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Additional Information



Research paper

Computational time reduction for sequential batch solutions in GNSS precise point positioning technique

Angel Martín^a, Ana Belén Anquela Julián^a, Alejandro Dimas-Pages^b, Fernando Cos-Gay^c

^a Department of Cartographic Engineering, Geodesy and Photogrammetry, Universitat Politècnica de València, C/ Camino de Vera s/n, 46022 Valencia, Spain

^b Institut Cartogràfic Valencià, Ciutat Administrativa 9 D' Octubre-Torre 1, Carrer de Castan Tobeas, 77, 46022 Valencia, Spain

^c Department of Architectural Constructions, Universitat Politècnica de València, C/ Camino de Vera s/n, Valencia 46022, Spain

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ABSTRACT

Precise point positioning (PPP) is a well established Global Navigation Satellite System (GNSS) technique that only requires information from the receiver (or rover) to obtain high-precision position coordinates. This is a very interesting and promising technique because eliminates the need for a reference station near the rover receiver or a network of reference stations, thus reducing the cost of a GNSS survey.

From a computational perspective, there are two ways to solve the system of observation equations produced by static PPP either in a single step (so-called batch adjustment) or with a sequential adjustment/filter. The results of each should be the same if they are both well implemented. However, if a sequential solution (that is, not only the final coordinates, but also those observed in previous GNSS epochs), is needed, as for convergence studies, finding a batch solution becomes a very time consuming task owing to the need for matrix inversion that accumulates with each consecutive epoch. This is not a problem for the filter solution, which uses information computed in the previous epoch for the solution of the current epoch. Thus filter implementations need extra considerations of user dynamics and parameter state variations between observation epochs with appropriate stochastic update parameter variances from epoch to epoch. These filtering considerations are not needed in batch adjustment, which makes it attractive.

The main objective of this research is to significantly reduce the computation time required to obtain sequential results using batch adjustment. The new method we implemented in the adjustment process led to a mean reduction in computational time by 45%.

1. Introduction

Since the pioneering work of Zumberge et al. (1997), precise point positioning (PPP) has been studied extensively (Kouba and Héroux, 2001; Gao and Shen, 2001; Colombo et al., 2004; Bisnath et al., 2008; Chen et al., 2009; Geng et al., 2010) and applied to a wide variety of potential applications both in static and kinematic environments, including not just positioning and navigation, but also plate tectonics studies, resource management in remote areas, aerial photogrammetry and sea-level measurements (Chen, 2004; Héroux et al., 2004; Kouba, 2005; Zhang and Andersen, 2006; Bisnath and Gao, 2008).

PPP uses high-precision carrier phase and pseudorange observations of a single receiver in processing undifferenced Global Naviga-

tion Satellite System (GNSS) algorithms, in which the most accurate satellite orbits and clock information as published by the International GNSS service (IGS), (Dow et al., 2009; Ray, 2010), are used.

The undifferenced GNSS observation equations for pseudorange and carrier phase measurements are based on the following simplified observation equations (Leick, 2004; Hofmann-Wellenhof et al., 2008):

$$\begin{aligned} P(L_i) &= \rho + c(dt - dT) + d_{trop} + d_{ion}^{L_i} + \epsilon \\ \phi(L_i) &= \rho + c(dt - dT) + d_{trop} - d_{ion}^{L_i} + \lambda_i N_i + \epsilon \end{aligned} \quad (1)$$

where $P(L_i)$ is the measured pseudorange on L_i frequency (m); $\phi(L_i)$ is the measured carrier phase on L_i (m); ρ is the geometric range between the receiver and the satellite (m); c is the speed of light (m/

Corresponding author.

Email addresses: aemartin@upvnet.upv.es (A. Martín); anquela@cgf.upv.es (A.B. Anquela Julián); aldimpa@gmail.com (A. Dimas-Pages); fcosgay@csa.upv.es (F. Cos-Gayón)

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s); dT is the satellite clock bias with respect to Global Positioning System (GPS) time (s); dt is the receiver clock bias respect to the GPS time (s); d_{trop} is the tropospheric delay (m); d_{ion/L_i} is the ionosphere delay on L_i (m); i_s is the carrier wavelength on L_i (m/cycle); N_i is the integer phase ambiguity on L_i (cycles); and ϵ represents all remaining biases including the measurement noise and multipath effect in pseudorange or carrier phase (m).

Finally:

$$\rho = \sqrt{(X_S - X)^2 + (Y_S - Y)^2 + (Z_S - Z)^2}$$

where (X_S, Y_S, Z_S) represents the satellite position and (X, Y, Z) represents the receiver position, both in the Earth-fixed geocentric reference frame.

Linear combinations from the two frequencies can reduce or even eliminate some of the parameters presented in the previous equations. One of these is the ionosphere-free combination, which eliminates the first-order delay (more than the 99% of the total delay) of the ionosphere (Seepard and Bisnath, 2014). This combination is traditionally used as a functional model for PPP. IGS precise satellite clocks are estimated from the ionosphere-free combination of carrier phase and pseudorange observations using a world wide network solution (Datch et al. 2007), therefore, they contain the ionosphere-free combination of the satellite electronic biases, so no differential code bias (DCB) correction is needed for the ionosphere-free linear combination of P(L1), P(L2) data.

The ionosphere-free pseudorange ($P(L_{IF})$) and carrier phase ($\phi(L_{IF})$) combinations can be formed as follows (Leick, 2004; Hofmann-Wellenhof et al., 2008):

$$P(L_{IF}) = \frac{f_1^2 P(L_1) - f_2^2 P(L_2)}{f_1^2 - f_2^2} = \rho + c(dt - dT) + d_{trop} + \epsilon$$

$$\phi(L_{IF}) = \frac{f_1^2 \phi(L_1) - f_2^2 \phi(L_2)}{f_1^2 - f_2^2} = \rho + c(dt - dT) + d_{trop} + N_{IF} + \epsilon$$

where f_1 and f_2 are the GPS frequencies on L_1 and L_2 (Hz), respectively, and N_{IF} is the ionosphere-free ambiguity term, further described below:

$$N_{IF} = \frac{cf_1 N_1 - cf_2 N_2}{f_1^2 - f_2^2}$$

It should be noted that the ionosphere-free ambiguity term is no longer an integer number. Ambiguity resolution is an important aspect in PPP, a review of the principal methods can be found in Shi and Gao (2015); in the present research it is treated as a floating-point number.

To transform point positioning to PPP, high-accuracy satellite coordinates and clock bias are needed. Therefore, the best approach to solving these system equations, in a post-process stage, is to use the IGS final orbit and clock products (Dow et al., 2009), which are freely available for public use. These products are made available to the user with a lag time of 12–18 days, (IGS 2013). In recent years, the accuracy of IGS orbit and clock products has improved drastically (Dow et al., 2009), the orbit accuracy of the final IGS products is better than 2.5 cm, and the clock accuracies are approximately 75 ps.

The system of observation equations outlined above should be solved with a least-squares adjustment. This can be done in a single step (batch adjustment) or with a sequential adjustment/filter. The disadvantage of batch adjustment is that it may become too computationally time consuming even for powerful computers, especially for a large set of equations.

Additionally, extra computational time is required if a sequential solution (coordinates of the receiver at each epoch) are required in

batch adjustment. This sequential solution is essential for studies of the convergence time required for PPP; thus, this research focuses on reducing the computation time required for a sequential PPP solution using batch adjustment to solve system equations.

As an example, a batch adjustment using 6 h of GNSS observation at 30-s intervals, 8 visible satellites per epoch, and 2 observation equations per epoch per satellite with no cycle slips will generate a final system with 11,520 observation equations and 3 station coordinate parameters (in the static case), 720 receiver clock parameters, 3 troposphere parameters (one every two hours) and the ambiguity terms for each observed satellite. If we consider only the ambiguity for the same 8 satellites for the entire observation period, this system of equations has a total of 732 parameters to solve; if a sequential solution is required, a different system equation need to be solved at each epoch, and so, in this example, 720 growing different system equations should be solved. Taking into account that the computing time for matrix inversion is increasing quadratically with the size of the matrix, a procedure to reduce the time in the matrix inversion is key for reducing the sequential processing time in batch adjustment solution.

This manuscript is written as follows: Section 2 explains the classical PPP software development, Section 3 introduces the new strategy to reduce the computational time in the sequential batch processing, Section 4 explains in detail how the design matrices should be formed in accordance with the new strategy, in Section 5 the experiments to check the proposed method are explained and, finally, a brief concluding section ends the paper.

2. PPP software

A PPP sequential batch approach was implemented using MATLAB software. To achieve the highest possible PPP positioning accuracy, phase wind-up, antenna phase offset and variation at the satellite and receiver, solid Earth tide, pole tide, relativistic corrections, and pseudorange- and carrier-phase biases are modelled in accordance with international standards (Kouba and H roux, 2001); if a cycle slip was detected, a new column for the satellite (a new ambiguity term) was introduced in the design matrix. Finally, the developed software (UPV software) can be considered to be the state-of-the-art in PPP. Fig. 1 shows the receiver coordinates differences between the BERNESE 5.2 software solution (Dach et al., 2015) and the final IGS weekly coordinates for JPLM permanent IGS station in static mode using the data from the first 6 h of October 1, 2013. Stacking equations into blocks of ten minutes are used for the coordinates comparisons; the same differences are plotted for the results of the developed software.

The core of the developed software is the establishment of the observation equations (the matrix definition A), because its solution is quite simple using MATLAB libraries and least-squares theory:

$$Ax = b + v \quad (3)$$

where A is the design matrix of coefficients; x is the unknowns or parameter vector, b is the vector of observations (observed minus computed terms) and v is observation noise and uncertainty (the so-called residuals), which are assumed to be Gaussian normally distributed with zero mean.

The strategy for solving Eq. (3) is to minimize the sum of the squares of the residuals, obtaining the common notation:

$$x = (A^T W A)^{-1} (A^T W b) \quad (4)$$

where W is introduced as the weight matrix, which is diagonal, considering uncorrelated observations.

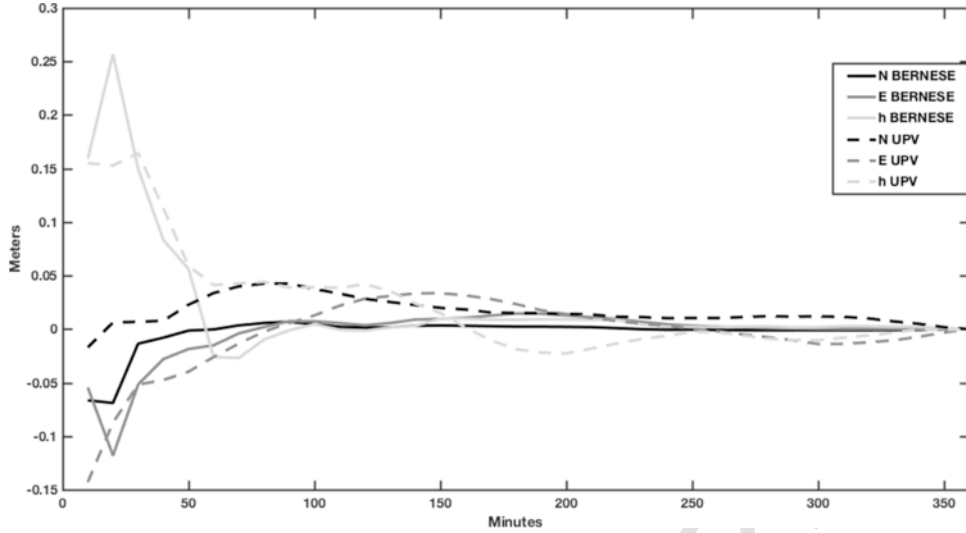


Fig. 1. Precise point positioning coordinate convergence (in comparison with the weekly IGS coordinates) of the JPLM IGS GNSS permanent station using BERNESSE 5.2 and UPV software. North (N), East (E) and Up (h) components are considered.

This system can be simplified to:

$$x = (A^T A)^{-1} (A^T b) \tag{5}$$

if the design matrices are renamed using the notation: $A = \sqrt{W}A$ and $b = \sqrt{W}b$. In this case W is a diagonal matrix, so the square root of the matrix is simply the square root of the elements of the diagonal.

The linearization of the observation equations is the basis for the construction of the design matrix (Hofmann-Wellenhof et al., 2008). An example of PPP design matrix from t_1 to t_n epochs and observed

satellites 1 to S , is:

Coordinate columns

$$A = \begin{pmatrix} \frac{(X_{t_1}^1 - X_0)}{\rho_{0(t_1)}^1} & \frac{(Y_{t_1}^1 - Y_0)}{\rho_{0(t_1)}^1} & \frac{(Z_{t_1}^1 - Z_0)}{\rho_{0(t_1)}^1} \\ \frac{(X_{t_1}^1 - X_0)}{\rho_{0(t_1)}^1} & \frac{(Y_{t_1}^1 - Y_0)}{\rho_{0(t_1)}^1} & \frac{(Z_{t_1}^1 - Z_0)}{\rho_{0(t_1)}^1} \\ \vdots & \vdots & \vdots \\ \frac{(X_{t_1}^S - X_0)}{\rho_{0(t_1)}^S} & \frac{(Y_{t_1}^S - Y_0)}{\rho_{0(t_1)}^S} & \frac{(Z_{t_1}^S - Z_0)}{\rho_{0(t_1)}^S} \\ \frac{(X_{t_1}^S - X_0)}{\rho_{0(t_1)}^S} & \frac{(Y_{t_1}^S - Y_0)}{\rho_{0(t_1)}^S} & \frac{(Z_{t_1}^S - Z_0)}{\rho_{0(t_1)}^S} \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \frac{(X_{t_n}^1 - X_0)}{\rho_{0(t_n)}^1} & \frac{(Y_{t_n}^1 - Y_0)}{\rho_{0(t_n)}^1} & \frac{(Z_{t_n}^1 - Z_0)}{\rho_{0(t_n)}^1} \\ \frac{(X_{t_n}^1 - X_0)}{\rho_{0(t_n)}^1} & \frac{(Y_{t_n}^1 - Y_0)}{\rho_{0(t_n)}^1} & \frac{(Z_{t_n}^1 - Z_0)}{\rho_{0(t_n)}^1} \\ \vdots & \vdots & \vdots \\ \frac{(X_{t_n}^S - X_0)}{\rho_{0(t_n)}^S} & \frac{(Y_{t_n}^S - Y_0)}{\rho_{0(t_n)}^S} & \frac{(Z_{t_n}^S - Z_0)}{\rho_{0(t_n)}^S} \\ \frac{(X_{t_n}^S - X_0)}{\rho_{0(t_n)}^S} & \frac{(Y_{t_n}^S - Y_0)}{\rho_{0(t_n)}^S} & \frac{(Z_{t_n}^S - Z_0)}{\rho_{0(t_n)}^S} \end{pmatrix}$$

There are two equations for every satellite per epoch, the first corresponding to the pseudorange observation and the second to the carrier phase observation (containing the ambiguity term). X_0 , Y_0 and Z_0 are the approximate coordinates of the station and m is the mapping function of the tropospheric wet delay. The troposphere can

be divided into two components, the hydrostatic component (dry) and wet component. The troposphere path delay can be expressed as a function of the dry and wet zenith path delay with an individual mapping function, relating the tropospheric delay to the elevation angle of the satellite. The hydrostatic component consists of mostly dry gases, is rather stable and can be modeled accurately with the Saastamoinen model. The wet component is a result of the water vapour, and it is difficult to model accurately, so in the linearization process of the observation equations, only the wet component is included as a parameter to be determined. The derivative with respect to the wet zenith path delay in the satellite direction is simply the mapping function of the wet part.

Matrix b has the form

$$b = \begin{pmatrix} P_{IF}^1(t_1) - \rho_0^1(t_1) + cdT^1(t_1) - cdt(t_1) - Trop^1(t_1) \\ L_{IF}^1(t_1) - \rho_0^1(t_1) + cdT^1(t_1) - cdt(t_1) - Trop^1(t_1) \\ \vdots \\ P_{IF}^s(t_1) - \rho_0^s(t_1) + cdT^s(t_1) - cdt(t_1) - Trop^s(t_1) \\ L_{IF}^s(t_1) - \rho_0^s(t_1) + cdT^s(t_1) - cdt(t_1) - Trop^s(t_1) \\ \vdots \\ P_{IF}^1(t_n) - \rho_0^1(t_n) + cdT^1(t_n) - cdt(t_n) - Trop^1(t_n) \\ L_{IF}^1(t_n) - \rho_0^1(t_n) + cdT^1(t_n) - cdt(t_n) - Trop^1(t_n) \\ \vdots \\ P_{IF}^s(t_n) - \rho_0^s(t_n) + cdT^s(t_n) - cdt(t_n) - Trop^s(t_n) \\ L_{IF}^s(t_n) - \rho_0^s(t_n) + cdT^s(t_n) - cdt(t_n) - Trop^s(t_n) \end{pmatrix} \quad (7)$$

where P and L are the observed ionosphere-free linear combinations of pseudorange and carrier phase respectively, ρ_0 is the geometric range receiver-satellite distance, cdT is the satellite clock delay (obtained from the IGS clock final file), $Trop$ is the tropospheric dry delay computed based on the Saastamoinen zenith path delay and Niell mapping functions, cdt is the a priori receiver clock correction, which is 0 for the first iteration of the first epoch, and, the computed value for the rest of the iterations; for the rest of the epochs, the value for the first iteration is the final value obtained in the previous epoch, and, the computed value for the rest of the iterations.

Finally, the weight can be obtained as a function of elevation angle e_l corresponding to a satellite 1 S and epoch t :

$$W_L = \frac{1}{e_L^2 \cos(e_l^S)^2} \\ W_P = \frac{1}{e_P^2 \cos(e_l^S)^2} \quad (8)$$

where e_p is the a priori ionosphere-free standard deviation of pseudorange observations and e_L is the ionosphere-free standard deviation of the phase observations. Typically, e_L is 5–20 mm and the ratio $e_p/e_L = 100$ for ionosphere-free undifferenced carrier phase and pseudorange observations is used.

Finally, the parameter vector is:

$$X = \begin{pmatrix} \delta X \\ \delta Y \\ \delta Z \\ ZWD_1 \\ \vdots \\ ZWD_k \\ c \Delta dt(t_1) \\ \vdots \\ c \Delta dt(t_n) \\ N_{IF}^1 \\ \vdots \\ N_{IF}^s \end{pmatrix} \quad (9)$$

where X , Y and Z are the corrections to the a priori receiver position, ZWD are the zenith components for the wet tropospheric delay (k different parameters, one every two hours), $c \Delta dt$ is the correction to the a priori receiver clock delay and N_{IF} is the ionosphere-free ambiguity term for the satellites.

3. Alternative method

As previously mentioned, the main problem for the sequential batch solution of the system equations, is the amount of computer time it consumes, especially in computing $(A^T A)^{-1}$ as the number of equations increases. The basis for the time reduction begins with the known expression for matrix inversion (Strang and Borre, 1997):

$$\begin{bmatrix} T & U \\ U^T & W \end{bmatrix}^{-1} = \begin{bmatrix} L & M \\ N & P \end{bmatrix} \quad (10)$$

This block diagonal condition is true if the inverse matrix is the one needed to solve the least square PPP method (Eq. (5))

The method for obtaining this inverse is:

$$L = T^{-1} + T^{-1} U P U^T T^{-1}$$

$$M = -T^{-1} U P$$

$$N = -P U^T T^{-1} U = (W - U^T T^{-1} U)^{-1} \quad (11)$$

If, in the U block, only parts of the normal equations coming from a new epoch are considered (i.e., clock delay; new troposphere delay, if it is necessary; or a new ambiguity term for any new satellites), the inverse can be obtained quickly from the inverse of the T block, which is known from the previous epoch.

For the first epoch (ep1), matrix T will be the usual system of normal equations $T_{ep1} = (A_{ep1}^T A_{ep1})$.

For the second epoch (ep2), the matrix $(A_{ep2}^T A_{ep2})$ of normal equations, includes only the observations for this new epoch. Some of the unknown parameters are repeated from the previous epoch (i.e., receiver coordinates; troposphere delay, if this is the case; and ambiguity terms for repeated satellites), but the new unknown parameters define the U matrix. The differential elements introduced for the new observations (epoch 2) in the system of normal equations defined in the first epoch, can be considered the sum of two matrices with the

same columns (Shen and Xu, 2003),

$$\begin{aligned}
 (A^T A)^{-1} &= (A_{ep1ep2}^T A_{ep1ep2})^{-1} \\
 &= \begin{pmatrix} A_{ep1}^T A_{ep1} \\ A_{ep2}^T A_{ep2} \end{pmatrix}^{-1} \\
 &= \left(A_{ep1}^T A_{ep1} + A_{ep2}^T A_{ep2} \right)^{-1}
 \end{aligned} \tag{12}$$

where the following expression can be used to solve the inverse:

$$\begin{aligned}
 (D + ECB)^{-1} &= D^{-1} - D^{-1}EKBD^{-1}K \\
 &= (C^{-1} + BD^{-1}E)^{-1}
 \end{aligned} \tag{13}$$

In this expression, C is the normal equation matrix generated for the second epoch, that is $C = (A_{ep2}^T A_{ep2})$, D is the normal equations matrix from the previous epoch, that is $D = (A_{ep1}^T A_{ep1})$, and E and B are defined such that they can be added to D . The only condition for this method is that C and D should be invertible matrices.

Obviously, T block for the second epoch is $T = (D + ECB)$.

With this procedure, to solve the inverse of the normal equations matrix considering both epochs, it is necessary the inverted matrix of the normal equations computed in the previous epoch, which is known, and the inversion of the normal equation matrix generated only considering the observations of the second epoch, which is a small matrix.

For the third (and subsequent) epochs, the new matrix D to add to the new matrix C for the third epoch is the block matrix $\begin{bmatrix} T & U \\ U^T & W \end{bmatrix}$, obtained from the computation of the second epoch.

4. Matrix design

The main problem of this method is defining the design matrix A and matrix B . In this section an example of the construction of the design matrix for the proposed method is described: imagine a matrix A with for 4 epochs, 3 satellites and a change in the troposphere column, it will have the following simplified form (where only phase equations are considered for simplicity):

Station coordinates			Troposphere		
X	Y	Z	Tr1	0	1
X	Y	Z	Tr1	0	1
X	Y	Z	Tr1	0	1
X	Y	Z	Tr1	0	0
X	Y	Z	Tr1	0	0
X	Y	Z	Tr1	0	0
X	Y	Z	0	Tr2	0
X	Y	Z	0	Tr2	0
X	Y	Z	0	Tr2	0
X	Y	Z	0	Tr2	0
X	Y	Z	0	Tr2	0
X	Y	Z	0	Tr2	0
X	Y	Z	0	Tr2	0

However, the order in our method is different: new columns with the new parameters to be adjusted are added at the end of the matrix for every epoch, so, the same example will produce the following matrix:

Station coordinates			Tropo	R.Clock	
X	Y	Z	Tr1	1	1
X	Y	Z	Tr1	1	0

X	Y	Z	Tr1	1	0
X	Y	Z	Tr1	0	1
X	Y	Z	Tr1	0	0
X	Y	Z	Tr1	0	0
X	Y	Z	0	0	1
X	Y	Z	0	0	0
X	Y	Z	0	0	0
X	Y	Z	0	0	1
X	Y	Z	0	0	0
X	Y	Z	0	0	0

where the original block structure disappears and becomes sequential, and a new column is added if a new satellite, epoch or troposphere parameter appears.

The following example is used to explain the new matrix design process: imagine a first epoch with five satellites (satellites 1-5) and a second epoch with only two observed satellites: satellite 1 from the previous epoch and a new input, satellite 6. The proposed design matrix A , considering only carrier-phase observations and one troposphere column, is:

$$A = \begin{pmatrix} \Delta X_1^1 & \Delta Y_1^1 & \Delta Z_1^1 & Tr & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \Delta X_1^2 & \Delta Y_1^2 & \Delta Z_1^2 & Tr & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \Delta X_1^3 & \Delta Y_1^3 & \Delta Z_1^3 & Tr & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \Delta X_1^4 & \Delta Y_1^4 & \Delta Z_1^4 & Tr & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \Delta X_1^5 & \Delta Y_1^5 & \Delta Z_1^5 & Tr & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline \Delta X_2^1 & \Delta Y_2^1 & \Delta Z_2^1 & Tr & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \Delta X_2^6 & \Delta Y_2^6 & \Delta Z_2^6 & Tr & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \tag{1}$$

where the subscript is the epoch and the superscript is the satellite. Weight parameters are not included for simplicity, although the matrix design process is the same, per Eq. (5).

In block form:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \tag{14b}$$

where the first block corresponds to the equations and parameters of the first epoch, and the remaining three blocks to the second epoch (the new receiver clock and ambiguity unknown for the satellite 6, which are reflected in the last two columns of the matrix).

The normal system of equations is:

$$N = A^T A =$$

$$\begin{pmatrix} \Sigma(\Delta X_1^i)^2 + \Sigma(\Delta X_2^i)^2 & \Sigma(\Delta X_1^i \Delta Y_1^i) + \Sigma(\Delta X_2^i \Delta Y_2^i) & \Sigma(\Delta X_1^i \Delta Z_1^i) + \Sigma(\Delta X_2^i \Delta Z_2^i) & \Sigma(\Delta X_1^i Tr) + \Sigma(\Delta X_2^i Tr) \\ \Sigma(\Delta X_1^i \Delta Y_1^i) + \Sigma(\Delta X_2^i \Delta Y_2^i) & \Sigma(\Delta Y_1^i)^2 + \Sigma(\Delta Y_2^i)^2 & \Sigma(\Delta Y_1^i \Delta Z_1^i) + \Sigma(\Delta Y_2^i \Delta Z_2^i) & \Sigma(\Delta Y_1^i Tr) + \Sigma(\Delta Y_2^i Tr) \\ \Sigma(\Delta X_1^i \Delta Z_1^i) + \Sigma(\Delta X_2^i \Delta Z_2^i) & \Sigma(\Delta Y_1^i \Delta Z_1^i) + \Sigma(\Delta Y_2^i \Delta Z_2^i) & \Sigma(\Delta Z_1^i)^2 + \Sigma(\Delta Z_2^i)^2 & \Sigma(\Delta Z_1^i Tr) + \Sigma(\Delta Z_2^i Tr) \\ \Sigma(\Delta X_1^i Tr) + \Sigma(\Delta X_2^i Tr) & \Sigma(\Delta Y_1^i Tr) + \Sigma(\Delta Y_2^i Tr) & \Sigma(\Delta Z_1^i Tr) + \Sigma(\Delta Z_2^i Tr) & \Sigma Tr_1^2 + \Sigma Tr_2^2 \end{pmatrix}$$

Thus, this is the matrix to invert per Eq. (10):

$$N = \begin{bmatrix} T & U \\ U^T & W \end{bmatrix}^{-1} = \begin{bmatrix} L & M \\ O & P \end{bmatrix}$$

Block T is formed by the elements of the first epoch and the elements of the second epoch with the same unknowns, blocks U and W correspond to the two new unknowns columns (receiver clock epoch and ambiguity for satellite 6, respectively). To solve this inversion, only the inverse of the T block is needed, as in Eq. (11).

T can be divided in two different matrices, one including only the elements of the first epoch and the second with elements of the second epoch with the same unknowns (the remaining elements are zero):

$$T = (T_1 + T_2)$$

$$= \begin{pmatrix} \sum (\Delta X_1^S)^2 & \sum (\Delta X_1^S \Delta Y_1^S) & \sum (\Delta X_1^S \Delta Z_1^S) & \sum (\Delta X_1^S Tr) & \sum \Delta X_1^S \\ \sum (\Delta X_1^S \Delta Y_1^S) & \sum (\Delta Y_1^S)^2 & \sum (\Delta Y_1^S \Delta Z_1^S) & \sum (\Delta Y_1^S Tr) & \sum \Delta Y_1^S \\ \sum (\Delta X_1^S \Delta Z_1^S) & \sum (\Delta Y_1^S \Delta Z_1^S) & \sum (\Delta Z_1^S)^2 & \sum (\Delta Z_1^S Tr) & \sum \Delta Z_1^S \\ \sum (\Delta X_1^S Tr) & \sum (\Delta Y_1^S Tr) & \sum (\Delta Z_1^S Tr) & \sum (\Delta Z_1^S Tr) & \sum \Delta Z_1^S \\ \Delta X_1^1 & \Delta Y_1^1 & \Delta Z_1^1 & \Delta X_1^2 & \Delta Y_1^2 & \Delta Z_1^2 & \Delta X_1^3 & \Delta Y_1^3 & \Delta Z_1^3 & \Delta X_1^4 & \Delta Y_1^4 & \Delta Z_1^4 & \Delta X_1^5 & \Delta Y_1^5 & \Delta Z_1^5 \end{pmatrix}$$

$$+ \begin{pmatrix} \sum (\Delta X_2^S)^2 & \sum (\Delta X_2^S \Delta Y_2^S) & \sum (\Delta X_2^S \Delta Z_2^S) & \sum (\Delta X_2^S Tr) & \sum \Delta X_2^S \\ \sum (\Delta X_2^S \Delta Y_2^S) & \sum (\Delta Y_2^S)^2 & \sum (\Delta Y_2^S \Delta Z_2^S) & \sum (\Delta Y_2^S Tr) & \sum \Delta Y_2^S \\ \sum (\Delta X_2^S \Delta Z_2^S) & \sum (\Delta Y_2^S \Delta Z_2^S) & \sum (\Delta Z_2^S)^2 & \sum (\Delta Z_2^S Tr) & \sum \Delta Z_2^S \\ \sum (\Delta X_2^S Tr) & \sum (\Delta Y_2^S Tr) & \sum (\Delta Z_2^S Tr) & \sum (\Delta Z_2^S Tr) & \sum \Delta Z_2^S \\ 0 & 0 & 0 & 0 & 0 \\ \Delta X_2^1 & \Delta Y_2^1 & \Delta Z_2^1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Eq. (13) can be used to solve the inverse, in this case:

$$L = (T_1 + T_2)^{-1}$$

$$L = (T_1 + BCE)^{-1}$$

$$= (T_1)^{-1} - (T_1)^{-1}BCE(T_1)^{-1}K \quad (17)$$

$$= (C^{-1} + E(T_1)^{-1}B)^{-1}$$

where C is the normal equations generated in the second epoch and matrices B and E are defined so that the multiplication BCE generates T_2 . The only limitation is that T_1 and C should be invertible matrices.

Following our example, matrix C is:

$$C = \begin{pmatrix} \sum (\Delta X_2^S)^2 & \sum (\Delta X_2^S \Delta Y_2^S) & \sum (\Delta X_2^S \Delta Z_2^S) & \sum (\Delta X_2^S Tr) & \sum \Delta X_2^S \\ \sum (\Delta X_2^S \Delta Y_2^S) & \sum (\Delta Y_2^S)^2 & \sum (\Delta Y_2^S \Delta Z_2^S) & \sum (\Delta Y_2^S Tr) & \sum \Delta Y_2^S \\ \sum (\Delta X_2^S \Delta Z_2^S) & \sum (\Delta Y_2^S \Delta Z_2^S) & \sum (\Delta Z_2^S)^2 & \sum (\Delta Z_2^S Tr) & \sum \Delta Z_2^S \\ \sum (\Delta X_2^S Tr) & \sum (\Delta Y_2^S Tr) & \sum (\Delta Z_2^S Tr) & \sum (\Delta Z_2^S Tr) & \sum \Delta Z_2^S \\ \Delta X_2^1 & \Delta Y_2^1 & \Delta Z_2^1 & \Delta X_2^6 & \Delta Y_2^6 & \Delta Z_2^6 & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

Thus, the design of matrices E and B are simple in this example:

$$E = \left(\begin{array}{cccccccccc|c} a & a & a & b & c & d & e & e & e & e & e & \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & d \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e \end{array} \right) \quad (19)$$

and matrix B is simply:

$$B = E^T$$

To clarify, the first three columns and rows of matrix E (a columns and rows of Eq. (19)) correspond to the station coordinates, so the position of the ones will be always the same; the fourth column and row represents the troposphere, (b column and row of Eq. (19)). When a new column for the troposphere is considered (every two hours), this column is populated with zeros and a new column appears. The position of the new column is determined by the position of the new troposphere column in design matrix A , Eq. (14), (at the end of the matrix in our design). The next column and row is filled with zero and corresponds to the previous epoch (c column and row of Eq. (19)), corresponding with the receiver clock parameter of the first epoch; for the next epochs, because new parameters are added at the end of the system of equations, a new column of zeros is added at the end of matrix E . The last columns reference the satellites that appear in the observed epoch, if the satellite is repeated from previous observation (and no cycle slip has been determined) a column with a one in the corresponding row is introduced (d column and row of Eq. (19)) and for the rest of the satellites, the same number of columns as satellites are included, where the columns are full of zeros, but only one row is included (e columns and row of Eq. (19)). In this example, is easy to confirm that multiplying the defined matrices B , C and E generates matrix T_2 .

A_{12} and A_{21} from Eq. (14b) can be easily obtained from matrix C if we consider only the position of the new columns from the new epoch and satellite (the marked columns in matrix C , Eq. (20), repeated below for clarity). Finally, A_{22} from Eq. (14b) is obtained using the intersecting elements of the columns and rows marked with a

rectangle in matrix C of Eq. (20).

$$C = \begin{pmatrix} \sum (\Delta X_2^s)^2 & \sum (\Delta X_2^s \Delta Y_2^s) & \sum (\Delta X_2^s \Delta Z_2^s) & \sum (\Delta X_2^s T_r) & \sum \Delta X \\ \sum (\Delta X_2^s \Delta Y_2^s) & \sum (\Delta Y_2^s)^2 & \sum (\Delta Y_2^s \Delta Z_2^s) & \sum (\Delta Y_2^s T_r) & \sum \Delta Y \\ \sum (\Delta X_2^s \Delta Z_2^s) & \sum (\Delta Y_2^s \Delta Z_2^s) & \sum (\Delta Z_2^s)^2 & \sum (\Delta Z_2^s T_r) & \sum \Delta Z \\ \sum (\Delta X_2^s T_r) & \sum (\Delta Y_2^s T_r) & \sum (\Delta Z_2^s T_r) & \sum T_r^2 & \sum T_r \\ \sum \Delta X & \sum \Delta Y & \sum \Delta Z & \sum T_r & \sum 1 \end{pmatrix}$$

$$\begin{pmatrix} \Delta X_2^1 & \Delta Y_2^1 & \Delta Z_2^1 & T_{r_2} & 1 \\ \Delta X_2^6 & \Delta Y_2^6 & \Delta Z_2^6 & T_{r_2} & 1 \end{pmatrix}$$

Therefore, the inverted matrix T_1 is needed to solve the inversion of Eq. (15), but this inversion has been done in the solution of the previous epoch in a sequential procedure. The inverse of C and K (Eq. (17)) must also be computed, but these are low-dimensional matrices (they contains only the satellites observed in the corresponding new epoch), so their inverse is not time consuming to compute.

5. Tests

The method described in the previous section has been implemented in the MATLAB source code of our PPP software.

In order to test the methodology, seven permanent IGS GNSS stations in North America (Fig. 2a) and four in Europe (Fig. 2b) were used, Table 1. Observation occurred during 6 h (from 00:00 to 6:00) on October 3, 2013, with a sample rate of 30 s. The experiments involve comparing the time required to reach a PPP sequential solution with the traditional and proposed methodologies.

The same receiver coordinates, troposphere, receiver clock and ambiguities parameters were obtained for every epoch in each comparison between the traditional and proposed methodologies and all IGS stations. This indicate good numerical performance of the proposed methodology in comparison with the traditional one.

The mean reduction in computational time is 46%, where the minimum is for SPT0 station with 36%, and the maximum for ALGO with 54%. The differences in the reduction time are related with the matrix dimension, a station with more observed satellites in comparison with another station, will produce a greater percentage of reduction in computational time.

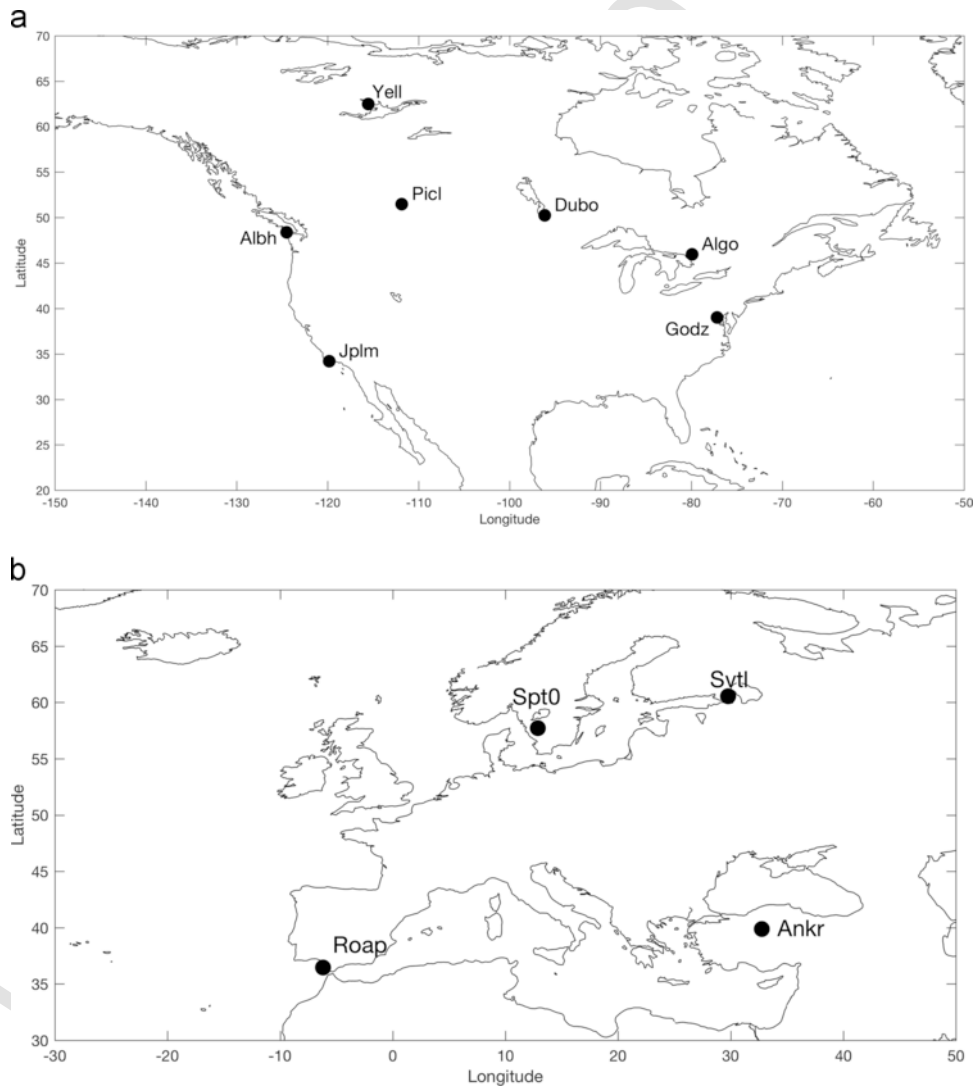


Fig. 2. (a) Location of the 7 North American IGS GNSS stations used in the study. Coastline file from the U.S. National Geophysical Data Center (2010). (b) Location of the 4 European IGS GNSS stations used in the study. Coastline file from the U.S. National Geophysical Data Center (2010).

Table 1
Location, receivers, antennas and coordinates for the IGS permanent sites.

Station	Location	Receiver	Antenna	Latitude (deg)	Longitude (deg)	H (m)
ALBH	Victoria (Canada)	AOA B. ACT	AOAD/M_T SCIS	48.389722	-123.487222	32.0
ALGO	Algonqui-Park (Canada)	TPS NET-G3A	AOAD/M_T NONE	45.958611	-78.071388	202.0
ANKR	Ankara (Turkey)	TPS E_GGD	TPSCR3_GGD NONE	39.887500	32.758333	974.8
DUBO	Lac Du Bonnet (Canada)	TPS NETG3	AOAD/M_T NONE	50.258611	-95.866111	251.0
GODZ	Greenbelt (USA)	JPS EGGDT	AOAD/M_T JPLA	39.021666	-76.826666	14.5
JPLM	Pasadena (USA)	JPS EGGDT	AOAD/M_T NONE	34.204722	-118.173055	423.9
PICL	Pickle Lake (Canada)	AOA SNR-12 ACT	AOAD/M_T NONE	51.479722	-90.161944	315.1
ROAP	San Fernando (Spain)	SEPT POLARX3ETR	LEIAR25.R4 NONE	36.464166	-6.206111	73.7
SPTO	Boras (Sweden)	JAVAD TRE_G3TH	AOAD/M_T OSOD	57.714722	12.891111	219.9
SVTL	Svetloe (Russian Federation)	TPS NETG3	TPSCR.G3 TPSH	60.532777	29.780833	77.1
YELL	Yellowknife (Canada)	JAVAD TRE_G3T	AOAD/M_T NONE	62.480833	-114.480555	181.0

6. Conclusions

A method to reduce the time required to compute a sequential batch solution for static PPP was developed in this study. Our research is focused on the reduction of computation time in calculating the inverse matrix of the system of equations needed in the least-square solution. After theoretical development, the main difficulties occurred in coding the method. The order of the columns, depending on new or previous parameters that arise, is the main factor to be considered in the matrix design process.

Finally, the developed method and software produce the same results as the traditional PPP method, but with a mean reduction in the computation time of 46%. The method was tested using 11 IGS GNSS permanent stations.

This reduction in computational time can be used too if the interest are troposphere parameters for meteorological purposes or receiver clock parameters for time transfer.

Uncited references

Shen and Xu (2013); Rai (2010).

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