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Work and Energy in plane-framed structures

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1 Summary of key ideas

Energy methods are commonly used in Structural Analysis. In this document we will explain the concept of Work and Energy and we will obtain the expressions for different types of work and forms of energy in static and linear-elastic plane-framed structures.

2 Introduction

Forces acting on a structure do different types of work, which is developed changing its own energy.

We can define work as the action of changing the system configuration under the application of one or more forces. Hence, work is done by force fields through displacement fields in their direction. In plane-framed structures with linear-elastic behavior we consider three types of work developed by the external forces: external work (work of the external forces), elastic work and complementary work

We can define energy as the capacity of a system for doing work, that is, to change its configuration. It is not an absolute concept, but relative between two system states. The forms of energy related with the structures are the potential energy, the elastic energy or strain energy, the complementary strain energy, the kinetic energy and the dissipation energy by friction.

The type of work and energy depend on the type of forces acting on the structure, the way of action of these forces and the characteristics of the structure.

There is always an energy balance considering all the work and energy types involved which can be expressed in an equation. Energy methods (Principle of Conservation of Energy, Principle of Virtual Work, among others) use this energy balance equation or equations to solve statically or kinematically the structure.

In this document we will explain the differences between the types of work and energy related with the plane-framed structures with linear-elastic behavior, obtaining their value with some examples.

3 Objectives

After reading this document, the student will be able to:

- Understand the difference between, total work of the external forces. elastic work. complementary work and obtain their value for a given plane-framed structure with a linear-elastic behaviour.

- Obtain the strain energy and the complementary strain energy for a given plane-framed structure.

4 Work and Energy in plane-framed structures

4.1 Total Work of the External Forces (W_{ext})

The work **W** done by a force of constant magnitude **F** along a straight-line path **s** is the scalar product of the vector force and the vector displacement (figure 1).

$$W = F \bullet s = |F| \cdot |s| \cdot \cos\alpha \qquad (1)$$

Both variables, force and displacement, have the same relevance, allowing us to take the displacement as the independent variable or the force as the independent variable.

a) Work of the external force: **s** independent variable; **F** dependent variable (projection of F in the direction of s)

$$W_{ext} = F \bullet s = \left[|F| \cdot \cos\alpha \right] \cdot |s| = |F| \cdot |s| \cdot \cos\alpha \qquad (2)$$

b) Complementary Work of the external force: **F** independent variable; **s** dependent variable (projection of s in the direction of F)

$$W^{*}_{ext} = F \bullet s = |F| \cdot \left[|s| \cdot \cos\alpha \right] = |F| \cdot |s| \cdot \cos\alpha = W_{ext} \qquad (3)$$

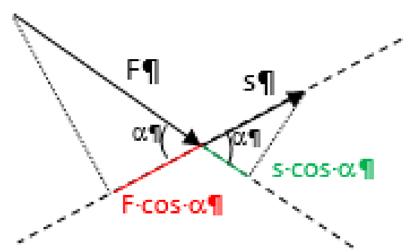

Figure 1. Work done by a force

Let's consider now a plane-framed structure with some external loads acting on it and producing displacements and deformations (figure 2).

Depending on the loading process and the variables which are considered independent we will define different types of work: the total work of the external forces, the elastic work and the complementary work.

Let's explain first the concept of total work of the external forces.

We consider the total work of the external forces (W_{ext}), when the loading process is sudden or instantaneous and, consequently, forces and displacements act with their final value (total)

These are the expressions of the total work of the external forces corresponding to:

- a point force P, being Δ the displacement of the application point in the direction of P:

$$W_{ext} = \pm P \cdot \Delta \tag{4}$$

- a moment M, being θ the rotation of the application point of M:

$$W_{ext} = \pm M \cdot \theta \tag{5}$$

- an axial linear force $q_x(x)$, being u(x) the corresponding displacement function along x local axis:

$$W_{ext} = \int_0^L \pm\, q_x(x) \cdot u(x) dx \tag{6}$$

- a transverse linear force $q_y(x)$, being v(x) the corresponding displacement function along y local axis (deflection function):

$$W_{ext} = \int_0^L \pm\, q_y(x) \cdot v(x) dx \tag{7}$$

When several forces act on a system, the total work of the external forces is given by the sum of the work done by each force.

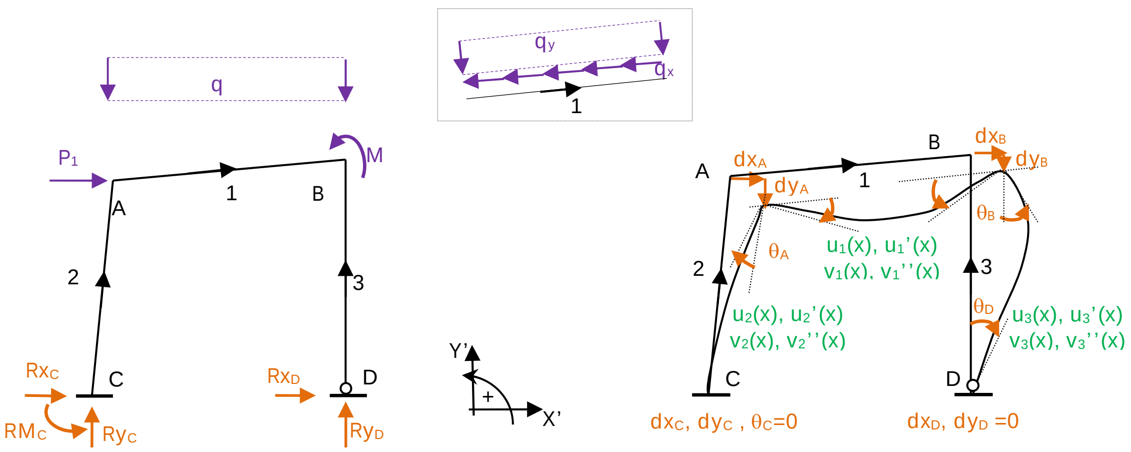

Figure 2. Example1: external loads and displacements.

In the example of figure 2, the total work of the external forces is the sum of the work done by the forces P and q and the moment M:

$$W_{ext} = W_{extP1} + W_{extq} + W_{extM} = P_1 \cdot dx_A + \int_0^{L_1} (-q_x) \cdot u_1(x) \cdot dx + \int_0^{L_1} (-q_y) \cdot v_1(x) \cdot dx + M \cdot \theta_B \tag{8}$$

(qx, qy are negatives according to the member local axes)

The linear force q, acting on member 1 must be transformed to the member local axes in order to obtain the corresponding work.

4.2 Elastic Work (W_e) and Complementary Work (W*) of the external forces

In an elastic structure, external forces will do an elastic work or a complementary work when the loading process is slow or quasi-static loading. Consequently, forces and displacements increase gradually from 0 to their final value. The relation between forces and displacements and vice versa must be known.

The difference between the elastic work and the complementary work is the variable which is taken as independent

1. Elastic Work (W_e)

In the Elastic Work (W_e) the Independent variables are the displacements, being the forces the dependent variables.

Considering a linear-elastic behavior, these are the expressions of the elastic work corresponding to:

- a displacement Δ, being P, the corresponding point force applied in the direction of Δ:

$$W_{ext} = \pm \frac{1}{2} P \cdot \Delta \qquad (9)$$

- a rotation θ, being M, the corresponding applied moment:

$$W_{ext} = \pm \frac{1}{2} M \cdot \theta \qquad (10)$$

- an axial displacement function u(x), being $q_x(x)$, the corresponding linear axial force:

$$W_{ext} = \int_0^L \pm \frac{1}{2} q_x(x) \cdot u(x) dx \qquad (11)$$

- a transverse displacement function v(x), deflection function, being $q_y(x)$, the corresponding linear transverse force

$$W_{ext} = \int_0^L \pm \frac{1}{2} q_y(x) \cdot v(x) dx \qquad (12)$$

When several forces act on a system, the elastic work is given by the sum of the work done by each force.

In the example of figure 2, the elastic work is the sum of the work done by the forces P and q and the moment M:

$$W_e = W_{eP1} + W_{eq} + W_{eM} = \frac{1}{2} \cdot P_1 \cdot dx_A + \int_0^{L_1} \frac{1}{2} \cdot (-q_x) \cdot u_1(x) \cdot dx + \int_0^{L_1} \frac{1}{2} \cdot (-q_y) \cdot v_1(x) \cdot dx + \frac{1}{2} \cdot M \cdot \theta_B \qquad (13)$$

2. Complementary Work (W*)

In the complementary work (W*) the Independent variables are the forces, being the displacements the dependent variables.

Considering a linear-elastic behavior, these are the expressions of the complementary work corresponding to:

- a point force P, being Δ the displacement of the application point in the direction of P:

$$W_{ext} = \pm \frac{1}{2} P \cdot \Delta \quad (14)$$

- a moment M, being θ the rotation of the application point of M:

$$W_{ext} = \pm \frac{1}{2} M \cdot \theta \quad (15)$$

- a linear axial force $q_x(x)$, being u(x) the corresponding axial displacement function (in x local axis):

$$W_{ext} = \int_0^L \pm \frac{1}{2} q_x(x) \cdot u(x) dx \quad (16)$$

- a linear transverse force $q_y(x)$, being v(x) the corresponding displacement function or deflection function (in y local axis):

$$W_{ext} = \int_0^L \pm \frac{1}{2} q_y(x) \cdot v(x) dx \quad (17)$$

When several forces act on a system, the complementary work is given by the sum of the work done by each force.

In the example of figure 2, the complementary work is the sum of the work done by the forces P and q and the moment M:

$$W^* = W^*_{P1} + W^*_q + W^*_M = \frac{1}{2} \cdot P_1 \cdot dx_A + \int_0^{L_1} \frac{1}{2} \cdot (-q_x) \cdot u_1(x) \cdot dx + \int_0^{L_1} \frac{1}{2} \cdot (-q_y) \cdot v_1(x) \cdot dx + \frac{1}{2} \cdot M \cdot \theta_B \quad (18)$$

When the structure has a linear elastic behavior the expressions of the elastic work and the complementary work are the same.

4.3 Strain Energy (U) and Complementary Strain Energy (U*)

While deformation of the structure is taking place, in a slow or quasi-static loading process and considering an elastic behaviour (linear or nonlinear), the internal forces do a work called internal work, which is always negative ($-W_{int}$).

The strain energy or internal potential energy (U) is the capacity of internal forces for doing work, under the deformed state of the structure. Thus, it is the elastic energy that is stored in the structure while undergoing deformation. It is always positive. ($U=-W_{int}$).

The independent variables are strains, ε while stresses (figure 3) are the dependent variables, $\sigma=f(\varepsilon)$.

The general expression of the Strain Energy is:

$$U = \int_V U_0 \, dV = \int_V \left(\int_0^{\varepsilon} \sigma(\varepsilon) \, d\varepsilon \right) dV \qquad (19)$$

being U_0 the strain energy density or strain energy per unit volume

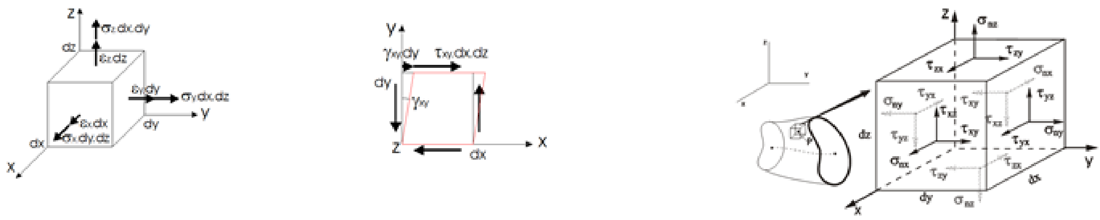

Figure 3. Stresses.

Given a plane-framed structure, the strain energy for the overall structure is the sum of the strain energy of each member. Neglecting the deformation due to shear, the strain energy will be then the sum of the strain energy due to axial loading and to bending moment of all the members.

The strain energy due to axial loading of a constant cross-section prismatic members, considering an elastic and linear behavior is:

AXIAL: $\varepsilon_x = u'(x)$

$$U = \int_V U_0 \, dV = \int_V \left(\int_0^{\varepsilon} \sigma(\varepsilon) \, d\varepsilon \right) dV = \int_V \left(\int_0^{\varepsilon} E \, \varepsilon_x \, d\varepsilon \right) dV = \int_V \frac{1}{2} E \, \varepsilon_x^2 \, dV = \int_0^L \left(\int_A \frac{1}{2} E \, [u'(x)]^2 \, dA \right) dx = \int_0^L \frac{1}{2} E A \, [u'(x)]^2 \, dx \qquad (20)$$

The strain energy due bending of a constant cross-section prismatic members, considering an elastic and linear behavior is:

BENDING: $\varepsilon_x = y \cdot v''(x)$

$$U = \int_V U_0 \, dV = \int_V \left(\int_0^{\varepsilon} \sigma(\varepsilon) \, d\varepsilon \right) dV = \int_V \left(\int_0^{\varepsilon} E \, \varepsilon_x \, d\varepsilon \right) dV = \int_V \frac{1}{2} E \, \varepsilon_x^2 \, dV = \int_0^L \left(\int_A \frac{1}{2} E y^2 \, [v''(x)]^2 \, dA \right) dx = \int_0^L \frac{1}{2} E I \, [v''(x)]^2 \, dx \qquad (21)$$

In the complementary Strain Energy (U^*) the independent variables are stresses, σ, while strains are the dependent variables, $\varepsilon = f(\sigma)$.

The general expression of the Complementary Strain Energy is:

$$U^* = \int_V U_0^* \, dV = \int_V \left(\int_0^{\sigma} \varepsilon(\sigma) \, d\sigma \right) dV \qquad (22)$$

being U_0^* the complementary strain energy density or complementary strain energy per unit volume

Given a plane-framed structure, the complementary strain energy for the overall structure is the sum of the complementary strain energy of each member. Neglecting the deformation due to shear, the complementary strain energy will be then the sum of the complementary strain energy due to axial loading and to bending moment of all the members.

The complementary strain energy due to axial loading of a constant cross-section prismatic members, considering an elastic and linear behavior is:

AXIAL: $\sigma_x = N(x)/A$

$$U^* = \int_V U_0^* \, dV = \int_V \left(\int_0^\sigma \varepsilon(\sigma)\, d\sigma \right) dV = \int_V \left(\int_0^\sigma \frac{\sigma_x}{E}\, d\sigma \right) dV = \int_V \frac{1}{2}\frac{\sigma_x^2}{E}\, dV = \int_0^L \left(\int_A \frac{1}{2}\frac{[N(x)]^2}{EA^2}\, dA \right) dx = \int_0^L \frac{1}{2}\frac{[N(x)]^2}{EA}\, dx \tag{23}$$

The complementary strain energy due bending of a constant cross-section prismatic members, considering an elastic and linear behavior is:

BENDING: $\sigma_x = [M(x)y]/I$

$$U^* = \int_V U_0^* \, dV = \int_V \left(\int_0^\sigma \varepsilon(\sigma)\, d\sigma \right) dV = \int_V \left(\int_0^\sigma \frac{\sigma_x}{E}\, d\sigma \right) dV = \int_V \frac{1}{2}\frac{\sigma_x^2}{E}\, dV = \int_0^L \left(\int_A \frac{1}{2}\frac{[M(x)]^2}{EI^2} y^2\, dA \right) dx = \int_0^L \frac{1}{2}\frac{[M(x)]^2}{EI}\, dx \tag{24}$$

Let's go back to the structure of figure 2. The strain energy and the complementary strain energy are:

Strain energy:

$$U = U_1 + U_2 + U_3 = U_{axial1} + U_{bending1} + U_{axial2} + U_{bending2} + U_{axial3} + U_{bending3} \tag{25}$$

Complementary strain energy:

$$U^* = U^*_1 + U^*_2 + U^*_3 = U^*_{axial1} + U^*_{bending1} + U^*_{axial2} + U^*_{bending2} + U^*_{axial3} + U^*_{bending3} \tag{26}$$

5 Closing

In this document we have explained the different types of work done by the external forces acting on a linear-elastic plane-framed structure (total work of the external forces, W_{ext}, elastic work, W_e, and complementary work, W^*) and the forms of energy (strain energy, U, and complementary strain energy, U^*).

We have formulated their expressions and have obtained them in an example.

As a practical application and self-training, we propose the student to obtain these values (W_{ext}, W_e, W^*, U and U^*) of the structure in figure 4.

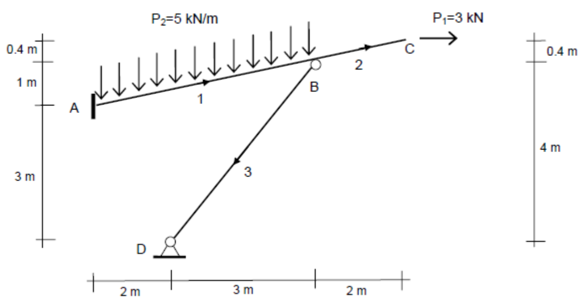

Figure 4. Self-training example

(Results: $U=U_{a1}+U_{a2}+U_{a3}+U_{b1}+U_{b2}$ $U^*=U^*_{a1}+U^*_{a2}+U^*_{a3}+U^*_{b1}+U^*_{b2}$

$W_{ext} = 3 \cdot dx_C + \int_0^{L1}(-5)\cos\alpha_1\, v_1(x)dx + \int_0^{L1}(-5)\,\mathrm{sen}\,\alpha_1\, u_1(x)dx$; $W_e=W^*=(W_{ext}/2)$)

6 Bibliography

6.1 Books:

[1] Abdilla E. "Fundamentos energéticos de la Teoría de Estructuras. Segunda parte-Aplicaciones. Volumen 1". Editorial UPV, ref.: 2003.718, 2003

[2] Gere J.M. "Mechanics of Materials", 6th edition. 2004 Thomson Learning, Inc

6.2 Figures: Author: Luisa Basset

Figure 1. Work done by a force
Figure 2. Example1: external loads and displacements
Figure 3. Stresses.
Figure 4. Self-training example