A dynamical comparison between iterative methods with memory: are the derivatives good for the memory? ☆

Alicia Cordero[a,*], Cristina Jordán[a], Juan R. Torregrosa[a]

[a] *Instituto Universitario de Matemática Multidisciplinar, Universitat Politècnica de València, València, Spain*

Abstract

The role of the derivatives at the iterative expression of methods with memory for solving nonlinear equations is analyzed in this manuscript. To get this aim, a known class of methods without memory is transformed into different families involving or not derivatives with an only accelerating parameter, then they are defined as discrete dynamical systems and the stability of the fixed points of their rational operators on quadratic polynomials are studied by means of real multidimensional dynamical tools, showing in all cases similar results. Finally, a different approach holding the derivatives, and by using different accelerating parameters, in the iterative methods involved present the most stable results, showing that the role of the appropriated accelerating factors is the most relevant fact in the design of this kind of iterative methods.

Keywords: Nonlinear equations, iterative method with memory, stability, bifurcation, basin of attraction, dynamical plane.

1. Introduction

The design of iterative methods for solving nonlinear equations or systems, $f(x) = 0$, is a challenging task that has proved to be bountiful in the last decades. From the Kung-Traub's conjecture [10], many authors have devoted their efforts in designing efficient optimal methods of increasing order of convergence. Kung and Traub also worked on iterative methods with memory, but it has been very recently when this kind of schemes have been re-discovered and many authors have dedicated their efforts in constructing new schemes with better convergence properties than their known partners. In this terms, the early works of Traub [17], Neta [12] and recent ones by Petković et al. [14, 15], Lotfi et al. [4, 11], Wang et al. [18], among others, give a close idea of the general interest on these methods. However, all of these researches focused their works on the design of the methods trying to improve their numerical aspects (convergence, number of functional evaluations, accelerators, etc.); only some recent works [1] approach this problem by means of the stability analysis, trying to find the anomalies and advantages of methods with memory. To get this aim, previous results in real discrete dynamics have been used (see, for example, [5, 7, 9, 16]).

Our goal in this paper is to carry out a dynamical study of some methods with memory with and without derivatives with a common without-memory partner. As the fixed point iteration functions have more than one variable, some auxiliary functions are introduced to facilitate the calculations. So, specific dynamical concepts are adapted to achieve the appropriate numerical sense.

On the other hand, we also analyze the local convergence of each method with memory under study. For it, we use the following result, that can be found in [13].

Theorem 1. *Let ψ be an iterative method with memory that generates a sequence $\{x_k\}$ of approximations to the root α, and let this sequence converges to α. If there exist a nonzero constant η and nonnegative numbers t_i, $i = 0, 1, \ldots, m$, such that the inequality*

$$|e_{k+1}| \leq \eta \prod_{i=0}^{m} |e_{k-i}|^{t_i}$$

☆This research was partially supported by Ministerio de Economía y Competitividad MTM2014-52016-C02-2-P.

*Corresponding author

Email addresses: acordero@mat.upv.es (Alicia Cordero), cjordan@mat.upv.es (Cristina Jordán), jrtorre@mat.upv.es (Juan R. Torregrosa)

holds, then the R-order of convergence of the iterative method ψ satisfies the inequality

$$O_R(\psi, \alpha) \geq s^*,$$

where s^* is the unique positive root of the equation

$$s^{m+1} - \sum_{i=0}^{m} t_i s^{m-i} = 0.$$

Once the order of convergence is stated, we focus our efforts in the main results of this manuscript, that is, the stability analysis and the study of the role of accelerators and the decision of holding the derivatives in their performance, or substituting them by divided differences.

1.1. Discrete dynamical systems

As it has been previously stated, the dynamical behavior of the operators associated to numerical methods is an efficient tool for analyzing the stability of the methods. In the following, we build the discrete dynamical system associated to an iterative method with memory in order to carry out its dynamical study.

The expression of an iterative method with memory, which uses two previous iterations to calculate the following estimation, is

$$x_{k+1} = g(x_{k-1}, x_k), \quad k \geq 1,$$

where x_0 and x_1 are the initial estimations. In order to obtain the fixed points of this method, we define [1] the *fixed point function* $G : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by means of:

$$\begin{aligned} G(x_{k-1}, x_k) &= (x_k, x_{k+1}), \\ &= (x_k, g(x_{k-1}, x_k)), \quad k = 1, 2, \ldots, \end{aligned}$$

being x_0 and x_1 the initial estimations. This definition can be extended in a natural way to adapt it to iterative schemes with memory that use more than two previous iterations per step.

As (x_{k-1}, x_k) is a fixed point of G if

$$G(x_{k-1}, x_k) = (x_{k-1}, x_k),$$

then, $x_{k+1} = x_k$ and $x_{k-1} = x_k$.

We have defined a discrete dynamical system in the real plane from function $G : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$G(z, x) = (x, g(z, x))$$

where g is the operator of the iterative method with memory. Fixed points (z, x) of G satisfy $z = x$ and $x = g(z, x)$.

In the following, we recall some basic real dynamical concepts. If a fixed point (z, x) of operator G is different from (r, r), where r is a zero of f, it is called *strange fixed point*. On the other hand, the orbit of a point $\bar{x} \in \mathbb{R}^2$ is defined as the set of successive images of $\bar{x}$ by the vector function, $\{\bar{x}, G(\bar{x}), \ldots, G^m(\bar{x}), \ldots\}$.

The dynamical behavior of the orbit of a point of $\mathbb{R}^2$ is classified depending on its asymptotical behavior. So, a point $x^* \in \mathbb{R}^2$ is a *k-periodic point* if $G^k(x^*) = x^*$ and $G^p(x^*) \neq x^*$, for $p = 1, 2, \ldots, k-1$. The stability of fixed points for multivariable nonlinear operators, see for example [16], satisfies the following statements:

Theorem 2. *Let G from $\mathbb{R}^n$ to $\mathbb{R}^n$ be $\mathcal{C}^2$. Assume x^* is a k-periodic point. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of $G'(x^*)$.*

a) If all the eigenvalues λ_j have $|\lambda_j| < 1$, then x^* is attracting.

b) If one eigenvalue λ_{j_0} has $|\lambda_{j_0}| > 1$, then x^* is unstable, that is, repelling or saddle.

c) If all the eigenvalues λ_j have $|\lambda_j| > 1$, then x^* is repelling.

In addition, a fixed point is called hyperbolic if all the eigenvalues λ_j of $G'(x^*)$ have $|\lambda_j| \neq 1$. In particular, if there exist an eigenvalue λ_i such that $|\lambda_i| < 1$ and an eigenvalue λ_j such that $|\lambda_j| > 1$, the hyperbolic point is called *s*addle point.

Moreover, a point x is a *c*ritical point of G if the associate Jacobian matrix $G'(x)$ satisfies $det(G'(x)) = 0$. One particular case of critical points, for iterative methods of convergence order higher than two, are those fixed points withassociated null eigenvalues $\lambda_j = 0, \forall j$. These points are called *s*uperattracting.

Then, if x^* is an attracting fixed point of function G, its *b*asin of attraction $\mathcal{A}(x^*)$ is defined as the set of pre-images of any order such that

$$\mathcal{A}(x^*) = \left\{ x^{(0)} \in \mathbb{R}^n : G^m(x^{(0)}) \to x^*, m \to \infty \right\}.$$

The set of the different basins of attraction define the *dynamical plane* of the system. The dynamical plane of a method is built by iterating a mesh of points and painting them in different colors depending on the attractor they converge to. The algorithms used appear in [2]; in this manuscript we have used a maximum number of iterations of 40, a mesh of 400×400 points and a tolerance of 10^{-3}.

The rest of this paper is summarized as follows: in Section 2, the design and convergence of some classes of iterative schemes with memory are presented; these families differ in the use (or not) of derivatives in their iterative expression and also on the number of accelerators used. Sections 3 and 4 are devoted to the real multidimensional analysis of these classes by using the stability of fixed points of the rational functions obtained when applying the methods on low-degree polynomials and also by means of bifurcation diagrams. Finally, some conclusions are stated.

2. Modified parametric family with memory

The family of fourth-order parametric methods under study was presented in [8] as an efficient class to estimate the solution of nonlinear systems of equations. Its iterative expression in the scalar case is

$$\begin{aligned} y_k &= x_k - \theta \frac{f(x_k)}{f'(x_k)}, \\ t_k &= x_k - \frac{f(y_k) + \theta f(x_k)}{f'(x_k)}, \\ x_{k+1} &= x_k - \frac{f(t_k) + f(y_k) + \theta f(x_k)}{f'(x_k)}, \quad k = 1, 2, \ldots \end{aligned}$$

and its local order of convergence is three, being fourth-order for $\theta = \pm 1$, under standard conditions. We denote this family by HMT.

In [1] this class was modified, by substituting the derivative $f'(x_k)$ appearing in all the steps by $f[x_k, w_k]$ (where $w_k = x_k + \gamma_k f(x_k)$), where γ_k is the acceleration parameter, to obtain a class of methods with memory denoted by MHMT. Its iterative expression is

$$\begin{aligned} \gamma_k &= -\frac{2}{f[x_k, x_{k-1}]}, \\ w_k &= x_k + \gamma_k f(x_k), \\ y_k &= x_k - \theta \frac{f(x_k)}{f[x_k, w_k]}, \\ t_k &= x_k - \frac{f(y_k) + \theta f(x_k)}{f[x_k, w_k]}, \\ x_{k+1} &= x_k - \frac{f(t_k) + f(y_k) + \theta f(x_k)}{f[x_k, w_k]}, \quad k = 1, 2, \ldots \end{aligned} \tag{1}$$

and its order of convergence was proved to be at least $\frac{1}{2}\left(3 + \sqrt{13}\right)$ if $\theta \neq 0$, and $2 + \sqrt{6}$ if $\theta = 1$. Its stability properties were studied on $x^2 - 1$, by using the tools of multidimensional real dynamics, and although it was showed to be stable in general, some unstable elements were found; in particular, two strange attractors were detected.

2.1. Design and local convergence

It is possible to design other classes with memory from HMT family, avoiding the use of divided differences (that is, preserving the derivatives). In this paper, we get two different ways to get this aim. In the first case, we add to the derivative appearing in each step the term $\gamma f(x_k)$. The iterative expression of the resulting class is

$$\begin{aligned} y_k &= x_k - \theta \frac{f(x_k)}{f'(x_k) + \gamma f(x_k)}, \\ t_k &= x_k - \frac{f(y_k) + \theta f(x_k)}{f'(x_k) + \gamma f(x_k)}, \\ x_{k+1} &= x_k - \frac{f(t_k) + f(y_k) + \theta f(x_k)}{f'(x_k) + \gamma f(x_k)}, \quad k = 1, 2, \ldots \end{aligned}$$

It is easy to prove that the third-order of convergence is held and the error equation is $e_{k+1} = (1-\theta)(\gamma + 2c_2)(\gamma + (1+\theta)c_2)e_k^3 + O(e_k^4)$, where $c_2 = \dfrac{f''(\alpha)}{2f'(\alpha)}$. Moreover, we get fourth-order of convergence for $\theta = 1$ (as in the original method), but not for $\theta = -1$.

To transform the iterative family in other one with memory increasing the order of convergence we need that $\gamma = -2c_2$ or $\gamma = -(1+\theta)c_2$. It can be done combining derivatives and divided differences,

$$\gamma_k^a = -\frac{f'[x_k, x_{k-1}]}{f'(x_k)} \text{ and } \gamma_k^b = -\frac{1+\theta}{2}\frac{f'[x_k, x_{k-1}]}{f'(x_k)},$$

or by using only divided differences,

$$\gamma_k^a = -\frac{f'[x_k, x_{k-1}]}{f[x_k, x_{k-1}]} \text{ and } \gamma_k^b = -\frac{1+\theta}{2}\frac{f'[x_k, x_{k-1}]}{f[x_k, x_{k-1}]}.$$

The first estimations are simpler and use a lower number of operations but, when they are applied on quadratic polynomials, the resulting rational functions no longer depend on the previous iterate, x_{k-1}, so they cannot be considered an iterative methods with memory. So, the second way to estimate the accelerating parameter is considered. Therefore, we get

$$\begin{aligned} \gamma_k^a &= -\frac{f'(x_k) - f'(x_{k-1})}{f(x_k) - f(x_{k-1})}, \\ y_k &= x_k - \theta \frac{f(x_k)}{f'(x_k) + \gamma_k^a f(x_k)}, \\ t_k &= x_k - \frac{f(y_k) + \theta f(x_k)}{f'(x_k) + \gamma_k^a f(x_k)}, \\ x_{k+1} &= x_k - \frac{f(t_k) + f(y_k) + \theta f(x_k)}{f'(x_k) + \gamma_k^a f(x_k)}, \quad k = 1, 2, \ldots \end{aligned} \tag{2}$$

denoted by MF1a, or

$$\begin{aligned} \gamma_k^b &= -\frac{1+\theta}{2}\frac{f'(x_k) - f'(x_{k-1})}{f(x_k) - f(x_{k-1})}, \\ y_k &= x_k - \theta \frac{f(x_k)}{f'(x_k) + \gamma_k^b f(x_k)}, \\ t_k &= x_k - \frac{f(y_k) + \theta f(x_k)}{f'(x_k) + \gamma_k^b f(x_k)}, \\ x_{k+1} &= x_k - \frac{f(t_k) + f(y_k) + \theta f(x_k)}{f'(x_k) + \gamma_k^b f(x_k)}, \quad k = 1, 2, \ldots \end{aligned} \tag{3}$$

denoted by MF1b. The local convergence of these schemes is analyzed in the following result. As MF1a has been studied in [3], we analyze the second option with derivatives, MF1b, but giving information about the coincidences and differences between them. In this case, both families have the same order of convergence, except for $\theta = -1$.

Theorem 3. *Let α be a simple zero of a sufficiently differentiable function $f : D \subset \mathbb{R} \to \mathbb{R}$ in an open interval D. If x_0 and x_1 are sufficiently close to α, then the order of convergence of methods with memory* (3) *is at least* $\frac{1}{2}\left(3+\sqrt{13}\right)$. *The error equation is,*

$$e_{k+1} = \frac{1}{2}(\theta-1)^2(\theta+1)c_2(2c_2^2-3c_3)e_{k-1}e_k^3 + O_4(e_{k-1}e_k),$$

where $c_j = \frac{1}{j!}\frac{f^{(j)}(\alpha)}{f'(\alpha)}$, $j = 2, 3, \ldots$ *and* $O_4(e_{k-1}e_k)$ *indicates that the sum of exponents of* e_{k-1} *and* e_k *in the rejected terms of the development is at least 4. However, if* $\theta = 1$, *the error equation is*

$$e_{k+1} = \frac{1}{2}(-8c_2^5 + 24c_2^3c_3 - 18c_2c_3^2)e_{k-1}^2e_k^4 + O_6(e_{k-1}e_k),$$

being the local order $2+\sqrt{6}$. *Finally, taking* $\theta = -1$ *in* (3), *the resulting error equation is of fourth-order of convergence,*

$$e_{k+1} = 2c_2(2c_2^2-2c_3)e_k^4 + O_4(e_{k-1}e_k).$$

Proof: By using Taylor series expansions, we obtain

$$\begin{aligned} y_k - \alpha &= e_k - \theta\frac{f(x_k)}{f'(x_k)+\gamma_k^b f(x_k)} \\ &= (1-\theta)e_k - \theta^2 c_2 e_k^2 + \frac{1}{2}\theta(1+\theta)\left(2c_2^2-3c_3\right)e_{k-1}e_k^2 + O_3(e_{k-1}e_k), \end{aligned}$$

then, the Taylor development of the second step is

$$\begin{aligned} t_k - \alpha &= e_k - \frac{f(y_k)+\theta f(x_k)}{f'(x_k)+\gamma_k^b f(x_k)} \\ &= -\frac{1}{2}\left(\left(-1+\theta^2\right)\left(2c_2^2-3c_3\right)\right)e_{k-1}e_k^2 + \frac{1}{2}\left(-1+\theta^2\right)\left(2c_2^3-5c_2c_3+4c_4\right)e_{k-1}^2e_k^2 + O_4(e_{k-1}e_k) \end{aligned}$$

and finally the error equation yields

$$e_{k+1} = -(\theta-1)^2(\theta+1)c_2(2c_2^2-3c_3)e_{k-1}e_k^3 + O_4(e_{k-1}e_k).$$

By using Theorem 1, the unique positive root of polynomial p^2-3p-1 gives us the R-order of the method, being in this case $p = \frac{1}{2}\left(3+\sqrt{13}\right)$. In the particular case $\theta = 1$, it can be checked that the polynomial whose positive root gives us the local order is p^2-4p-2, that yields $p = 2+\sqrt{6}$. □

On the other hand, returning to the initial class (2), it is not necessary to hold the same accelerating parameter at the denominator of all the steps in the process; if different accelerating parameters, γ_1, γ_2 and γ_3, are used in the iterative expression, then the analysis of the convergence of the resulting family without memory shows that, fixing $\gamma_2 = \gamma_3 = \theta\gamma_1$, the fourth order of convergence is held for both original values, $\theta = \pm 1$, being the error equation in this case

$$e_{k+1} = -(\theta^2-1)(\theta\gamma_1+2c_2)c_2e_k^3 + O(e_k^4).$$

By estimating $\gamma_1 = -\frac{2}{\theta}c_2$, we get

$$\begin{aligned} \gamma_{1k} &= -\frac{1}{\theta}\frac{f'[x_k,x_{k-1}]}{f[x_k,x_{k-1}]} \\ y_k &= x_k - \theta\frac{f(x_k)}{f'(x_k)+\gamma_{1k}f(x_k)}, \\ t_k &= x_k - \frac{f(y_k)+\theta f(x_k)}{f'(x_k)+\theta\gamma_{1k}f(x_k)}, \\ x_{k+1} &= x_k - \frac{f(t_k)+f(y_k)+\theta f(x_k)}{f'(x_k)+\theta\gamma_{1k}f(x_k)}, \quad k = 1, 2, \ldots \end{aligned} \tag{4}$$

that is another class of iterative schemes with memory, denoted by MF2. The local convergence of this scheme is analyzed in the following result, whose proof is similar to that of Theorem 3.

Theorem 4. *Let α be a simple zero of a sufficiently differentiable function $f : D \subset \mathbb{R} \to \mathbb{R}$ in an open interval D. If x_0 and x_1 are sufficiently close to α, then the order of convergence of methods with memory* (4) *is at least* $\frac{1}{2}\left(3+\sqrt{13}\right)$. *The error equation is*

$$e_{k+1} = -2(\theta^2-1)c_2(2c_2^2-3c_3)e_{k-1}e_k^3 + O_4(e_{k-1}e_k),$$

where $c_j = \frac{1}{j!}\frac{f^{(j)}(\alpha)}{f'(\alpha)}$, $j = 2, 3, \ldots$ *However, if* $\theta = 1$, *the error equation is*

$$e_{k+1} = -(4c_2^5 + 12c_2^3c_3 - 9c_2c_3^2)e_{k-1}^2e_k^4 + O_6(e_{k-1}e_k),$$

being the local order $2+\sqrt{6}$ *in this case and if* $\theta = -1$, *the resulting error equation is*

$$e_{k+1} = 2(-2c_2^2c_3 + 3c_3^2)e_{k-1}^2e_k^4 + O_6(e_{k-1}e_k),$$

and the local order is in this case $2+\sqrt{5}$.

In the following sections, the rational functions obtained when the proposed families with memory are applied on a set of real quadratic polynomials are analyzed by using the dynamical tools described in the Introduction.

As our aim is to analyze the dynamical behavior of the proposed families on real quadratic polynomials, we will study the fixed point operator associated to the presented families on $p_1(x) = x^2-1$, $p_2(x) = x^2+1$ and $p_3(x) = x^2$, that will be denoted by $M_1^j(z,x,\theta)$, $M_2^j(z,x,\theta)$ and $M_3^j(z,x,\theta)$, respectively, where $j = 1a, 1b, 2$ depending on the analyzed family is (2), (3), or (4). We choose these polynomials since it is known (see [6]) that any quadratic polynomial, by an affine change of variables reduces to one of them. The dynamics of operators associated to affine conjugate functions are equivalent. Let us observe that each one of the previous operator is a function of two variables: the last iteration, x_k (denoted by x), the previous one x_{k-1} denoted by z and one parameter, θ.

In order to analyze the stability of the members of the class MF (including MF1a, MF1b and MF2), we study the asymptotic behavior of the fixed points of the respective rational functions obtained by applying the fixed point operator MF on each one of the polynomials, $p_i(x)$ $i = 1, 2, 3$. In previous analysis, it was found that the only attracting strange fixed point of method MHMT on quadratic polynomials was $(0, 0)$, in the interval $-4 < \theta < -2$ (see [1]) and strange attractors were found. A similar behavior is found in [3] for MF1a on quadratic polynomials, where period-doubling bifurcations appear for some values of parameter θ, including an area of chaotic performance. In the following, we complete the analysis with MF1b and MF2.

3. Multidimensional dynamical analysis of family MF1b

Now, we study the behavior of the operator associated to family MF1b on quadratic polynomials. Firstly, we study the associate fixed point operator on $p_1(x) = x^2-1$,

$$M_1^{1b}(z,x,\theta) \quad = \quad \left(x, x - C - D - \frac{(x+z)((x-C-D)^2-1)}{-\theta x^2+\theta+x^2+2xz+1}\right),$$

where

$$C = \frac{\theta\left(x^2-1\right)(x+z)}{-\theta x^2+\theta+x^2+2xz+1}$$

and

$$D = \frac{\left(x^2-1\right)(x+z)\left(\theta(x+1)(2x+z-1)-x^2-2xz-1\right)\left(\theta(x-1)(2x+z+1)-x^2-2xz-1\right)}{\left(-\theta x^2+\theta+x^2+2xz+1\right)^3}.$$

In an analogous way as in the Introduction, all the fixed points have two equal components.

Proposition 1. *The fixed points (and their stability) of the operator associated to MF1b on quadratic polynomial* $p_1(x)$ *are:*

a) Points $(1,1)$ *and* $(-1,-1)$ *associated to the roots of* $p_1(x)$, *being both superattracting.*

b) The origin $(z, x) = (0, 0)$*, which is an attracting fixed point for* $-2 < \theta < -\frac{3}{2}$*, it is repulsive if* $-\frac{3}{2} < \theta < -1$ *and it is a saddle point in other cases.*

c) The real roots of polynomial $s(x) = 2+13\theta+36\theta^2+55\theta^3+50\theta^4+27\theta^5+8\theta^6+\theta^7+(28+134\theta+240\theta^2+174\theta^3-4\theta^4-78\theta^5-40\theta^6-6\theta^7)x^2+(166+543\theta+516\theta^2-15\theta^3-186\theta^4+33\theta^5+80\theta^6+15\theta^7)x^4+(552+1092\theta+192\theta^2-604\theta^3-152\theta^4+108\theta^5-80\theta^6-20\theta^7)x^6+(1134+891\theta-1236\theta^2-663\theta^3+862\theta^4-147\theta^5+40\theta^6+15\theta^7)x^8+(1404-810\theta-1584\theta^2+1998\theta^3-804\theta^4+66\theta^5-8\theta^6-6\theta^7)x^{10}+(810-1863\theta+1836\theta^2-945\theta^3+234\theta^4-9\theta^5+\theta^7)x^{12}$*, whose number varies depending on the range of parameter* θ*: there are four real saddle points if* $\theta < -7.72447$*, two saddle points if* $-7.72447 \leq \theta < -2$*, two saddle points if* $\theta = -2$*, four saddle points if* $-2 \leq \theta < -1$*, two saddle points if* $\theta = -1$*, four saddle points when* $-1 < \theta < -0.846682$ *four, two saddle points if* $\theta = -0.846682$*, none if* $-0.846682 < \theta < 9.70662$*, two non-hyperbolic points if* $\theta = 9.70662$ *and four (two saddle and two attracting points) if* $\theta > 9.70662$*.*

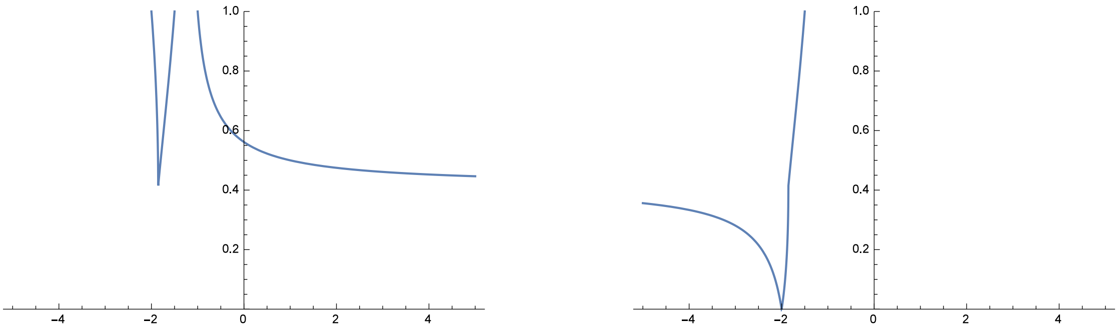
Proof: In order to obtain the fixed points of MF1b on $p_1(x)$ it is necessary to solve the equation

$$M_1^{1b}(z, x, \theta) = (z, x),$$

that is, $z = x$ and

$$\frac{2x\left(x^2-1\right)s(x)}{\left(\theta\left(x^2-1\right)-3x^2-1\right)^7} = 0.$$

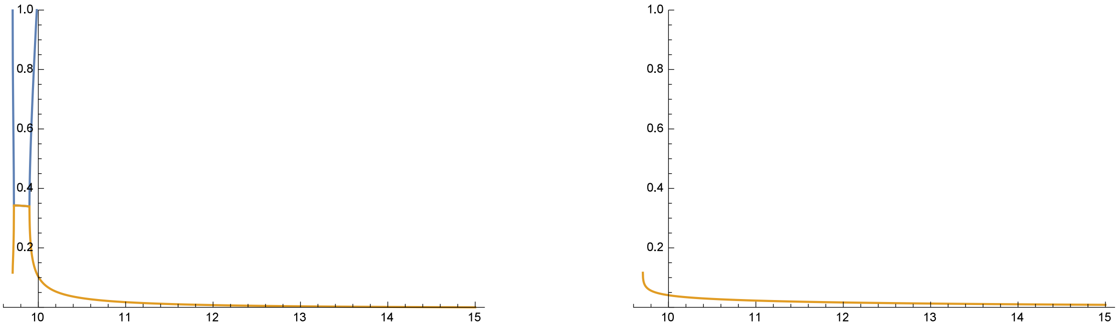
Obviously, points $(1, 1)$ and $(-1, -1)$ satisfy the previous equation and their associate eigenvalues are null; so, they are superattracting. It is clear that $(0, 0)$ is also a fixed point whose associate eigenvalues are $\lambda_1 = \frac{2\theta+3-\sqrt{8\theta^2+24\theta+17}}{2(\theta+1)}$ and $\lambda_2 = \frac{2\theta+3+\sqrt{8\theta^2+24\theta+17}}{2(\theta+1)}$. It can be checked that, for $-2 < \theta < -\frac{3}{2}$, both eigenvalues are lower than one in absolute value (see Figure 1), are higher than one (repulsive) if $-\frac{3}{2} < \theta < -1$ and are saddle points in the rest of real values of θ.



(a) $|\lambda_1|$ (b) $|\lambda_2|$

Figure 1: Absolute value of the eigenvalues of $M_1^{1b\prime}(0, 0, \theta)$

The rest of strange fixed points are the roots of polynomial $s(x)$, denoted by s_i, $i = 1, 2, \ldots, 12$. By analyzing the number of real roots of $s(x)$, we remark a clear dependence of the value of parameter θ. If $\theta < -7.72447$, the first four roots of $s(x)$ are real and are saddle points. When $-7.72447 \leq \theta < -2$, s_1 and s_2 are real saddle points and $\theta = -2$ yields $x = \pm 0.639037$ as saddle fixed points. For $-2 < \theta < -1$, s_i $i = 1, 2, 3, 4$ are real and saddle fixed points. In the limit value $\theta = -1$ there are only two real saddle points, $x = \pm 0.353123$. This situation is repeated in the range $-1 < \theta < -0.846682$ where s_i $i = 1, 2, 3, 4$ are saddle points, but s_2 and s_4 become complex for $\theta = -0.846682$. There are no strange fixed points different from $(0, 0)$ for values of θ in $]-0.846682, 9.70662[$. For $\theta = 9.70662$, only s_1 and s_3 are real and the eigenvalues of both points coincide, $\lambda_1 = 1$, $\lambda_2 = 0.116417$, so they are not hyperbolic. If $\theta > 9.70662$, s_2 and s_3 are real saddle points, meanwhile s_1 and s_4 are simultaneously attractive in $]9.7066, 9.72[\bigcup]9.9, 9.98[$, approximately; for other values of θ both are simultaneously saddle points (they can be seen in Figure 2). $\square$

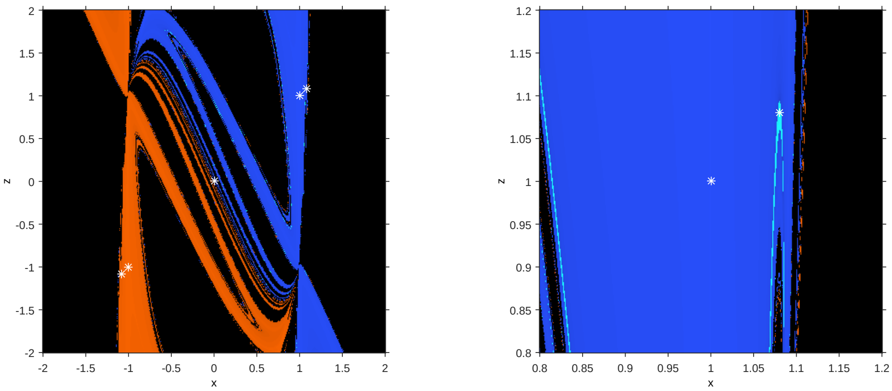


(a) $|\lambda_1|$ and $|\lambda_2|$ of r_i, $i = 1, 4$

(b) $|\lambda_1|$ and $|\lambda_2|$ of r_2

Figure 2: Absolute value of the eigenvalues of$M_1^{1b'}(r_i, r_i, \theta)$,$i = 1, 2$

An example of the behavior stated at Proposition 1 is presented at Figure 3, where the basins of attraction for $\theta = 9.71$ are showed. As can be observed at Figure 3a, the basins of the roots of polynomial $p_1(x)$ are much bigger than the ones of the strange fixed points, that can only be seen at its detail (Figure 3b).



(a) $\theta = 9.71$

(b) $\theta = 9.71$, a detail

Figure 3: Dynamical plane of MF1b method on $p_1(x) = x^2 - 1$

The corresponding results on polynomials $p_2(x)$ and $p_3(x)$ are shown below, whose proofs are similar to the previous ones; for this reason, they will be omitted. The associate fixed point operator on $p_2(x) = x^2 + 1$ is expressed as

$$M_2^{1b}(z, x, \theta) = \left(x, -\frac{F^2+1}{2x - \frac{(\theta+1)(x^2+1)}{x+z}} - \frac{\left(F - \frac{F^2+1}{2x - \frac{(\theta+1)(x^2+1)}{x+z}}\right)^2 + 1}{2x - \frac{(\theta+1)(x^2+1)}{x+z}} + F \right),$$

where

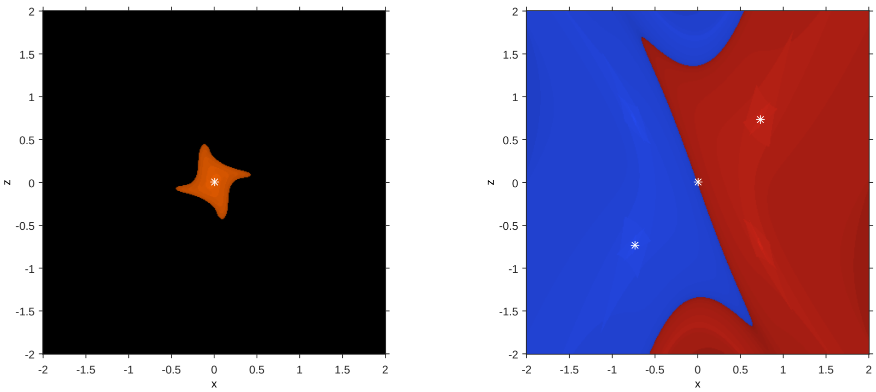
$$F = \frac{\theta\left(x^2+1\right)(x+z)}{\theta\left(x^2+1\right) - x(x+2z) + 1} + x.$$

Proposition 2. *The fixed points (and their stability) of the operator associated to MF1b on quadratic polynomial $p_2(x)$ are:*

a) The origin $(z, x) = (0, 0)$, with undetermined stability if $\theta \neq -1$ (for $\theta = -1$, there no exist real strange fixed points).

b) If $-7.72447 < \theta < -2$, *there exist two real roots of* $r(x) = 2 + 13\theta + 36\theta^2 + 55\theta^3 + 50\theta^4 + 27\theta^5 + 8\theta^6 + \theta^7 - (28 + 134\theta + 240\theta^2 + 174\theta^3 - 4\theta^4 - 78\theta^5 - 40\theta^6 - 6\theta^7)x^2 + (166 + 543\theta + 516\theta^2 - 15\theta^3 - 186\theta^4 + 33\theta^5 + 80\theta^6 + 15\theta^7)x^4 - (552 + 1092\theta + 192\theta^2 - 604\theta^3 - 152\theta^4 + 108\theta^5 - 80\theta^6 - 20\theta^7)x^6 + (1134 + 891\theta - 1236\theta^2 - 663\theta^3 + 862\theta^4 - 147\theta^5 + 40\theta^6 + 15\theta^7)x^8 - (1404 - 810\theta - 1584\theta^2 + 1998\theta^3 - 804\theta^4 + 66\theta^5 - 8\theta^6 - 6\theta^7)x^{10} + (810 - 1863\theta + 1836\theta^2 - 945\theta^3 + 234\theta^4 - 9\theta^5 + \theta^7)x^{12}$ *that are attracting fixed points. For any other value of parameter* θ*, there are no strange fixed points different from* $(0, 0)$.

Some of the results in Proposition 2 can be visualized at Figure 4, where the origin is seen as an attracting fixed point when $\theta = -1.8$ (Figure 4a), but it is a saddle point if $\theta = -5$, when other two fixed points are attracting, as can be seen at Figure 4b.



(a) $\theta = -1.8$

(b) $\theta = -5$

Figure 4: Dynamical planes of MF1b method on $p_2(x) = x^2 + 1$

Finally, the associate fixed point operator to MF1b on $p_3(x) = x^2$ is expressed as

$$M_3^{1b}(z, x, \theta) = \left(x, \frac{(Gx)\left(\frac{G(x+z)}{(\theta-1)x-2z} + ((\theta-1)x - 2z)^3\right)}{(-\theta x + x + 2z)^6} \right),$$

being

$$G = ((2\theta - 1)x + (\theta - 2)z)\left(\theta^2 x^2 + \theta z(z - x) + z(x + 2z)\right).$$

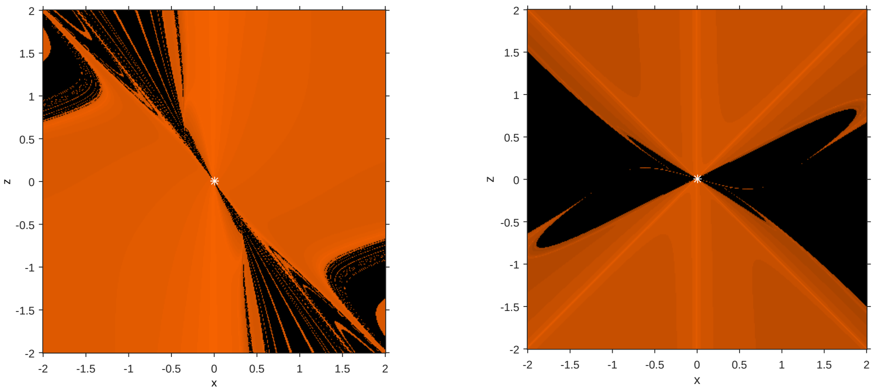
Let us remark that the only fixed point of the operator associated to MF1b on quadratic polynomial $p_3(x)$ is $(z, x) = (0, 0)$, except for $\theta = 3$, where the rational operator is not defined. Moreover, an indetermination appears again when its stability is analyzed, so the stability of $(0, 0)$ it cannot be determined by using Robinson's Theorem. In Figure 5, the origin has a wide basin of convergence, but it is located at its boundary (being repulsive in this direction), so it is a saddle point.

Let us remark that, although the dynamics of the family is quite stable, for particular values of the parameter there exist attracting fixed points different from the solutions of the problem. Now we analyze how this stability varies in terms of the parameter and if there exist some other bifurcations, leading not only to attracting points from repulsive ones, but also to periodic orbits or other kind of attractors.

3.1. Bifurcation diagrams

To study the bifurcation phenomena, we use Feigenbaum diagrams of the map associated to the family MF1b on quadratic polynomials $p_i(x)$, $i = 1, 2, 3$ by using as a starting point each one of the strange fixed points of the map and observing the ranges of the parameter θ where changes of stability or other behaviors happen.

By using the strange fixed point $(0, 0)$ as initial estimation, a Feigenbaum diagram can be seen in Figure 6a. As in this case the stability cannot be determined by using the Theorem 2, (see Proposition 2), the bifurcation diagram gives us clear information, that is, $(0, 0)$ is an attracting fixed point for $\theta \in]-2, -1.5[$. Out of this interval, periodic orbits bifurcations and two chaotic regions appear for values of parameter slightly higher than -1.5, see a detail in Figure 6. Moreover, convergence to the roots of $p_1(x)$ is observed. When the roots of $r(x)$ are used as

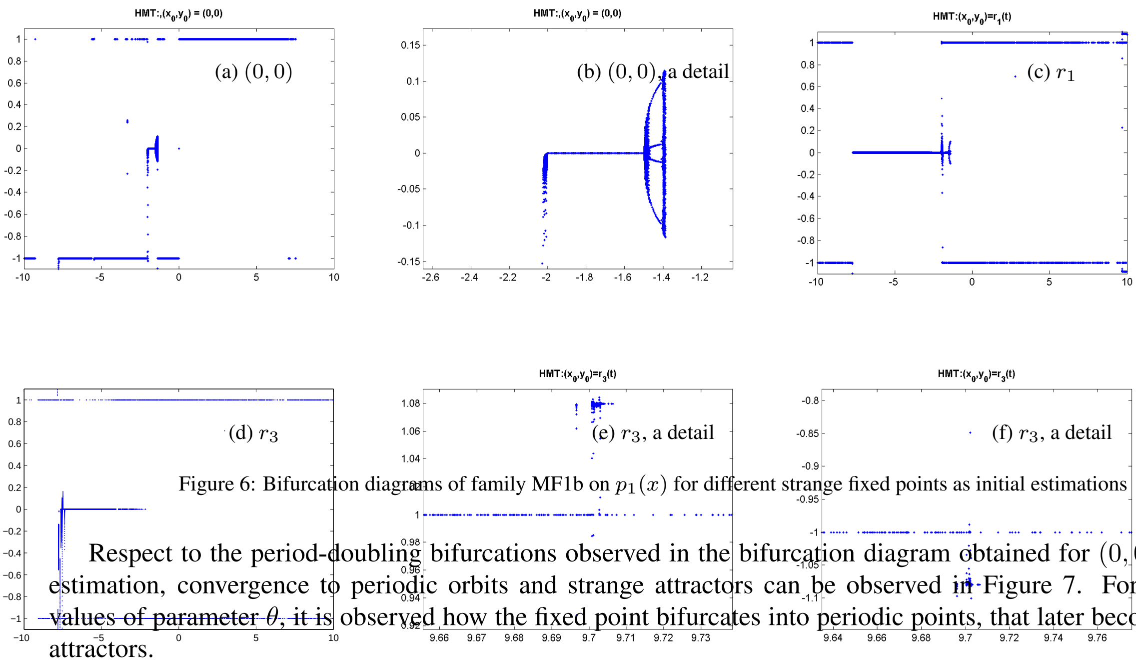


(a) $\theta = -2$

(b) $\theta = 1$

Figure 5: Dynamical plane of MF1b method on $p_3(x) = x^2$

initial estimations, the obtained bifurcation diagrams show convergence to zero or to the roots (even divergence, for values of $\theta \in]-8,-7.4]$), and also convergence to two attracting strange points close to 1 and -1, for values of θ in a small interval around 9.7.



(a) $(0,0)$

(b) $(0,0)$, a detail

(c) r_1

(d) r_3

(e) r_3, a detail

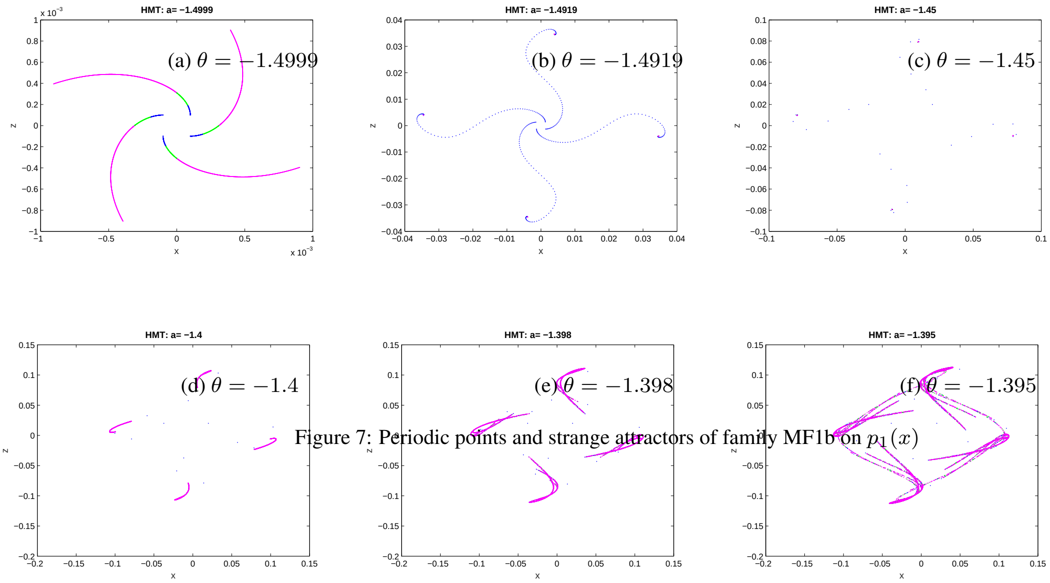
(f) r_3, a detail

Figure 6: Bifurcation diagrams of family MF1b on $p_1(x)$ for different strange fixed points as initial estimations

Respect to the period-doubling bifurcations observed in the bifurcation diagram obtained for $(0,0)$ as initial estimation, convergence to periodic orbits and strange attractors can be observed in Figure 7. For increasing values of parameter θ, it is observed how the fixed point bifurcates into periodic points, that later become strange attractors.

As it has been stated in Proposition 2, only two strange fixed points are stable in the interval $-7.72447 < \theta < -2$, when MF1b is applied on $p_2(x)$ (see Figure 8c). Moreover, the strange point $(0,0)$ can behave as an attracting point (see for example Figure 8b), but it also is immerse in a chaotic region where orbits are dense, see Figure 8a.

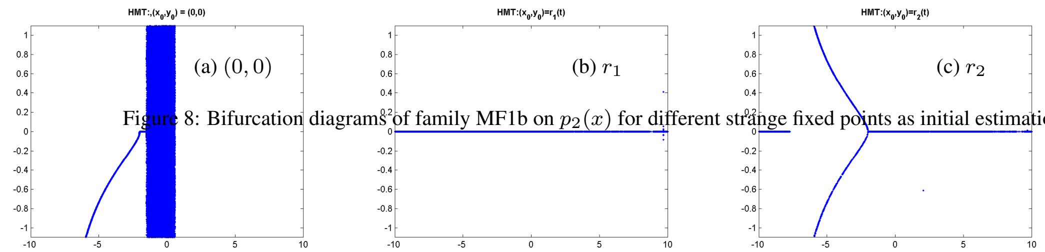
In order to better understand the behavior of family MF1b with memory, we plot in (z,x)-space the iteration of



(a) $\theta = -1.4999$ (b) $\theta = -1.4919$ (c) $\theta = -1.45$

(d) $\theta = -1.4$ (e) $\theta = -1.398$ (f) $\theta = -1.395$

Figure 7: Periodic points and strange attractors of family MF1b on $p_1(x)$



(a) $(0, 0)$ (b) r_1 (c) r_2

Figure 8: Bifurcation diagrams of family MF1b on $p_2(x)$ for different strange fixed points as initial estimations

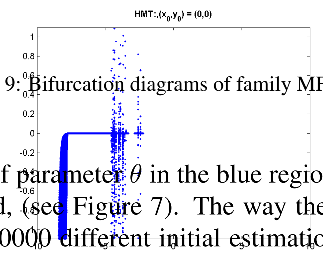


Figure 9: Bifurcation diagrams of family MF1b on $p_3(x)$ from $(0, 0)$

operator $M_2^2(z, x, \theta)$, for values of parameter θ in the blue region of Figure 6b close to $\theta = -1.4$. So, symmetric strange attractors have been found, (see Figure 7). The way these pictures have been obtained is the following: fixing the value of parameter θ, 10000 different initial estimations have been taken in a small rectangle close to the origin. The method has been used on each of them, plotting one point per iteration. The code color used is as follows: the first 2000 iterations appear in blue color, the following 2000 in green and the rest of them in magenta color. The resulting images show how the four attracting strange fixed points appearing in the bifurcation diagrams change into attracting regions, being disjoint or not depending of the value of the parameter. However, the set of initial estimations that belong to their respective basins of attraction is very reduced, as well as the interval of real values of θ that induces this behavior.

4. Dynamical analysis of family MF2

Finally, we analyze the rational function corresponding to family MF2 on quadratic polynomials, starting from $p_1(x) = x^2 - 1$. The fixed point operator is, in this case,

$$M_1^2(z,x,\theta) \quad = \quad \left(x, x - M - \frac{(x+z)(-1+x-M)(1+x-M)}{2+2xz}\right),$$

where

$$M = \frac{KL\left(-1+x^2\right)(x+z)}{8(1+xz)\left(1+(-1+\theta)x^2+\theta xz\right)^2} + \frac{\theta^2\left(-1+x^2\right)(x+z)}{2\left(1+(-1+\theta)x^2+\theta xz\right)},$$

$$K = \left(-2+2x^2+\theta^2(-1+x)(x+z)-2\theta x(x+z)\right)$$

and

$$L = \left(-2+2x^2-2\theta x(x+z)+\theta^2(1+x)(x+z)\right).$$

As in previous analysis, due to the construction of the multidimensional operator, all the fixed points of $M_1^2(z,x,\theta)$ have two equal components.

Proposition 3. *The fixed points (and their stability) of the operator associated to MF2 on quadratic polynomial $p_1(x)$ are:*

a) Points $(1,1)$ and $(-1,-1)$ associated to the roots, being both superattracting.

b) The origin $(z,x) = (0,0)$, which is a saddle point for any value of θ.

c) The real roots of polynomial $q(x) = 2+\theta^2+(-7+16\theta-6\theta^2+6\theta^3-2\theta^4)x^2+(9-40\theta+63\theta^2-30\theta^3+22\theta^4-8\theta^5+2\theta^6)x^4+(-6+32\theta-92\theta^2+124\theta^3-68\theta^4+40\theta^5-16\theta^6+4\theta^7-\theta^8)x^6+(4-16\theta+39\theta^2-92\theta^3+124\theta^4-72\theta^5+36\theta^6-12\theta^7+3\theta^8)x^8+(-3+16\theta-30\theta^2+30\theta^3-42\theta^4+56\theta^5-32\theta^6+12\theta^7-3\theta^8)x^{10}+(1-8\theta+25\theta^2-38\theta^3+30\theta^4-16\theta^5+10\theta^6-4\theta^7+\theta^8)x^{12}$, whose number varies depending on the range of parameter θ: there are four real saddle points if $\theta < 0$ or $\theta > 4.12098$ and two saddle points if $\theta = 4.12098$.

When MF2 is applied on polynomial $p_2(x) = x^2 + 1$, the resulting rational function is

$$M_2^2(z,x,\theta) = \left(x, N - \frac{(x+z)\left(1+N^2\right)}{2(-1+xz)} - (x+z)\frac{1+\left(N-\frac{(x+z)\left(1+N^2\right)}{2(-1+xz)}\right)^2}{2(-1+xz)}\right),$$

where $N = x - \frac{\theta\left(1+x^2\right)}{2x-\frac{2\left(1+x^2\right)}{\theta(x+z)}}$. As the fixed points must satisfy $z = x$, the analysis of the rational function yields that the only real fixed point is $(0,0)$, whose stability depends of the values of the eigenvalues of its associate Jacobian matrix, $\lambda_1 = \frac{1}{4}\left(4+\theta^2-\sqrt{32+16\theta^2+\theta^4}\right)$ and $\lambda_2 = \frac{1}{4}\left(4+\theta^2+\sqrt{32+16\theta^2+\theta^4}\right)$. It can be shown that $|\lambda_1| < 1$ for all real θ and simultaneously $|\lambda_2| > 1$. So, the origin is a saddle point. This result is summarized in the following proposition.

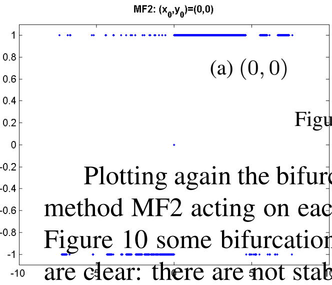
Proposition 4. *The only real fixed point of the operator associated to MF2 on quadratic polynomial $p_2(x)$ is $(0,0)$ and it is a saddle point.*

In the same manner, the following result can be stated, where the stability of the only fixed point for the fixed point operator associated to MF2 on quadratic polynomial $p_3(x) = x^2$ can not be determined by means of Robinson's Theorem.

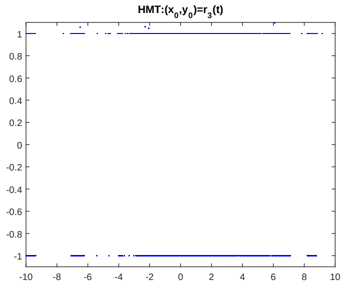
Proposition 5. *The fixed point operator associated to MF2 on quadratic polynomial $p_3(x) = x^2$ is*

$$M_3^2(z,x,\theta) = \left(x, \frac{(Px + (-2+\theta)\theta z)\left(Px^2 + 2\left(-1+\theta^2\right)xz + \theta(2+\theta)z^2\right)x}{64z^2((-1+\theta)x+\theta z)^4} \right.$$
$$\left. \left(-8z((-1+\theta)x+\theta z)^2 - \frac{(x+z)\,(Px + (-2+\theta)\theta z)\left(Px^2 + 2\left(-1+\theta^2\right)xz + \theta(2+\theta)z^2\right)}{2z} \right) \right),$$

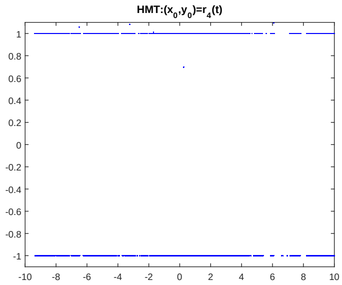
where $P = 2 - 2\theta + \theta^2$, being $(0,0)$ the only fixed point if $\theta \neq \frac{1}{2}$.



(a) $(0,0)$



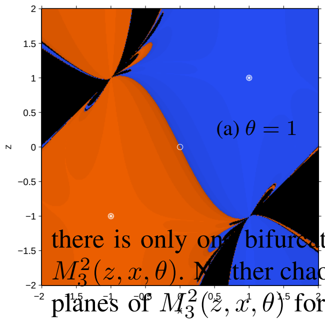
(b) r_3



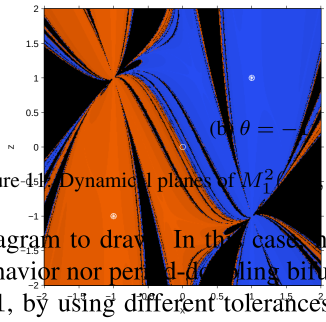
(c) r_4

Figure 10: Bifurcation diagrams of family MF2 on $p_1(x)$ from strange fixed points

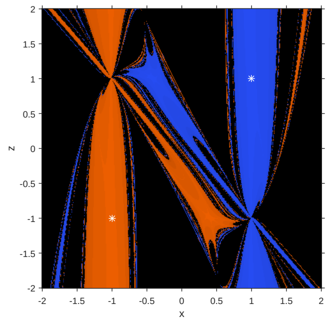
Plotting again the bifurcation diagrams by using the strange fixed points of each operator (corresponding to the method MF2 acting on each of the polynomials $p_i(x)$, $i = 1, 2, 3$, have shown only convergence to the roots. In Figure 10 some bifurcation diagrams are showed for $p_1(x)$. The conclusions about the stability of method MF2 are clear: there are not stable strange fixed points and strange attractors do not appear in the bifurcation diagrams (see some dynamical planes in Figure 11). As the only fixed point of the rational operator $M_2^2(z,x,\theta)$ is $(0,0)$,



(a) $\theta = 1$



(b) $\theta = -1$



(c) $\theta = -3$

Figure 11: Dynamical planes of $M_1^2(z,x,\theta)$ for different values of θ

there is only one bifurcation diagram to draw. In this case, no convergent behavior is observed, as well as for $M_3^2(z,x,\theta)$. Neither chaotic behavior nor period-doubling bifurcation are showed. In Figure 12, some dynamical planes of $M_3^2(z,x,\theta)$ for $\theta = 1$, by using different tolerances are presented, showing that the stability of fixed point appearing in the dynamical plane does not depend of the tolerance used in the calculations. In fact, the orange lines $z = x$ and $z = -x$ are pre-images of the stable manifold $x = 0$, that is the quicker way to converge to the fixed point.

5. Conclusions

In this paper, some classes of iterative methods with memory have been obtained from the same without memory partner and their qualitative behavior have been analyzed by using real multidimensional dynamical tools.

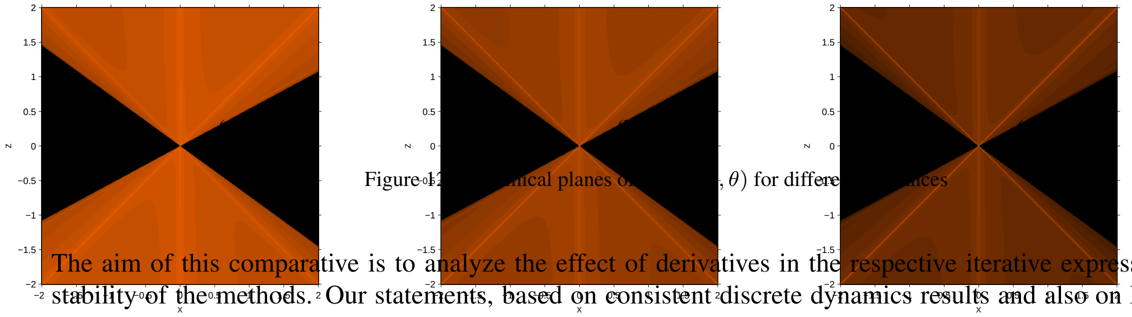


Figure 12: ... ical planes o... , θ) for differe... nces

The aim of this comparative is to analyze the effect of derivatives in the respective iterative expressions on the stability of the methods. Our statements, based on consistent discrete dynamics results and also on Feigenbaum diagrams of the families, allow us to affirm that, in this case, the role of the accelerators (in number and in the way they are defined) is greater than that the use or not of derivatives.

[1] B. Campos, A. Cordero, J.R. Torregrosa, P. Vindel, A multidimensional dynamical approach to iterative methods with memory, Appl. Math. Comp. 271 (2015) 701–715.

[2] F.I. Chicharro, A. Cordero, J.R. Torregrosa, Drawing dynamical and parameters planes of iterative families and methods, Sci. World, Volume 2013, Article ID 780153, 11 pages.

[3] A. Cordero, J.R. Torregrosa, On the dynamics of an iterative method with memory, Proceedings of the 16th International Conference Computational and Mathematical Methods in Science and Engineering, to appear.

[4] A. Cordero, T. Lotfi, P. Bakhtiari, J.R. Torregrosa, An efficient two-parametric family with memory for nonlinear equations, Numer. Algor. 68 (2015) 323–335.

[5] Z. Elhadj, J. C. Sprott, On the Dynamics of a New Simple 2-d Rational Discrete Mapping, Int. Bifur. Chaos 21(1) (2011) 1–6.

[6] S. Plaza, J.M. Gutiérrez, Dinámica del método de Newton, Servicio de Publicaciones Universidad de La Rioja, 2013.

[7] W.F. Hassan Al-Shameri, Dynamical properties of the Hénon mapping, Int. Math. Anal. 6(49) (2012) 2419–2430.

[8] J.L. Hueso, E. Martínez, J.R. Torregrosa, New modifications of Potra-Pták's method with optimal fourth and eighth orders of convergence, Comput. Appl. Math. 234 (2010) 2969–2976.

[9] V. G. Ivancevic, T. T. Ivancevic, High-Dimensional Chaotic and Attractor Systems: A Comprehensive Introduction, Springer Science & Business Media, 2007.

[10] H. T. Kung, J. F. Traub, Optimal order of one-point and multi-point iteration, Assoc. Comput. Math. 21 (1974) 643–651.

[11] T. Lotfi, F. Soleymani, S. Shateyi, P. Assari, F. Khaksar Haghani, New Mono- and Biaccelerator Iterative Methods with Memory for Nonlinear Equations, Abstr. Appl. Anal., Volume 2014 (2014), Article ID 705674, 8 pages.

[12] B. Neta, A new family of high order methods for solving equations, Int. Comput. Math. 14, (1983) 191–195.

[13] J.M. Ortega, W.C. Rheinboldt, Iterative solution of nonlinear equations in several variables, Academic Press, 1970.

[14] M.S. Petković, J. Džunić, L.D. Petković, A family of two-point methods with memory for solving nonlinear equations, Appl. Anal. Discrete Math. 5 (2011) 298–317.

[15] M. Petković, B. Neta, L. Petković, J. Džunić, Multipoint methods for solving nonlinear equations, Academic Press, 2013.

[16] R.C. Robinson, An Introduction to Dynamical Systems, Continous and Discrete, American Mathematical Society, Providence, 2012.

[17] J.F. Traub, Iterative methods for the solution of equations, Chelsea Publishing Company, New York, 1982.

[18] X. Wang, T. Zhang, Efficient n-point iterative methods with memory for solving nonlinear equations, Numer. Algor. 70(2) (2015) 357–375.