

Document downloaded from:

<http://hdl.handle.net/10251/103784>

This paper must be cited as:

Palma, R.; Moliner, E.; Pérez-Aparicio, JL. (2017). Elasto-thermoelectric beam formulation for modeling thermoelectric devices. *Finite Elements in Analysis and Design*. 129:32-41.  
doi:10.1016/j.finel.2017.02.001



The final publication is available at

<https://doi.org/10.1016/j.finel.2017.02.001>

Copyright Elsevier

Additional Information

# Elasto-thermoelectric beam formulation for modeling thermoelectric devices

R. Palma<sup>a</sup>, E. Moliner<sup>a</sup>, J.L. Pérez-Aparicio<sup>b</sup>

<sup>a</sup>Department of Mechanical Engineering and Construction, Av. Sos Baynat, s/n - Universitat Jaume I, 12071 Castellón de la Plana (Spain), e-mail: rpalma@uji.es

<sup>b</sup>Mecánica de Medios Continuos y Teoría de Estructuras, Universitat Politècnica de València, Spain

---

## Abstract

The present paper provides a dynamic, non-linear and fully coupled Finite Element (FE) formulation based on the *Timoshenko* beam theory to study elasto-thermoelectric responses in thermoelectric devices. The two main motivations of this work are: i) to study mechanical responses in thermoelectric devices, which must be taken into account in the design of *Peltier* cells due to the fragility and relative low strength of the semiconductors, and ii) to provide a numerical tool that decreases the CPU time to allow the introduction of designs based on optimization processes and on sensitivity analyses that could require many evaluations. In order to undertake the objectives of this work, the general three-dimensional governing equations are reduced to one-dimensional ones by means of several assumptions. Then, a set of five multi-coupled partial differential equations is obtained. The resultant expressions are thermodynamically consistent and form a multi-coupled monolithic FE formulation, differently to stagger formulations that require two separated steps to reach the final result. Numerically, this set of multi-coupled equations is discretized using the FE method and implemented into FEAP [1]. For a proper validation of the code, four benchmarks are performed using one-dimensional dynamic analytical solutions developed by the authors. Finally, this formulation is compared with a three-dimensional FE formulation also developed by the authors in [2] to model a commercial *Peltier* cell. This comparison reveals that: i) relative errors are lower than 13% and ii) CPU times decrease significantly, more than one order of magnitude. In conclusion, the beam thermoelectric formulation is an accurate model that reduces CPU time and could be used in future design of thermoelectric devices.

**Keywords:** Thermoelectric, *Timoshenko* beam model, Finite Element Method, Thermodynamics, *Peltier* cells

---

## 1. Introduction

Thermoelectric materials, which couple thermal and electric fields, are used in many applications for cooling/heating and energy generation, see [3] for a complete review of thermoelectric applications.

The study of the thermoelectric coupling has been widely addressed from analytical, experimental and numerical points of view. However, the inclusion of the mechanical field (elasto-thermoelectric) in order to take into account thermal stresses has not been adequately studied. The main reason of this lack of research works could be that, traditionally, the thermoelectric problem has been dealt by electric engineers. The electrical engineering community usually uses one-dimensional (1D) analytical solutions that, at least, make it very difficult to couple the mechanical field. Notice that, from a mathematical point of view, the mechanical field magnitudes are represented by tensors, while electric and thermal ones by vectors. In short, the mechanical behavior of

thermoelectric devices has not been well understood to date, and one of the aim of this work is to provide a simple numerical tool to study this coupling in future works.

From a numerical point of view, the authors of the present work have published several works on thermoelectric modeling, using the Finite Element (FE) method. In [4], a three-dimensional (3D) non-linear FE formulation for thermoelectric modeling was developed. The non-linearities emerge from the *Joule* term and the temperature dependency of the material properties. The previous formulation was applied to model a commercial *Peltier* device in [5]. Subsequently, in [6] the FE formulation was extended by including a temperature relaxation time based on the *Cattaneo* model. In [7], four relaxation times were introduced and it was numerically verified that the hysteresis-like response of thermoelectrics was due to a coupled relaxation time [8]. In the previous FE formulations, the mechanic field

39 was not included. In [2] a thermodynamically consistent  
 40 3D, non-linear and fully coupled formulation (including  
 41 thermal, electric, magnetic and mechanic fields) was  
 42 addressed under static and dynamic situations. Small  
 43 displacements, rotations and strains were assumed for  
 44 the mechanical field. Other authors have used commercial  
 45 FE codes to study thermoelectricity. For instance,  
 46 *Peltier* devices were simulated using the commercial FE  
 47 code ANSYS in [9, 10] and COMSOL in [11]. These works  
 48 do not take into account the mechanical field. Conversely,  
 49 in [12] a *Peltier* cell was analyzed using an elasto-thermo-  
 50 electric FE implemented in COMSOL. This last work concludes  
 51 that thermal stresses cause a mechanical bending of the  
 52 thermocouples that compose the *Peltier* cell. Recently,  
 53 the authors of [13, 14, 15] have analyzed thermal stresses  
 54 in thermoelectric power generators using FE formulations.  
 55 For this purpose, a stagger procedure consisting of two  
 56 steps is developed. First, temperature distributions are  
 57 obtained using a pure thermoelectric model implemented  
 58 in ANSYS; then, these thermal distributions are introduced  
 59 in a pure mechanical model. This is not a monolithic FE  
 60 approach (fully coupled stiffness matrix) derived from  
 61 a thermodynamically consistent formulation, as in [2].  
 62 The authors in [15] report that thermal stresses should  
 63 be considered to improve the mechanical reliability of  
 64 these generators.

66 The literature review shows that there exist still few  
 67 works dealing with the elasto-thermoelectric behavior  
 68 in thermoelectric devices, despite the fact that thermal  
 69 stresses may significantly affect their mechanical  
 70 performance and, consequently, their service life.  
 71 Among others, one drawback to numerically study thermal  
 72 stresses is the higher computational cost, [16]. For  
 73 instance, the two numerical alternatives (iterative  
 74 solutions as in [15] and consistent fully coupled as in  
 75 [2]) to compute thermal stresses increase the CPU time  
 76 and the formulation difficulty. An alternative approach  
 77 to consider thermal stresses, reducing CPU time and  
 78 ensuring a consistent formulation could be derived from  
 79 classic mechanics of materials. For example, in [17] and  
 80 [18] multiphysics beam formulations to model piezoelectrics  
 81 and composites, respectively, are reported.

82 The present work presents a dynamic, non-linear  
 83 and fully coupled FE formulation based on the *Timo-*  
 84 *shenko* beam theory to study elasto-thermoelectric  
 85 responses in thermoelectric devices. In order to obtain  
 86 the FE formulation, several assumptions such as small  
 87 strains, displacements and rotations, two-dimensional  
 88 slender beam-like structures and the absence of mag-  
 89 netic fields, convection, or radiation phenomena are  
 90 introduced. Then, this beam formulation is implemented

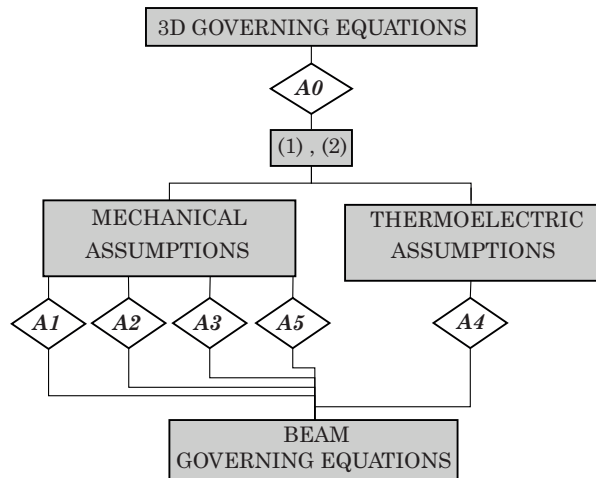


Figure 1: Starting from a 3D formulation, several assumptions are introduced to derive a thermodynamically consistent beam formulation. All these assumptions are justified and highlighted through the work.

91 into FEAP [1], a FE analysis program developed by the  
 92 University of Berkeley at California (USA). For the val-  
 93 idation of the implementation, four benchmarks are per-  
 94 formed using 1D dynamic analytical solutions of a sin-  
 95 gle thermoelement. Despite the fact that bending is not  
 96 present in thermoelectric thermoelements, the bending  
 97 could be relevant in *Peltier* devices due to the frame-  
 98 like behavior of these devices. For this purpose, the  
 99 present formulation is compared with the 3D one de-  
 100 veloped by the authors in [2] to model a commercial  
 101 *Peltier* device. This comparison reveals that: i) relative  
 102 errors are lower than 13% and ii) CPU times decrease  
 103 significantly, more than one order of magnitude. In con-  
 104 clusion, the beam thermoelectric formulation is an ac-  
 105 curate model and could be used in future optimizations  
 106 and sensitivity analyses that require many evaluations.

## 2. Governing equations

108 The aim of this section is to establish the governing  
 109 equations that are used for the FE formulation. Start-  
 110 ing from a 3D set of equations, several assumptions or  
 111 simplifications are considered to derive a thermodynam-  
 112 ically consistent beam formulation. The assumptions  
 113 through the document are highlighted in text-box and  
 114 are denoted by *A1*, *A2*, etc. Furthermore, for the sake  
 115 of clarity the complete set of assumptions is represented  
 116 in the flowchart shown in Figure 1.

### 2.1. Outline of 3D governing equations

118 This section outlines the 3D elasto-thermoelectric  
 119 governing equations, composed of three balance equa-

120 tions, three constitutive (also called transport) equations  
121 and six boundary conditions. In addition, one compati-  
122 bility equation is considered for the mechanical field.

123 **Assumption A0:** Small strains and displacements.

124 In most applications, **A0** is a first and good approxi-  
125 mation due to the high stiffness of typical thermoelectric  
126 materials. Considering this assumption, the non-local  
127 strong forms of the balance equations are expressed as  
128 follow:

$$\begin{aligned}
0 &= \int_{\Omega} (\nabla \cdot \boldsymbol{\sigma} + \mathbf{b} - \rho_m \ddot{\mathbf{u}}) \, d\Omega, \\
0 &= \int_{\Omega} (-\nabla \cdot \mathbf{q} - \mathbf{j} \cdot \nabla V - T_0 \boldsymbol{\beta} : \dot{\boldsymbol{\epsilon}} - \rho_m c \dot{T}) \, d\Omega, \\
0 &= \int_{\Omega} \nabla \cdot \mathbf{j} \, d\Omega,
\end{aligned} \tag{1}$$

129 where  $\Omega$  denotes 3D domain,  $\rho_m$  mass density,  $\mathbf{u}$  dis-  
130 placement vector,  $\boldsymbol{\sigma}$  stress tensor,  $\mathbf{b}$  vector of body  
131 forces,  $c$  heat capacity,  $T$  temperature,  $\mathbf{q}$  heat flux,  $\mathbf{j}$   
132 electric flux,  $V$  voltage,  $T_0$  reference temperature,  $\boldsymbol{\beta}$   
133 thermal expansion tensor (in *Lamé* form) and  $\boldsymbol{\epsilon}$  denotes  
134 strain tensor. In addition,  $(\dot{\cdot})$  and  $(\ddot{\cdot})$  represent first and  
135 second time derivatives. The first equation is the linear  
136 momentum balance required to ensure the mechanical  
137 equilibrium. Furthermore, the angular momentum bal-  
138 ance requires that  $\boldsymbol{\sigma}$  is symmetric. The second equation  
139 is the energy balance and takes into account the energy  
140 of the three fields: the first term on the right side repre-  
141 sents the thermal energy that flows across the boundary;  
142 the second and third terms are sources due to electric  
143 field (*Joule* heating) and mechanical field (*Biot* or two-  
144 way effect, see [2], [19], [20] and [21]), respectively.  
145 Finally, the third equation states the balance of electric  
146 charge and is obtained by combining the *Ampère* and  
147 *Gauss* laws of electromagnetism. Free electric charges  
148 are not considered in the present work and, therefore,  
149 the left term is zero.

The constitutive equations are a set of three coupled  
equations given by:

$$\begin{aligned}
\boldsymbol{\sigma} &= \mathbf{C} : \boldsymbol{\epsilon} - \boldsymbol{\beta} (T - T_0), \\
\mathbf{q} &= -\kappa(T) \nabla T + \alpha(T) T \mathbf{j}, \\
\mathbf{j} &= -\gamma(T) \nabla V - \alpha(T) \gamma(T) \nabla T,
\end{aligned} \tag{2}$$

150 where  $\mathbf{C}$  denotes elastic tensor,  $\kappa$  thermal conductivity,  
151  $\alpha$  *Seebeck* coefficient and  $\gamma$  denotes electric conductiv-  
152 ity. The first equation describes the thermoelastic cou-  
153 pling; electric and mechanic fields are not coupled since

154 polarization effects (such as piezoelectric interactions)  
155 are not usually present in thermoelectric devices. The  
156 second and third equations couple thermal and electric  
157 fields by two separate effects, *Seebeck* and *Peltier*, both  
158 measured by the *Seebeck* coefficient.

159 Although there is not an explicit relationship between  
160 electric and mechanic fields, the problem is fully cou-  
161 pled since both depend on temperature.

162 In most practical situations, the materials are  
163 isotropic and homogeneous. Furthermore,  $\kappa$ ,  $\gamma$ ,  $\alpha$  typi-  
164 cally depend on temperature, resulting material non-  
165 linearity. According to [5], the temperature depen-  
166 dency of material properties can be fitted using second-  
167 order polynomials to obtain:

$$\begin{aligned}
\kappa(T) &= \kappa_0 + \kappa_1 T + \kappa_2 T^2, \\
\gamma(T) &= \gamma_0 + \gamma_1 T + \gamma_2 T^2, \\
\alpha(T) &= \alpha_0 + \alpha_1 T + \alpha_2 T^2,
\end{aligned} \tag{3}$$

168 where  $\kappa_i$ ,  $\gamma_i$ ,  $\alpha_i$  are coefficients reported in [5].

169 As commented, the mechanical field requires a com-  
170 patibility equation that relates displacement vector and  
171 the strain tensor and is given by:

$$\boldsymbol{\epsilon} = \nabla^{sy} \mathbf{u}, \tag{4}$$

172 where  $(\cdot)^{sy}$  denotes the symmetric part of the displace-  
173 ment gradient. Notice that the skew-symmetric part rep-  
174 represents the rigid body rotations and, therefore, do not  
175 contribute to the strain measure.

176 Finally, a set of six boundary conditions (3 *Neumann*-  
177 type and 3 *Dirichlet*-type) must be considered:

$$\begin{aligned}
\boldsymbol{\sigma} \cdot \mathbf{n} &= \mathbf{t} & ; & & \mathbf{u} &= \bar{\mathbf{u}} & ; \\
\mathbf{q} \cdot \mathbf{n} &= q_c & ; & & T &= \bar{T} & ; \\
\mathbf{j} \cdot \mathbf{n} &= j_c & ; & & V &= \bar{V} & ,
\end{aligned} \tag{5}$$

178 where  $\mathbf{n}$  denotes outward normal to the boundary,  $\mathbf{t}$  trac-  
179 tion vector,  $q_c$  and  $j_c$  denote heat and electric fluxes  
180 at boundary, respectively, and  $\bar{\mathbf{u}}$ ,  $\bar{T}$ ,  $\bar{V}$  denote the pre-  
181 scribed displacement, temperature and voltage, respec-  
182 tively.

## 2.2. Elasto-thermoelectric beam equations

184 In this section, the 3D governing equations are sim-  
185 plified to the *Timoshenko* beam model. The choice of  
186 this model intends to find a compromise between gen-  
187 erality and simplicity of implementation. The former  
188 is achieved since this beam model is more general than  
189 that of *Bernoulli*: it takes into account shear stresses and  
190 strains by considering an extra rotation, a degree of free-  
191 dom coupled with displacements. The simplicity arises  
192 from the fully coupled formulation of the present work

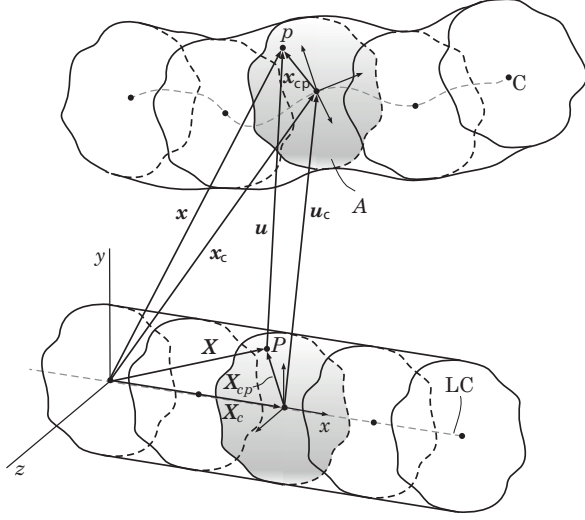


Figure 2: Beam before (bottom) and after (top) deformation. The beam is represented as a long body composed of a succession of 2D cross sections, of area  $A$ , attached at their centroid ( $C$ ) to a longitudinal axis called line of centroids ( $LC$ ).

in which the inclusion of the extra rotation is comparable in difficulty to that of temperature and voltage.

The *Timoshenko* beam is a well established model from a century ago, [22], but it is here briefly reviewed to introduce the coupled formulations: in particular, the understanding of the coupling between rotational and translational degrees of freedom (dof) is a good introduction for the understanding of the coupling between mechanical, on one side, and voltage and temperature dof's the other.

### 2.2.1. Kinematics and thermoelectric distributions

A beam is a long body composed of a succession of 2D planar surfaces called cross section, of area  $A$ . The surfaces are attached in their centroid to a longitudinal axis perpendicular to each cross section, called line of centroids ( $LC$ ).

The beam motion is constrained by the *kinematic beam hypothesis* (assumption 1, denoted by **A1**):

**Assumption A1:** A cross section that is plane before deformation remains plane after.

According to **A1**, the cross section moves as a rigid body: neither changes its shape nor deviates from flatness. Therefore, the motion of beams can be described as the deformation of the  $LC$  plus the rigid rotation of the cross section.

Figure 2 shows a beam before and after deformation with  $LC$  along the  $x$  axis and with cross section of area

$A$  and contour  $\Gamma$ . The position of the cross section centroid  $C$  and of one point  $P$  inside are described by  $X_c(x)$ ,  $X_{cp}(y, z)$  before and by  $x_c(x)$ ,  $x_{cp}(y, z)$  after deformation, respectively. The origin of the first is the coordinate center, of the second the centroid itself.

From vector calculus:

$$\begin{aligned} X &= X_c(x) + X_{cp}(y, z), \\ x &= x_c(x) + x_{cp}(y, z). \end{aligned} \quad (6)$$

According to the *Chasles'* theorem, the rigid rotation of  $X_{cp}$  is expressed by:

$$x_{cp} = R \cdot X_{cp}, \quad (7)$$

where  $R$  is the rotational operator given by  $R = I + \Theta + hot$ ; *hot* is the abbreviation of high order terms,  $\Theta$  is the skew-symmetric spin tensor that can be expressed as  $\Theta = \epsilon \cdot \theta$ , where  $\epsilon$  is the *Levi-Civita* symbol and  $\theta$  is an axial vector of rotations (also called spin vector).

**Assumption A2:** Small rotations are considered:  $R \approx I + \Theta$ .

In the community of Continuum Mechanics, the displacement vector is given by  $u = x - X$ . Therefore, the beam displacement vector is obtained by using (6), (7) and applying **A2** to read:

$$u = u_c(x) + \theta(x) \times X_{cp}(y, z), \quad (8)$$

where the spin vector depends on the position of the cross section,  $x$ . As commented, in (8) the displacement is composed of an  $LC$  deformation (first term on the right side) and a rigid rotation of the cross section (second term).

**Assumption A3:** 2D beams are considered.

In most thermoelectric applications, the geometry of the cross section is constant and symmetric respect to both axes  $y$ ,  $z$  and the loads are only applied along  $x$  and/or  $y$  directions. According to **A3** the displacements in (8) are reduced to:

$$\begin{cases} u = u_c(x) - \theta y, \\ v = v_c(x). \end{cases} \quad (9)$$

Regarding the kinematic-like description of temperature and voltage distributions, *Taylor* series expansions evaluated at the centroid  $C$  are used [17]:

$$\begin{aligned} T(x, y, z) &= T_c(x) + y \partial_y T + z \partial_z T + hot, \\ V(x, y, z) &= V_c(x) + y \partial_y V + z \partial_z V + hot, \end{aligned} \quad (10)$$

250 where  $\partial_y, \partial_z$  denote partial derivatives with respect to  
 251  $y, z$ , respectively. Notice that the thermoelectric beam  
 252 simplification is similar to the mechanic one given by  
 253 (8). Temperature and voltage distributions are composed  
 254 of centroidal terms  $T_c, V_c$  and across the section  
 255 ones, given by second and third terms on the right side  
 256 in (10). The terms across the section are not considered  
 257 in the present work due to **A4**.

258 **Assumption A4:** Voltage and temperature distributions through cross section are constant.

259 The assumption **A4** implies that  $T(x, y, z) =$   
 260  $T_c(x), V(x, y, z) = V_c(x)$  and it is a good approximation  
 261 for *Peltier* devices in the absence of magnetic field, con-  
 262 vection and radiation phenomena, as can be extracted  
 263 from the 3D FE model reported in [2].

### 264 2.2.2. Strain-like measures

265 The strain measures are obtained by operating the 3D  
 266 compatibility equation (4) and using the displacement  
 267 vector (9). Then, the non-zero entries in  $\epsilon$  are given by:

$$\begin{cases} \epsilon_{xx} = \partial_x u_c - \partial_x \theta, \\ \epsilon_{xy} = \frac{1}{2} (\partial_x v_c - \theta), \end{cases} \quad (11)$$

268 where  $\partial_x$  denote partial derivative with respect to  $x$ . As  
 269 is typical in beam theory, three new strain measures  $\epsilon_x,$   
 270  $\epsilon_y, \epsilon_\theta$  are defined as:

$$\begin{cases} \epsilon_x = \partial_x u_c, \\ \epsilon_y = \partial_x v_c - \theta, \\ \epsilon_\theta = -\partial_x \theta. \end{cases} \quad (12)$$

271 Using these new definitions, (11) becomes:

$$\begin{cases} \epsilon_{xx} = \epsilon_x + \epsilon_\theta y, \\ \epsilon_{xy} = \frac{1}{2} \epsilon_y. \end{cases} \quad (13)$$

272 Regarding temperature and voltage strain-like mea-  
 273 sures, the general 3D gradients are reduced to:

$$\begin{aligned} \nabla T &\approx \partial_x T_c, \\ \nabla V &\approx \partial_x V_c. \end{aligned} \quad (14)$$

### 274 2.2.3. Equilibrium equations

275 From an equilibrium point of view, the difference be-  
 276 tween 3D and beam formulations arises from the con-  
 277 cept of stress resultant. That is, the tractions  $\mathbf{t}$  in any  
 278 point of the cross section are expressed by the force  
 279  $\mathbf{F}$  and the momentum  $\mathbf{M}$  resultants at the center of the  
 280 cross section, see Figure 3 (left). Mathematically, con-  
 281 sidering that  $\mathbf{n} \equiv (1, 0, 0)$  is the outward normal to the

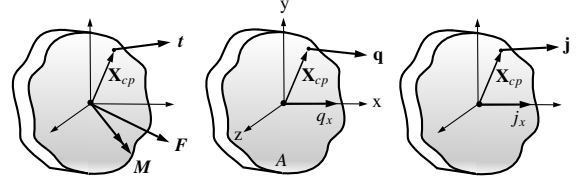


Figure 3: From 3D to 1D. Beam models use resultants integrating through cross section area: force and moment (left), amount of heat along  $x$  (middle) and electric intensity along  $x$  (right).

282 cross section, the force and momentum resultants are  
 283 given by:

$$\begin{aligned} \mathbf{F} &= \int_A \mathbf{t} \, dA = \int_A \boldsymbol{\sigma} \cdot \mathbf{n} \, dA = N \hat{\mathbf{x}} + Q \hat{\mathbf{y}} + Q_z \hat{\mathbf{z}}, \\ \mathbf{M} &= \int_A \mathbf{x}_{cp} \times (\boldsymbol{\sigma} \cdot \mathbf{n}) \, dA = M_x \hat{\mathbf{x}} + M_y \hat{\mathbf{y}} + M \hat{\mathbf{z}}, \end{aligned} \quad (15)$$

284 where  $N, Q, Q_z$  are axial and shear forces and  $M_x, M_y,$   
 285  $M$  are the moment respect to the unit vectors  $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$  axis,  
 286 respectively. Notice that  $Q_z, M_x, M_y$  are neglected due  
 287 to **A3** and since the cross section is symmetric respect  
 288 to both axes  $y, z$  in most thermoelectric applications.

289 In order to ensure the mechanical equilibrium, linear  
 290 and angular momenta must be stated by particularizing  
 291 (1) to the beam model, see [23]. Then, a set of three  
 292 partial differential equations is obtained:

$$0 = \int_x \left[ \frac{\partial}{\partial x} \begin{Bmatrix} N \\ Q \\ M \end{Bmatrix} + \begin{Bmatrix} f_u - \rho_m A \ddot{u}_c \\ f_y - \rho_m A \ddot{v}_c \\ Q + m - \rho_m I \ddot{\theta} \end{Bmatrix} \right] dx, \quad (16)$$

293 where  $f_u, f_y, m$  are distributed applied forces and moment  
 294 and  $I$  denotes second moment of area.

295 Thermal and electric fluxes must be reduced to 1D  
 296 following a similar procedure to that described to obtain  
 297 force and momentum resultants. Denoting by  $Q$  the  
 298 amount of heat transferred per unit time and by  $I$  the  
 299 electric intensity that flows along  $x, q_x, j_x$  are expressed  
 300 as:

$$Q = \int_A \mathbf{q} \cdot \mathbf{n} \, dA = A q_x, \quad I = \int_A \mathbf{j} \cdot \mathbf{n} \, dA = A j_x, \quad (17)$$

where to solve the integral it is considered that cross  
 sections are constant, Figure 3 middle and right. Intro-  
 ducing this 1D approximation in the second, third equa-  
 tions of (1):

$$0 = \int_x \left[ \frac{\partial}{\partial x} \begin{Bmatrix} -Q \\ I \end{Bmatrix} - \begin{Bmatrix} \mathcal{J} + \mathcal{T} + A \rho_m c \dot{T}_c \\ 0 \end{Bmatrix} \right] dx, \quad (18)$$

where, for the sake of clarity, the terms due to *Joule*  $\mathcal{J}$  and to two-way  $\mathcal{T}$  have been denoted by:

$$\mathcal{J} = I \partial_x V_c, \quad \mathcal{T} = T_0 A E \alpha_T \dot{\epsilon}_x, \quad (19)$$

respectively. Notice that  $F$ ,  $M$  are vectors since the stress tensor is a second order tensor and  $Q$ ,  $I$  are scalars due to the fact that heat and electric fluxes are vectors.

#### 2.2.4. Constitutive equations

**Assumption A5:** The *Poisson* effect is neglected.

Due to the kinematic approximation of the *Timoshenko* beam model, the non-zero components of  $\epsilon$  are given in (11). This simplification results in non-zero spurious stresses  $\sigma_{yy}, \sigma_{zz}$  using constitutive equations. In order to fix this inconsistency, both stress components are forced to zero. Using the *Hooke's* constitutive equations in *Lamé* form, see [23], the result is  $\sigma_{xx} = E \epsilon_{xx}$ . Comparing the values of all stress entries,  $(2G + \lambda) = E$ ; this expression is equivalent to neglect the *Poisson* coefficient in the definition of  $G$ .

Using (13), the first equation in (2) and taking into account **A5**, the 1D thermoelastic constitutive equation becomes:

$$\begin{cases} \sigma_{xx} = E (\epsilon_x + \epsilon_\theta y - \alpha_T \Delta T), \\ \sigma_{xy} = G \epsilon_y, \end{cases} \quad (20)$$

where  $\Delta T = T_c - T_0$ ,  $\alpha_T$  is the thermal expansion coefficient. Finally, using (15) and, again, considering constant cross sections, the 1D thermoelastic constitutive equations are given by:

$$\begin{Bmatrix} N \\ Q \\ M \end{Bmatrix} = \int_A \begin{Bmatrix} T_{xx} \\ T_{xy} \\ -T_{xx} y \end{Bmatrix} dA = \begin{Bmatrix} A E [\epsilon_x - \alpha_T \Delta T] \\ k_s A G \epsilon_y \\ -E I \epsilon_\theta \end{Bmatrix}, \quad (21)$$

where the first moment of area  $\int_A y dA = 0$  since  $y, z$  cross the centre of gravity. The *Timoshenko* shear factor, which is typically 5/6 for rectangular cross-sections [24], is denoted by  $k_s$ .

Similarly, the second and third 3D thermoelectric constitutive equations (2) are simplified to 1D relationships by considering (17), to give:

$$\begin{aligned} Q &= -A \kappa(T_c) \partial_x T_c + \alpha(T_c) T_c I, \\ I &= -A \gamma(T_c) \partial_x V_c - A \alpha(T_c) \gamma(T_c) \partial_x T_c. \end{aligned} \quad (22)$$

### 3. Finite Element formulation

The current section presents the discretisations based on the FE method to solve the set of five fully coupled partial differential equations described in Section 2. As commented and since the problem is multi-coupled, there are five dof's:  $u_c, v_c, \theta, T_c$  and  $V_c$ .

#### 3.1. Weak forms

The balance equations given in (16) and (18) are expressed in weak form following the standard FE procedure, namely, multiplying by test functions and applying the *divergence* theorem to obtain:

$$- \int_x \frac{\partial}{\partial x} \begin{Bmatrix} \delta u_c N \\ \delta v_c Q \\ \delta \theta M \\ \delta T_c Q \\ \delta V_c I \end{Bmatrix} - \begin{Bmatrix} \delta u_c (n - \rho_m A \ddot{u}_c) \\ \delta v_c (q - \rho_m A \dot{v}_c) \\ \delta \theta (Q + m - \rho_m I \ddot{\theta}) \\ \delta T_c (A \rho_m c \dot{T}_c + \mathcal{J} + \mathcal{T}) \\ 0 \end{Bmatrix} dx = 0. \quad (23)$$

#### 3.2. Residual forms

Since the problem is non-linear due to the presence of the *Joule* term and the temperature dependency of the material properties given in (3), the FE formulation is expressed in residual forms to use the *Newton-Raphson* algorithm.

Firstly, the continuum weak form calculated in (23) is discretized by using the same standard 1D shape functions of *Lagrange* type for all the degrees of freedom, since the problem holds continuity  $C^0$ :

$$\begin{Bmatrix} \delta u_c \\ \delta v_c \\ \delta \theta \\ \delta T_c \\ \delta V_c \end{Bmatrix} \approx \mathcal{N}_a \begin{Bmatrix} \delta a_a^u \\ \delta a_a^v \\ \delta a_a^\theta \\ \delta a_a^T \\ \delta a_a^V \end{Bmatrix}; \quad \partial_x \begin{Bmatrix} \delta u_c \\ \delta v_c \\ \delta \theta \\ \delta T_c \\ \delta V_c \end{Bmatrix} \approx \mathcal{B}_a \begin{Bmatrix} \delta a_a^u \\ \delta a_a^v \\ \delta a_a^\theta \\ \delta a_a^T \\ \delta a_a^V \end{Bmatrix}, \quad (24)$$

where  $\delta a_a^i$  denotes the virtual nodal value of the dofs  $i = u_c, v_c, \theta, T_c, V_c$  at node  $a$ , and  $\mathcal{B}_a = \partial_x \mathcal{N}_a$ .

Secondly, the residuals at node  $a$  are obtained by introducing (24) in (23):

$$\left\{ \begin{array}{c} \mathcal{R}_a^u \\ \mathcal{R}_a^v \\ \mathcal{R}_a^\theta \\ \mathcal{R}_a^T \\ \mathcal{R}_a^V \end{array} \right\} = - \int_{x^e} \left[ \mathcal{B}_a \left\{ \begin{array}{c} N \\ Q \\ M \\ Q \\ I \end{array} \right\} - \mathcal{N}_a \left\{ \begin{array}{c} n - \rho_m A \ddot{u}_c \\ q - \rho_m A \dot{v}_c \\ Q + m - \rho_m I \ddot{\theta} \\ A \rho_m c \dot{T}_c + \mathcal{J} + \mathcal{T} \\ 0 \end{array} \right\} \right] dx^e \quad (25)$$

361 Notice that the residuals hold the zero, first and second  
362 derivatives with respect to time of the degrees of free-  
363 dom. In addition, the residuals are a set of five equations  
364 that are used to obtain the five unknowns of the elasto-  
365 thermoelectric beam problem.

$$\frac{\partial N}{\partial \mathbf{a}_b^u} = A E \mathcal{B}_b, \quad \frac{\partial N}{\partial \mathbf{a}_b^T} = -A E \alpha_T \mathcal{N}_b,$$

$$\frac{\partial Q}{\partial \mathbf{a}_b^v} = k_s A G \mathcal{B}_b, \quad \frac{\partial Q}{\partial \mathbf{a}_b^\theta} = -k_s A G \mathcal{N}_b,$$

$$\frac{\partial M}{\partial \mathbf{a}_b^\theta} = -E I \mathcal{B}_b,$$

$$\frac{\partial I}{\partial \mathbf{a}_b^V} = -A \gamma(T_c) \mathcal{B}_b, \quad \frac{\partial Q}{\partial \mathbf{a}_b^V} = \alpha(T_c) T_c \frac{\partial I}{\partial \mathbf{a}_b^V},$$

$$\frac{\partial \mathcal{J}}{\partial \mathbf{a}_b^V} = \frac{\partial I}{\partial \mathbf{a}_b^V} \mathcal{B}_b \mathbf{a}_b^V + I \mathcal{B}_b,$$

$$\frac{\partial \mathcal{J}}{\partial \mathbf{a}_b^T} = \frac{\partial I}{\partial \mathbf{a}_b^T} \mathcal{B}_b \mathbf{a}_b^V,$$

$$\frac{\partial I}{\partial \mathbf{a}_b^T} = -A \frac{\partial \gamma}{\partial T} \mathcal{N}_b \mathcal{B}_b \mathbf{a}_b^V - A \frac{\partial \alpha}{\partial T} \mathcal{N}_b \gamma(T_c) \mathcal{B}_b \mathbf{a}_b^T$$

$$-A \alpha(T_c) \frac{\partial \gamma}{\partial T} \mathcal{N}_b \mathcal{B}_b \mathbf{a}_b^T - A \alpha(T_c) \gamma(T_c) \mathcal{B}_b,$$

$$\frac{\partial Q}{\partial \mathbf{a}_b^T} = -A \frac{\partial \kappa}{\partial T} \mathcal{N}_b \mathcal{B}_b \mathbf{a}_b^T - A \kappa(T_c) \mathcal{B}_b$$

$$+ \frac{\partial \alpha}{\partial T} \mathcal{N}_b T_c I + \alpha(T_c) \mathcal{N}_b I + \alpha(T_c) T_c \frac{\partial I}{\partial \mathbf{a}_b^T}.$$

### 366 3.3. Tangent matrices

367 The tangent matrices are obtained by deriving the  
368 residuals with respect to the dof, in compact form:

$$\mathcal{K}_{ab}^{ij} = -\frac{\partial \mathcal{R}_a^i}{\partial \mathbf{a}_b^j}, \quad \mathcal{C}_{ab}^{ij} = -\frac{\partial \mathcal{R}_a^i}{\partial \dot{\mathbf{a}}_b^j}, \quad \mathcal{M}_{ab}^{ij} = -\frac{\partial \mathcal{R}_a^i}{\partial \ddot{\mathbf{a}}_b^j},$$

369 where, again,  $i, j$  denote the dof;  $a, b$  the nodes and  $\mathcal{K}$ ,  
370  $\mathcal{C}$ ,  $\mathcal{M}$  the stiffness, capacity and mass matrices, respec-  
371 tively.

372 The stiffness matrices are calculated by:

$$\left\{ \begin{array}{c} \mathcal{K}_{ab}^{uj} \\ \mathcal{K}_{ab}^{vj} \\ \mathcal{K}_{ab}^{\theta j} \\ \mathcal{K}_{ab}^{Tj} \\ \mathcal{K}_{ab}^{Vj} \end{array} \right\} = \int_{x^e} \mathcal{B}_a \left\{ \begin{array}{c} \frac{\partial N}{\partial \mathbf{a}_b^j} \\ \frac{\partial Q}{\partial \mathbf{a}_b^j} \\ \frac{\partial M}{\partial \mathbf{a}_b^j} \\ \frac{\partial Q}{\partial \mathbf{a}_b^j} \\ \frac{\partial I}{\partial \mathbf{a}_b^j} \\ \frac{\partial Q}{\partial \mathbf{a}_b^j} \\ \frac{\partial I}{\partial \mathbf{a}_b^j} \end{array} \right\} dx^e - \int_{x^e} \mathcal{N}_a \left\{ \begin{array}{c} 0 \\ 0 \\ \frac{\partial Q}{\partial \mathbf{a}_b^j} \\ \frac{\partial \mathcal{J}}{\partial \mathbf{a}_b^j} \\ \frac{\partial I}{\partial \mathbf{a}_b^j} \\ 0 \end{array} \right\} dx^e, \quad (26)$$

373 where the non-zero derivatives are:

374 The capacity matrices are due to the *Biot* term and  
375 to the heat transient. For this reason, they are closely  
376 related to the thermal residual and are given by:

$$\left\{ \begin{array}{c} \mathcal{C}_{ab}^{Tu} \\ \mathcal{C}_{ab}^{TT} \end{array} \right\} = - \int_{x^e} \mathcal{N}_a \left\{ \begin{array}{c} T_0 A E \alpha_T \mathcal{B}_b \\ \rho_m c A \mathcal{N}_b \end{array} \right\} dx^e. \quad (27)$$

377 Finally, the mass matrices emerge from the mechan-  
378 ical inertia and they are obtained from the mechanical  
379 residuals, three first equations in (25):

$$\left\{ \begin{array}{c} \mathcal{M}_{ab}^{uu} \\ \mathcal{M}_{ab}^{vv} \\ \mathcal{M}_{ab}^{\theta\theta} \end{array} \right\} = \int_{x^e} \mathcal{N}_a \rho_m \left\{ \begin{array}{c} A \\ A \\ I \end{array} \right\} \mathcal{N}_b dx^e. \quad (28)$$

380 This FE formulation is implemented into the research  
381 code FEAP [1]. This code provides several dummy rou-  
382 tines (user elements) that can be used for the implemen-  
383 tation of new developed modular elements written in  
384 Fortran. In order to solve the non-linear problem, the



385 *Newton-Raphson* algorithm is used and the *Newmark- $\beta$*   
386 for time integration of the dynamic. Notice that since  
387 the problem is multi-coupled, several orders of the time  
388 derivatives are present. This drawback is solved by using  
389 a special command implemented in FEAP. Finally,  
390 the shear locking, which is typically present in the *Tim-*  
391 *oshenko* beam models [25, 26], is solved due to the facil-  
392 ities of the FEAP code.

#### 393 4. Validations

394 This section presents four validations (called cases I,  
395 II, III and IV) to ensure a proper implementation of the  
396 numerical formulation developed in Sec. 3. For this pur-  
397 pose, the numerical results are compared with dynamic  
398 1D analytical solutions developed by the authors; most  
399 of the analytical solutions are reported in [6].

Property	Value	Units
$\kappa_0, \kappa_1, \kappa_2$	1.663,-3.58e-3,-3.19e-5	[W/m $^\circ$ C]
$\gamma_0, \gamma_1, \gamma_2$	1.09e5,-5.59e2,2.49	[A/Vm]
$\alpha_0, \alpha_1, \alpha_2$	1.98e-4,3.53e-7,7.52e-10	[V/ $^\circ$ C]
$E, G$	4.70e10,1.68e10	[N/m $^2$ ]
$\rho_m$	7.53e3	[Kg/m $^3$ ]
$c$	544	[J/KgK]
$\alpha_T$	5.37e-6	[ $^\circ$ C $^{-1}$ ]

Table 1: p-type bismuth telluride thermoelement properties. For the n-type thermoelement properties are equal, except for  $\alpha_0, \alpha_1, \alpha_2$  with negative sign.

400 For all the validations, a single p-type bismuth tel-  
401 luride pulsed thermoelement as that described in [16] of  
402 dimensions 5.8 $\times$ 1.4 $\times$ 1.4 [mm] and properties given in  
403 Table 1 is modeled. Figure 4 shows the p-type 3D thermo-  
404 element, its 1D beam representation and the bound-  
405 ary conditions used for the validations. From a mechan-  
406 ical point of view, the boundary conditions correspond  
407 to a cantilever beam. Thermally, the temperatures are  
408 fixed at both ends of the thermoelement: at cold and  
409 sides  $T_{co} = 30$  [ $^\circ$ C] and  $T_{ho} = 80$  [ $^\circ$ C], respectively. Fi-  
410 nally, for the electric field the voltage is set to zero at the  
411 cold side. For cases II, III and IV, an electric intensity  
412 of  $I = 2$  [A] is applied.

413 Table 2 summarizes all validations. For case I, the  
414 prescribed intensity is zero, resulting in a linear distri-  
415 bution of temperature along the thermoelement due to  
416 the *Fourier* effect (denoted by F in the table). In addi-  
417 tion, a voltage distribution is generated due to the *See-*  
418 *beck* effect (denoted by S). For case II, an electric in-  
419 tensity is prescribed and the *Joule* effect (J) appears.  
420 Then, the temperature distribution is quadratic due to

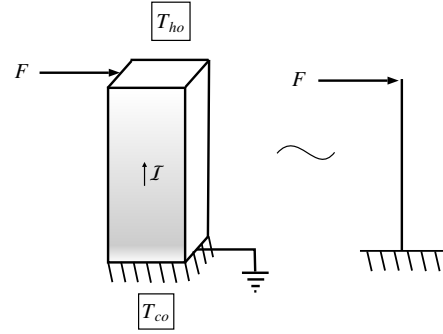


Figure 4: Geometric model and boundary conditions for validations. The 3D model (left) is reduced to a 1D beam model (right).

421 the heat source. For case III the temperature depen-  
422 dency of the  $\alpha(T_c)$  is considered and the *Thomson* effect  
423 (Th) emerges. Finally, for case IV a force is applied at  
424 the end of the beam; this force represents the thermal  
425 expansion of the copper bar, see Figure 8 (top), and is  
426 proportional to:  $E\alpha_T\Delta T$ . Consequently, a bending (Bn)  
427 is observed. Due to J and Th effects, cases II, III and IV  
428 are non-linear.

Case	Simplifications	Effects
I	$I = 0; \alpha, \gamma, \kappa = ct$	F, S
II	$I = 2$ [A]; $\alpha, \gamma, \kappa = ct$	F, S, J
III	$I = 2$ [A]; $\alpha(T_c); \gamma, \kappa = ct$	F, S, J, Th
IV	$I = 2$ [A]; $\alpha, \gamma, \kappa = ct; F = 1$ [N]	F, S, J, Bn

Table 2: One-dimensional validation cases, simplifications and considered effects: F - *Fourier*, S - *Seebeck*, J - *Joule*, Th - *Thomson* and Bn - bending. Constant ( $ct$ ) properties are obtained by using (3) with  $T=(T_{co} + T_{ho})/2$ .

429 Figure 5 compares analytical (lines) and numerical  
430 solutions (solid circles) for cases I to III and for three  
431 time instants  $t = 1, 5, 40$  [s]. Temperature distribu-  
432 tions (left), voltage distributions (middle) and axial dis-  
433 placements (right) along the thermoelement (denoted as  
434 *Distance* in the figure) are represented. Each case cor-  
435 responds to a row.

436 For case I (first row in Figure 5), the temperature dis-  
437 tribution (left) parabolically goes from 0 [ $^\circ$ C] at  $t = 0$   
438 [s] to the linear distribution at  $t = 40$  [s]. Notice that  
439 at cold and hot sides the temperature is fixed and the  
440 steady state is achieved at approximately  $t = 40$  [s].  
441 Due to the parabolic nature of the energy balance, sec-  
442 ond in (1), the velocity of the temperature wave is infi-  
443 nite. For this reason, smooth curves without front wave  
444 are observed. At the steady state, a linear distribution  
445 is reached according to the *Fourier* law. Due to the *See-*  
446 *beck* effect, voltage distributions (middle) are generated.

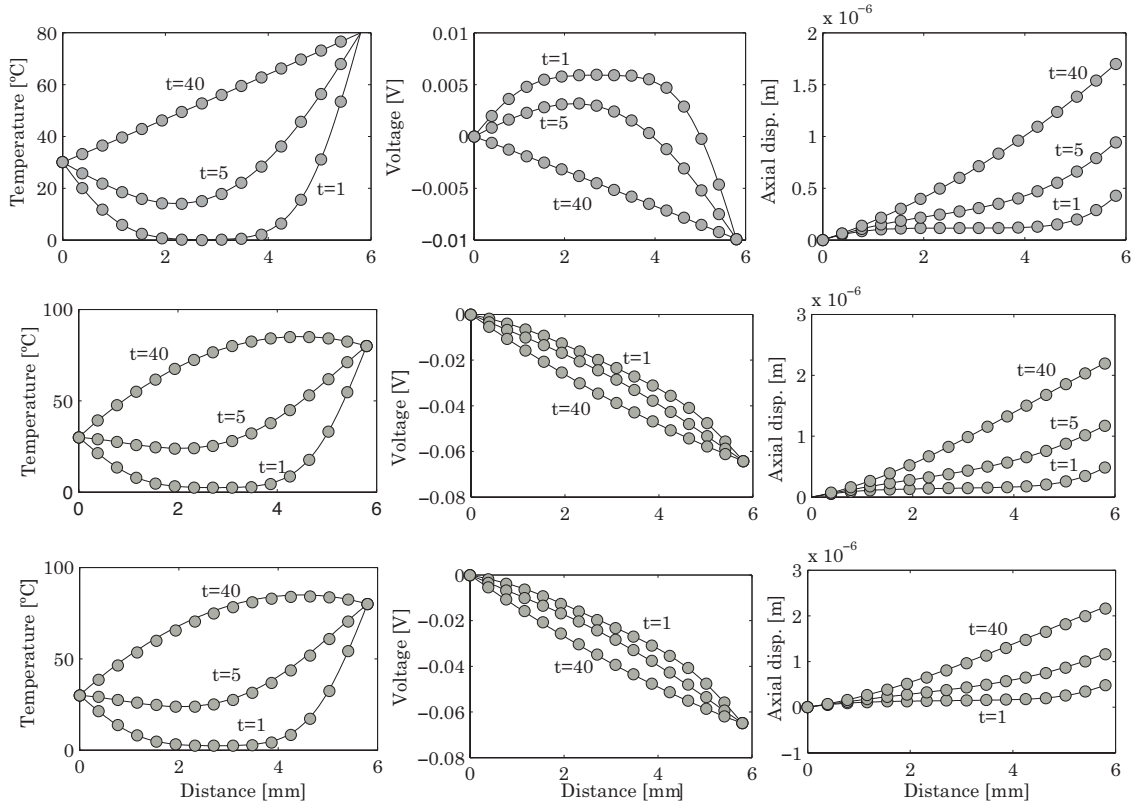


Figure 5: Analytical (lines) and numerical solutions (solid circles) for cases I to III (each case represented in a row) and for three time instants  $t = 1, 5, 40$  [s]. Temperature (left), voltage (middle) and axial displacements (right) vs. distance of thermoelement.

447 These distributions are proportional to the temperature 469  
 448 curves, in which the *Seebeck* coefficient given in Table 1 470  
 449 is the constant of proportionality. Finally, the axial dis- 471  
 450 placement is represented in the right figure for the three 472  
 451 instants. At the steady state, a quadratic displacement 473  
 452 is obtained since it depends on the spatial integration 474  
 453 of the linear temperature distribution. In addition, the 475  
 454 displacement is proportional to the thermal expansion 476  
 455 coefficient given in Table 1. The resultant axial force is 477  
 456 zero since the mechanical boundary conditions allow a 478  
 457 free expansion of the thermoelement.

458 For case II (second row in Figure 5), again parabolic 480  
 459 distributions are obtained. However, the temperature 481  
 460 and, consequently, voltage distributions are quadratic at 482  
 461  $t = 40$  [s] since the *Joule* effect is present. This effect 483  
 462 can be understood as a volumetric heat source that in- 484  
 463 creases the temperature inside the thermoelement. For 485  
 464 instance, the maximum temperature is 85 [°C] (5 de- 486  
 465 grees greater than the temperature at the hot side). Fi- 487  
 466 nally, for the axial displacement a cubic distribution is 488  
 467 reached at the steady state and, again, the resultant axial 489  
 468 force is zero.

The distributions for case III (third row in Figure 5) 489  
 490 are similar to those obtained in case II since the *Thomson* 491  
 492 effect involves a correction of about 1-2% of the 493  
 494 results, as was reported in [27]. For instance, the max- 495  
 496 imum temperature inside the thermoelement is 84.24 496  
 497 [°C]. The *Thomson* effect decreases the maximum tem- 498  
 499 perature since it is an extra heat flux with opposite sign 499  
 500 to the *Joule* heating. This reduction of the temperature 500  
 also slightly decreases the generated voltage and axial 501  
 displacement. For the temperature, there exists a small 502  
 difference between numerical and analytical results due 503  
 to the fact that for the analytical solution is assumed a 504  
 constant *Thomson* coefficient in order to solve the par- 505  
 tial differential equation.

Figure 6 shows the axial  $u$  and vertical  $v$  displace- 506  
 507 ments and rotation  $\theta$  for case IV, in which bending is 508  
 509 present due to the application of a force  $F = 1$  [N], see 509  
 510 Figure 4; only the steady state solutions are represented. 510  
 Both for displacements (left axis in Figure 6) and for 511  
 rotation (right axis), the agreement between closed and 512  
 numerical solutions is very good.

In conclusion, Table 3 shows the maximum relative 513

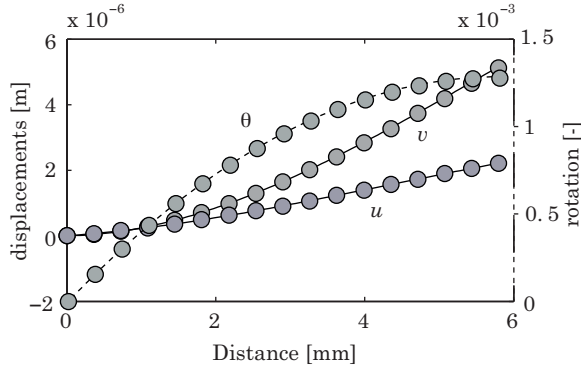


Figure 6: Analytical (lines and dashed line) and numerical solutions (solid circles) for case IV. Vertical  $v$  and axial  $u$  displacements (left axis) and rotation  $\theta$  (right axis) vs. distance of thermoelement.

491 errors at  $t = 40$  [s] (steady state). For cases I, II and  
 492 IV, the relative errors are lower than 0.1%; for case III  
 493 the errors become greater than 2%. As commented, this  
 494 fact could be due to the calculation of the *Thomson* co-  
 495 efficient: from an analytical point of view it is assumed  
 496 to be constant and, numerically, is obtained from (3).

Magnitude	Case I	Case II	Case III	Case IV
Temperature	0.037	0.033	2.450	0.033
Voltage	0.087	0.021	3.010	0.021
Axial disp.	0.027	0.019	2.241	0.019
Vertical disp.	–	–	–	0.061
Rotation	–	–	–	0.062

Table 3: Maximum relative errors in [%] between analytical and numerical results at  $t = 40$  [s] (steady-state solution), see Figure 5.

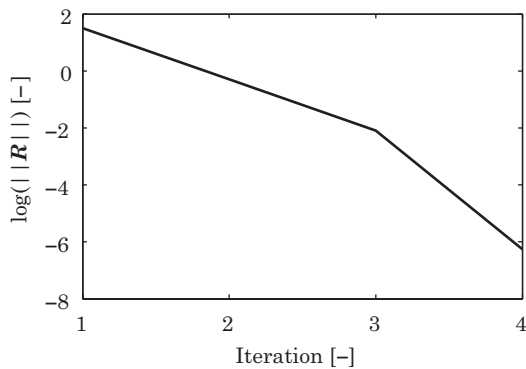


Figure 7: Study of the *Newton-Raphson* convergence: logarithmic residual norm vs. iteration.

497 As commented, cases II and III are non-linear due  
 498 to the presence of the *Joule* effect. Figure 7 shows the  
 499 logarithm of the residual norm versus the number of it-

500 erations  $i$  of the *Newton-Raphson* algorithm. Consid-  
 501 ering  $\|\mathcal{R}_{i+1}\| \leq p_1 \|\mathcal{R}_i\|^{p_2}$ , where  $p_1$  and  $p_2$  are positive  
 502 constants, it is observed that  $p_2 > 1$  and a super-linear  
 503 convergence is achieved.

## 5. Comparisons with 3D FE model

504 The main purpose of this section is to compare the  
 505 present beam formulation with the 3D one reported in  
 506 [5]. Then, CPU times and relative errors are calculated  
 507 and the advantages and drawbacks of the proposed beam  
 508 formulation are highlighted.

509 In order to perform this comparison, a pulsed *Peltier*  
 510 device as that described in [16] is simulated. This de-  
 511 vice is composed of 127 thermocouples electrically con-  
 512 nected in series and thermally in parallel. A thermocou-  
 513 ple is a pair of two p- and n-type thermoelements con-  
 514 nected by copper *Cu* bars and with tin *Sn-Pb* solders,  
 515 as shown in Figure 8 (top). The material properties of  
 516 the thermoelements are given in Table 1; the ones of  
 517 *Cu*, *Sn-Pb* and the dimensions of the thermocouple are  
 518 reported in [16].

519 For the 3D model, the full thermocouple is modeled  
 520 using the structured mesh shown in Figure 8 (top). For  
 521 the beam model, only the line of centroids of each mat-  
 522 erial is modeled. This geometry is highlighted in the  
 523 figure by superposing solid black lines in the 3D mesh.  
 524 Obviously, the number of finite elements required to  
 525 model the thermocouple is highly reduced using the  
 526 beam formulation. Boundary conditions are also repre-  
 527 sented in the figure: mechanically, the device is fixed at  
 528 the hot side (bottom in the figure); thermally, the tem-  
 529 peratures at both sides are  $T_{co} = 20$ ,  $T_{ho} = 50$  [°C];  
 530 electrically, the voltage is set to zero at the middle of  
 531 the horizontal *Cu* beams and an electric intensity  $\mathcal{I} = 1$   
 532 [A] is prescribed.

533 Figure 9 shows the h-convergence of the Coefficient  
 534 Of Performance (COP), top figure, and maximum *Von*  
 535 *Mises* (VM) stress inside the thermocouple (middle)  
 536 versus the number of FE nodes, for both 3D (solid line)  
 537 and beam (dashed line) steady-state solutions. Notice  
 538 that the calculation of COP and VM involves thermo-  
 539 electric and thermoelastic variables, respectively: all  
 540 the couplings are required. Finally, the bottom figure  
 541 represents the CPU time for each calculation.

542 For the 3D model, the COP converges with approx-  
 543 imately 4500 nodes whereas the VM requires 6000  
 544 nodes. This difference in the number of nodes is due  
 545 to the vectorial nature of the thermoelectric variables  
 546 and the second rank of the stress tensor. For a proper  
 547 calculation of both variables a CPU time of 50 [s] is  
 548 employed.

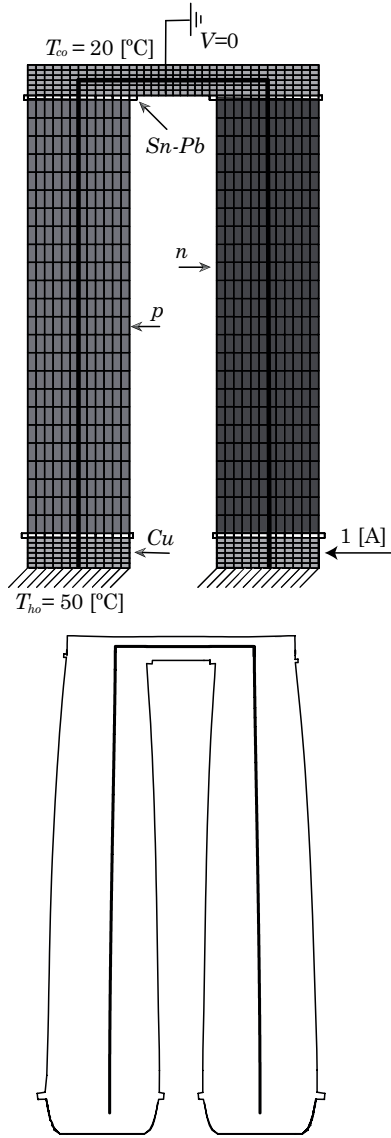


Figure 8: Top: 3D and beam (highlighted in solid black lines) meshes of a thermocouple, composed of four materials: n- and p-type thermoelements, copper *Cu* bars and tin *Sn-Pb* solders. Mechanical, thermal and electrical boundary conditions also represented. Bottom: Outline of the deformed 3D and beam meshes (zoom  $\times 500$ ).

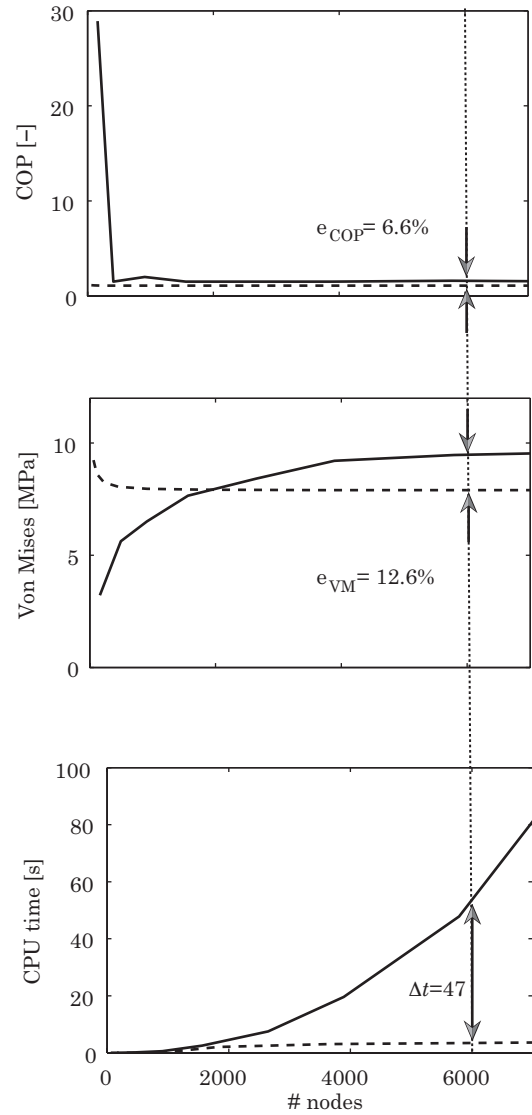


Figure 9: h-convergence and CPU time for both 3D (solid line) and beam (dashed line) steady-state solutions. COP (top), *Von Mises* stress (middle) and CPU time (bottom) vs. number of nodes. Relative errors between 3D and beam models denoted by  $e$ .

550 For the beam model, the requirement of nodes  
 551 (approximately 500) and consequently the CPU time  
 552 (lower than 3 [s]) decreases since a 1D mesh is used.  
 553 The main advantage of the beam formulation is the reduction  
 554 of CPU time; in contrast, the beam model has  
 555 a lower accuracy. The relative errors between 3D and  
 556 beam formulations are  $e_{COP} = 6.6\%$  and  $e_{VM} = 12.6\%$ .

557 To sum up, the lower CPU time for the beam model  
 558 could permit the application of the present formulation

559 in sensitivity analyses and optimizations, which require  
 560 many evaluations to obtain the final results.

561 Figure 10 shows the distributions of voltage (top-left),  
 562 temperature (top-right), horizontal (bottom-left) and  
 563 vertical (bottom-right) displacements versus the  
 564 distance. Now, solid lines represent the 3D calculations  
 565 and, again, solid circles the beam solutions.

566 Regarding voltage, the agreement between 3D and  
 567 beam results is very good except in the bottom *Cu* bar.  
 568 Obviously, the beam model shows an idealization for  
 569 which all the horizontal electric flux along the bar flows

vertically through the thermoelement. On the contrary, the 3D model captures the *rotational* effects that are present at both bottom corner of the thermoelement. This limitation of the beam model is the cause of the larger relative error in the COP calculation, as commented in the previous paragraphs.

In terms of temperature, a good agreement between both solutions is achieved. Minor differences are observed due to the higher potential drop (consequently, higher *Joule* heating) predicted by the 3D model.

On the one hand, the horizontal displacement (bottom-left) is due to the thermal contraction of the copper bar at the top of the devices that results in a bending of both thermoelements, see Figure 8 (bottom) where an outline of the deformed 3D and beam meshes are represented. On the other hand, the vertical displacement is due to the thermal expansion of the device. Notice that the reference temperature is assumed to be 25 [°C] and, therefore, both thermal expansion and contraction are present. The comparison between both, 3D and beam solutions, shows a reasonable agreement taking into account the limitations of the 1D formulation.

Nevertheless, the remarkable changes in the cross sections observed for the 3D model in Figure 8 (bottom) can not be captured by the beam model since, according to *AI*, the cross sections in the beam model are assumed as rigid bodies and, consequently, they can not be deformed.

## 6. Conclusions

This work presents a dynamic, non-linear and fully coupled finite element formulation based on the *Timoshenko* beam theory to study elasto-thermoelectricity. The formulation is implemented in the research code FEAP and is validated using 1D analytical solutions. Then, this formulation is used to model a pulsed *Peltier* device and the results are compared with 3D FE solutions.

The main advantage of the beam model is the low computational cost. For instance, a reduction of 47 [s] with respect to a 3D model can be achieved for the modeling of a pulsed *Peltier* devices. In contrast, the main drawback is the inherent error due to the 1D limitations of the beam theory. In this regard, a maximum relative COP error of 6.6% between both models is found.

In conclusion, the combination of the beam element to obtain coarse results and the 3D model to calculate details of the *Peltier* devices is a methodology that could provide good results with a comprehensive CPU time. This combination could be used in future optimizations and sensitivity analyses that require many evaluations.

## 7. Acknowledgments

The authors of the present work would like to acknowledge to the students Jorge Segarra Tamarit and Alfredo Martí García for their collaboration.

## 8. References

- [1] R.L. Taylor. *FEAP A Finite Element Analysis Program: User Manual*. University of California, Berkeley, 2010. <http://www.ce.berkeley.edu/feap>.
- [2] J.L. Pérez-Aparicio, R. Palma, and R.L. Taylor. Multiphysics and thermodynamic formulations for equilibrium and non-equilibrium interactions: non-linear finite elements applied to multi-coupled active materials. *Arch Comput Methods Eng*, accepted:–, 2015.
- [3] D. Zhao and G. Tan. A review of thermoelectric cooling: Materials, modeling and applications. *Applied Thermal Engineering*, 66:15–24, 2014.
- [4] J.L. Pérez-Aparicio, R.L. Taylor, and D. Gavela. Finite element analysis of nonlinear fully coupled thermoelectric materials. *Computational Mechanics*, 40:35–45, 2007.
- [5] J.L. Pérez-Aparicio, R. Palma, and R.L. Taylor. Finite element analysis and material sensitivity of Peltier thermoelectric cells coolers. *International Journal of Heat and Mass Transfer*, 55:1363–1374, 2012.
- [6] R. Palma, J.L. Pérez-Aparicio, and R.L. Taylor. Non-linear finite element formulation applied to thermoelectric materials under hyperbolic heat conduction model. *Computer Method in Applied Mechanics and Engineering*, 213-216:93–103, 2012.
- [7] R. Palma, J.L. Pérez-Aparicio, and R. Bravo. Study of hysteretic thermoelectric behavior in photovoltaic materials using the finite element method, extended thermodynamics and inverse problems. *Energy Conversion and Management*, 65:557–563, 2013.
- [8] I.J. Ferrer, P. Díaz-Chao, A. Pascual, and C. Sánchez. Hysteresis-like behaviour of the thermoelectric voltage in photovoltaic materials. *Thin Solid Films*, 511-512:177–181, 2006.
- [9] E.E. Antonova and D.C. Looman. Finite elements for thermoelectric device analysis in ansys. In *International Conference on Thermoelectrics*, 2005.
- [10] Y. Shi, Z. Zhu, Y.n Deng, W. Zhu, X. Chen, and Y. Zhao. A real-sized three-dimensional numerical model of thermoelectric generators at a given thermal input and matched load resistance. *Energy Conversion and Management*, 101:713–720, 2015.
- [11] G. Wu and X. Yu. A holistic 3D finite element simulation model for thermoelectric power generator element. *Energy Conversion and Management*, 86:99–110, 2014.
- [12] M. Jaegle. Multiphysics simulation of thermoelectric systems - modeling of peltier - cooling and thermoelectric generation. In *Proceedings of the COMSOL*, 2008.
- [13] S. Turenne, T. Clin, D. Vasilevskiy, and R.A. Masut. Finite element thermomechanical modeling of large area thermoelectric generators based on bismuth telluride alloys. *J Electron Mater*, 39:1926–1933, 2010.
- [14] M. Picard, S. Turenne, D. Vasilevskiy, and R.A. Masut. Numerical simulation of performance and thermomechanical behavior of thermoelectric modules with segmented bismuth-telluride-based legs. *J Electron Mater*, 42:2343–2349, 2013.
- [15] Y. Wu, T. Ming, X. Li, T. Pan, K. Peng, and X. Luo. Numerical simulations on the temperature gradient and thermal stress of a thermoelectric power generator. *Energy Conversion and Management*, 88:915–927, 2014.

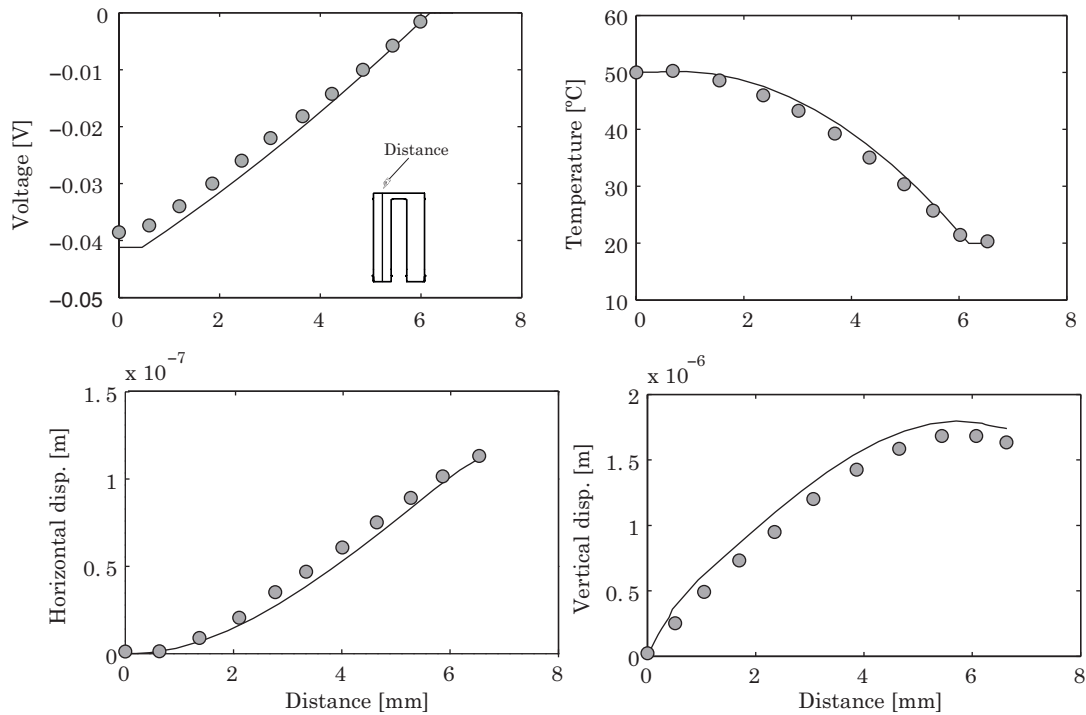


Figure 10: Distributions of voltage (top-left), temperature (top-right), horizontal (bottom-left) and vertical (bottom-right) displacements vs. distance. 3D calculations in solid lines and beam in solid circles.

- 679 [16] J.L. Pérez-Aparicio, R. Palma, and P. Moreno-Navarro. Elasto- 710  
680 thermoelastic non-linear, fully coupled, and dynamic finite 711  
681 element analysis of pulsed thermoelectrics. *Applied Thermal En-* 712  
682 *gineering*, 107:398–409, 2016. 713
- 683 [17] R. Poya, A.J. Gil, and P.D. Ledger. A computational frame- 714  
684 work for the analysis of linear piezoelectric beams using hp- 715  
685 FEM. *Computers & Structures*, 152:155–172, 2015. 716
- 686 [18] V. Kutis, J. Murín, R. Belák, and J. Paulech. Beam element with 717  
687 spatial variation of material properties for multiphysics analy-  
688 sis of functionally graded materials. *Computers & Structures*,  
689 89:1192–1205, 2011.
- 690 [19] M.A. Biot. Thermoelasticity and irreversible thermodynamics.  
691 *Journal of Applied Physics*, 27(3):240–253, 1956.
- 692 [20] X. Guo, Y.-B. Yi, and S. Pourkamali. A finite element analy-  
693 sis of thermoelastic damping in vented MEMS beam resonators.  
694 *International Journal of Mechanical Sciences*, 74:73–82, 2013.
- 695 [21] D.V. Parayil, S.S. Kularni, and D.N. Pawaskar. Analytical and  
696 numerical solutions for thick beams with thermoelastic damp-  
697 ing. *International Journal of Mechanical Sciences*, 94-95:10–  
698 19, 2015.
- 699 [22] S. Timoshenko. *Strength of Materials: Elementary Theory and*  
700 *Problems*. D. Van Nostrand Company, Inc., 1951.
- 701 [23] K.D. Hjelmstad, editor. *Fundamentals of Structural Mechanics*.  
702 Springer Science + Business Media, Inc., 2005.
- 703 [24] R.A. Sauer and J.C. Mergel. A geometrically exact finite beam  
704 element formulation for thin film adhesion and debonding. *Fi-*  
705 *nite Elements in Analysis and Design*, 86:120–135, 2014.
- 706 [25] S. Gao, B. Liang, and E. Vidal-Salle. Development of a new  
707 3D beam element with section changes: The first step for large  
708 scale textile modelling. *Finite Elements in Analysis and Design*,  
709 104:80–88, 2015.
- [26] R.E. Erkmén and M.M. Attard. Displacement-based finite ele-  
ment formulations for material-nonlinear analysis of composite  
beams and treatment of locking behaviour. *Finite Elements in*  
*Analysis and Design*, 47:1293–1305, 2011.
- [27] M.J. Huang, R.H. Yen, and A.B. Wang. The influence of the  
Thomson effect on the performance of a thermoelectric cooler.  
*International Journal of Heat and Mass Transfer*, 48:413–318,  
2005.