Full solution of random autonomous first-order linear systems of difference equations. Application to construct random phase portrait for planar systems

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Abstract

This paper deals with the explicit determination of the first probability density function of the solution stochastic process to random autonomous first-order linear systems of difference equations under very general hypotheses. This finding is applied to extend the classical stability classification of the zero-equilibrium point based on phase portrait to the random scenario. An example illustrates the potentiality of the theoretical results established and their connection with their deterministic counterpart.

Keywords: Random autonomous linear difference systems, first probability density function, random phase portrait

1. Introduction

The aim of this paper is twofold. Firstly, to determine the first probability density function (1-PDF) of the solution stochastic process (SP) of random autonomous first-order linear systems of difference equations of arbitrary size, say m. Secondly, to extend the main deterministic results on stability for planar systems ($m = 2$) to the random scenario. This paper is heavily inspired in our previous contribution [1], but having two main differences. Firstly, we deal here with random difference equations instead of random differential equations. Secondly, we will provide a comprehensive probabilistic stability classification of the zero-equilibrium point of random planar homogeneous systems rather than just to illustrate the classification with one example. The interest of our analysis is expected to reach a large audience for two main reasons. On the one hand, our study provides a generalization to the random framework of deterministic autonomous difference equations, which have mathematical interest by themselves. Indeed, for instance, these class of equations become after discretizating autonomous differential equations. On the other hand, it is well-known that autonomous difference equations are adequate choice when modelling many real phenomena. Therefore, it is expected that the consideration of randomness into autonomous difference equations will provide more realistic models in applications. To justify the latter assertion, it is convenient to point out that in dealing with real models the data (initial conditions and coefficients) are usually fixed after physical experiments, thus containing measurement errors. Therefore, it is more appropriate to consider data as random variables (RVs) rather than deterministic constants. To the best of our knowledge, most of the extant literature has focussed on the study of stochastic difference equations where the uncertainty is considered by means of special classes of SPs like markovian processes or, even more specific as the white noise process (the formal derivative of the Wiener process, also termed brownian motion). This latter case restricts itself the uncertainty to gaussian processes with irregular sample behaviour since the trajectories of the Wiener process are nowhere differentiable. Important results in this respect are included in the recent book [2]. In [3] the mean square exponential stability of impulsive stochastic difference equations are studied. Under this approach, the 1-PDF of the solution SP is rarely computed but its mean and variance. The case where randomness is considered by means of a wide class of probability distributions, including the gaussian but having milder sample behaviour,

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leads to random difference equations. Some recent contributions that consider this class of uncertainty in difference equations are [4, 5, 6], for example. It is worthy to point out that in [5, 6], besides computing the mean and the variance, the 1-PDF of the solution SP is also determined.

As it shall be seen later, our approach is based on the application of the Random Variable Transformation (RVT) technique [7]. This technique has also been successfully applied to determine the solution SP of scalar random differential equations [8, 9], for example. It is important to emphasize that solving a random differential/difference equation means to compute not only its solution SP but also to determine its main statistical functions, such as the mean and the variance. Although many important biparametric distributions like gaussian, beta, gamma, etc., are characterized by these two statistical moments, another as λ-distributions, Pareto distributions, etc., are not. Thus, it is more desirable to determine the 1-PDF, since from it all one-dimensional statistical moments of the solution SP can be straightforwardly computed. This allows us to compute significant statistical information including the symmetry and kurtosis and confidence intervals as well.

The outline of the paper is as follows. In Section 2 the 1-PDF of general random autonomous first-order linear homogeneous systems of difference equations is determined explicitly. In Section 3, a comprehensive probabilistic classification of the equilibrium point for planar systems is given. In Section 4 an example illustrating the main theoretical results established in Section 3 is exhibited. Conclusions are drawn in Section 5.

2. Computing the 1-PDF

The goal of this section is to compute an explicit formula for the 1-PDF of random autonomous first-order homogeneous linear systems of difference equations

$$\mathbf{X}_{n+1} = \mathbf{A}\mathbf{X}_n, \quad n \geq 0, \qquad \mathbf{A} = [A_{ij}], \ 1 \leq i, j \leq m, \tag{1}$$

where A_{ij}, $1 \leq i, j \leq m$, and X_{i0}, $1 \leq i \leq m$, that define the starting seed $\mathbf{X}_0 = [X_{10}, \ldots, X_{m0}]^\top$, are $h = m + m^2$ absolutely continuous RVs defined in a complete probabilistic space $(\Omega, \mathfrak{F}, \mathbb{P})$. It is assumed that these RVs have the following joint PDF

$$f_0(\mathbf{x}_0, \mathbf{a}) = f_0(x_{10}, \ldots, x_{m0}, a_{11}, \ldots, a_{m1}, \ldots, a_{1m}, \ldots, a_{mm}).$$

As usually, in this latter expression we have written the deterministic quantities, like a, in lower case to avoid any confusions with RV, which will be denoted by capital letters, A. For random vectors/matrices we will use bold letters, $\mathbf{A}$. Hereinafter, we will assume that the random matrix $\mathbf{A}$ is invertible in the probabilistic sense, i.e., $\mathbb{P}\left[\{\omega \in \Omega : \det(A(\omega)) \neq 0\}\right] = 1$. Notice that this hypothesis is not restrictive since A_{ij} are assumed to be absolutely continuous RVs.

Let us observe that the solution of (1) is given by $\mathbf{X}_n = \mathbf{A}^n\mathbf{X}_0$. For the sake of consistency with the notation of our previous work, we will apply the RVT technique as stated in Th.1 of [1]. With this aim, let us fix $n > 0$ and denote by $\mathbf{e}_i = [0, \ldots, 0, 1, 0, \ldots, 0]^\top$ the i-th canonical vector of size m, $1 \leq i \leq m$. Additionally, let us also define the transformation $\mathbf{r} : \mathbb{R}^h \to \mathbb{R}^h$, and the inverse mapping of $\mathbf{r}$, $\mathbf{s} = \mathbf{r}^{-1}$, whose components are given by

$$\left\{\begin{array}{lllllll} \mathbf{y}_1 & = & \mathbf{r}_1(\mathbf{x}_0, \mathbf{a}) & = & \mathbf{a}^n\mathbf{x}_0, & & \\ \mathbf{y}_2 & = & \mathbf{r}_2(\mathbf{x}_0, \mathbf{a}) & = & \mathbf{a}\mathbf{e}_1 & = & \mathbf{a}_1, \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{y}_m & = & \mathbf{r}_m(\mathbf{x}_0, \mathbf{a}) & = & \mathbf{a}\mathbf{e}_{m-1} & = & \mathbf{a}_{m-1}, \\ \mathbf{y}_{m+1} & = & \mathbf{r}_{m+1}(\mathbf{x}_0, \mathbf{a}) & = & \mathbf{a}\mathbf{e}_m & = & \mathbf{a}_m. \end{array}\right. \Rightarrow \left\{\begin{array}{lllll} \mathbf{x}_0 & = & \mathbf{s}_1(\mathbf{y}_1, \ldots, \mathbf{y}_{m+1}) & = & [\mathbf{y}_2, \ldots, \mathbf{y}_{m+1}]^{-n}\,\mathbf{y}_1, \\ \mathbf{a}_1 & = & \mathbf{s}_2(\mathbf{y}_1, \ldots, \mathbf{y}_{m+1}) & = & \mathbf{y}_2, \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{a}_{m-1} & = & \mathbf{s}_m(\mathbf{y}_1, \ldots, \mathbf{y}_{m+1}) & = & \mathbf{y}_m, \\ \mathbf{a}_m & = & \mathbf{s}_{m+1}(\mathbf{y}_1, \ldots, \mathbf{y}_{m+1}) & = & \mathbf{y}_{m+1}. \end{array}\right. \tag{2}$$

Notice that in (2) $\mathbf{a}_i$ represents the i-th column of the matrix $\mathbf{a} = \mathbf{A}(\omega)$, $\omega \in \Omega$. Now, we compute the Jacobian, which is defined by the following determinant

$$J_h = \det\begin{bmatrix} [\mathbf{y}_2, \ldots, \mathbf{y}_{m+1}]^{-n} & \mathbf{0}_m & \cdots & \cdots & \mathbf{0}_m \\ \mathbf{0}_m & \mathbf{I}_m & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \mathbf{0}_m \\ \mathbf{0}_m & \cdots & \cdots & \mathbf{0}_m & \mathbf{I}_m \end{bmatrix}_{h\times h} = \det\left([\mathbf{y}_2, \ldots, \mathbf{y}_{m+1}]^{-n}\right) = \left(\det\left([\mathbf{y}_2, \ldots, \mathbf{y}_{m+1}]\right)\right)^{-n} = (\det(\mathbf{a}))^{-n} \neq 0,$$

where $\mathbf{0}_m$ and $\mathbf{I}_m$ are the null and the identity matrix of size m, respectively. Therefore, applying Th.1 of [1], we obtain the joint PDF of the random vector $[\mathbf{Y}_1, \ldots, \mathbf{Y}_{m+1}]$

$$f_{\mathbf{Y}_1,\ldots,\mathbf{Y}_{m+1}}(\mathbf{y}_1, \ldots, \mathbf{y}_{m+1}) = f_0\left([\mathbf{y}_2, \ldots, \mathbf{y}_{m+1}]^{-n}\, \mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_{m+1}\right)\left|\det\left([\mathbf{y}_2, \ldots, \mathbf{y}_{m+1}]\right)\right|^{-n}. \tag{3}$$

As the solution of the IVP (1) is given by the first component of the random vector $[\mathbf{Y}_1, \ldots, \mathbf{Y}_{m+1}]$, in order to compute the PDF of the solution, $\mathbf{X}_n$, first we must marginalize expression (3) with respect to $\mathbf{Y}_2, \ldots, \mathbf{Y}_{m+1}$, and secondly to express the result in terms of the data. This yields

$$f_1(\mathbf{x}, n) = \int_{\mathbb{R}^{m^2}} f_0\left(\mathbf{a}^{-n}\mathbf{x}, \mathbf{a}_1, \ldots, \mathbf{a}_m\right)|\det(\mathbf{a})|^{-n}\, \mathrm{d}a_{11}\cdots \mathrm{d}a_{m1}\cdots \mathrm{d}a_{1m}\cdots \mathrm{d}a_{mm}. \tag{4}$$

3. Random planar autonomous first-order linear systems: A probabilistic classification of the equilibrium point

This section is devoted to classify, from a probabilistic standpoint, the equilibrium point of the random system of difference equations (1), when $m = 2$. Observe that our analysis is restricted to the homogeneous case where the only equilibrium point is the random null vector, $\mathbf{X}_\mathrm{e} = \mathbf{0}$. Nevertheless, the non-homogeneous case, $\mathbf{X}_{n+1} = \mathbf{A}\mathbf{X}_n + \mathbf{B}$, can be reduced to the homogeneous one, just taking into account the equation (1) can be centered about $\mathbf{X}_\mathrm{e} = (\mathbf{I}_m - \mathbf{A})^{-1}\mathbf{B}$. Indeed, this can be done because the probability that the RV $\lambda = 1$ be an eigenvalue of $\mathbf{A}$ is zero. The case $m = 2$ in the random matrix difference equation (1) corresponds to random planar systems. In the deterministic context, it is well-known that important results related to stability of the zero-equilibrium point have been established for planar systems. This section is addressed to extend the classical stability classification of the zero-equilibrium $\mathbf{X}_\mathrm{e} = [0, 0]^\top$ to the random scenario. As it shall see later, our approach leads to a nice generalization, in a probabilistic sense, that retains the well-known deterministic results when probabilistic events associated to that classification happens with probability one. Naturally, as it also occurs in the deterministic case, the classification depends on the characteristic roots, λ_1 and λ_2, associated to the random matrix equation (1) with $m = 2$. These roots can be expressed in terms of the trace and the determinant of the random matrix $\mathbf{A}$:

$$\lambda^2 - \mathrm{tr}(\mathbf{A})\lambda + \det(\mathbf{A}) = 0, \qquad \lambda_i = \frac{\mathrm{tr}(\mathbf{A}) \pm \sqrt{(\mathrm{tr}(\mathbf{A}))^2 - 4\det(\mathbf{A})}}{2}, \quad i = 1, 2.$$

If we denote by $V_1(\omega) = \mathrm{tr}(\mathbf{A}(\omega))$ and $V_2(\omega) = \det(\mathbf{A}(\omega))$, for each $\omega \in \Omega$, the classification can be represented by means of Figure 1. Then taking advantage of the previous determination of the 1-PDF of the solution SP of (1), given by (4), we can compute the probability that the zero-equilibrium or critical point belongs to one of the following states: a stable (node or sink/spiral), an unstable (node or source/spiral) or a saddle point. Observe the case that the zero-equilibrium point be a center, an improper stable node or an improper unstable node have been neglected because they are associated to the events A_1, A_2 and A_3 defined by

$$\begin{aligned}
A_1 &= \{\omega \in \Omega : \det(\mathbf{A}(\omega)) = 1, |\mathrm{tr}(\mathbf{A}(\omega))| \leq 2\},\\
A_2 &= \{\omega \in \Omega : (\mathrm{tr}(\mathbf{A}(\omega)))^2 = 4\det(\mathbf{A}(\omega)), |\mathrm{tr}(\mathbf{A}(\omega))| < 2\},\\
A_3 &= \{\omega \in \Omega : (\mathrm{tr}(\mathbf{A}(\omega)))^2 = 4\det(\mathbf{A}(\omega)), |\mathrm{tr}(\mathbf{A}(\omega))| > 2\},
\end{aligned}$$

respectively, which can happens with probability zero since A_{ij}, $1 \leq i, j \leq 2$, are assumed to be absolutely continuous RVs. Below, these probabilities are completely specified. It is important to point out that, in contrasts to what happens in the deterministic scenario where the equilibrium point can only belong to one state of the previous list, in the random context the situation is different since the zero-equilibrium point can have different states but each one with different probabilities.

- Stable node: $\mathrm{P}_{\mathrm{sn}} = \displaystyle\int_{-2}^{0}\int_{-1-v_1}^{\frac{v_1^2}{4}} f_{V_1,V_2}(v_1, v_2)\,\mathrm{d}v_2\,\mathrm{d}v_1 + \int_{0}^{2}\int_{-1+v_1}^{\frac{v_1^2}{4}} f_{V_1,V_2}(v_1, v_2)\,\mathrm{d}v_2\,\mathrm{d}v_1.$
- Stable spiral: $\mathrm{P}_{\mathrm{ss}} = \displaystyle\int_{-2}^{2}\int_{\frac{v_1^2}{4}}^{1} f_{V_1,V_2}(v_1, v_2)\,\mathrm{d}v_2\,\mathrm{d}v_1.$

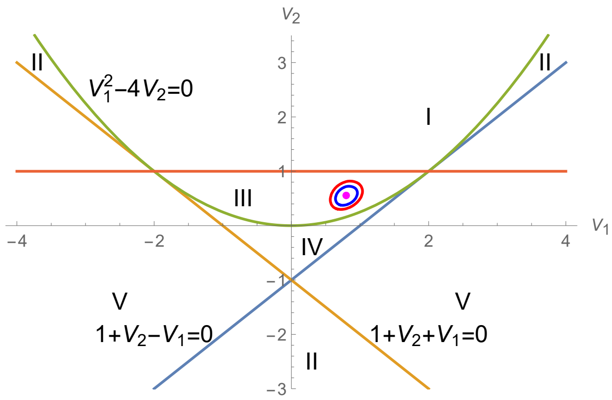

Figure 1: Graphical representation of the stability classification of the zero-equilibrium point to the random matrix difference equation (1) with $m = 2$ (random planar systems). Each relevant region has a label corresponding to: I (unstable spiral); II (unstable node or source); III (stable spiral); IV (stable node or sink) and V (saddle point). The magenta point has been obtained from the matrix $\bar{\mathbf{a}}$ of the averaged system (12) associated to the random system (1) with $m = 2$ and the random inputs given by (7)–(8). The blue-red confidence regions have been determined from (6).

- Unstable node: $\mathrm{P}_{\mathrm{un}} = \displaystyle\int_{-\infty}^{-2}\int_{-1-v_1}^{\frac{v_1^2}{4}} f_{V_1,V_2}(v_1,v_2)\,\mathrm{d}v_2\,\mathrm{d}v_1 + \int_{2}^{\infty}\int_{-1+v_1}^{\frac{v_1^2}{4}} f_{V_1,V_2}(v_1,v_2)\,\mathrm{d}v_2\,\mathrm{d}v_1 + \int_{-\infty}^{-1}\int_{1+v_2}^{-1-v_2} f_{V_1,V_2}(v_1,v_2)\,\mathrm{d}v_1\,\mathrm{d}v_2.$
- Unstable spiral: $\mathrm{P}_{\mathrm{us}} = \displaystyle\int_{1}^{\infty}\int_{-2\sqrt{v_2}}^{2\sqrt{v_2}} f_{V_1,V_2}(v_1,v_2)\,\mathrm{d}v_1\,\mathrm{d}v_2.$
- Saddle point: $\mathrm{P}_{\mathrm{s}} = \displaystyle\int_{-\infty}^{0}\int_{-1+v_1}^{-1-v_1} f_{V_1,V_2}(v_1,v_2)\,\mathrm{d}v_2\,\mathrm{d}v_1 + \int_{0}^{\infty}\int_{-1-v_1}^{-1+v_1} f_{V_1,V_2}(v_1,v_2)\,\mathrm{d}v_2\,\mathrm{d}v_1.$

All these probabilities depend on the PDF, $f_{V_1,V_2}(v_1,v_2)$, of the random vector $(V_1,V_2) = \mathrm{tr}((\mathbf{A}), \det(\mathbf{A}))$. Applying Th1. of [1], that is to say, the RVT method, we can determine $f_{V_1,V_2}(v_1,v_2)$. To this end, let us define the following mapping, $\mathbf{r} : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$, whose inverse is the mapping $\mathbf{s} : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ given by

$$\left\{\begin{array}{lllll} v_1 & = & r_1(a_{11},a_{21},a_{12},a_{22}) & = & a_{11}+a_{22}, \\ v_2 & = & r_2(a_{11},a_{21},a_{12},a_{22}) & = & a_{11}a_{22}-a_{12}a_{21}, \\ v_3 & = & r_3(a_{11},a_{21},a_{12},a_{22}) & = & a_{12}, \\ v_4 & = & r_4(a_{11},a_{21},a_{12},a_{22}) & = & a_{22}, \end{array}\right. \Rightarrow \left\{\begin{array}{lllll} a_{11} & = & s_1(v_1,v_2,v_3,v_4) & = & v_1-v_4, \\ a_{21} & = & s_2(v_1,v_2,v_3,v_4) & = & -\dfrac{v_2-(v_1-v_4)v_4}{v_3}, \\ a_{12} & = & s_3(v_1,v_2,v_3,v_4) & = & v_3, \\ a_{22} & = & s_4(v_1,v_2,v_3,v_4) & = & v_4. \end{array}\right.$$

It is easy to check that the Jacobian of $\mathbf{s}$ is $J_4 = -1/v_3 \neq 0$. Therefore, the joint PDF of random vector $(V_1,V_2) = (\mathrm{tr}(\mathbf{A}), \det(\mathbf{A}))$ is given by

$$f_{V_1,V_2}(v_1,v_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{A_{11},A_{21},A_{12},A_{22}}\left(v_1-a_{22}, -\frac{v_2-(v_1-a_{22})a_{22}}{a_{12}}, a_{12}, a_{22}\right)\frac{1}{|a_{12}|}\,\mathrm{d}a_{12}\,\mathrm{d}a_{22}, \tag{5}$$

being

$$f_{A_{11},A_{21},A_{12},A_{22}}(a_{11},a_{21},a_{12},a_{22}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_0\left(x_{10},x_{20},a_{11},a_{21},a_{12},a_{22}\right)\mathrm{d}x_{10}\mathrm{d}x_{20}. \tag{6}$$

4. An illustrative example

The aim of this section is to illustrate the theoretical results previously established by means an example. This includes the interpretation of the phase portrait in the context of random planar systems of the form (1) with $m = 2$

and the previous probabilistic stability classification of the zero-equilibrium point. We will consider that the random vector input

$$\mathbf{Z} = (X_{10}, X_{20}, A_{11}, A_{21}, A_{12}, A_{22}), \quad \mathbf{Z} \sim \mathrm{N}(\vec{\mu}, \Sigma) \tag{7}$$

where the mean, $\vec{\mu}$, and the variance-covariance matrix, Σ, are,

$$\vec{\mu} = \begin{bmatrix} 2 \\ 2 \\ -0.125 \\ -0.962 \\ 0.692 \\ 0.925 \end{bmatrix}, \quad \Sigma = \frac{1}{4000} \begin{bmatrix} 55 & 5 & 20 & 1 & 1 & 4 \\ 5 & 20 & 5 & 10 & 2 & 4 \\ 20 & 5 & 10 & 7 & 1 & 4 \\ 1 & 10 & 7 & 30 & 2 & 4 \\ 1 & 2 & 1 & 2 & 25 & 5 \\ 4 & 4 & 4 & 4 & 5 & 10 \end{bmatrix}. \tag{8}$$

In Fig. 2 it is shown the phase portrait for different fixed times instants, $n \in \{0, 1, 2, 3\}$, together with the PDF of the solution SP in two of these time instants ($n \in \{2, 3\}$). In the planar phase portrait, the mean $\mathbb{E}[X_{in}]$, $i = 1, 2$, and the confidence regions $\mathcal{D}_{\mathbf{X}_n}(1-\alpha) \subset \mathbb{R}^2$ at different fixed levels of confidence, $\alpha \in \{0.50, 0.90\} \in (0, 1)$, for $n = \{2, 3\}$, have been plotted. These statistical quantities have been computed by means of the following expressions

$$\mathbb{E}[X_{1n}] = \int_{\mathbb{R}^2} x_1 f_1(x_1, x_2; n)\, \mathrm{d}x_1 \mathrm{d}x_2, \quad \mathbb{E}[X_{2n}] = \int_{\mathbb{R}^2} x_2 f_1(x_1, x_2; n)\, \mathrm{d}x_2 \mathrm{d}x_1, \tag{9}$$

$$1 - \alpha = \iint_{\mathcal{D}_{\mathbf{X}_n}(1-\alpha)} f_1(x_1, x_2; n)\, \mathrm{d}x_1 \mathrm{d}x_2, \quad \mathcal{D}_{\mathbf{X}_n}(1-\alpha) = \{(x_1, x_2) : \, f_1(x_1, x_2; n) = k\}, \tag{10}$$

where, in agreement with (4),

$$f_1(x_1, x_2; n) = \int_{\mathbb{R}^4} f_0\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-n} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, a_{11}, a_{21}, a_{12}, a_{22} \right) |a_{11}a_{22} - a_{12}a_{21}|^{-n}\, \mathrm{d}a_{11}\mathrm{d}a_{12}\mathrm{d}_{21}\mathrm{d}_{22}. \tag{11}$$

As it has been indicated previously, the zero-equilibrium point can behaves different classification in the random context with different probabilities. Based upon our analysis, these probabilities can be quantified. In our example, the probability that the null random point $[0, 0]^\top$ be either a saddle point, a stable node/sink or a unstable node is 0. While the probability of being a stable spiral is 0.999995 and the probability of being a unstable spiral is 0.000005. The fact that the equilibrium point is more likely a stable spiral than an unstable spiral is heavily connected with the deterministic theory. Indeed, let us consider the averaged problem associated to (1)

$$\overline{\mathbf{x}}_{n+1} = \overline{\mathbf{a}}\, \overline{\mathbf{x}}_n, \quad n \geq 0, \quad \overline{\mathbf{a}} = \begin{bmatrix} -0.125 & 0.692 \\ -0.962 & 0.925 \end{bmatrix}, \tag{12}$$

and starting value $\overline{\mathbf{x}}_0 = [2, 2]^\top$. Observe that $\overline{\mathbf{a}}$ and $\overline{\mathbf{x}}_0$ correspond to the expectation or average defined in (8). For the deterministic problem (12) one gets

$$\mathrm{tr}(\overline{\mathbf{a}})^2 - 4\det(\overline{\mathbf{a}}) = 0.8^2 - 4\, 0.55 = -1.56 < 0, \quad -1 + \det(\overline{\mathbf{a}}) = -1 + 0.55 = -0.45 < 0.$$

Therefore, the equilibrium point $[0, 0]^\top$ is a stable spiral. In Fig. 1 we have plotted the magenta point $(\det(\overline{\mathbf{a}}), \mathrm{tr}(\overline{\mathbf{a}})) = (0.55, 0.8)$ and the confidence regions (blue-red rings) at 50% and 90% confidence levels, respectively. We observe this plot is in agreement with the probabilistic results previously shown.

5. Conclusions

Taking advantage of the so-called Random Variable Transformation technique, in this paper we have determined the first probability density function of the solution stochastic process of a random autonomous first-order linear systems of difference equations under very general hypotheses (statistical dependence among the random input data and a wide class of randomness are allowed). From this key information, we have provided a nice probabilistic generalization of classical results for the important particular case of planar systems. This includes the exact quantification

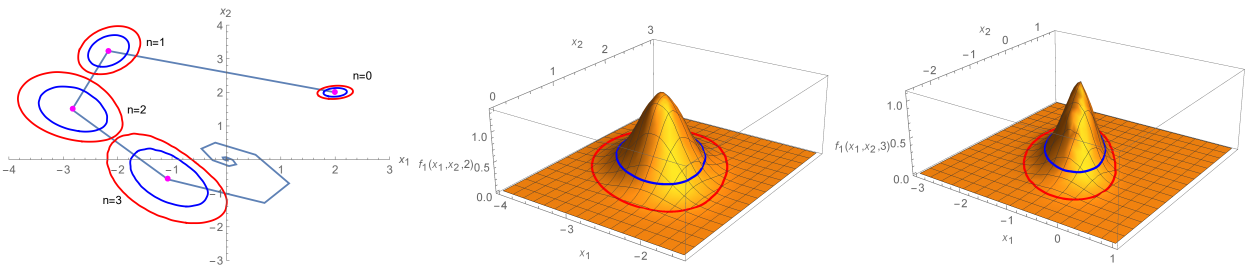

Figure 2: Left: Phase portrait of random system of difference equations (1) with $m = 2$ where the random input vector $\mathbf{Z}$ has the gaussian multivariate distribution given in (7)–(8). The solid line connects the mean and the rings represent confidence regions at 50% and 90% confidence levels at every value of $n \in \{0, 1, 2, 3\}$. Center and Right: PDFs of the solution SP at $n = 2$ and $n = 3$, respectively, together with the corresponding confidence regions.

of the probabilities associated to each possible states to the zero-equilibrium point. The study comprises the important case of random autonomous linear difference equations of order m as a particular case just taking the random coefficient matrix as the so-called companion matrix. Besides, the results established in this paper have a great potential regarding applications since many physical models can be properly described by random linear systems of difference equations. In addition, the study of many random autonomous nonlinear models require the application of linearization to conduct their mathematical analysis.

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Conflict of Interest Statement

The authors declare that there is no conflict of interests regarding the publication of this article.

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