals**

Let $I_1, \dots, I_n$ be monomial ideals of $\mathcal{O}_n$ such that $\sigma(I_1, \dots, I_n) < \infty$. We denote by $\mathcal{S}(I_1, \dots, I_n)$ the family of maps $g = (g_1, \dots, g_n) : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ for which $g^{-1}(0) = \{0\}$, $g_i \in I_i$, for all $i = 1, \dots, n$, and $\sigma(I_1, \dots, I_n) = e(g_1, \dots, g_n)$. The elements of $\mathcal{S}(I_1, \dots, I_n)$ were characterized in [2, Theorem 3.10]. We denote by $\mathcal{S}_0(I_1, \dots, I_n)$ the maps $g \in \mathcal{S}(I_1, \dots, I_n)$ for which $\Gamma_+(g_i) = \Gamma_+(I_i)$, for all $i = 1, \dots, n$.

Theorem 4.10. *Let $I_1, \dots, I_n, J$ be monomial ideals of $\mathcal{O}_n$ such that $\sigma(I_1, \dots, I_n) < \infty$ and J has finite colength. Then the following conditions are equivalent:*

(a) *$(I_1, \dots, I_n)$ is J-non-degenerate.*
(b) *$(g_1, \dots, g_n)$ is J-non-degenerate, for every $(g_1, \dots, g_n) \in \mathcal{S}(I_1, \dots, I_n)$;*
(c) *$(g_1, \dots, g_n)$ is J-non-degenerate, for some $(g_1, \dots, g_n) \in \mathcal{S}_0(I_1, \dots, I_n)$;*
(d) *for any compact face Δ of $\Gamma_+(J)$ we have*

$$\left|\{i : \mathrm{p}_{J,\Delta}(I_i) \neq 0\}\right| \geqslant \dim(\Delta) + 1.$$

Proof. Let $M = M_J$. As remarked before relation (11), given an element $(g_1, \dots, g_n) \in \mathcal{S}(I_1, \dots, I_n)$, we have that

$$\sigma(I_1, \dots, I_n) = e(g_1, \dots, g_n) \geqslant \frac{\nu_J(g_1) \cdots \nu_J(g_n)}{M^n} e(J) \geqslant A_J(I_1, \dots, I_n). \tag{15}$$

Let is prove (a) $\Rightarrow$ (b). So, let us suppose that $\sigma(I_1, \dots, I_n) = A_J(I_1, \dots, I_n)$. If $(g_1, \dots, g_n) \in \mathcal{S}(I_1, \dots, I_n)$, then (15) shows that $e(g_1, \dots, g_n) = A_J(g_1, \dots, g_n)$, which is to say that g is J-non-degenerate.

The implication (b) $\Rightarrow$ (c) is obvious. Let us prove (c) $\Rightarrow$ (d). Let $(g_1, \dots, g_n) \in \mathcal{S}_0(I_1, \dots, I_n)$ such that $(g_1, \dots, g_n)$ is J-non-degenerate and let us fix a compact face Δ of $\Gamma_+(J)$. Let $r = \dim \Delta$. The ideal of $\mathcal{R}_\Delta$ generated by $\mathrm{p}_\Delta(g_1), \dots, \mathrm{p}_\Delta(g_n)$ has finite colength in $\mathcal{R}_\Delta$, by Theorem 4.2. Since $\dim \mathcal{R}_\Delta = r + 1$, at least $r + 1$ elements of $\{\mathrm{p}_\Delta(g_1), \dots, \mathrm{p}_\Delta(g_n)\}$ are not zero. By definition, the condition $(g_1, \dots, g_n) \in \mathcal{S}_0(I_1, \dots, I_n)$ implies that $\Gamma_+(g_i) = \Gamma_+(I_i)$, for all $i = 1, \dots, n$. Thus $\nu_J(g_i) = \nu_J(I_i)$, for all $i = 1, \dots, n$. Then condition (d) follows, by the definition of the ideals $\mathrm{p}_{J,\Delta}(I_1), \dots, \mathrm{p}_{J,\Delta}(I_n)$ (see (8)).

Let us prove (d) $\Rightarrow$ (a). Let G_i be a fixed minimal generating system of I_i formed by monomials, for all $i = 1, \dots, n$, and let $\mathbf{G} = (G_1, \dots, G_n)$. Let us fix a compact face Δ of $\Gamma_+(J)$ and let $r = \dim(\Delta)$. By hypothesis, there exist a set of indices $B_\Delta \subseteq \{1, \dots, n\}$ such that $|B_\Delta| = r + 1$ and $\mathrm{p}_{J,\Delta}(I_j) \neq 0$, for all $j \in B_\Delta$. Let us write $B_\Delta = \{j_{\Delta,1}, \dots, j_{\Delta,r+1}\}$, for some $1 \leqslant j_{\Delta,1} < \cdots < j_{\Delta,r+1} \leqslant n$. Let us denote by G_j^Δ be the set of monomials x^k of G_j such that $k \in C(\Delta)$ and $\phi_J(k) = \nu_J(I_j)$, for all $j = 1, \dots, n$. By the definition of B_Δ, we have that $G_j^\Delta \neq \emptyset$, for all $j \in B_\Delta$. Let $\mathbf{G}^\Delta = (G^\Delta_{j_{\Delta,1}}, \dots, G^\Delta_{j_{\Delta,r+1}})$.

As in the proof of Theorem 3.2, we can apply Theorem 2.3 to the $(r+1)$-tuple of ideals $(\mathrm{p}_{J,\Delta}(I_{j_{\Delta,1}}),\dots,\mathrm{p}_{J,\Delta}(I_{j_{\Delta,r+1}}))$. That is, let us identify the set of $\mathbf{G}^{\Delta}$-maps with $\mathbb{C}^N$, where $N=|G^{\Delta}_{j_{\Delta,1}}|+\cdots+|G^{\Delta}_{j_{\Delta,r+1}}|$, via the vector of coefficients of the $\mathbf{G}^{\Delta}$-maps. Then, there exist a Zariski-open subset $U_\Delta\subseteq\mathbb{C}^N$, such that any $\mathbf{G}^{\Delta}$-map whose vector of coefficients belongs to U_Δ is a joint reduction of $(\mathrm{p}_{J,\Delta}(I_{j_{\Delta,1}}),\dots,\mathrm{p}_{J,\Delta}(I_{j_{\Delta,r+1}}))$.

Since $\Gamma_+(J)$ has a finite number of compact faces, the above discussion shows that there exists a $\mathbf{G}$-map $(g_1,\dots,g_n):(\mathbb{C}^n,0)\to(\mathbb{C}^n,0)$ with the property that, for any compact face Δ of $\Gamma_+(J)$, the map $(\mathrm{p}_{J,\Delta}(g_{j_{\Delta,1}}),\dots,\mathrm{p}_{J,\Delta}(g_{j_{\Delta,\dim(\Delta)+1}}))$ is a joint reduction of $(\mathrm{p}_{J,\Delta}(I_{j_{\Delta,1}}),\dots,\mathrm{p}_{J,\Delta}(I_{j_{\Delta,\dim(\Delta)+1}}))$.

Let us fix a compact face Δ of $\Gamma_+(J)$ and let $r=\dim(\Delta)$. In order to simplify the notation, let us suppose that $B_\Delta=\{1,\dots,r+1\}$. Then, by (6), we obtain that

$$\overline{\mathrm{p}_{J,\Delta}(g_1)\,\mathrm{p}_\Delta(I_2)\cdots\mathrm{p}_\Delta(I_{r+1})+\cdots+\mathrm{p}_{J,\Delta}(g_{r+1})\,\mathrm{p}_\Delta(I_1)\cdots\mathrm{p}_\Delta(I_r)}=\overline{\mathrm{p}_\Delta(I_1)\cdots\mathrm{p}_\Delta(I_{r+1})}.$$

In particular,

$$\overline{\mathrm{p}_\Delta(I_1)\cdots\mathrm{p}_\Delta(I_{r+1})}\subseteq\overline{\big\langle\,\mathrm{p}_{J,\Delta}(g_1),\dots,\mathrm{p}_{J,\Delta}(g_{r+1})\big\rangle}$$

and this implies that

$$\mathbf{V}\left(\mathrm{p}_{J,\Delta}(g_1),\dots,\mathrm{p}_{J,\Delta}(g_{r+1})\right)\subseteq\mathbf{V}(\mathrm{p}_\Delta(I_1)\cdots\mathrm{p}_\Delta(I_{r+1}))\subseteq\mathbf{H},$$

where the last inclusion is a consequence of the fact that $\mathrm{p}_\Delta(I_1)\cdots\mathrm{p}_\Delta(I_{r+1})$ are monomial ideals of $\mathcal{O}_n$.

Therefore, we conclude that, for each compact face Δ of $\Gamma_+(J)$, the set of common zeros of the polynomials $\mathrm{p}_{J,\Delta}(g_1),\dots,\mathrm{p}_{J,\Delta}(g_n)$ is contained in $\mathbf{H}$, which means that $(g_1,\dots,g_n)$ is J-non-degenerate, by virtue of Theorem 4.2. Then, all inequalities of (15) become equalities. Hence (a) holds and we have completed the proof. ☐

Corollary 4.11. *Let $I_1,\dots,I_n$ be monomial ideals of $\mathcal{O}_n$ such that $\sigma(I_1,\dots,I_n)<\infty$. Then $\sigma(I_1,\dots,I_n)\geqslant\mathrm{ord}(I_1)\cdots\mathrm{ord}(I_n)$ and equality holds if and only if for all non-empty $\mathtt{L}\subseteq\{1,\dots,n\}$ we have that*

$$\big|\{i:\mathrm{ord}(I_i^{\mathtt{L}})=\mathrm{ord}(I_i)\}\big|\geqslant|\mathtt{L}|.$$

Proof. Let $\mathbf{m}=\mathbf{m}_n$. Let us apply Theorem 4.10 in the case $J=\mathbf{m}$. The filtrating map $\phi_{\mathbf{m}}:\mathbb{R}^n_{\geqslant0}\to\mathbb{R}_{\geqslant0}$ is given by $\phi_{\mathbf{m}}(k)=|k|$, for all $k\in\mathbb{R}^n_{\geqslant0}$. Hence $\nu_{\mathbf{m}}(I)=\mathrm{ord}(I)$. Moreover, if $\Delta\subseteq\Gamma_+(\mathbf{m})$ and $j\in\{0,1,\dots,n-1\}$, then Δ is a compact face of dimension j of $\Gamma_+(\mathbf{m})$ if and only if there exists some $\mathtt{L}\subseteq\{1,\dots,n\}$ such that $|\mathtt{L}|=j+1$ and $\Delta=\Gamma(\mathbf{m})^{\mathtt{L}}$, where we recall that $\Gamma(\mathbf{m})$ denotes the Newton boundary of $\Gamma_+(\mathbf{m})$.

Let us fix a non-empty subset $\mathtt{L}\subseteq\{1,\dots,n\}$ and let $\Delta=\Gamma(\mathbf{m})^{\mathtt{L}}$. If $i\in\{1,\dots,n\}$, then $\mathrm{p}_{\mathbf{m},\Delta}(I_i)\neq0$ if and only if there exists some $k\in\mathrm{supp}(I_i^{\mathtt{L}})$ such that $|k|=\mathrm{ord}(I_i)$, which is to say that $\mathrm{ord}(I_i)=\mathrm{ord}(I_i^{\mathtt{L}})$. Then the result follows as a consequence of Theorem 4.10. ☐

5. Central maps with respect to pairs of ideals

Given two monomial ideals I and J of $\mathcal{O}_n$ of finite colength, it is obvious that it is always possible to find a J-non-degenerate n-tuple $(K_1,\dots,K_n)$ of monomial ideals contained in I. It suffices to take each K_i equal to some power of J contained in I. However, we will see that, by replacing I by suitable powers I^s, such an n-tuple can be constructed so that the Newton boundary of $\Gamma_+(K_i)$ intersects the Newton boundary of $\Gamma_+(I^s)$, for all $i=1,\dots,n$ (roughly speaking this means that the ideals $K_1,\dots,K_n$ will not be far away from I^s). This fact will lead to a characterization of when the integral closure of some power of I is equal to the integral closure of the ideal generated by the components of a J-non-degenerate map $g=(g_1,\dots,g_n):(\mathbb{C}^n,0)\to(\mathbb{C}^n,0)$.

Definition 5.1. Let I and J be monomial ideals of $\mathcal{O}_n$ of finite colength. For any $i\in\{1,\dots,n\}$, we define the following number:

$$a_{i,J}(I)=\max\big\{\nu_J\big(\Gamma_+(I)\cap C(\Delta)\big):\ \Delta \text{ is a compact face of } \Gamma_+(J) \text{ of dimension } n-i\big\},$$

where we recall that if A is a closed subset of $\mathbb{R}^n_{\geqslant 0}$, then $\nu_J(A)=\min\{\phi_J(k):k\in A\}$. Therefore $a_{i,J}(I)\in\mathbb{Q}_{>0}$, for all $i=1,\dots,n$. It easily follows that $a_{1,J}(I)\leqslant\cdots\leqslant a_{n,J}(I)$. We denote the vector $(a_{1,J}(I),\dots,a_{n,J}(I))$ by $\mathbf{a}_J(I)$.

Under the conditions of the above definition, since $\Gamma_+(I)=\Gamma_+(\overline{I})$, it follows immediately that $a_{i,J}(I)=a_{i,J}(\overline{I})$, for all $i=1,\dots,n$.

Remark 5.2. Let I be a monomial ideal of $\mathcal{O}_n$ and let $\mathbf{m}=\mathbf{m}_n$. We denote $a_{i,\mathbf{m}}(I)$ simply by $a_i(I)$, for all $i=1,\dots,n$. Since the set of compact faces of $\Gamma_+(\mathbf{m})$ is given by $\{\Gamma(\mathbf{m})^{\mathtt{L}}:\mathtt{L}\subseteq\{1,\dots,n\},\mathtt{L}\neq\emptyset\}$ and $\phi_{\mathbf{m}}(k)=|k|$, for all $k\in\mathbb{R}^n_{\geqslant 0}$, then

$$a_i(I)=\max\big\{\mathrm{ord}(I^{\mathtt{L}}):\mathtt{L}\subseteq\{1,\dots,n\},\ |\mathtt{L}|=n-i+1\big\}\tag{16}$$

and we recover the definition of the integers $a_i(I)$ given in [4, p. 197], which in turn was motivated by the expression for the sequence of mixed Łojasiewicz exponents of I given in [6, Corollary 4.2]. We will show some connections between the numbers $a_{i,J}(I)$ and mixed Łojasiewicz exponents (in the sense of [3, 5, 11]) in a subsequent work.

Let I and J be monomial ideals of $\mathcal{O}_n$ of finite colength and let $u\in\mathbb{Z}^n_{\geqslant 0}$, $u\neq 0$. We denote by k^I_u the point of intersection of $\Gamma(I)$ with the half-line $\{\lambda u:\lambda\in\mathbb{R}_{\geqslant 0}\}$. Therefore

$$a_{n,J}(I)=\max\big\{\phi_J(k^I_u):\ u\in\mathbf{v}(\Gamma_+(J))\big\}.$$

We also observe that, under the conditions of Definition 5.1, the maximum that leads to the computation of $a_{i,J}(I)$ is attained at some point of $\mathbf{v}(\Gamma_+(I))\cup\{k^I_u:u\in\mathbf{v}(\Gamma_+(J))\}$.

The point k^I_u has rational coordinates, for all $u\in\mathbb{Z}^n_{\geqslant 0}$, $u\neq 0$. So, we define

$$c_J(I)=\min\big\{c\in\mathbb{Z}_{\geqslant 1}:ck^I_u\in\mathbb{Z}^n_{\geqslant 0},\text{ for all } u\in\mathbf{v}(\Gamma_+(J))\big\}.$$

Theorem 5.3. *Let I and J be monomial ideals of $\mathcal{O}_n$ of finite colength. Let $c = c_J(I)$ and let $M = M_J$. For any $i \in \{1, \dots, n\}$, let us consider the ideal*

$$K_i = \big\langle x^k : k \in \operatorname{supp}\big(\overline{I^{cM}}\big),\ \phi_J(k) = a_{i,J}(\overline{I^{cM}})\big\rangle. \tag{17}$$

Then $(K_1, \dots, K_n)$ is J-non-degenerate.

Proof. Let $\Gamma_+ = \Gamma_+(J)$ and let $\phi = \phi_J$. We recall that $c\Gamma_+(I) = \Gamma_+(I^c)$ and hence $c\Gamma(I) = \Gamma(I^c)$. Thus ck^I_u is the point where $\Gamma(I^c)$ meets the half-line $\{\lambda u : \lambda \in \mathbb{R}_{\geqslant 0}\}$, for all $u \in \mathbf{v}(\Gamma_+(J))$. Let us remark that ck^I_u has integer coordinates, for all $u \in \mathbf{v}(\Gamma_+(J))$.

Let us define the ideal $K = \overline{I^{cM}}$. Hence $\operatorname{supp}(K) = (cM\Gamma_+(I)) \cap \mathbb{Z}^n_{\geqslant 0}$. Let us remark that, since $a_{i,J}(I)$ is attained at some point belonging to $\mathbf{v}(\Gamma_+(I)) \cup \{k^I_u : u \in \mathbf{v}(\Gamma_+(J))\}$, then $a_{i,J}(K) = cMa_{i,J}(I) \in \mathbb{Z}_{\geqslant 1}$, for all $i = 1, \dots, n$. Moreover, by the definition of c we have that, if Δ denotes any compact face of $\Gamma_+(J)$, then the intersection of $C(\Delta)$ with $\Gamma_+(K)$ has integer vertices. Hence $\nu_J(K_{C(\Delta)}) = \nu_J(\Gamma_+(K) \cap C(\Delta))$, where by (9) and the fact that K is integrally closed, the ideal $K_{C(\Delta)}$ is generated by the monomials x^k such that $k \in \Gamma_+(K) \cap C(\Delta)$.

From the definition of K_i we obtain that $\nu_J(K_i) = a_{i,J}(K) = cMa_{i,J}(I)$, for all $i = 1, \dots, n$. In order to prove the result we will check that condition (d) of Theorem 4.10 applied to $(K_1, \dots, K_n)$ holds. Let Δ be a compact face of Γ_+ and let $r = \dim(\Delta)$, $r \in \{0, 1, \dots, n-1\}$. Then the objective is to prove the inequality $|\{i : \mathrm{p}_{J,\Delta}(K_i) \neq 0\}| \geqslant r + 1$.

Let $i \in \{n-r, \dots, n\}$. Then $r \geqslant n-i$, and this implies that there exists some face $\Delta' \subseteq \Delta$ of dimension $n - i$.

From the definition of $a_{i,J}(K)$, we have that $\nu_J(K_{C(\Delta')}) = \nu_J(\Gamma_+(K) \cap C(\Delta')) \leqslant a_{i,J}(K)$. Let us consider a point $k \in \operatorname{supp}(K_{C(\Delta')})$ such that $\phi(k) = \nu_J(K_{C(\Delta')})$. By the definition of K, it follows that $\nu_J(K_{C(\Delta')}) = cM\nu_J(\Gamma_+(I) \cap C(\Delta'))$. Let u be any vertex of Δ' and let us consider a point $k' \in \mathbb{R}^n_{\geqslant 0}$ of the form $k' = k + \gamma u$, for some $\gamma \in \mathbb{R}_{\geqslant 0}$. Since k and u belong to the same cone $C(\Delta')$, then $\phi(k + \gamma u) = \phi(k) + \phi(\gamma u)$. Hence we obtain the following equivalences

$$\begin{aligned}
\nu_J(x^{k'}) = a_{i,J}(K) &\Longleftrightarrow \phi(k + \gamma u) = a_{i,J}(K) \\
&\Longleftrightarrow \phi(k) + \gamma M = cMa_{i,J}(I) \\
&\Longleftrightarrow \gamma = \frac{cMa_{i,J}(I) - \nu_J(K_{C(\Delta')})}{M} = ca_{i,J}(I) - c\nu_J(\Gamma_+(I) \cap C(\Delta')).
\end{aligned} \tag{18}$$

Since $\dim(\Delta') = n - i$, then $\nu_J(\Gamma_+(I) \cap C(\Delta') \leqslant a_{i,J}(I)$, by the definition $a_{i,J}(I)$. Let us assign to γ the value determined in (18). Then γ is a non-negative integer. Therefore $k' \in \mathbb{Z}^n_{\geqslant 0}$. Since $k, u \in C(\Delta')$ and $k' = k + \gamma u$, then $k' \in C(\Delta')$ and thus $x^{k'} \in K_{C(\Delta')} \subseteq K_{C(\Delta)} \subseteq K$. In particular, $k' \in \operatorname{supp}(K)$ and, by the definition of γ, we have that $\nu_J(x^{k'}) = a_{i,J}(K)$. This means that $x^{k'} \in \mathrm{p}_{J,\Delta}(K_i)$. Hence we have proved that $\mathrm{p}_{J,\Delta}(K_i) \neq 0$, for all $i \in \{n-r, \dots, n\}$. Therefore $|\{i : \mathrm{p}_{J,\Delta}(K_i) \neq 0\}| \geqslant r+1 = \dim(\Delta)+1$ and thus $(K_1, \dots, K_n)$ is J-non-degenerate, by Theorem 4.10. □

Corollary 5.4. *Let I and J be monomial ideals of $\mathcal{O}_n$ of finite colength. Let $c = c_J(I)$ and let $M = M_J$. Then there exists a J-homogeneous polynomial map $F = (F_1, \dots, F_n) : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that F is J-non-degenerate, $\nu_J(F_i) = a_{i,J}(I^{cM})$ and $F_i \in \overline{I^{cM}}$, for all $i \in \{1, \dots, n\}$.*

Proof. As in the proof of Theorem 5.3, let $K = \overline{I^{cM}}$. Let $K_1, \dots, K_n$ be the ideals defined in (17). By Theorem 5.3, $(K_1, \dots, K_n)$ is J-non-degenerate. Then, by Theorem 4.10, there exists a polynomial map $F = (F_1, \dots, F_n) : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $F \in \mathcal{S}_0(K_1, \dots, K_n)$, F is J-non-degenerate and $\nu_J(F_i) = \nu_J(K_i) = cMa_{i,J}(I)$, for all $i = 1, \dots, n$. The set $\{x^k : k \in K, \phi_J(k) = a_{i,J}(K)\}$ is a generating system of K_i, for all $i = 1, \dots, n$. Therefore, by Proposition 2.2, we can take each polynomial F_i as a generic $\mathbb{C}$-linear combination of this generating system of K_i. Then we conclude that F_i can be taken as a J-homogeneous polynomial, for all $i = 1, \dots, n$, and the result follows. □

If F is any map satisfying the thesis of Corollary 5.4, then we say that F is a *central map with respect to the pair* (I, J).

Theorem 5.5. *Let $I, J \subseteq \mathcal{O}_n$ be monomial ideals of $\mathcal{O}_n$ of finite colength. Let $M = M_J$. Then*

$$\frac{e(I)}{e(J)} \leqslant \frac{a_{1,J}(I) \cdots a_{n,J}(I)}{M^n} \tag{19}$$

and the following conditions are equivalent:

(a) *equality holds in* (19)*;*
(b) *there exists a polynomial map $F = (F_1, \dots, F_n) : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ and some $s \in \mathbb{Z}_{\geqslant 1}$ such that F is J-non-degenerate and J-homogeneous, $\nu_J(F_i) = sa_{i,J}(I)$, for all $i = 1, \dots, n$, and $\overline{I^s} = \overline{\langle F_1, \dots, F_n \rangle}$;*
(c) *there exists a polynomial map $F = (F_1, \dots, F_n) : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ and some $s \in \mathbb{Z}_{\geqslant 1}$ such that F is J-non-degenerate and J-homogeneous and $\overline{I^s} = \overline{\langle F_1, \dots, F_n \rangle}$;*
(d) *there exists a polynomial map $g = (g_1, \dots, g_n) : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ and some $s \in \mathbb{Z}_{\geqslant 1}$ such that g is J-non-degenerate and $\overline{I^s} = \overline{\langle g_1, \dots, g_n \rangle}$.*

Proof. By Corollary 5.4, there exists a central polynomial map $F : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ with respect to the pair (I, J). Let $c = c_J(I)$ and let $K = \overline{I^{cM}}$. Hence F is J-non-degenerate and J-homogeneous with $\nu_J(F_i) = a_{i,J}(K)$, for all $i = 1, \dots, n$. Moreover, $F_i \in K$, for all $i = 1, \dots, n$. Hence, if we define the ideals $K_1, \dots, K_n$ as in (17), then

$$\sigma(K_1, \dots, K_n) = e(F_1, \dots, F_n) = \frac{a_{1,J}(K) \cdots a_{n,J}(K)}{M^n} e(J) \tag{20}$$

$$= (cM)^n \frac{a_{1,J}(I) \cdots a_{n,J}(I)}{M^n} e(J) \tag{21}$$

$$= c^n a_{1,J}(I) \cdots a_{n,J}(I) e(J). \tag{22}$$

Since $K_i \subseteq K = \overline{I^{cM}}$, for all $i = 1, \dots, n$, then $\sigma(K_1, \dots, K_n) \geqslant e(K) = (cM)^n e(I)$. Joining this with (20) and (22) we obtain that

$$c^n a_{1,J}(I) \cdots a_{n,J}(I) e(J) \geqslant (cM)^n e(I).$$

Thus inequality (19) follows.

Let us prove (a) $\Rightarrow$ (b). If equality holds in (19), then (20)-(22) imply that $e(F_1, \dots, F_n) = e(K)$. Then, by the Rees' Multiplicity Theorem (see [10, p. 147] or [12, p. 222]), we conclude that $\overline{\langle F_1, \dots, F_n \rangle} = \overline{K} = \overline{I^{cM}} = K$ and thus (b) follows by taking $s = cM$.

Let us prove (b) $\Rightarrow$ (a). If $F = (F_1, \dots, F_n) : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ denotes any map satisfying the conditions of (b), for some $s \in \mathbb{Z}_{\geqslant 1}$, then $e(F_1, \dots, F_n) = e(I^s) = s^n e(I)$. Moreover $e(F_1, \dots, F_n) = \nu_J(F_1) \cdots \nu_J(F_n) \frac{e(J)}{M^n} = s^n a_{1,J}(I) \cdots a_{n,J}(I) \frac{e(J)}{M^n}$, by Theorem 4.2. Then equality holds in (19).

The implications (b) $\Rightarrow$ (c) $\Rightarrow$ (d) are obvious. Let us prove (c) $\Rightarrow$ (b). Let $F = (F_1, \dots, F_n) : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ be a complex analytic map such that F is J-non-degenerate, F is J-homogeneous and $\overline{I^s} = \overline{\langle F_1, \dots, F_n \rangle}$, for some $s \in \mathbb{Z}_{\geqslant 1}$. Therefore, we have that $e(I^s) = e(F_1, \dots, F_n) = A_J(F_1, \dots, F_n)$. Let $d_i = \nu_J(F_i)$, for all $i = 1, \dots, n$. By reordering the components of F, we can assume that $d_1 \leqslant \cdots \leqslant d_n$. Since $e(I^s) = s^n e(I)$ and we have already proved relation (19), we have

$$\frac{d_1 \cdots d_n}{s^n M^n} = \frac{e(I)}{e(J)} \leqslant \frac{a_{1,J}(I) \cdots a_{n,J}(I)}{M^n}. \tag{23}$$

Then it suffices to prove that $a_{i,J}(I) \leqslant d_i/s$, for all $i = 1, \dots, n$. Thus (23) would imply that $a_{i,J}(I) = d_i/s$, for all $i = 1, \dots, n$, and hence (b) follows by considering the same map F coming from assuming (c).

Let us fix an index $i \in \{1, \dots, n\}$ and let Δ be a face of $\Gamma_+(J)$ of dimension $n - i$. The condition $\overline{I^s} = \overline{\langle F_1, \dots, F_n \rangle}$ implies that $\Gamma_+(I^s) = \Gamma_+(F_1, \dots, F_n)$. In particular,

$$\begin{aligned} \nu_J\big(\Gamma_+(I^s) \cap C(\Delta)\big) &= \nu_J\big(\Gamma_+(F_1, \dots, F_n) \cap C(\Delta)\big) \\ &= \min\big\{d_i : \operatorname{supp}(\mathrm{p}_\Delta(F_i)) \neq 0\big\}. \end{aligned} \tag{24}$$

Since F is J-non-degenerate and $\dim \mathcal{R}_\Delta = n - i + 1$, there exist at least $n - i + 1$ non-zero elements in the set $\{\mathrm{p}_{J,\Delta}(F_1), \dots, \mathrm{p}_{J,\Delta}(F_n)\}$, by Theorem 4.2. This implies, by (24), that

$$\nu_J\big(\Gamma_+(I^s) \cap C(\Delta)\big) \leqslant d_{n-(n-i+1)+1} = d_i.$$

Therefore $s a_{i,J}(I) = a_{i,J}(I^s) \leqslant d_i$, for all $i = 1, \dots, n$. That is, $a_{i,J}(I) \leqslant d_i/s$, for all $i = 1, \dots, n$.

Finally, let us prove (d) $\Rightarrow$ (c). Let $g = (g_1, \dots, g_n)$ be a J-non-degenerate map and let $s \geqslant 1$ such that $\overline{I^s} = \overline{\langle g_1, \dots, g_n \rangle}$. Let $F_i = \mathrm{p}_J(g_i)$, for all $i = 1, \dots, n$, and let $F = (F_1, \dots, F_n)$. We observe that F is J-non-degenerate and J-homogeneous. In particular, $e(I^s) = e(g_1, \dots, g_n) = e(F_1, \dots, F_n)$. Moreover $\Gamma_+(F_i) \subseteq \Gamma_+(g_i) \subseteq \Gamma_+(I^s)$, for all $i = 1, \dots, n$. Since I is a monomial ideal, the integral closure of I^s is also a monomial ideal. Therefore $\overline{\langle F_1, \dots, F_n \rangle} \subseteq \overline{I^s}$. Then we obtain the equality $\overline{\langle F_1, \dots, F_n \rangle} = \overline{I^s}$, by the Rees' multiplicity theorem, and thus item (c) follows. $\square$

Remark 5.6. As we showed in the proof of Theorem 5.5, if equality holds in (19), then we can take $s = c_J(I)M_J$ in items (b), (c) and (d) of Theorem 5.5.

Let I be a monomial ideal of $\mathcal{O}_n$ of finite colength. Let us observe that $c_{\mathbf{m}_n}(I) = 1$ and $M_{\mathbf{m}_n} = 1$. Hence, by Theorem 5.5 and (16) we obtain that $e(I) \leqslant a_1(I) \cdots a_n(I)$ and equality holds if and only if there exist polynomials $F_1, \dots, F_n \in \mathbb{C}[x_1, \dots, x_n]$ such that F_i is homogeneous of degree $a_i(I)$, for all $i \in \{1, \dots, n\}$, and $\overline{I} = \overline{\langle F_1, \dots, F_n \rangle}$. Hence, by using different techniques, we deduce Theorem 3.5 of [4] as the case $J = \mathbf{m}_n$ of Theorem 5.5.

Let I and J be monomial ideals of $\mathcal{O}_n$ of finite colength. Then we define

$$\mathrm{C}_J(I) = \frac{a_{1,J}(I) \cdots a_{n,J}(I)}{M_J^n}.$$

In (25) we will see that $1 \leqslant \mathrm{C}_I(J)\,\mathrm{C}_J(I)$, where equality does not hold in general, as we show in Example 5.8.

Corollary 5.7. *Let $I, J \subseteq \mathcal{O}_n$ be monomial ideals of $\mathcal{O}_n$ of finite colength. Then*

$$\frac{1}{\mathrm{C}_I(J)} \leqslant \frac{e(I)}{e(J)} \leqslant \mathrm{C}_J(I) \tag{25}$$

and the following conditions are equivalent:

(a) *equality holds in some part of* (25)*;*
(b) *equality holds in both parts of* (25)*;*
(c) $\mathrm{C}_I(J)\,\mathrm{C}_J(I) = 1$.

Proof. Relation (25) is an immediate application of relation (19), in Theorem 5.5.

Let us see the implication (a) $\Rightarrow$ (b). Let us assume that $e(I)/e(J) = \mathrm{C}_J(I)$. By Theorem 5.5, there exists a J-non-degenerate map $g = (g_1, \dots, g_n) : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ and an integer $s \geqslant 1$ such that $\overline{I^s} = \overline{\langle g_1, \dots, g_n \rangle}$. By Proposition 4.6, there exist $r, a_1, \dots, a_n \in \mathbb{Z}_{\geqslant 1}$ such that $\overline{J^r} = \overline{\langle g_1^{a_1}, \dots, g_n^{a_n} \rangle}$.

Since I is a monomial ideal, then $\overline{I^s}$ is also a monomial ideal and hence the equality $\overline{I^s} = \overline{\langle g_1, \dots, g_n \rangle}$ implies that $\langle g_1, \dots, g_n \rangle$ is a Newton non-degenerate ideal whose Newton polyhedron is equal to $s\Gamma_+(I)$, by [7, Corollary 2.6] or [2, Proposition 3.6]. Then $(g_1^{\alpha_1}, \dots, g_n^{\alpha_n})$ is I-non-degenerate, for any $\alpha_1, \dots, \alpha_n \in \mathbb{Z}_{\geqslant 1}$. In particular, $(g_1^{a_1}, \dots, g_n^{a_n})$ is I-non-degenerate. Joining this fact with the equality $\overline{J^r} = \overline{\langle g_1^{a_1}, \dots, g_n^{a_n} \rangle}$, we conclude that $e(J)/e(I) = \mathrm{C}_I(J)$, by Theorem 5.5. Following an analogous argument, we obtain that if $e(J)/e(I) = \mathrm{C}_I(J)$ then $e(I)/e(J) = \mathrm{C}_J(I)$.

The implication (b) $\Rightarrow$ (a) is trivial. The equivalence between (b) and (c) is an immediate consequence of (25). $\square$

If I, J is any pair of monomial ideals in $\mathcal{O}_n$ of finite colength, then we write $I \sim J$ if $\mathrm{C}_I(J)\,\mathrm{C}_J(I) = 1$. Let us observe that $\sim$ is a reflexive and symmetric relation. However $\sim$ is not a transitive relation, as the following example shows.

Example 5.8. Let us consider the ideals of $\mathcal{O}_2$ given by $I = \langle xy, x^5, y^5 \rangle$, $J = \mathbf{m}_2 = \langle x, y \rangle$ and $K = \langle x, y^2 \rangle$. We observe that $\mathbf{a}_I(J) = (1, \frac{5}{2})$, $\mathbf{a}_J(K) = (1, 2)$ and $\mathbf{a}_I(K) = (2, \frac{10}{3})$.

Moreover $M_I = 5$, $M_J = 1$, $M_K = 2$, $e(I) = 10$, $e(J) = 1$ and $e(K) = 2$. Therefore $\mathrm{C}_I(J)\,\mathrm{C}_J(I) = 1$ and $\mathrm{C}_J(K)\,\mathrm{C}_K(J) = 1$. However $\mathrm{C}_I(K) = \frac{4}{15}$ and $\mathrm{C}_K(I) = \frac{15}{2}$ and hence $1 < \mathrm{C}_I(K)\,\mathrm{C}_K(I) = 2$.

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