ON THE CONVERGENCE OF A HIGHER ORDER FAMILY OF METHODS AND ITS DYNAMICS

IOANNIS K. ARGYROS, ALICIA CORDERO, ÁNGEL A. MAGREÑÁN, AND JUAN R. TORREGROSA

Abstract. In this paper, we present the study of the local convergence of a higher-order family of methods. Moreover, the dynamical behavior of this family of iterative methods applied to quadratic polynomials is studied. Some anomalies are found in this family be means of studying the dynamical behavior. Parameter spaces are shown and the study of the stability of all the fixed points is presented.

1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution x^* of equation

$$F(x) = 0, \tag{1.1}$$

where F is a differentiable function defined on a convex subset D of S with values in S, where S is $\mathbb{R}$ or $\mathbb{C}$.

Many problems from Applied Sciences including engineering can be solved by means of finding the solutions to equations in a form like (1.1) using Mathematical Modelling [9, 12]. For example, dynamic systems are mathematically modeled by difference or differential equations, and their solutions usually represent the states of the systems. Except in special cases, the solutions of these equations can be found in closed form. This is the main reason why the most commonly used solution methods are usually iterative. The convergence analysis of iterative methods is usually divided into two categories: semilocal and local convergence analysis. The semilocal convergence matter is, based on the information around an initial point, to give criteria ensuring the convergence of iteration procedures. A very important problem in the study of iterative procedures is the convergence domain. In general the convergence domain is small. Therefore, it is important to enlarge the convergence domain without additional hypothesis. Another important problem is to find more precise error estimates on the distances $\|x_{n+1} - x_n\|$, $\|x_n - x^*\|$. These are with the study of the dynamical behavior our objectives in this paper.

The dynamical properties related to an iterative method applied to polynomials give important information about its stability and reliability. In recently studies, authors such us Cordero et al. [11, 12, 13, 14], Amat et al [1, 2, 3], Gutiérrez et al. [17], Chun et al. [10] and many others [6, 12, 10] have found interesting dynamical planes, including periodical behavior and others anomalies. One of our main interests in this paper is the study of the parameter spaces associated to a

2010 *Mathematics Subject Classification.* 68Q20,68W40.

Key words and phrases. Banach space, majorizing sequence, local/semilocal convergence, Complex dynamics.

This paper is in final form and no version of it will be submitted for publication elsewhere.

family of iterative methods, which allow us to distinguish between the good and bad methods in terms of its numerical properties.

In this work, we consider the following optimal fourth-order family of methods defined by Sharma in [24] for each $n = 0, 1, 2, \ldots$ by

$$\begin{aligned} y_n &= x_n - \frac{2}{3}\frac{F(x_n)}{F'(x_n)} \\ z_n &= x_n - \frac{F'(x_n) + 3F'(y_n)}{-2F'(x_n) + 6F'(y_n)}\frac{F(x_n)}{F'(x_n)} \\ x_{n+1} &= x_n - \frac{F'(x_n) + aF'(y_n)}{bF'(x_n) + cF'(y_n)}\frac{F(z_n)}{F'(x_n)} \end{aligned} \tag{1.2}$$

where a, b, c are parameters. In this paper, the dynamics of this family applied to an arbitrary quadratic polynomial $p(z) = (z - A)(z - B)$ will be analyzed, characterizing the stability of all the fixed points. The graphic tool used to obtain the parameter space and the different dynamical planes have been introduced by Magreñán in [18, 19], but there exist other techniques such us the one given by Chicharro et al in [9].

The rest of the paper is organized as follows: in Section 2 the study of the local convergence is studied and in Section 3 some of the basic dynamical concepts related to the complex plane are presented, the stability of the fixed points of the family and the dynamical behavior of the family is analyzed, where the parameter space and some selected dynamical planes are presented. Finally, the conclusions drawn to this study are presented in the concluding Section 4.

2. Local convergence

In this Section $F : \mathbb{D} \subseteq \mathbb{X} \to \mathbb{Y}$ is a Fréchet-differentiable operator, where $\mathbb{X}$, $\mathbb{Y}$ are Banach spaces and $\mathbb{D}$ is a convex set. Let $U(v, \rho)$, $\bar{U}(v, \rho)$ stand for the open and closed balls in $\mathbb{X}$ respectively with center $v \in \mathbb{X}$ and of radius $\rho > 0$. We shall study the local convergence analysis of method defined for each $x = 0, 1, 2 \ldots$

$$\begin{aligned} y_n &= x_n - \xi F'(x_n)^{-1}F(x_n) \\ z_n &= x_n - \lambda A_n^{-1}F'(x_n)^{-1}F(x_n) \\ x_{n+1} &= z_n - \mu B_n^{-1}F'(x_n)^{-1}F(z_n) \end{aligned} \tag{2.1}$$

where x_0 is an initial point, ξ, λ, μ are parameters,

$$A_n = 2(F'(x_n) + 3F'(y_n))^{-1}(F'(x_n) - 3F'(y_n))$$

and

$$B_n = (F'(x_n) + aF'(y_n))^{-1}(bF'(x_n) + cF'(y_n)).$$

Notice that if $\xi = \frac{2}{3}$, $\lambda = -1$ and $\mu = 1$ method (2.1) reduces to Newton's method [4, 5]. Furthermore, if $\xi = \frac{2}{3}$, $\lambda = -1$ and $\mu = 0$, we get the Jarratt's method [1, **?**]. Many other choices of the parameters ξ, λ, μ are possible [4, 5].

The convergence of method (1.2) was shown in [25] under hypotheses up to the seventh derivative of function F. These hypotheses limit the applicability of method (1.2). For a motivational example, define function F on $\mathbb{X} = \mathbb{Y} = \mathbb{R}$, $D = \bar{U}(0, 1)$ by

$$F(x) = \begin{cases} c_1 x^3 lnx^2 + c_2 x^5 + c_3 x^4, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

where $c_1 \neq 0$, c_2 and c_3 are real parameters. Then, we have that

$$F'(x) = 3c_1 x^2 lnx^2 + 5c_2 x^4 + 4c_3 x^3 + 2c_1 x^2,$$

$$F''(x) = 12c_1 x lnx^2 + 20c_2 x^3 + 12c_3 x^2 + 10c_1 x$$

and

$$F'''(x) = 12c_1 lnx^2 + 60c_2x^2 + 12c_3x + 22c_1.$$

Then, obviously, function $F'''(x)$ is unbounded on D. Hence, the results in [24], cannot apply to show the convergence of method (2.1) or its special cases requiring hypotheses on the third derivative of function F or higher. In this Section we present the local convergence analysis of method (2.1) using hypotheses only on the first derivative of function F. Hence, the applicability of these methods is expanded under less restrictive conditions. Moreover, the radius of convergence and computable error bounds on the distances $\|x_n - x^*\|$ (not given in [25]) are also given in this Section.

Let $L_0 > 0$, $L > 0$, $M_0 > 0$, $M \geq 1$, $\alpha > 0$, $a \in S \setminus \{-1\}$,b, c, ξ, λ, $\mu \in S$ with $L_0 \leq L$ be given parameters. It is convenient for the local convergence analysis that follows to introduce some functions and parameters. Define function on the interval $[0, \frac{1}{L_0})$ by

$$g_1(t) = \frac{1}{2(1 - L_0t)}\,[Lt + 2M|1 - \xi|]$$

and parameters by

$$r_1 = \frac{2(1 - M|1 - \xi|)}{2L_0 + L} \quad \text{and} \quad r_A = \frac{2}{L_0 + L}. \tag{2.2}$$

Suppose that

$$M|1 - \xi| < 1. \tag{2.3}$$

Then, we have that

$$0 < r_1 < r_A < \frac{1}{L_0} \tag{2.4}$$

and

$$0 \leq g_1(t) < 0 \quad \text{for each } t \in [0, r_1).$$

Define function on the interval $[0, \frac{1}{L_0})$ by

$$g_0(t) = \frac{L_0}{4}(1 + 3g_1(t))$$

and set

$$h_0(t) = g_0(t)t - 1.$$

We have that $h_0(0) = -1 < 0$ and $h_0(t) \to +\infty$ when $t \to \frac{1}{L_0}^-$. It follows from the Intermediate Value Theorem that function h_0 has zeros in the interval $(0, \frac{1}{L_0})$. Denote by r_0 the smallest such zero. Then, we have that

$$0 \leq g_0(t) < 1 \quad \text{for each } t \in [0, r_0).$$

We have that $h_0(r_1) = L_0r_1 - 1 < 0$, so $r_1 < r_0$.

Define functions on the interval $[0, \frac{1}{L_0})$ by

$$p(t) = 2L_0t + 3L_0g_1(t)t + 4 + \alpha|\lambda|(L_0t + 3L_0g_1(t)t + 4)(1 + L_0t),\ \text{if } \lambda \neq 0$$

or

$$p(t) = 4M(2 + M_0|\lambda|),\ \text{if } \lambda \neq 0$$

or

$$p(t) = 0,\ \text{if } \lambda = 0,$$

$$\gamma(t) = \frac{3L_0(1 + g_1(t))t}{4(1 - g_0(t)t)}$$

and

$$\delta(t) = 3L_0(1 + g_1(t))t + 4g_0(t)t - 4.$$

We have that $\delta(0) = -4 < 0$ and $\delta(t) \to \infty$ as $t \to \frac{1}{L_0}^-$. Hence, function δ has zeros in the interval $(0, \frac{1}{L_0})$. Denote by r_δ the smallest such zero. Then, we have that

$$0 \le \gamma(t) < 1 \quad \text{for each } t \in [0, r_\delta).$$

Define function on the interval $[0, \frac{1}{L_0})$ by

$$g_2(t) = \frac{1}{2(1-L_0 t)}\left[Lt + \frac{M_0 p(t)}{2(1-g_0(t)t)(1-\gamma(t))}\right]$$

and set

$$h_2(t) = g_2(t) - 1.$$

Suppose that

$$p(0)M_0 < 4.$$

Then, we have that $h_2(0) = \frac{p(0)M_0}{4} - 1 < 0$ as $h_2(t) \to +\infty$ as $t \to \frac{1}{L_0}^-$. Hence, function h_2 has zeros in the interval $(0, \frac{1}{L_0})$. Denote by r_2 the smallest such zero.

Then, we have that

$$0 \le g_2(t) < 1 \quad \text{for each } t \in [0, r_2).$$

Define function on the interval $(0, \frac{1}{L_0})$ by

$$g(t) = \frac{L_0}{|1+a|}(1 + |a| g_1(t))t$$

and set

$$h(t) = g(t) - 1.$$

We have that $h(0) = -1 < 0$ and $h(t) \to +\infty$ as $t \to \frac{1}{L_0}^-$. Hence, function h has zeros in the interval $(0, \frac{1}{L_0})$. Denote by r_h the smallest such zero.

Then, we have that

$$0 \le g(t) < 1 \quad \text{for each } t \in [0, r_h).$$

Define function on the interval $[0, \frac{1}{L_0})$ by

$$\varphi(t) = \frac{|1+b| L_0 (1 + g_1(t))t}{|1+a|(1-g(t)t)}, \quad \text{if } a+b+c+1 = 0$$

or

$$\varphi(t) = \frac{(|1+b| + |a+c|)M}{|1+a|(1-g(t)t)}, \quad \text{if } (|1+b| + |a+c|)M < |1+a|$$

and set

$$\psi(t) = |1+b| L_0 (1 + g_1(t))t + |1+a| g(t)t - |1+a|.$$

We have that $\psi(0) = -|1+a| < 0$ and $\psi(t) \to +\infty$ as $t \to \frac{1}{L_0}^-$. Similarly for the second definition of function ψ.

Hence, function ψ has zeros in the interval $(0, \frac{1}{L_0})$. Denote by r_ψ the smallest such zero. Then, we have that

$$0 \le \varphi(t) < 1 \quad \text{for each } t \in [0, r_\psi).$$

Finally, define function on the interval $(0, \frac{1}{L_0})$ by

$$g_3(t) = g_2(t)\left[1 + \frac{|\mu| M}{(1-L_0 t)(1-\varphi(t))}\right]$$

and set

$$h_3(t) = g_3(t) - 1.$$

Suppose that

$$\frac{p(0)M_0}{4}(1 + |\mu| M) < 1. \tag{2.5}$$

Then, we have that $h_3(0) = \frac{p(0)M_0}{4} + |\mu|M - 1 < 0$ and $h_3(t) \to +\infty$ as $t \to \frac{1}{L_0}^-$. Hence, function h_3 has zeros in the $(0, \frac{1}{L_0})$. Denote by r_3 the smallest such zero. Then, we have that

$$0 \le g_3(t) < 1 \quad \text{for each } t \in [0, r_3).$$

Set

$$r = \min\{r_1, r_\delta, r_2, r_h, r_\psi, r_3\}. \tag{2.6}$$

Then, we have that for each $t \in [0, r)$

$$0 \le g_1(t) < 1, \tag{2.7}$$

$$0 \le g_0(t)t < 1, \tag{2.8}$$

$$0 \le \gamma(t) < 1, \tag{2.9}$$

$$0 \le g_2(t) < 1, \tag{2.10}$$

$$0 \le g(t) < 1, \tag{2.11}$$

$$0 \le \varphi(t) < 1 \tag{2.12}$$

and

$$0 \le g_3(t) < 1. \tag{2.13}$$

Next, using the preceding notation we can show the main local convergence result for method (2.1).

Theorem 1. *Let $F : \mathbb{D} \subset \mathbb{X} \to \mathbb{Y}$ be a Fréchet-differentiable operator. Suppose that there exist $x^* \in \mathbb{D}$, $L_0 > 0$, $L > 0$, $M_0 > 0$, $M \ge 1$, $\alpha > 0$, $a \in S \setminus \{-1\}$,b, c, ξ, λ, $\mu \in S$ such that for all $x, y \in D$ the following conditions hold:*

$$M|1 - \xi| < 1,$$

$$\frac{p(0)M_0}{4}(1 + |\mu|M) < 1,$$

$a + b + c + 1 = 0$, *if the first definition of function φ is used or* $M(|1+b| + |a+c|) < |1+a|$, *if the second definition of function φ is used,*

$$F(x^*) = 0, \quad F'(x^*)^{-1} \in L(\mathbb{Y}, \mathbb{X}), \quad \|F'(x^*)\| \le \alpha, \tag{2.14}$$

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*)\| \le L_0\|x - x^*\|, \tag{2.15}$$

$$\|F'(x^*)^{-1}(F'(x) - F'(y)\| \le L\|x - y\|, \tag{2.16}$$

$$\|F'(x)\| \le M_0, \tag{2.17}$$

$$\|F'(x^*)^{-1}F'(x)\| \le M, \tag{2.18}$$

and

$$\bar{U}(x^*, r) \subseteq \mathbb{D}, \tag{2.19}$$

where functions p and φ are defined previously and r is given by (2.6). Then, sequence $\{x_n\}$ generated for $x_0 \in U(x^*, r) \setminus \{x^*\}$ by method (2.1) is well defined, remains in $\bar{U}(x^*, r)$ for each $n = 0, 1, 2, \ldots$ and converges to x^*. Moreover, the following estimates hold

$$\|F'(x_n)^{-1}F'(x^*)\| \le \frac{1}{1 - L_0\|x_n - x^*\|}, \tag{2.20}$$

$$\|y_n - x^*\| \le g_1(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\| < r, \tag{2.21}$$

$$\|(F'(x_n) + 3F'(y_n))^{-1}F'(x^*)\| \le \frac{1}{4(1 - g_0(\|x_n - x^*\|)\|x_n - x^*\|)}, \tag{2.22}$$

$$\|(F'(x^*)^{-1}(A_n + \lambda F'(x_n)\| \le \frac{p(\|x_n - x^*\|)}{4(1 - g_0(\|x_n - x^*\|)\|x_n - x^*\|)}, \tag{2.23}$$

$$\|A_n^{-1}\| \le \frac{1}{1 - \gamma(\|x_n - x^*\|)}, \tag{2.24}$$

$$\|z_n - x^*\| \le g_2(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\|, \tag{2.25}$$

$$\|(F'(x_n)+aF'(y_n))^{-1}F'(x^*)\| \le \frac{1}{|1+a|(1-g(\|x_n-x^*\|)\|x_n-x^*\|)}, \tag{2.26}$$

$$\|B_n^{-1}\| \le \frac{1}{1-\varphi(\|x_n-x^*\|)}, \tag{2.27}$$

and

$$\|x_{n+1}-x^*\| \le g_3(\|x_n-x^*\|)\|x_n-x^*\| < \|x_n-x^*\|, \tag{2.28}$$

where the functions "g", γ and φ are defined above Theorem 2.1. Furthermore, for $T \in [r, \frac{2}{L_0})$ the limit point x^* is the only solution of equation $F(x) = 0$ in $\bar{U}(x^*, T) \cap D$.

Proof. We shall use induction to show estimates (2.20)–(2.28). Using (2.6), (2.15) and the hypothesis $x_0 \in U(x^*, r)$, we have that

$$\|F'(x^*)^{-1}(F'(x_0)-F'(x^*))\| \le L_0\|x_0-x^*\| < L_0 r < 1., \tag{2.29}$$

It follows from (2.29) and the Banach lemma on invertible operators [] that $F'(x_0)^{-1} \in L(\mathbb{Y}, \mathbb{X})$ and

$$\|F'(x_0)^{-1}F'(x^*)\| \le \frac{1}{1-L\|x_0-x^*\|-l} < \frac{1}{1-L_0 r}, \tag{2.30}$$

which shows (2.20) for $n = 0$. Hence, y_0 is well defined by the first substep of method (2.1) for $n = 0$. Using the first substep of method (2.1) for $n = 0$ and (2.14), we get the identity

$$\begin{aligned} & y_0 - x^* = x_0 - x^* - F'(x_0)^{-1}F(x_0) + (1-\xi)F'(x_0)F(x_0) \\ & = -F'(x_0)^{-1}F'(x^*)\textstyle\int_0^1 F'(x^*)[F'(x^*+\theta(x_0-x^*)) - F'(x_0)](x_0-x^*)d\theta \\ & +(1-\xi)F'(x_0)^{-1}F'(x^*)F'(x^*)^{-1}F(x_0) \end{aligned} \tag{2.31}$$

We also have that

$$F(x_0) = F(x_0) - F(x^*) = \int_0^1 F'(x^*+\theta(x_0-x^*))(x_0-x^*)d\theta. \tag{2.32}$$

Using (2.6), (2.7), (2.16), (2.18), (2.30)–(2.32), we get that

$$\begin{aligned} & \|y_0-x^*\| \le \|F'(x_0)^{-1}F'(x^*)\|\|\textstyle\int_0^1 F'(x^*)[F'(x^*+\theta(x_0-x^*)) - F'(x_0)]\|d\theta\|x_0-x^*\| \\ & +|1-\xi|\|F'(x_0)^{-1}F'(x^*)\|\|\textstyle\int_0^1 F'(x^*)[F'(x^*+\theta(x_0-x^*)) - F'(x_0)]\|d\theta\|x_0-x^*\| \\ & \le \frac{L\|x_0-x^*\|^2}{2(1-L_0\|x_0-x^*\|)} + \frac{|1-\xi|M\|x_0-x^*}{1-L_0\|x_0-x^*\|} \\ & = g_1(\|x_0-x^*\|)\|x_0-x^*\| < \|x_0-x^*\| < r, \end{aligned}$$

which shows (2.21) for $n = 0$ and $y_0 \in U(x^*, r)$.

We must show that (2.22) holds for $n = 0$. Using (2.6), (2.8), (2.15), (2.21) (for $n = 0$), we get that

$$\begin{aligned}
&\|(4F'(x^*))^{-1}(F'(x_0) + 3F'(y_0) - 4F'(x^*))\| \leq \frac{1}{4}(\|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\| \\
&+3\|F'(x^*)^{-1}(F'(y_0) - F'(x^*))\|) \\
&\leq \frac{1}{4}(L_0\|x_0 - x^*\| + 3L_0\|y_0 - x^*\|) \\
&\leq \frac{1}{4}(L_0\|x_0 - x^*\| + g_1(\|x_0 - x^*\|)\|x_0 - x^*\|) \\
&= g_0(\|x_0 - x^*\|)\|x_0 - x^*\| < g_0(r)r < 1.
\end{aligned} \tag{2.33}$$

It follows from (2.33) that $(F'(x_0) + 3F'(y_0))^{-1} \in L(\mathbb{Y}, \mathbb{X})$ and

$$\|(F'(x_0) + 3F'(y_0))^{-1}F'(x^*)\| \leq \frac{1}{4(1 - g_0(\|x_0 - x^*\|)\|x_0 - x^*\|)},$$

which shows (2.22) for $n = 0$. By using a instead of "3" in the preceding estimate (2.33) we also show estimate (2.26) for $n = 0$. We also need an estimate on $F'(x^*)^{-1}(A_0 + \lambda F'(x_0))$.

We have that

$$A_0+\lambda F'(x_0) = (F(x_0)+3F'(y_0))^{-1}[2(F'(x_0)-3F'(y_0))+\lambda(F'(x_0)+3F'(y_0))F'(x_0)] \tag{2.34}$$

and

$$\begin{aligned}
&2(F'(x_0) - 3F'(y_0)) + \lambda(F'(x_0) + 3F'(y_0))F'(x_0) \\
&= 2(F'(x_0) - F'(x^*) - 3(F'(y_0) - F'(x^*)) - 2F'(x^*)) \\
&+\lambda(F'(x_0) - F'(x^*) + 3(F'(y_0) - F'(x^*) + 4F'(x^*))) \\
&\times F'(x^*)F'(x^*)^{-1}(F'(x_0) - F'(x^*) + F'(x^*)).
\end{aligned} \tag{2.35}$$

Then, by (2.14), (2.15), (2.21) and (2.35) we get that

$$\begin{aligned}
&\|F'(x^*)^{-1}(2(F'(x_0) - 3F'(y_0))) + \lambda(F'(x_0) + 3F'(y_0))F'(x_0)\| \\
&\leq 2\|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\| + 3\|F'(x^*)^{-1}(F'(y_0) - F'(x^*))\| + 4 \\
&+|\lambda| \left[\|F'(x^*)^{-1}(F'(x_0) - F'(x^*)\| + 3\|F'(x^*)^{-1}(F'(y_0) - F'(x^*))\| + 4\right] \|F'(x^*)\| \\
&\left[\|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\| + 1\right] \\
&\leq 2L_0\|x_0 - x^*\| + 3L_0\|y_0 - x^*\| + 4 \\
&+\alpha|\lambda|(L_0\|x_0 - x^*\| + 3L_0\|y_0 - x^*\| + 4)(1 + L_0\|x_0 - x^*\|) \\
&\leq 2L_0\|x_0 - x^*\| + 3L_0 g_1(\|x_0 - x^*\|)\|x_0 - x^*\| + 4 \\
&+\alpha|\lambda|(L_0\|x_0 - x^*\| + 3L_0 g_1(\|x_0 - x^*\|)\|x_0 - x^*\| + 4)(1 + L_0\|x_0 - x^*\|) = p(\|x_0 - x^*\|),
\end{aligned} \tag{2.36}$$

leading to the first definition of function p. Alternatively, from the first line of (2.35), (2.17), (2.32) and (2.18) we get that

$$\begin{aligned}&\|F'(x^*)^{-1}((2F'(x_0)-3F'(y_0)\|)+\lambda(F'(x_0)+3F'(y_0))F'(x_0))\|\\&\leq 2(\|F'(x^*)^{-1}F'(x_0)\|+3\|F'(x^*)^{-1}F'(y_0)\|)\\&+|\lambda|(\|F'(x^*)^{-1}F'(x_0)\|+3\|F'(x^*)^{-1}F'(y_0)\|)\|F'(x_0)\|\\&\leq 8M+4|\lambda|MM_0=4M(2+|\lambda|M_0)=p(t),\end{aligned}\tag{2.37}$$

which is the second definition of function p. Clearly, if $\lambda=0$, we obtain the third definition of function p. Then, we have by (2.22), (2.34)–(2.37) that

$$\|F'(x^*)^{-1}(A_0+\lambda F'(x_0)\|\leq\|F'(x_0)+3F'(y_0))^{-1}F'(x^*)\|\|F'(x^*)^{-1}(A_0+\lambda F'(x_0))\|$$

$$\leq\frac{p(\|x_0-x^*\|)}{4(1+g_0(\|x_0-x^*\|)\|x_0-x^*\|)},$$

which shows (2.23) for $n=0$. Next, we show (2.24) for $n=0$. We can write

$$\|A_0+I\|=3\|(F'(x_0)+3F'(y_0))^{-1}F'(x^*)F'(x^*)^{-1}(F'(x_0)-F'(y_0))\|.\tag{2.38}$$

Using (2.6), (2.9), (2.15), (2.22) and (2.38), we get in turn that

$$\begin{aligned}&\|A_0+I\|\leq 3\|(F'(x_0)+3F'(y_0))^{-1}F'(x^*)\|\|F'(x^*)^{-1}(F'(x_0)-F'(x^*))\|+\|F'(x^*)(F'(y_0)-F'(x^*))\|\\&\leq\frac{3L_0(\|x_0-x^*\|+\|y_0-x^*\|)}{4(1-g_0(\|x_0-x^*\|)\|x_0-x^*\|)}\\&\leq\frac{3L_0(1+g_1(\|x_0-x^*\|))\|x_0-x^*\|}{4(1-g_0(\|x_0-x^*\|)\|x_0-x^*\|)}=\gamma(\|x_0-x^*\|)<\gamma(r)<1.\end{aligned}\tag{2.39}$$

It follows from (2.39) that $A_0^{-1}\in L(\mathbb{Y},\mathbb{X})$ and

$$\|A_0^{-1}\|\leq\frac{1}{1-\gamma(\|x_0-x^*\|)},$$

which shows (2.25) for $n=0$. Hence, z_0 is well defined by the second substep of method (2.1) for $n=0$. Then, we can write using method (2.1) for $n=0$ that

$$z_0-x^*=x_0-x^*-F'(x_0)F(x_0)+(F'(x_0)^{-1}+\lambda A_0^{-1})F(x_0).\tag{2.40}$$

We have that

$$\begin{aligned}F'(x_0)^{-1}+\lambda A_0^{-1}&=F'(x_0)^{-1}(I+\lambda F'(x_0)A_0^{-1})\\&=F'(x_0)^{-1}F'(x^*)F'(x^*)^{-1}(I+\lambda F'(x_0))A_0^{-1}.\end{aligned}\tag{2.41}$$

Then, by (2.6), (2.10), (2.17), (2.21)–(2.24) (for $n=0$),(2.30), (2.31), (2.32), (2.40) and (2.41) we get in turn that

$$\begin{aligned}&\|z_0-x^*\|=\|x_0-x^*-F'(x_0)F(x_0)\|+\|F'(x_0)^{-1}F'(x^*)\|\\&\times\|F'(x^*)^{-1}(A_0+\lambda F'(x_0))\|\|A_0^{-1}\|\|F(x_0)\|\\&\leq g_1(\|x_0-x^*\|)\|x_0-x^*\|+\\&\frac{M_0p(\|x_0-x^*\|)\|x_0-x^*\|}{4(1-L_0\|x_0-x^*\|)(1-g_0(\|x_0-x^*\|)\|x_0-x^*\|)(1-\gamma(\|x_0-x^*\|))}\\&=g_2(\|x_0-x^*\|)\|x_0-x^*\|<g_2(r)\|x_0-x^*\|<\|x_0-x^*\|<r,\end{aligned}$$

which shows (2.25) for $n = 0$ and $z_0 \in U(x^*, r)$. We need an estimate on $\|B_0+I\|$. If $a+b+c+1 = 0$, we get by the first definition of function φ, (2.6), (2.12), (2.15), (2.21) and (2.26) that

$$\begin{aligned}
&\|B_0 + I\| = \|(F'(x_0) + aF'(y_0))^{-1}F'(x^*)F'(x^*)^{-1}((b+1)F'(x_0) + (a+c)F'(y_0))\| \\
&\leq |1+b|\|(F'(x_0) + aF'(y_0))^{-1}F'(x^*)\| + \|F'(x^*)^{-1}((F'(x_0) - F'(x^*)) + (F'(x^*) - F'(y_0)))\| \\
&\leq \frac{|1+b|(L_0\|x_0 - x^*\| + L_0\|y_0 - x^*\|)}{|1+a|(1 - g(\|x_0 - x^*\|)\|x_0 - x^*\|)} \\
&\leq \frac{|1+b|L_0(1 + g_1(\|x_0 - x^*\|))\|x_0 - x^*\|}{|1+a|(1 - g(\|x_0 - x^*\|)\|x_0 - x^*\|)} \\
&= \varphi\|x_0 - x^*\| < \varphi(r) < 1.
\end{aligned} \tag{2.42}$$

It follows from (2.42) that $B_0^{-1} \in L(\mathbb{Y}, \mathbb{X})$ and

$$\|B_0^{-1}\| \leq \frac{1}{1 - \varphi(\|x_0 - x^*\|)},$$

which shows (2.27) for $n = 0$ if the first definition of function φ is used. However, if the second definition of function φ is used we get instead from the first line of (2.42) that

$$\begin{aligned}
\|B_0 + I\| &\leq \frac{|1+b|\|F'(x^*)^{-1}F'(x_0)\| + |a+c|\|F'(x^*)^{-1}F'(y_0)\|}{|1+a|(1 - g(\|x_0 - x^*\|)\|x_0 - x^*\|)} \\
&\leq \frac{(|1+b| + |a+c|)M}{|1+a|(1 - g(\|x_0 - x^*\|)\|x_0 - x^*\|)} \\
&= \varphi\|x_0 - x^*\| < \varphi(r) < 1.
\end{aligned} \tag{2.43}$$

Then, again we arrive at estimate (2.27). Hence, x_1 is well defined by the third substep of method (2.1) for $n = 0$. We also have

$$x_1 - x^* = z_0 - x^* - \mu B_0^{-1}(F'(x_0)^{-1}F'(x^*))(F'(x^*)^{-1}F(z_0)). \tag{2.44}$$

Then, using (2.6), (2.13), (2.18), (2.21), (2.25), (2.27) (for $n = 0$), (2.30), (2.31), (2.32) (for $z_0 = x_0$) and (2.44) we obtain that

$$\begin{aligned}
\|x_1 - x^*\| &= \|z_0 - x^*\| + \|B_0^{-1}\|\|F'(x_0)^{-1}F'(x^*)\|\|F'(x^*)^{-1}F(z_0)\| \\
&\leq g_2(\|x_0 - x^*\|)\|x_0 - x^*\| + \frac{M|\mu|\|z_0 - x^*\|}{(1 - L_0\|x_0 - x^*\|)(1 - \varphi(\|x_0 - x^*\|))} \\
&= g_3(\|x_0 - x^*\|)\|x_0 - x^*\| < g_3(r)\|x_0 - x^*\| < \|x_0 - x^*\| < r,
\end{aligned}$$

which shows (2.28) for $n = 0$ and $x_1 \in U(x^*, r)$. By simply replacing x_0, y_0, x_1 by x_k, y_k, x_{k+1} in the preceding estimates we arrive at estimates (2.20)–(2.28). Using the estimate $\|x_{k+1} - x^*\| \leq c\|x_k - x^*\| < r$, where $c = g_3(r) \in [0, 1)$ we deduce that $\lim_{k\to\infty} x_k = x^*$ and $x_{k+1} \in U(x^*, r)$. Finally, to show the uniqueness part, let $y^* \in U(x^*, T)$ be such that $F(y^*) = 0$. Let $Q = \int_0^1 F'(y^* + \theta(x^* - y^*))d\theta$. In view of (2.15), we get in turn that

$$\begin{aligned}
&\|F'(x^*)^{-1}(Q - F'(x^*))\| \leq \| \textstyle\int_0^1 L_0\|y^* + \theta(x^* - y^*)\|d\theta \\
&= \frac{L_0\|x^* - y^*\|}{2} = \frac{L_0}{2}T < 1.
\end{aligned} \tag{2.45}$$

It follows from (2.45) that $Q^{-1} \in L(\mathbb{Y}, \mathbb{X})$. Therefore, from the identity $0 = F(x^*) - F(y^*) = Q(x^* - y^*)$, we conclude that $x^* = y^*$. □

Remark 1. (1) *In view of* (2.15) *and the estimate*

$$\begin{aligned} \|F'(x^*)^{-1}F'(x)\| &= \|F'(x^*)^{-1}(F'(x) - F'(x^*)) + I\| \\ &\leq 1 + \|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \\ &\leq 1 + L_0\|x_0 - x^*\| \end{aligned}$$

condition (2.18) *can be dropped and* M *can be replaced by*

$$M(t) = 1 + L_0 t \quad \text{or simply} \quad M(t) = M = 2, \text{ since } t \in [0, \frac{1}{L_0}).$$

(2) *The results can be also be used to solve equations where the operator* F' *satisfies the autonomous differential equation*

$$F'(x) = P(F(x)),$$

where P *is a known continuous operator. Since* $F'(x^*) = P(F(x^*)) = P(0)$, *we can apply the results without actually knowing the solution* x^*. *Let as an example* $F(x) = e^x - 1$. *Then, we can choose* $P(x) = x + 1$ *and* $x^* = 0$.

(3) *The local results can be used for projection methods such as Arnoldi's method, the generalized minimum residual method(GMRES), the generalized conjugate method(GCR) for combined Newton/finite projection methods and in connection to the mesh independence principle in order to develop the cheapest and most efficient mesh refinement strategy* [4, 5].

(4) *The radius* r_A *defined in* (2.2) *was given by authors in* [4, 5] *as the convergence radius for Newton's method under condition* (2.15) *and* (2.16)

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n), \quad \text{for each } n = 0, 1, 2 \ldots. \tag{2.46}$$

It follows from (2.2) *and* (2.6) *that the convergence radius* r *of the method* (2.1) *cannot be larger than the convergence radius* r_A *of the second order Newton's method* (2.46). *As already noted in* r_A *is at least as the convergence ball give by Rheinboldt* [23]

$$r_R = \tfrac{2}{3L}. \tag{2.47}$$

In particular, for $L_0 < L$ *we have that*

$$r_R < r_A$$

and

$$\frac{r_R}{r_A} \to \frac{1}{3} \quad extas \quad \frac{L_0}{L} \to 0.$$

That is our convergence ball r_A *is at most three times larger than Rheinboldt's. The same value for* r_R *given by Traub* [27].

(5) *It is worth noticing that method* (2.1) *is not changing if we use the conditions of Theorem 2.1 instead of the stronger conditions given in* [24]. *Moreover, for the error bounds in practice we can use the computational order of convergence (COC)* [17]

$$\xi = \sup \frac{ln \frac{\|x_{n+2} - x_{n+1}\|}{\|x_{n+1} - x_n\|}}{ln \frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|}}, \quad \text{for each } n = 1, 2, \ldots$$

or the approximate computational order of convergence (ACOC)

$$\xi^* = \sup \frac{ln \frac{\|x_{n+2}-x^*\|}{\|x_{n+1}-x^*\|}}{ln \frac{\|x_{n+1}-x^*\|}{\|x_n-x^*\|}}, \quad \text{for each } n = 0, 1, 2, \dots$$

This way we obtain in practice the order of convergence in a way that avoids the bounds involving estimates higher than the first Fréchet derivative.

(6) *Let us define new method for each* $n = 0, 1, 2, \dots$ *by*

$$\begin{aligned} \bar{y}_n &= \bar{x}_n - \xi F'(\bar{x}_n)^{-1} F(\bar{x}_n) \\ \bar{z}_n &= \bar{y}_n - \lambda \bar{A}_n^{-1} F'(\bar{x}_n)^{-1} F(\bar{x}_n) \\ \bar{x}_{n+1} &= \bar{z}_n - \mu B_n^{-1} F'(\bar{x}_n)^{-1} F(\bar{z}_n) \end{aligned} \tag{2.48}$$

where $\bar{x}_0 = x_0$ *is an initial point,* ξ, λ, μ *are parameters,* $\bar{A}_n = 2(F'(\bar{x}_n) + 3F'(\bar{y}_n))^{-1}(F'(\bar{x}_n) - 3F'(\bar{y}_n))$ *and* $\bar{B}_n = 2(F'(\bar{x}_n) + aF'(\bar{y}_n))^{-1}(bF'(\bar{x}_n) + cF'(\bar{y}_n))$*. Method* (2.48) *can be better than method* (2.1) *since* $\bar{y}_n$ *is used instead of "*$\bar{x}_n$*" (i.e. "*x_n*") in the second substep. Let us define functions* $\bar{g}_2$, $\bar{h}_2$, $\bar{g}_3$ *and* $\bar{h}_3$ *(instead of* g_2, h_2, g_3 *and* h_3*, respectively given above Theorem 2.1) by*

$$\bar{g}_2(t) = g_1(t) + \frac{|\lambda| M}{1 - L_0 t)(1 - \gamma(t))},$$

$$\bar{h}_2(t) = \bar{g}_2(t) - 1,$$

$$\bar{g}_3(t) = \bar{g}_2(t) \left[1 + \frac{|\mu| M}{(1 - L_0 t)(1 - \varphi(t))} \right]$$

and

$$\bar{h}_3(t) = \bar{g}_3(t) - 1.$$

Moreover, replace condition (2.5) *by*

$$M(|1-\xi| + |\lambda|)(1 + |\mu| M) < 1. \tag{2.49}$$

$$\bar{h}_2(0) = M(|1-\xi| + |\lambda|) - 1 < 0,$$

$$\bar{h}_3(0) = M(|1-\xi| + |\lambda|)(1 + |\mu| M) - 1 < 0,$$

Then, we have that $\bar{h}_2(t) \to +\infty$, $\bar{h}_3(t) \to +\infty$ *as* $t \to \frac{1}{L_0}^-$.

Hence, functions $\bar{h}_2$, $\bar{h}_3$ *has zeros in the interval* $(0, \frac{1}{L_0})$*. Denote by* $\bar{r}_2$, $\bar{r}_3$ *the minimal such zeros. Set*

$$\bar{r} = \min\{r, r_\delta, \bar{r}_2, r_h, r_\psi, \bar{r}_3\}. \tag{2.50}$$

Then, we have for each $t \in [0, \bar{r})$

$$0 \le g_1(t) < 1,$$

$$0 \le g_0(t) t < 1,$$

$$0 \le \gamma(t) < 1,$$

$$0 \le \bar{g}_2(t) < 1,$$

$$0 \le g(t) < 1,$$

$$0 \le \varphi(t) < 1$$

and

$$0 \le \bar{g}_3(t) < 1.$$

Then, we can show the following local convergence result for method (2.48).

Proposition 1. *Let* $F : \mathbb{D} \subset \mathbb{X} \to \mathbb{Y}$ *be a Fréchet-differentiable operator. Suppose that there exist* $x^* \in \mathbb{D}$, $L_0 > 0$, $L > 0$, $M_0 > 0$, $M \ge 1$, $\alpha > 0$, $a \in S \setminus \{-1\}$, b, c, ξ, λ, $\mu \in S$ *such that for all* $x, y \in D$ *the following hold:*

- (2.14)–(2.18)

- (2.49) *and*

$$\bar{U}(x^*,\bar{r}) \subseteq D, \tag{2.51}$$

Then, sequence $\{\bar{x}_n\}$ *generated for* $\bar{x}_0 \in U(x^*,\bar{r}) \setminus \{x^*\}$ *by method* (2.48) *is well defined, remains in* $\bar{U}(x^*,r)$ *for each* $n = 0,1,2,\dots$ *and converges to* x^**. Moreover, the following estimates hold for each* $n = 0,1,2,\dots$

$$\|F'(\bar{x}_n)^{-1}F'(x^*)\| \leq \frac{1}{1-L_0\|\bar{x}_n - x^*\|},$$

$$\|\bar{y}_n) - x^*\| \leq g_1(\|\bar{x}_n - x^*\|)\|\bar{x}_n - x^*\| < \|\bar{x}_n - x^*\| < \bar{r},$$

$$\|(F'(\bar{x}_n)+3F'(\bar{y}_n))^{-1}F'(x^*)\| \leq \frac{1}{4(1-g_0(\|\bar{x}_n - x^*\|)\|\bar{x}_n - x^*\|)},$$

$$\|\bar{A}_n^{-1}\| \leq \frac{1}{1-\gamma(\|\bar{x}_n - x^*\|)},$$

$$\|\bar{z}_n - x^*\| \leq \bar{g}_2(\|\bar{x}_n - x^*\|)\|\bar{x}_n - x^*\| < \|\bar{x}_n - x^*\|,$$

$$\|(F'(\bar{x}_n)+aF'(\bar{y}_n))^{-1}F'(x^*)\| \leq \frac{1}{|1+a|(1-g(\|\bar{x}_n - x^*\|)\|\bar{x}_n - x^*\|)},$$

$$\|\bar{B}_n^{-1}\| \leq \frac{1}{1-\varphi(\|\bar{x}_n - x^*\|)},$$

and

$$\|\bar{x}_{n+1} - x^*\| \leq \bar{g}_3(\|\bar{x}_n - x^*\|)\|\bar{x}_n - x^*\| < \|\bar{x}_n - x^*\|.$$

Furthermore, for $T \in [\bar{r}, \frac{2}{L_0})$ *the limit point* x^* *is the only solution of equation* $F(x) = 0$ *in* $\bar{U}(x^*,T) \cap D$.

Proof. Simply replace function g_2, g_3 by $\bar{g}_2$, $\bar{g}_3$, respectively in the proof of Theorem 2.1 and notice that instead of the old estimates on $\|z_0 - z^*\|$, $\|x_1 - x^*\|$ we have from

$$\bar{z}_0 = \bar{y}_0 - \lambda\bar{A}_0^{-1}F'(\bar{x}_0)^{-1}F(\bar{x}_0)$$

that

$$\|\bar{z}_0 - x^*\| \leq \|\bar{y}_0 - x^*\| + |\lambda|\|\bar{A}_0^{-1}\|\|F'(\bar{x}_0)^{-1}F'(x^*)\|\|F'(x^*)^{-1}F(\bar{x}_0)\|$$

$$\leq [g_1(\|\bar{x}_0 - x^*\|)] + \frac{|\lambda|M}{(1-L_0\|\bar{x}_0 - x^*\|)(1-\gamma(\|\bar{x}_0 - x^*\|))}\|\bar{x}_0 - x^*\|$$

$$= \bar{g}_2(\|\bar{x}_0 - x^*\|)\|\bar{x}_0 - x^*\| < \|\bar{x}_0 - x^*\| < \bar{r},$$

and from

$$\bar{x}_1 = \bar{z}_0 - \bar{B}_0^{-1}F'(\bar{x}_0)^{-1}F(\bar{z}_0),$$

we have that

$$\|\bar{x}_1 - x^*\| \leq \|\bar{z}_0 - x^*\| + \|\bar{B}_0^{-1}\|\|F'(\bar{x}_0)^{-1}F'(x^*\|\|F'(x^*)^{-1}F(\bar{z}_0)\|$$

$$\leq \bar{g}_2(\|\bar{z}_0 - x^*\|)\|\bar{x}_0 - x^*\| + \frac{|\mu|M\|\bar{z}_0 - x^*\|}{(1-L_0\|\bar{x}_0 - x^*\|)(1-\varphi(\|\bar{x}_0 - x^*\|))}$$

$$\leq \bar{g}_2(\|\bar{x}_0 - x^*\|)\|\bar{x}_0 - x^*\| + \frac{|\mu|M\bar{g}_2(\|\bar{x}_0 - x^*\|)\|\bar{x}_0 - x^*\|}{(1-L_0\|\bar{x}_0 - x^*\|)(1-\varphi(\|\bar{x}_0 - x^*\|))}$$

$$= \bar{g}_3(\|\bar{x}_0 - x^*\|)\|\bar{x}_0 - x^*\| < \|\bar{x}_0 - x^*\| < \bar{r}.$$

□

3. Dynamical study of a special case of the method (1.2)

In this section we are going to study the complex dynamics of a special case of the method (1.2) in which we will fix two parameters as it appears in [24]:

$$b = -\frac{3a+1}{2},$$

and

$$c = \frac{5a+3}{2}.$$

With this values method (1.2) has the following form:

$$\begin{aligned} y_n &= x_n - \frac{2}{3}\frac{F(x_n)}{F'(x_n)} \\ z_n &= x_n - \frac{F'(x_n)+3F'(y_n)}{-2F'(x_n)+6F'(y_n)}\frac{F(x_n)}{F'(x_n)} \\ x_{n+1} &= x_n - \frac{F'(x_n)+F'(y_n)}{-\frac{3a+1}{2}F'(x_n)+\frac{5a+3}{2}F'(y_n)}\frac{F(z_n)}{F'(x_n)} \end{aligned} \tag{3.1}$$

By applying this operator on a quadratic polynomial with two different roots A and B, $p(z) = (z-A)(z-B)$. Using the Möebius map $h(z) = \frac{z-A}{z-B}$, which carries root A to 0, root B to ∞ and ∞ to 1, we obtain the rational operator associated to the family of iterative schemes is finally

$$G(z,\alpha) = -\frac{z^6\left(6-2a+12z+4az+9z^2+az^2+6z^3+2az^3+3z^4+3az^4\right)}{-3-3a-6z-2az-9z^2-az^2-12z^3-4az^3-6z^4+2az^4}. \tag{3.2}$$

3.1. **Study of the fixed points and their stability.** It is clear that $z=0$ and $z=\infty$ are fixed points of $G(z,\alpha)$. Moreover, there exist some strange fixed which are:

- $z=1$ related to divergence to ∞
- The roots of

$$\begin{aligned} p(z) &= 3+3a+9z+5az+18z^2+6az^2+30z^3+10az^3+36z^4+8az^4+30z^5+10az^5 \\ &\quad +18z^6+6az^6+9z^7+5az^7+3z^8+3az^8 \end{aligned}$$

These solutions of this polynomial depend on the value of the parameter α.

In Figure 2 the bifurcation diagram of the fixed points is shown

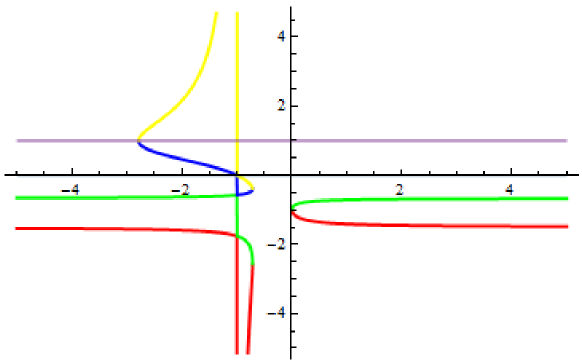

FIGURE 1. Bifurcation diagram of the fixed points.

3.2. **Study of the critical points and parameter spaces.** It is a well-known fact that there is at least one critical point associated with each invariant Fatou component. The critical points of the family are the solutions of is $G'(z,\alpha)=0$, where

$$G'(z,\alpha)= -\frac{4z^5(1+z)^2(1+z^2)}{(-3-3a-6z-2az-9z^2-az^2-12z^3-4az^3-6z^4+2az^4)^2} \times \frac{-27-18a+9a^2-54z-48az-34a^2z-54z^2+12az^2+34a^2z^2-54z^3-48az^3-34a^2z^3-27z^4-18az^4+9a^2z^4}{(-3-3a-6z-2az-9z^2-az^2-12z^3-4az^3-6z^4+2az^4)^2}.$$

It is clear that $z=0$ and $z=\infty$ are critical points. Furthermore, the free critical points are the roots of the polynomial:

$$\begin{aligned} q(z) &= -27-18a+9a^2-54z-48az-34a^2z-54z^2+12az^2+34a^2z^2-54z^3-48az^3 \\ &\quad -34a^2z^3-27z^4-18az^4+9a^2z^4 \end{aligned}$$

which will be called $cr_i(a)$ for $i=1,2,3,4$ and

$$\begin{aligned} cr_5(a) &= -1 \\ cr_6(a) &= i \end{aligned}$$

In Figure **??** the bifurcation diagram of the critical points is shown

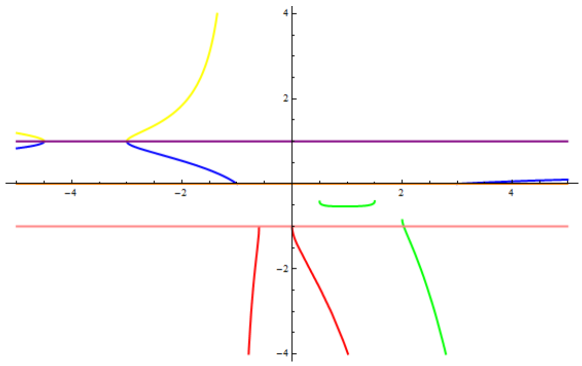

FIGURE 2. Bifurcation diagram of the critical points.

So, there are four independent free critical points, without loss of generality, we consider in this paper the free critical point $cr_3(a)$, since it behavior is representative. Now, we are going to look for the best members of the family by means of using the parameter space associated to the free critical points.

In Figures 5–9, the parameter spaces associated to $cr_3(a)$ are shown and in Figures 3-4 and Figures 6-8 the parameter spaces associated to the other free critical points are shown. A point is painted in cyan if the iteration of the method starting in $z_0=cr_1(\alpha)$ converges to the fixed point 0 (related to root A), in magenta if it converges to ∞ (related to root B) and in yellow if the iteration converges to 1 (related to ∞). Moreover, it appears in red the convergence, after a maximum of 2000 iterations and with a tolerance of 10^{-6}, to any of the strange fixed points, in orange the convergence to 2-cycles, in light green the convergence to 3-cycles, in dark red to 4-cycles, in dark blue to 5-cycles, in dark green to 6-cycles, dark yellow to 7-cycles, and in white the convergence to 8-cycles. The regions in black correspond to zones of convergence to other cycles. As a consequence, every point

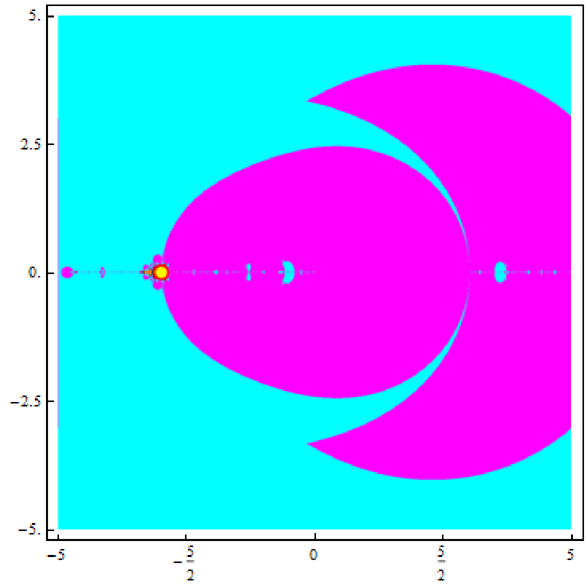

FIGURE 3. Parameter space associated to the free critical point $cr_1(a)$.

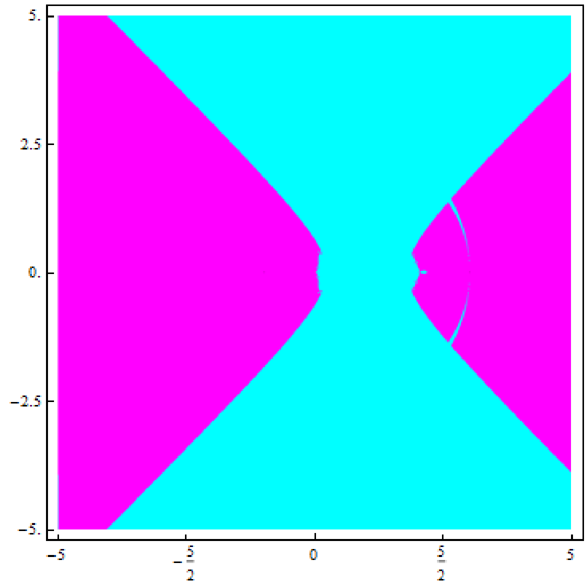

FIGURE 4. Parameter space associated to the free critical point $cr_2(a)$.

of the plane which is neither cyan nor magenta is not a good choice of α in terms of numerical behavior.

In these dynamical planes we have painted in magenta the convergence to 0, in cyan the convergence to ∞ and in black the zones with no convergence to the roots.
Then, focussing the attention in the region shown in Figure 5 it is evident that there exist members of the family with complicated behavior. In Figure 10, the dynamical planes of a member of the family with regions of convergence to any of the strange fixed points is shown

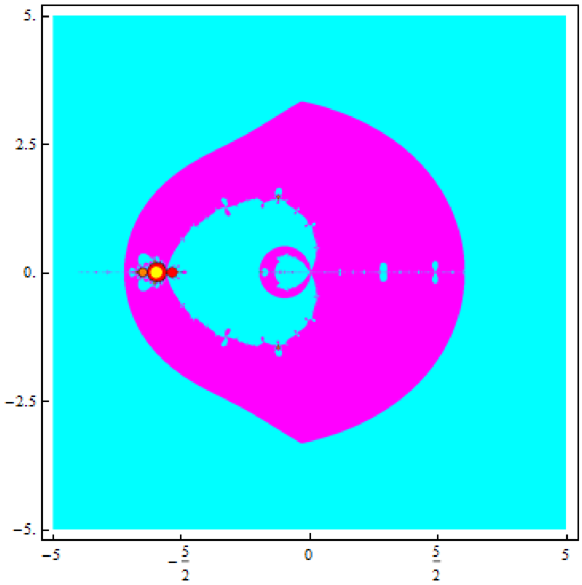

Figure 5. Parameter space associated to the free critical point $cr_3(a)$.

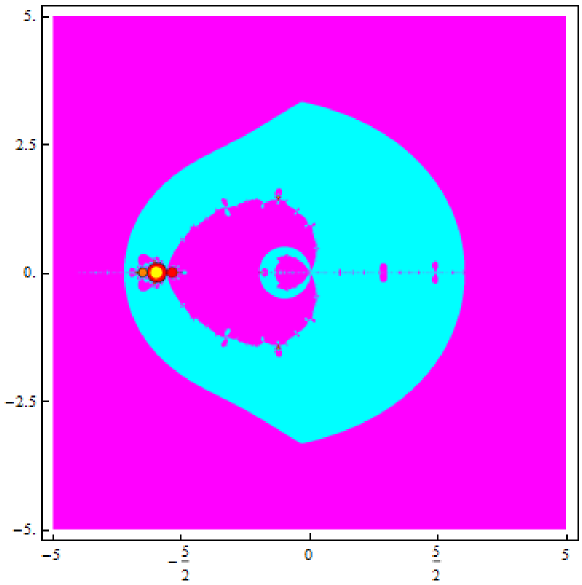

Figure 6. Parameter space associated to the free critical point $cr_4(a)$.

In Figures 11, 12 dynamical planes of members of the family with regions of convergence to an attracting 2-cycle is shown.

On the other hand, in Figure 13, the dynamical plane of a member of the family with regions of convergence to $z = 1$, related to ∞ is presented.

Other special cases are shown in Figures 14, 15 and 16.

4. Numerical example

We present numerical examples in this section.

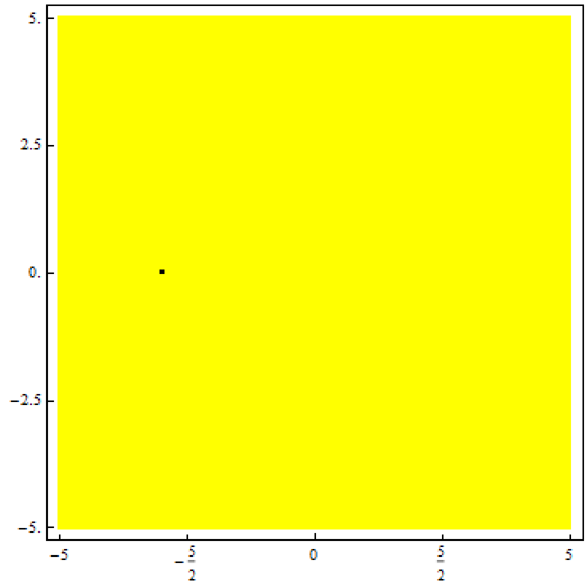

FIGURE 7. Parameter space associated to the free critical point $cr_5(a)$.

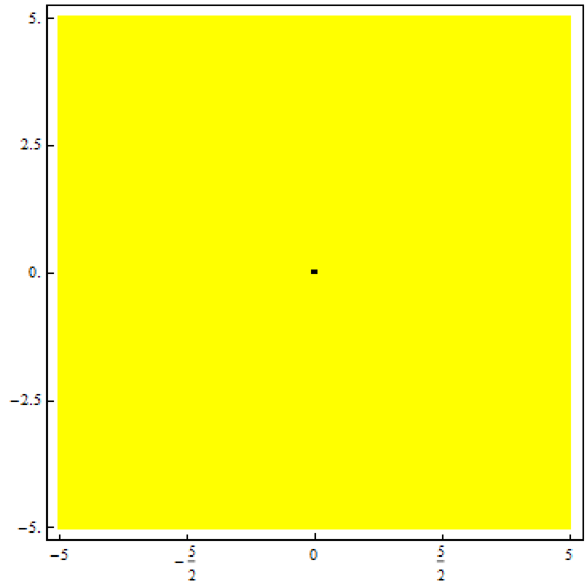

FIGURE 8. Parameter space associated to the free critical point $cr_6(a)$.

Example 4.1 Let $S = \mathbb{R}, D = [-2, 2], x^* = 0$ and define function F on D by

$$F(x) = x^3 - 0.1. \tag{4.1}$$

Then, choosing

$$\lambda = 0.025,$$
$$\mu = 0.1,$$
$$\xi = 1,$$

and

$$a = \frac{1}{2}, \quad b = c = -\frac{3}{4}$$

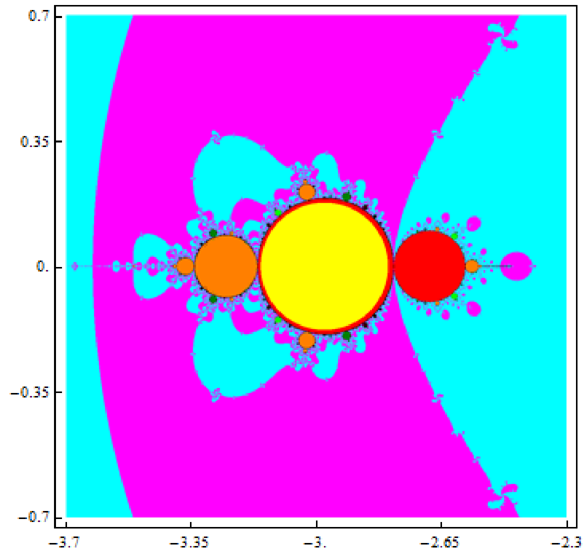

FIGURE 9. Detail of the parameter space associated to the free critical point $cr_3(a)$.

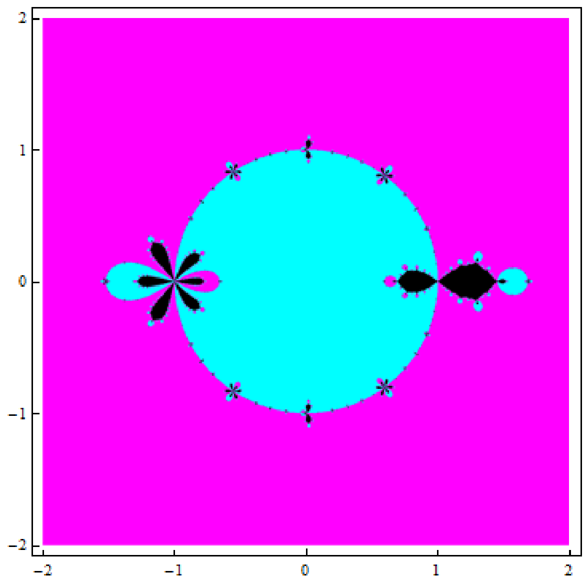

FIGURE 10. Basins of attraction associated to the method with $a = -2.65$.

we get

$$L_0 = 2.32079\ldots,$$
$$L = 4.64159\ldots$$
$$M = 2,$$
$$\alpha = 0.64633\ldots,$$
$$M_0 = 0.64633\ldots,$$

Then, by the definition of the "g" functions we obtain

$$r_1 = 0.215443\ldots \quad , r_\delta = 0.193679\ldots \quad , r_2 = 0.158717\ldots \quad ,$$

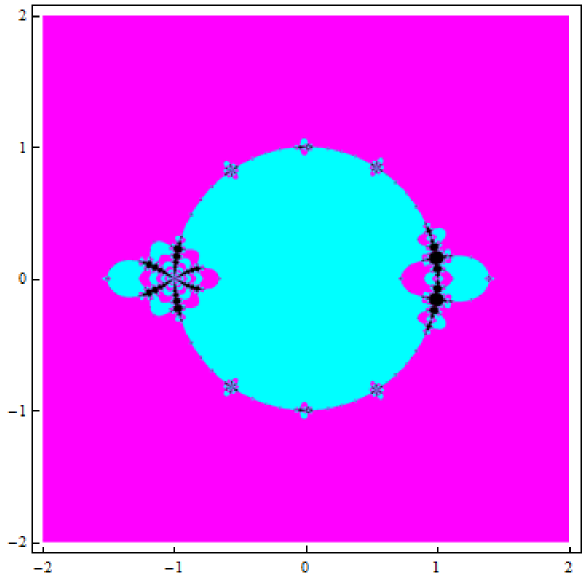

FIGURE 11. Basins of attraction associated to the method with $a = -3.35$.

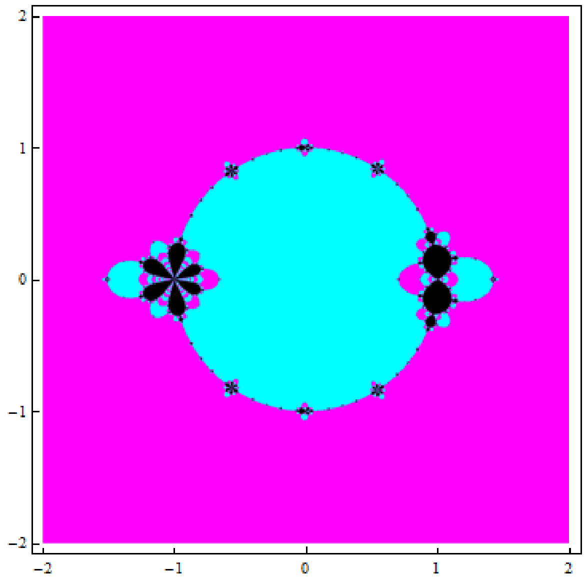

FIGURE 12. Basins of attraction associated to the method with $a = -3.25$.

$$r_h = 0.270928\ldots \quad , r_\psi = 0.205636\ldots \text{ and } r_3 = 0.156463\ldots.$$

and as a consequence

$$r = r_3 = 0.156463\ldots.$$

So we can ensure the convergence of the method (1.2) by Theorem 1.

Example 4.2

Let $\mathbb{X} = [-1/2, 1/2]$, $\mathbb{Y} = \mathbb{R}$, $x_0 = 0$ and $F : \mathbb{X} \to \mathbb{Y}$ the polynomial:

$$F(x) = \frac{1}{6}x^3 + \frac{1}{6}x^2 - \frac{5}{6}x + \frac{1}{3}.$$

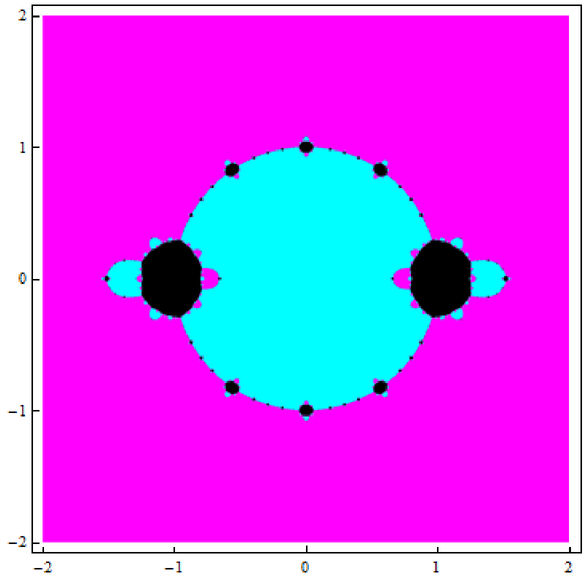

FIGURE 13. Basins of attraction associated to the method with $a = -3$.

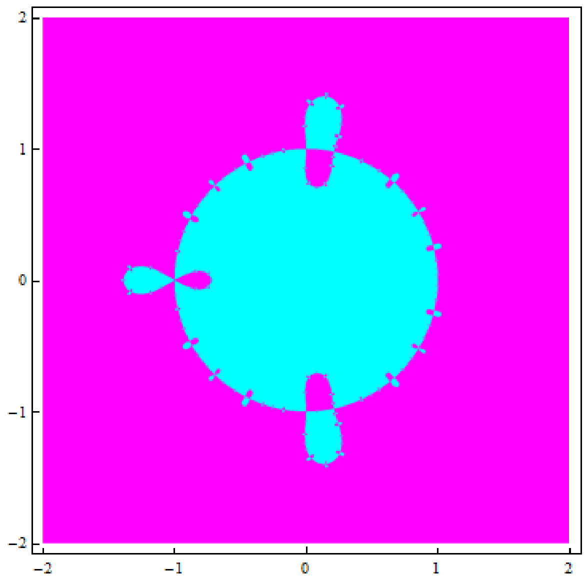

FIGURE 14. Basins of attraction associated to the method with $a = 1$.

Then, choosing

$$\lambda = 0.25,$$
$$\mu = 0.25,$$
$$\xi = 0.75,$$

and

$$a = b = c = -\frac{1}{3}$$

we get

$$L_0 = 1.86081\ldots,$$

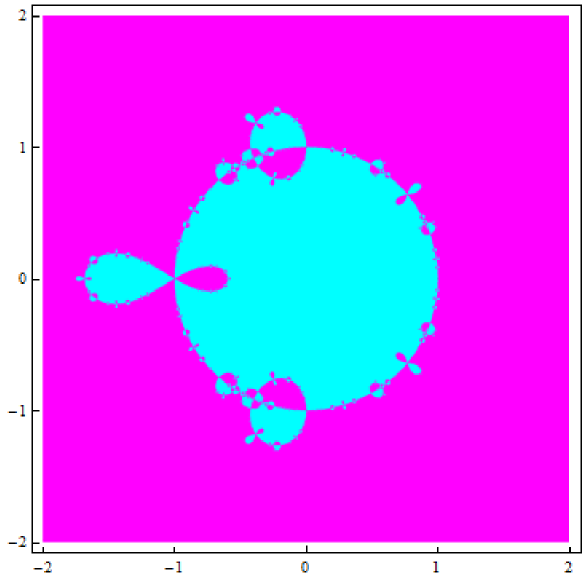

FIGURE 15. Basins of attraction associated to the method with $a = -1$.

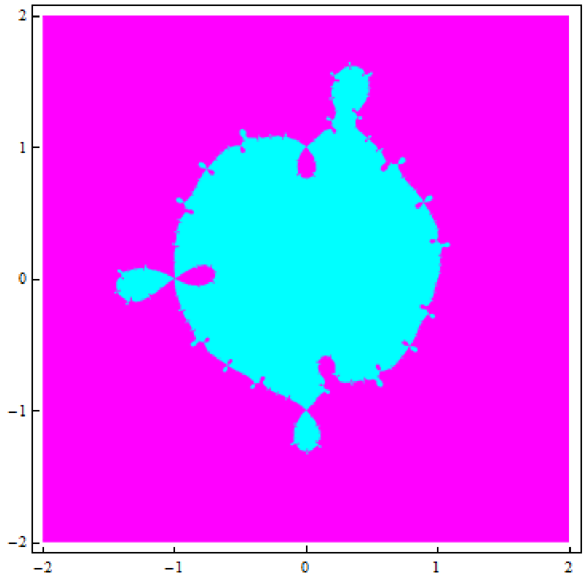

FIGURE 16. Basins of attraction associated to the method with $a = i$.

$$L = 2.33045\ldots$$
$$M = 2,$$
$$\alpha = 0.572135\ldots,$$
$$M_0 = 0.541667\ldots,$$

Then, by the definition of the "g" functions we obtain

$$r_1 = 0.165233\ldots \quad , r_\delta = 0.197013\ldots \quad , r_2 = 0.203915\ldots \quad ,$$
$$r_h = 0.242742\ldots \quad , r_\psi = 0.297044\ldots \text{ and } r_3 = 0.194199\ldots.$$

and as a consequence

$$r = r_1 = 0.165233\ldots.$$

So we can ensure the convergence of the method (1.2) by Theorem 1.

Acknowledgements

This activity has been partially supported by the Universidad Internacional de La Rioja (UNIR, http://www.unir.net), under the Plan Propio de Investigación, Desarrollo e Innovación [2013–2015]. Research group: Matemática aplicada al mundo real (MAMUR) and by the grant MTM2014-52016-C2-1-P.

References

[1] S. Amat, S. Busquier, S. Plaza, Dynamics of the King and Jarratt iterations, Aequationes Math. 69 (2005) 3, 212223.

[2] S. Amat, S. Busquier, S. Plaza, Chaotic dynamics of a third-order Newton-type method, J. Math. Anal. Appl. 366 (2010) 1, 2432.

[3] S. Amat, M.A. Hernández, N. Romero, A modified Chebyshev's iterative method with at least sixth order of convergence, Appl. Math. Comput. 206(1), 164-174 (2008).

[4] I.K. Argyros, "Convergence and Application of Newton-type Iterations," Springer, 2008.

[5] I.K. Argyros, S. Hilout, Numerical methods in Nonlinear Analysis, World Scientific Publ. Comp. New Jersey, 2013.

[6] I. K. Argyros, Á. A. Magreñán, On the convergence of an optimal fourth-order family of methods and its dynamics, Appl. Math. Comput. 252(1), 336–346 (2015).

[7] I. K. Argyros, D. González, Local convergence for an improved Jarratt-type method in Banach space., International Journal of Artificial Intelligence and Interactive Multimedia. 3(Special Issue on Teaching Mathematics Using New and Classic Tools):20-25 (2015).

[8] I. K: Argyros, S. George, Ball Convergence for Steffensen-type Fourth-order Methods. International Journal of Artificial Intelligence and Interactive Multimedia. 3(Special Issue on Teaching Mathematics Using New and Classic Tools): 37-42(2015).

[9] F. Chicharro, A. Cordero, J.R. Torregrosa, Drawing dynamical and parameters planes of iterative families and methods, The Scientific World Journal Volume 2013 Article ID 780153.

[10] C. Chun, Some improvements of Jarratt's method with sixth-order convergence, Appl. Math. Comput. 190(2), 1432-1437 (1990).

[11] A. Cordero, J. García-Maimó, J.R. Torregrosa, M.P. Vassileva, P. Vindel, Chaos in King's iterative family, Applied Mathematics Letters 26 (2013) 842–848.

[12] A. Cordero, J.R. Torregrosa, P. Vindel, Dynamics of a family of Chebyshev-Halley type methods, Applied Mathematics and Computation 219 (2013) 8568–8583.

[13] A. Cordero, J.R. Torregrosa, Variants of Newton's method using fifth-order quadrature formulas, Applied Mathematics and Computation 190 (2007) 686–698.

[14] J. A. Ezquerro, M.A. Hernández, Recurrence relations for Chebyshev-type methods, Appl. Math. Optim. 41(2), 227-236 (2000).

[15] J. A. Ezquerro, M.A. Hernández, New iterations of R-order four with reduced computational cost. BIT Numer. Math. 49, 325- 342 (2009).

[16] J. A. Ezquerro, M.A. Hernández, On the R-order of the Halley method, J. Math. Anal. Appl. 303, 591-601 (2005).

[17] J.M. Gutiérrez, M.A. Hernández, Recurrence relations for the super-Halley method, Computers Math. Applic. 36(7), 1-8(1998).

[18] Á. A. Magreñán, Different anomalies in a Jarratt family of iterative root-finding methods, Applied Mathematics and Computation 233 (2014), 29–38.

[19] Á. A. Magreñán, A new tool to study real dynamics: The convergence plane Applied Mathematics and Computation, 248 (2014), 215–224.

[20] S. K. Parhi, D.K. Gupta, Recurrence relations for a Newton-like method in Banach spaces, J. Comput. Appl. Math. 206(2), 873-887(2007).

[21] L. B. Rall, Computational solution of nonlinear operator equations, Robert E. Krieger, New York(1979).

[22] H. Ren, Q. Wu, W. Bi, New variants of Jarratt method with sixth-order convergence, Numer. Algorithms 52(4), 585-603(2009).

[23] W.C. Rheinboldt, An adaptive continuation process for solving systems of nonlinear equations, Polish Academy of Science, Banach Ctr. Publ. 3 (1978) 129–142.

[24] R. Sharma, Iterative methods for the solution of nonlinear equations (PhD Thesis), Punjab University, 2011.

[25] J. R. Sharma, Improved Chebyshev-Halley methods with six and eight order of convergence, to appear in Applied Mathematics and Computation.

[26] J.F. Traub, Iterative methods for the solution of equations, Prentice- Hall Series in Automatic Computation, Englewood Cliffs, N. J., 1964.

[27] X. Wang, J. Kou, C. Gu, Semilocal convergence of a sixth-order Jarratt method in Banach spaces, Numer. Algorithms 57, 441-456(2011).

Cameron University, Department of Mathematics Sciences Lawton, OK 73505, USA
E-mail address: iargyros@cameron.edu

Instituto de Matemáticas Multidisciplinar, Universitat Politècnica de València, 46022 València, Spain
E-mail address: acordero@mat.upv.es

Universidad Internacional de La Rioja, Escuela de Ingeniería, C/Gran Vía 41, 26005, Logroño (La Rioja), Spain
E-mail address: alberto.magrenan@gmail.com

Instituto de Matemáticas Multidisciplinar, Universitat Politècnica de València, 46022 València, Spain
E-mail address: jrtorre@mat.upv.es