INVARIANTS FOR BI-LIPSCHITZ EQUIVALENCE OF IDEALS

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Abstract. We introduce the notion of bi-Lipschitz equivalence of ideals and derive numerical invariants for such equivalence. In particular, we show that the log canonical threshold of ideals is a bi-Lipschitz invariant. We apply our method to several deformations $f_t : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ and show that they are not bi-Lipschitz trivial, specially focusing on several known examples of non μ^*-constant deformations.

1. Introduction

In 1970, O. Zariski posed in [53, p. 483] the following celebrated question: let f and g be two analytic function germs $(\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ such that there is a homeomorphism $\varphi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ so that $\varphi(f^{-1}(0)) = g^{-1}(0)$, then do the germs f and g have the same multiplicity?

We recall that the *multiplicity* or *order* of a function $f \in \mathcal{O}_n$, denoted by $\mathrm{ord}(f)$, is defined as the maximum of those $r \in \mathbb{Z}_{\geqslant 1}$ such that $f \in \mathbf{m}_n^r$, where $\mathbf{m}_n$ denotes the maximal ideal of the ring $\mathcal{O}_n$ of analytic function germs $(\mathbb{C}^n, 0) \to \mathbb{C}$. B. Teissier [45, p. 300] introduced the sequence $\mu^*(f) = (\mu^{(n)}(f), \mu^{(n-1)}(f), \dots, \mu^{(1)}(f))$, where $\mu^{(i)}(f)$ denotes the Milnor number of the restriction of f to a generic linear i-dimensional subspace of $\mathbb{C}^n$, for $i \in \{1, \dots, n\}$, and started a systematic study on topology of complex hypersurfaces (see for instance [45, 46]). We remark that $\mu^{(1)}(f) = \mathrm{ord}(f) - 1$. Teissier's works have significant impact, but the question above is still unsolved except for the case $n = 2$, and is known as the *Zariski's multiplicity conjecture* (see the survey [20]).

In [39], J.-J. Risler and D. Trotman showed that if $f, g \in \mathcal{O}_n$ are bi-Lipschitz right-left equivalent, then they have the same multiplicity. Since the concept of bi-Lipschitz homeomorphism is substantially more fruitful than just talking about homeomorphisms, the article [39] has been a motivation for several researchers to investigate singularities from the viewpoint of bi-Lipschitz equivalence in several contexts (see for instance [21, 22, 23]).

In this article we introduce the notion of bi-Lipschitz equivalence of ideals (see Definition 2.1) and derive numerical invariants for such equivalence. This notion is motivated by a particular relation between the respective Jacobian ideals of any two given function germs $(\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ which are bi-Lipschitz right-left equivalent (see (3.3)). We show that the

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order, the Łojasiewicz exponent and the log canonical threshold of a given ideal $I \subseteq \mathcal{O}_n$ are invariant in the bi-Lipschitz class of I. As a consequence, we show that the Briançon-Speder example [14] and a modification of another example of [14] (see Examples 4.13 and 4.16) are not bi-Lipschitz right-left trivial. We do not know any reference where this fact is shown, despite the fact that S. Koike [28] showed that the Briançon-Speder example (Example 4.13) is not bi-Lipschitz trivial in the real case.

The paper is organized as follows. In Section 2 we introduce some notation and recall preliminary concepts needed in the article. In Section 3, we show that the order and the Łojasiewicz exponent of a given ideal $I \subseteq \mathcal{O}_n$ are bi-Lipschitz invariant. Moreover, we show that if I and J are ideals of $\mathcal{O}_n$ such that the integral closure $\overline{I}$ of I is equal to $\mathbf{m}_n^{\mathrm{ord}(I)}$ and J is bi-Lipschitz equivalent to I, then $\overline{I} = \overline{J}$ (see Corollary 3.5). We also prove that, if $f \in \mathcal{O}_n$ has an isolated singularity at the origin, then the Łojasiewicz exponent of $J(f)$ is invariant in the class of bi-Lipschitz right-left equivalence of f (see Theorem 3.4), where $J(f)$ denotes the Jacobian ideal of $\mathcal{O}_n$.

In Section 4 we show that the log canonical threshold $\mathrm{lct}(I)$ of an ideal I is also a bi-Lipschitz invariant. This fact has several consequences. One of them is that many known examples of non μ^*-constant deformations $f_t : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$, like the Briançon-Speder example, are also examples of non bi-Lipschitz right-left trivial deformations. The key observation is stated as Corollary 4.9, which is a consequence of the results of Veys and Zúñiga-Galindo in [51] (see Theorem 4.8). Using this result, we have computed the value of $\mathrm{lct}(J(f_t))$, for generic t close enough to $0 \in \mathbb{C}$, for several deformations $f_t : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. Hence, combining the bi-Lipschitz invariance of the log canonical threshold of ideals and Corollary 4.9, we obtain a way to conclude the non bi-Lipschitz right-left triviality of deformations $f_t : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$.

We conjecture that $\mu^*(f)$ is a bi-Lipschitz invariant of f, but we do not know how to prove it. So we consider special ideals called *diagonal ideals* in Section 5. One consequence is Corollary 5.3, which shows the μ^*-constancy of bi-Lipschitz right-left trivial families $f_t : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ if $J(f_0)$ is diagonal. We also explore the connections between the bi-Lipschitz equivalence of ideals, diagonal ideals and the sequence $\mathcal{L}_0^*(I) = (\mathcal{L}_0^{(n)}(I), \dots, \mathcal{L}_0^{(1)}(I))$ of mixed Łojasiewicz exponents (see [7, 11]). At the end of the paper, we also study a special class of ideals, that we call *Hickel ideals*, which arises as a consequence of an inequality proved by Hickel [24] (see (2.3)) relating the multiplicity of I and the sequence $\mathcal{L}_0^*(I)$.

2. **Preliminaries**

We start by recalling notational conventions. Let $a(x)$ and $b(x)$ be two function germs $(\mathbb{C}^n, x_0) \to \mathbb{R}$, where $x_0 \in \mathbb{C}^n$. Then

- $a(x) \lesssim b(x)$ *near* x_0 means that there exists a positive constant $C > 0$ and an open neighbourhood U of x_0 in $\mathbb{C}^n$ such that $a(x) \leqslant C\, b(x)$, for all $x \in U$.
- $a(x) \sim b(x)$ *near* x_0 means that $a(x) \lesssim b(x)$ near x_0 and $b(x) \lesssim a(x)$ near x_0.

For an n-tuple $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$, we write $\|x\| = \sqrt{|x_1|^2 + \cdots + |x_n|^2}$.

We say that a given condition depending on a parameter $t \in \mathbb{C}$ holds for all $|t| \ll 1$ when there exists some open neighbourhood U of $0 \in \mathbb{C}$ such that the said condition holds for all $t \in U$.

J. Mather [34] defined the notions of right equivalence, right-left equivalence and contact equivalence for map germs (see also [52]). The corresponding equivalence classes are the orbits of the action of the groups $\mathcal{R}$, $\mathcal{A}$ and $\mathcal{K}$ respectively, where

- $\mathcal{R}$ is the group of diffeomorphism germs of the source,
- $\mathcal{A}$ is the direct product of the group of diffeomorphism germs of the source and the target,
- $\mathcal{K}$ is the group that is formed by the elements $(\varphi(x), \phi_x(y))$ so that
  - $x \mapsto \varphi(x)$ is a diffeomorphism germ of the source, and
  - $y \mapsto \phi_x(y)$ are diffemorphism germs of the target for any x.

For shortness, we often refer to right equivalence, right-left equivalence and contact equivalence as $\mathcal{R}$-equivalence, $\mathcal{A}$-equivalence, and $\mathcal{K}$-equivalence, respectively.

It is natural to consider the bi-Lipschitz analogue of these notions, which we expose in the following subsection.

2.1. **Bi-Lipschitz equivalences.** We start with recalling the definition of bi-Lipschitz map. A map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is said to be *Lipschitz* if

$$\|f(x) - f(x')\| \lesssim \|x - x'\| \text{ near } 0.$$

We say that a homeomorphism $\varphi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is *bi-Lipschitz* if φ and φ^{-1} are Lipschitz. Now we can state obvious bi-Lipschitz analogues for several equivalence relations. Let us consider two map germs $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, then

- we say that f and g are *bi-Lipschitz $\mathcal{R}$-equivalent* if there is a bi-Lipschitz homeomorphism $\varphi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ so that $f = g \circ \varphi$.
- we say that f and g are *bi-Lipschitz $\mathcal{A}$-equivalent* if there are a bi-Lipschitz homeomorphism $\varphi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ and a bi-Lipschitz homeomorphism $\phi : (\mathbb{C}^p, 0) \to (\mathbb{C}^p, 0)$ so that $\phi \circ f = g \circ \varphi$.
- we say that f and g are *bi-Lipschitz $\mathcal{K}$-equivalent* if there are a bi-Lipschitz homeomorphism $\varphi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ and a bi-Lipschitz homeomorphism $\Phi : (\mathbb{C}^n \times \mathbb{C}^p, 0) \to (\mathbb{C}^n \times \mathbb{C}^p, 0)$, written as $(x, y) \mapsto (\varphi(x), \phi_x(y))$, so that $\Phi(\mathbb{C}^n \times \{0\}) = \mathbb{C}^n \times \{0\}$ and $\phi_x(f(x)) = g(\varphi(x))$, for all x belonging to some open neighbourhood of $0 \in \mathbb{C}^n$.
- we say that f and g are *bi-Lipschitz $\mathcal{K}^*$-equivalent* if there are a bi-Lipschitz homeomorphism $\varphi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ and a map $A : (\mathbb{C}^n, 0) \to \mathrm{GL}(\mathbb{C}^p)$ so that A and $A^{-1} : (\mathbb{C}^n, 0) \to \mathrm{GL}(\mathbb{C}^p)$ are Lipschitz and that $A(x)f(x) = g(\varphi(x))$, for all x belonging to some open neighbourhood of $0 \in \mathbb{C}^n$.

Two given subsets X_1 and X_2 of $(\mathbb{C}^n, 0)$ are called *bi-Lipschitz equivalent* if there is a bi-Lipschitz homeomorphism $\varphi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ so that $\varphi(X_1) = X_2$.

The definition of bi-Lipschitz $\mathcal{K}$-equivalence is used in [4]. It is possible to consider a weaker version of the definition of $\mathcal{K}$-equivalence by replacing the condition that Φ is bi-Lipschitz by the condition that ϕ_x is bi-Lipschitz, for all x belonging to some open neighbourhood of $0 \in \mathbb{C}^n$. We only need this condition in the proof of Theorem 4.2.

The definition of $\mathcal{K}^*$-equivalence is inspired by the condition (iii) of the first proposition in paragraph (2.3) in [34].

If I is an ideal of $\mathcal{O}_n$, then we denote by $\overline{I}$ the integral closure of I. Given a bi-Lipschitz homeomorphism $\varphi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$, we do not have the induced map $\varphi^* : \mathcal{O}_n \to \mathcal{O}_n$, since $f \circ \varphi$ may not be holomorphic for $f \in \mathcal{O}_n$. So we introduce the following definition.

Definition 2.1. Let I and J be ideals of $\mathcal{O}_n$. We say that I and J are *bi-Lipschitz equivalent* if there exist two families $f_1, \dots, f_p$ and $g_1, \dots, g_q$ of functions of $\mathcal{O}_n$ such that

(a) $\langle f_1, \dots, f_p \rangle \subseteq I$ and $\overline{\langle f_1, \dots, f_p \rangle} = \overline{I}$,
(b) $\langle g_1, \dots, g_q \rangle \subseteq J$ and $\overline{\langle g_1, \dots, g_q \rangle} = \overline{J}$,
(c) there is a bi-Lipschitz homeomorphism $\varphi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that

$$\|(f_1(x), \dots, f_p(x))\| \sim \|(g_1(\varphi(x)), \dots, g_q(\varphi(x)))\| \qquad \text{near } 0.$$

We remark that, under the conditions of item (a), the ideal $\langle f_1, \dots, f_p \rangle$ is usually called a *reduction* of I (see [26, p. 6]).

Let us consider an analytic map $F : (\mathbb{C} \times \mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$. Let $f_t : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be the map given by $f_t(x) = F(t, x)$, for all $|t| \ll 1$. Let I_t denote the ideal of $\mathcal{O}_n$ generated by the component functions of f_t for all $|t| \ll 1$. We say that the family of ideals I_t is *bi-Lipschitz trivial* when I_0 is bi-Lipschitz equivalent to I_t, for all $|t| \ll 1$. We say that the deformation f_t is *bi-Lipschitz $\mathcal{A}$-trivial* when f_0 is bi-Lipschitz $\mathcal{A}$-equivalent to f_t, for all $|t| \ll 1$. The notions of bi-Lipschitz $\mathcal{R}$, $\mathcal{K}$ or $\mathcal{K}^*$-triviality of deformations f_t are defined analogously.

Remark 2.2. Since $\mathcal{O}_n$ is a normal ring, any principal ideal of $\mathcal{O}_n$ is integrally closed (see [26, Proposition 1.5.2]). Therefore, if $f, g \in \mathbf{m}_n$, then the ideals $\langle f \rangle$ and $\langle g \rangle$ are bi-Lipschitz equivalent if and only if there exists some homeomorphism $\varphi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $|f| \sim |g \circ \varphi|$ near 0.

Let $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be analytic map germs. Here we remark some obvious consequences:

- If f and g are bi-Lipschitz $\mathcal{R}$-equivalent, then they are bi-Lipschitz $\mathcal{A}$ (and $\mathcal{K}^*$)-equivalent.
- If f and g are bi-Lipschitz $\mathcal{A}$-equivalent or $\mathcal{K}^*$-equivalent, then they are bi-Lipschitz $\mathcal{K}$-equivalent.
- If f and g are bi-Lipschitz $\mathcal{K}$-equivalent, then the ideals generated by their components are bi-Lipschitz equivalent.
- If two ideals are bi-Lipschitz equivalent, then their zero loci are bi-Lipschitz equivalent.

The following questions seem to be open.

Question 2.3.
- If f and g are bi-Lipschitz $\mathcal{K}$-equivalent, are f and g bi-Lipschitz $\mathcal{K}^*$-equivalent?
- If f and g are bi-Lipschitz $\mathcal{A}$-equivalent, are f and g bi-Lipschitz $\mathcal{K}^*$-equivalent?

Question 2.4. Let X and Y be germs of complex analytic subvarieties at 0 in $\mathbb{C}^n$. If there exist a bi-Lipschitz homeomorphism $h : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ so that $h(X) = Y$, are the respective defining ideals of X and Y bi-Lipschitz equivalent?

Let $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be two reduced holomorphic functions. Assume that there is a bi-Lipschitz homeomorphism $\varphi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ so that $f^{-1}(0) = \varphi(g^{-1}(0))$. The authors do not know whether $g(\varphi(x))/f(x)$ is bounded away from 0 and infinity, or not.

2.2. **Łojasiewicz exponent of ideals.** Let I and J be ideals of $\mathcal{O}_n$. Let $\{f_1, \dots, f_p\}$ be a generating system of I and let $\{g_1, \dots, g_q\}$ be a generating system of J. Let us consider the maps $f = (f_1, \dots, f_p) : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ and $g = (g_1, \dots, g_q) : (\mathbb{C}^n, 0) \to (\mathbb{C}^q, 0)$. We define the *Łojasiewicz exponent of I with respect to J*, denoted by $\mathcal{L}_J(I)$, as the infimum of the set

$$\big\{\alpha \in \mathbb{R}_{\geqslant 0} : \|g(x)\|^\alpha \lesssim \|f(x)\| \text{ near } 0\big\}. \tag{2.1}$$

By convention, we set $\inf \emptyset = \infty$. So if the above set is empty, then $\mathcal{L}_J(I) = \infty$.

It is well known that $\mathcal{L}_J(I)$ is finite if and only if $V(I) \subseteq V(J)$. When $\mathcal{L}_J(I)$ is finite, then this is a rational number (see [33] or [48]).

Let us suppose that the ideal I has finite colength. When $J = \mathbf{m}_n$, then we denote the number $\mathcal{L}_J(I)$ by $\mathcal{L}_0(I)$. That is

$$\mathcal{L}_0(I) = \inf\big\{\alpha \in \mathbb{R}_{\geqslant 0} : \|x\|^\alpha \lesssim \|f(x)\| \text{ near } 0\big\}.$$

We refer to $\mathcal{L}_0(I)$ as the *Łojasiewicz exponent of I*.

If $I_1, \dots, I_n$ are ideals of $\mathcal{O}_n$ of finite colength, then we denote by $e(I_1, \dots, I_n)$ the mixed multiplicity of $I_1, \dots, I_n$ defined by Teissier and Risler in [45, §2]. We also refer to [26, §17.4] and [44] for the definition and fundamental properties of mixed multiplicities of ideals.

Let I be an ideal of $\mathcal{O}_n$ of finite colength. Given an index $i \in \{1, \dots, n\}$, we define $e_i(I) = e(I, \dots, I, \mathbf{m}, \dots, \mathbf{m})$, where I is repeated i times and $\mathbf{m}$ is repeated $n - i$ times. In particular $e_1(I) = \mathrm{ord}(I)$ and $e_n(I) = e(I)$, where $e(I)$ denotes the multiplicity of I (see [26] or [50]).

If f has an isolated singularity at the origin, then we denote by $\mu(f)$ the Milnor number of f, that is, $\mu(f) = \dim_{\mathbb{C}} \mathcal{O}_n/J(f)$. It is proven in [45] that $\mu^{(i)}(f) = e_i(J(f))$, for all $i = 1, \dots, n$, where $\mu^{(i)}(f)$ denotes the Milnor number of the restriction of f to a generic linear i-dimensional subspace of $\mathbb{C}^n$, $i = 1, \dots, n$. By the results of Teissier [45, p. 334] and Briançon-Speder [15, p. 159] we know that, if $f_t : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ denotes an analytic family of function germs such that f_t have simultaneously isolated singularities at 0, then the constancy of $\mu^*(f_t)$ is equivalent to the Whitney equisingularity of the deformation f_t.

In [46, p. 287] Teissier asked whether $\mathcal{L}_0(J(f_t))$ remains constant in μ-constant analytic deformations $f_t : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. There is still no general answer to this question. However, as a consequence of [46, 1.7] and [46, Théorème 6] it follows that, if $f_t : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ denotes a μ^*-constant analytic deformation, then $\mathcal{L}_0(J(f_t))$ is also constant. As a consequence of a more general result, we will see that if the deformation f_t is bi-Lipschitz trivial, then $\mathcal{L}_0(J(f_t))$ is constant.

Analogously to mixed multiplicities, there is a notion of mixed Łojasiewicz exponent $\mathcal{L}_0(I_1, \dots, I_n)$, where $I_1, \dots, I_n$ are ideals of $\mathcal{O}_n$ (see [7, 11, 12] for details). In particular, if I is an ideal of finite colength, we can speak about the sequence $\mathcal{L}^*_0(I) = (\mathcal{L}^{(n)}_0(I), \dots, \mathcal{L}^{(1)}_0(I))$, where $\mathcal{L}^{(i)}_0(I) = \mathcal{L}_0(I, \dots, I, \mathbf{m}, \dots, \mathbf{m})$, with I repeated i times and $\mathbf{m}$ repeated $n - i$ times, for all $i = 1, \dots, n$. By [12, Lemma 3.9], if we fix an index $i \in \{1, \dots, n\}$, then $\mathcal{L}^{(i)}_0(I)$ is equal to the Łojasiewicz exponent, in the usual sense, of the restriction of I to a generic linear subspace of $\mathbb{C}^n$ of dimension i (see also [24]). We recall that $\mathcal{L}^{(1)}_0(I) = \mathrm{ord}(I)$.

By [12, Corollary 3.2] or [24, p. 644], if I is an ideal of $\mathcal{O}_n$ of finite colength, then

$$\frac{e_i(I)}{e_{i-1}(I)} \leqslant \mathcal{L}^{(i)}_0(I) \tag{2.2}$$

for all $i = 1, \dots, n$. In particular

$$e(I) \leqslant \mathcal{L}^{(1)}_0(I) \cdots \mathcal{L}^{(n)}_0(I). \tag{2.3}$$

We say that I is a *Hickel ideal* if equality holds in (2.3) (see Lemma 5.5). We refer to [9] for a characterization of this property for monomial ideals. Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be an analytic function germ with an isolated singularity at the origin. We say that f is a *Hickel singularity* when the Jacobian ideal $J(f)$ is a Hickel ideal, that is, when the Milnor number of f is written as $\mu(f) = \mathcal{L}^{(1)}_0(J(f)) \cdots \mathcal{L}^{(n)}_0(J(f))$.

3. The bi-Lipschitz invariance of the Łojasiewicz exponent

In this section we show the bi-Lipschitz invariance of the Łojasiewicz exponent and the order of ideals. Moreover we also show that $\mathcal{L}_0(J(f))$ is bi-Lipschitz $\mathcal{A}$-invariant and bi-Lipschitz $\mathcal{K}^*$-invariant, for any $f \in \mathcal{O}_n$ with an isolated singularity at the origin.

We start with a general result about bi-Lipschitz equivalence of ideals.

Proposition 3.1. *Let I and J be ideals of $\mathcal{O}_n$. If I and J are bi-Lipschitz equivalent, then $\mathbf{m}^r I^s$ and $\mathbf{m}^r J^s$ are bi-Lipschitz equivalent, for all $r, s \in \mathbb{Z}_{\geqslant 1}$.*

Proof. It is enough to show that I^r and J^r are bi-Lipschitz equivalent, for all $r \in \mathbb{Z}_{\geqslant 1}$, and $\mathbf{m}I$ and $\mathbf{m}J$ are bi-Lipschitz equivalent.

Since I and J are bi-Lipschitz equivalent, there exist elements $f_1, \dots, f_p \in I$, $g_1, \dots, g_q \in J$ and a bi-Lipschitz homeomorphism $\varphi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that

$$\|f(x)\| \sim \|(g \circ \varphi)(x)\| \qquad \text{near } 0, \tag{3.1}$$

where $f = (f_1, \ldots, f_p)$ and $g = (g_1, \ldots, g_q)$. Let $r \in \mathbb{Z}_{\geqslant 1}$. Then (3.1) implies that

$$\|f(x)\|^r \sim \|(g \circ \varphi)(x)\|^r \qquad \text{near } 0. \tag{3.2}$$

We know that $\overline{I^r} = \overline{\langle f_1^r, \ldots, f_p^r \rangle}$ and $\overline{J^r} = \overline{\langle g_1^r, \ldots, g_q^r \rangle}$ (see for instance [26, Proposition 8.15] or [50, Corollary 1.40]). Therefore

$$\|(f_1^r(x), \ldots, f_p^r(x))\| \sim \|f(x)\|^r \sim \|(g \circ \varphi)(x)\|^r \sim \| \left(g_1^r(\varphi(x)), \ldots, g_q^r(\varphi(x))\right) \| \qquad \text{near } 0,$$

which says that I^r and J^r are bi-Lipschitz equivalent.

Applying (3.1) and the fact that φ is a bi-Lipschitz map, we obtain

$$\|x\|\|f(x)\| \sim \|x\|\|g(\varphi(x))\| \sim \|\varphi(x)\|\|g(\varphi(x))\| \qquad \text{near } 0.$$

It is straightforward to see that

$$\|x\|\|f(x)\| \sim \|(x_1 f_1(x), \ldots, x_1 f_p(x), \ldots, x_n f_1(x), \ldots, x_n f_p(x))\| \qquad \text{near } 0$$

and an analogous relation holds by replacing $(f_1, \ldots, f_p)$ by $(g_1, \ldots, g_q)$. Therefore we conclude that $\mathbf{m}I$ and $\mathbf{m}J$ are bi-Lipschitz equivalent. □

In Example 4.5 we show an application of the previous result.

Theorem 3.2. *Let I and J be ideals of $\mathcal{O}_n$ such that I and J are bi-Lipschitz equivalent. Then* $\operatorname{ord}(I) = \operatorname{ord}(J)$. *If, moreover, I and J have finite colength, then $\mathcal{L}_0(I) = \mathcal{L}_0(J)$.*

Proof. Since I and J are bi-Lipschitz equivalent, there exist analytic map germs $f = (f_1, \ldots, f_p) : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ and $g = (g_1, \ldots, g_q) : (\mathbb{C}^n, 0) \to (\mathbb{C}^q, 0)$ such that $\overline{I} = \overline{\langle f_1, \ldots, f_p \rangle}$, $\overline{J} = \overline{\langle g_1, \ldots, g_q \rangle}$ and there exists a bi-Lipschitz homeomorphism $\varphi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ so that $\|g(\varphi(x))\| \sim \|f(x)\|$ near 0. By symmetry, it is enough to show that $\mathcal{L}_0(I) \leqslant \mathcal{L}_0(J)$ and $\operatorname{ord}(I) \leqslant \operatorname{ord}(J)$.

Let $\theta \in \mathbb{R}_{\geqslant 0}$ such that $\|x\|^\theta \lesssim \|g(x)\|$ near 0. Then

$$\|x\|^\theta \sim \|\varphi(x)\|^\theta \lesssim \|g(\varphi(x))\| \sim \|f(x)\| \qquad \text{near } 0$$

and we obtain that $\mathcal{L}_0(I) \leqslant \mathcal{L}_0(J)$.

We remark that

$$\operatorname{ord}(J) = \max\{s : J \subseteq \mathbf{m}_n^s\} = \max\{s : J \subseteq \overline{\mathbf{m}_n^s}\} = \max\{s : \|g(x)\| \lesssim \|x\|^s \text{ near } 0\}.$$

If $\|f(x)\| \lesssim \|x\|^s$ near 0, then we have

$$\|g(x)\| \sim \|f(\varphi(x))\| \lesssim \|\varphi(x)\|^s \sim \|x\|^s \qquad \text{near } 0$$

and we obtain $\operatorname{ord}(I) \leqslant \operatorname{ord}(J)$. □

Remark 3.3. If I is an ideal of $\mathcal{O}_n$ of finite colength, then an elementary computation shows that

$$\mathcal{L}_0(\mathbf{m}^r I^s) = r + s\mathcal{L}_0(I).$$

for all $r, s \in \mathbb{Z}_{\geqslant 1}$. Hence if J is another ideal of $\mathcal{O}_n$ of finite colength, then saying that $\mathcal{L}_0(\mathbf{m}^r I^s) = \mathcal{L}_0(\mathbf{m}^r J^s)$, for all $r, s \in \mathbb{Z}_{\geqslant 1}$, is equivalent to just saying that $\mathcal{L}_0(I) = \mathcal{L}_0(J)$. In the next section, we introduce another bi-Lipschitz invariant associated to any ideal I

of $\mathcal{O}_n$ that, when computed for the ideals of $\{\mathbf{m}^r I^s : r, s \in \mathbb{Z}_{\geqslant 1}\}$, gives rise to a significant infinite set of bi-Lipschitz invariants of I (see Remark 4.4).

Theorem 3.4. *Let $f, g \in \mathcal{O}_n$ with an isolated singularity at the origin. Let us suppose that f and g are bi-Lipschitz $\mathcal{A}$-equivalent or bi-Lipschitz $\mathcal{K}^*$-equivalent. Then $J(f)$ and $J(g)$ are bi-Lipschitz equivalent. In particular,* $\mathrm{ord}(f) = \mathrm{ord}(g)$ *and* $\mathcal{L}_0(J(f)) = \mathcal{L}_0(J(g))$.

Proof. Let us consider a bi-Lipschitz homeomorphism $\varphi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ and a bi-Lipschitz homeomorphism $\phi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ so that $g(\varphi(x)) = \phi(f(x))$, for all x belonging to some open neighbourhood of $0 \in \mathbb{C}^n$. By Rademacher's theorem (see for instance [30, Theorem 5.1.11]), the first order partial derivatives of φ and φ^{-1} exist in some open neighbourhood of $0 \in \mathbb{C}^n$ except in a thin set. The bi-Lipschitz property of φ implies that the first order partial derivatives of φ and φ^{-1} are bounded. Then we conclude that

$$\|(\nabla g)(\varphi(x))\| \lesssim \|(\nabla g)(\varphi(x))D\varphi(x)\| = \|D\phi(f(x))\nabla f(x)\| \lesssim \|\nabla f(x)\| \tag{3.3}$$

almost everywhere, where $D\varphi(x)$ denotes the Jacobian matrix of φ at x. By continuity, we have $\|(\nabla g)(\varphi(x))\| \lesssim \|\nabla f(x)\|$ near 0. Similarly, we have $\|(\nabla f)(\varphi^{-1}(x))\| \lesssim \|\nabla g(x)\|$ near 0. Hence we conclude that the ideals $J(f)$ and $J(g)$ are bi-Lipschitz equivalent.

Let us suppose that f and g are bi-Lipschitz $\mathcal{K}^*$-equivalent. Let $A : (\mathbb{C}^n, 0) \to \mathbb{C}^*$ be a Lipschitz map such that the map $A^{-1} : (\mathbb{C}^n, 0) \to \mathbb{C}^*$ defined by $A^{-1}(x) = A(x)^{-1}$ is Lipschitz and $g(\varphi(x)) = A(x)f(x)$, for all x belonging to some open neighbourhood of the origin. Then we obtain that

$$\begin{aligned}
\|(\nabla g)(\varphi(x))\| &\lesssim \|(\nabla g)(\varphi(x))D\varphi(x)\| && \text{(since } \varphi^{-1} \text{ is Lipschitz)}\\
&= \|\nabla(g \circ \varphi)(x)\| \\
&= \|\nabla A(x)f(x) + A(x)\nabla f(x)\| && \text{(since } g(\varphi(x)) = A(x)f(x))\\
&\leq \|\nabla A(x)\||f(x)| + |A(x)|\|\nabla f(x)\| \\
&\lesssim |f(x)| + \|\nabla f(x)\| && \text{(since } A(x) \text{ is Lipschitz)}\\
&\lesssim \|x\|\|\nabla f(x)\| + \|\nabla f(x)\| && \text{(since } |f(x)| \lesssim \|x\|\|\nabla f(x)\|)\\
&\lesssim \|\nabla f(x)\|,
\end{aligned}$$

almost everywhere. Similarly, we have $\|(\nabla f)(\varphi^{-1}(x))\| \lesssim \|\nabla g(x))\|$ near 0 and hence we obtain that the ideals $J(f)$ and $J(g)$ are bi-Lipschitz equivalent. □

Let I be an ideal of $\mathcal{O}_n$ of finite colength. The Łojasiewicz exponent $\mathcal{L}_0(I)$ of I is always a rational number. Let us write $\mathcal{L}_0(I) = \frac{p}{q}$, where $p, q \in \mathbb{Z}_{\geqslant 1}$. Then, by [33], we have that $\mathbf{m}^p \subseteq \overline{I^q}$. Thus $e(\mathbf{m}^p) \geqslant e(\overline{I^q}) = e(I^q)$, which implies that $p^n \geqslant q^n e(I)$. Moreover, the inclusion $I \subseteq \mathbf{m}^{\mathrm{ord}(I)}$ implies that $e(I) \geqslant \mathrm{ord}(I)^n$. Then we have

$$\mathcal{L}_0(I)^n \geqslant e(I) \geqslant \mathrm{ord}(I)^n. \tag{3.4}$$

We refer to [12, Corollaries 3.2 and 3.4] for more general inequalities.

Corollary 3.5. *Let I and J be ideals of $\mathcal{O}_n$ of finite colength. Let us suppose that $\overline{I} = \mathbf{m}^{\mathrm{ord}(I)}$. Then I and J are bi-Lipschitz equivalent if and only if $\overline{I} = \overline{J}$.*

Proof. The *if* part is obvious. Let us suppose that I and J are bi-Lipschitz equivalent. Then $\mathrm{ord}(I) = \mathrm{ord}(J)$ and $\mathcal{L}_0(I) = \mathcal{L}_0(J)$, by Theorem 3.2. Since $\overline{I} = \mathbf{m}^{\mathrm{ord}(I)}$, we have $\mathcal{L}_0(I) = \mathrm{ord}(I)$. By relation (3.4) we obtain

$$e(I) = \mathrm{ord}(I)^n = \mathcal{L}_0(I)^n = \mathcal{L}_0(J)^n \geqslant e(J) \geqslant \mathrm{ord}(J)^n = \mathrm{ord}(I)^n.$$

Which implies that $\overline{J} = \mathbf{m}^{\mathrm{ord}(J)} = \mathbf{m}^{\mathrm{ord}(I)}$, by the Rees' multiplicity theorem (see for instance [26, p. 222]). □

Corollary 3.6. *Let $f \in \mathcal{O}_n$ such that $\overline{J(f)} = \mathbf{m}^{\mathrm{ord}(f)-1}$. Then, if $g \in \mathcal{O}_n$ verifies that f and g are bi-Lipschitz $\mathcal{A}$-equivalent or bi-Lipschitz $\mathcal{K}^*$-equivalent, then $\overline{J(g)} = \overline{J(f)}$.*

Proof. This is an immediate application of Theorem 3.4 and Corollary 3.5. □

4. Log canonical threshold

The main purpose of this section is to show in Theorem 4.2 that the log canonical threshold $\mathrm{lct}(I)$ is bi-Lipschitz invariant and to apply this fact in several known examples. We refer to the survey [36] for fundamental information about the notion of log canonical threshold.

The *log canonical threshold* of a non-zero function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, denoted by $\mathrm{lct}(f)$, is the supremum of those $s \in \mathbb{R}_{\geqslant 0}$ so that $|f(x)|^{-2s}$ is locally integrable at 0, that is, integrable on some compact neighbourhood of 0. This definition is generalized for ideals as follows.

Definition 4.1. Let I be a proper ideal of $\mathcal{O}_n$. Let us consider a generating system $\{g_1, \dots, g_r\}$ of I. The *log canonical threshold* of I, denoted by $\mathrm{lct}(I)$, is defined as follows:

$$\mathrm{lct}(I) = \sup\left\{s \in \mathbb{R}_{\geqslant 0} : \left(|g_1(x)|^2 + \cdots + |g_r(x)|^2\right)^{-s} \text{ is locally integrable at } 0\right\}.$$

The *Arnold index* of I, denoted by $\mu(I)$, is defined as $\mu(I) = \frac{1}{\mathrm{lct}(I)}$ (we follow the notation used in [19]).

It is straightforward to see that the definition of $\mathrm{lct}(I)$ does not depend on the choice of a generating system of I and $\mathrm{lct}(I) \geqslant \mathrm{lct}(g)$, for all $g \in I$. More generally, if J and I are proper ideals of $\mathcal{O}_n$ such that $J \subseteq I$, then $\mathrm{lct}(J) \leqslant \mathrm{lct}(I)$. If $I \subseteq \mathbf{m}_n^r$, then

$$\mathrm{lct}(I) \leqslant \mathrm{lct}(\mathbf{m}_n^r) = \frac{\mathrm{lct}(\mathbf{m}_n)}{r} = \frac{n}{r}$$

by [36, Property 1.14]. In particular $\mathrm{lct}(I)\,\mathrm{ord}(I) \leqslant n$. We also remark that $\mathrm{lct}(I) = \mathrm{lct}(\overline{I})$ and that $\mathrm{lct}(I)$ is a positive rational number (see [36]).

Theorem 4.2. *Let $f, g \in \mathbf{m}_n$ and let I and J be proper ideals of $\mathcal{O}_n$.*

(a) *If f and g are bi-Lipschitz $\mathcal{K}$-equivalent, then* $\mathrm{lct}(f) = \mathrm{lct}(g)$.
(b) *If I and J are bi-Lipschitz equivalent, then* $\mathrm{lct}(I) = \mathrm{lct}(J)$.

Proof. (a): By the definition of bi-Lipschitz $\mathcal{K}$-equivalence, let us consider bi-Lipschitz homeomorphisms $\varphi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$, $x \mapsto x' = \varphi(x)$, and $\phi_x : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$,

$y \mapsto y' = \phi_x(y)$, for all x belonging to some open neighbourhood U of $0 \in \mathbb{C}^n$, such that $g(\varphi(x)) = \phi_x(f(x))$, for all $x \in U$.

By Rademacher's theorem (see [30, Theorem 5.1.11]), φ is differentiable almost everywhere in the sense of Lebesgue measure, and its jacobian $J(\varphi)$ is measurable. By the Lipschitz property, we have $|J(\varphi)| \lesssim 1$ and $|\phi_x(y)| \sim |y|$. So we have

$$\begin{aligned}\int_{\varphi(K)} |g(x')|^{-2s} \frac{dx' \wedge d\bar{x}'}{\sqrt{-1}^n} &= \int_K |g(\varphi(x))|^{-2s} |J(\varphi)| \frac{dx \wedge d\bar{x}}{\sqrt{-1}^n} \\ &\lesssim \int_K |\phi_x(f(x))|^{-2s} \frac{dx \wedge d\bar{x}}{\sqrt{-1}^n} \\ &\lesssim \int_K |f(x)|^{-2s} \frac{dx \wedge d\bar{x}}{\sqrt{-1}^n}\end{aligned}$$

where K is a compact neighbourhod of 0. This implies $\mathrm{lct}(f) \leqslant \mathrm{lct}(g)$ and vice versa.
(b): Let $f = (f_1, \dots, f_p) : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ and $g = (g_1, \dots, g_q) : (\mathbb{C}^n, 0) \to (\mathbb{C}^q, 0)$ be analytic map germs and let $\varphi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ be a germ of bi-Lipschitz homeomorphism such that $\overline{I} = \overline{\langle f_1, \dots, f_p \rangle}$, $\overline{J} = \overline{\langle g_1, \dots, g_q \rangle}$ and $\|f(x)\| \sim \|g(\varphi(x))\|$ near $0 \in \mathbb{C}^n$. We have

$$\int_{\varphi(K)} \|g(x')\|^{-2s} \frac{dx' \wedge d\bar{x}'}{\sqrt{-1}^n} = \int_K \|g(\varphi(x))\|^{-2s} |J(\varphi)| \frac{dx \wedge d\bar{x}}{\sqrt{-1}^n} \lesssim \int_K \|f(x)\|^{-2s} \frac{dx \wedge d\bar{x}}{\sqrt{-1}^n}$$

where K is a compact neighbourhod of 0. This implies $\mathrm{lct}(I) \leqslant \mathrm{lct}(J)$ and vice versa. □

In the rest of this section we show some results about the computation of the log canonical threshold of an ideal by means of Newton polyhedra. We will apply these results in some examples illustrating Theorem 4.2. First we need to introduce some definitions.

Let $A \subseteq \mathbb{Z}^n_{\geqslant 0}$, $A \neq \emptyset$, then we define the *Newton polyhedron determined by* A, denoted by $\Gamma_+(A)$, as the convex hull in $\mathbb{R}^n_{\geqslant 0}$ of the set $\{k + v : k \in A, v \in \mathbb{R}^n_{\geqslant 0}\}$. We say that a given subset $\Gamma_+ \subseteq \mathbb{R}^n_{\geqslant 0}$ is a *Newton polyhedron* when there exists a non-empty subset $A \subseteq \mathbb{Z}^n_{\geqslant 0}$ such that $\Gamma_+ = \Gamma_+(A)$.

Let $h \in \mathcal{O}_n$, $h \neq 0$, and let us suppose that the Taylor expansion of h around the origin is given by $h = \sum_k a_k x^k$. The *support of* h, denoted by $\mathrm{supp}(h)$, is defined as $\mathrm{supp}(h) = \{k : a_k \neq 0\}$. We also set $\mathrm{supp}(0) = \emptyset$. Let Δ be a compact subset of $\mathbb{R}^n$. Then we denote by h_Δ the polynomial obtained as the sum of all terms $a_k x^k$ such that $k \in \Delta$. If $\mathrm{supp}(h) \cap \Delta = \emptyset$, then we set $h_\Delta = 0$.

If $h \neq 0$, we define the *Newton polyhedron of* h as $\Gamma_+(h) = \Gamma_+(\mathrm{supp}(h))$. When $h = 0$, then we set $\Gamma_+(h) = \emptyset$. Given an ideal I of $\mathcal{O}_n$, then we define the *support of* I as $\mathrm{supp}(I) = \cup_{h \in I} \mathrm{supp}(h)$. The *Newton polyhedron of* I is defined as $\Gamma_+(I) = \Gamma_+(\mathrm{supp}(I))$.

If $f = (f_1, \dots, f_p) : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is a complex analytic map, then we denote $\Gamma_+(\langle f_1, \dots, f_p \rangle)$ indistinctly by $\Gamma_+(f)$ or by $\Gamma_+(f_1, \dots, f_p)$.

Let $\Gamma_+ \subseteq \mathbb{R}^n_{\geqslant 0}$ be a Newton polyhedron and let $v \in \mathbb{R}^n_{\geqslant 0}$. We define $\ell(v, \Gamma_+) = \min\{\langle v, k \rangle : k \in \Gamma_+\}$, where $\langle\, , \rangle$ denotes the standard inner product in $\mathbb{R}^n$. We also define $\Delta(v, \Gamma_+) = \{k \in \Gamma_+ : \langle v, k \rangle = \ell(v, \Gamma_+)\}$. Given a subset $\Delta \subseteq \Gamma_+$, we say that Δ is a *face* of Γ_+ when there exists some $v \in \mathbb{R}^n$, $v \neq 0$, such that $\Delta = \Delta(v, \Gamma_+)$.

Given an ideal $I = \langle g_1, \dots, g_r \rangle \subseteq \mathcal{O}_n$, we recall that I is called *Newton non-degenerate* when, for any compact face Δ of $\Gamma_+(I)$, the set of solutions of the system $(g_1)_\Delta(x) = \cdots = (g_r)_\Delta(x) = 0$ is contained in $\{x \in \mathbb{C}^n : x_1 \cdots x_n = 0\}$ (see [6, 13]). Given a function $f \in \mathcal{O}_n$, we denote by $I(f)$ the ideal of $\mathcal{O}_n$ generated by $x_1 \frac{\partial f}{\partial x_1}, \dots, x_n \frac{\partial f}{\partial x_n}$. Then f is called *Newton non-degenerate* when $I(f)$ is Newton non-degenerate (see [29]). In Definition 4.6 we expose a generalization of this notion to analytic maps $(\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ due Veys and Zúñiga-Galindo [51].

Let $\Gamma_+ \subseteq \mathbb{R}^n_+$ be a Newton polyhedron. We define $\mu(\Gamma_+) = \min\{\mu \in \mathbb{R}_{\geqslant 0} : \mu(1, \dots, 1) \in \Gamma_+\}$ and $P_{\Gamma_+} = \mu(\Gamma_+)(1, \dots, 1)$. That is, P_{Γ_+} is the point belonging to the boundary of Γ_+ where the half line $\mu(1, \dots, 1)$, $\mu \in \mathbb{R}_{\geqslant 0}$, first meets Γ_+. Let J be a proper monomial ideal of $\mathcal{O}_n$. By a result of Howald [25, p. 2667] (see also [36, p. 315]), we know that

$$\operatorname{lct}(J) = \frac{1}{\mu(\Gamma_+(J))}. \tag{4.1}$$

That is, $\mu(J) = \mu(\Gamma_+(J))$. Let us define $P_J = P_{\Gamma_+(J)}$.

As we see in the following example, Theorem 4.2(a) is useful to prove the non bi-Lipschitz $\mathcal{K}$-equivalence between functions of $\mathbf{m}_n$ whose Jacobian ideal does not have finite colength.

Example 4.3. For any $a \in \mathbb{Z}_{\geqslant 3}$, let us consider the function of $\mathcal{O}_2$ given by $f_a = x^3y^2 + y^a$ and the ideal $I_a = \langle x^3y^2, y^a \rangle \subseteq \mathcal{O}_2$. We observe that f_a is a Newton non-degenerate function in the sense of Kouchnirenko [29], for all $a \in \mathbb{Z}_{\geqslant 3}$. Thus, following [36, Example 1.10], we have that $\operatorname{lct}(f_a) = \min\{1, \operatorname{lct}(I_a)\} = \frac{a+1}{3a}$, for all $a \in \mathbb{Z}_{\geqslant 3}$. If $a, b \in \mathbb{Z}_{\geqslant 3}$, then we conclude that f_a is bi-Lipschitz $\mathcal{K}$-equivalent to f_b if and only if $a = b$, by Theorem 4.2(a). We remark that if $a \geqslant 5$, then the ideal $J(f_a)$ is Newton non-degenerate, that is, $\overline{J(f_a)}$ is a monomial ideal (see [6] or [13]). Thus $\operatorname{lct}(J(f_a)) = \frac{1}{2}$, for all $a \in \mathbb{Z}_{\geqslant 5}$, by (4.1).

Remark 4.4. We recall that if $f_t : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is an analytic family such that f_t has an isolated singularity, for all $|t| \ll 1$, and the Milnor number $\mu(f_t)$ is constant along this family, then $\operatorname{lct}(f_t)$ is also constant (see [42] and [49]). As we will see in Examples 4.13 and 4.14, the constancy of $\mu(f_t)$ does not imply the constancy of $\operatorname{lct}(J(f_t))$.

We also point out that, as a consequence of Proposition 3.1 and Theorem 4.2, if I and J are ideals of $\mathcal{O}_n$ such that I and J are bi-Lipschitz equivalent, then $\operatorname{lct}(\mathbf{m}^r I^s) = \operatorname{lct}(\mathbf{m}^r J^s)$, for all $r, s \in \mathbb{Z}_{\geqslant 1}$.

Example 4.5. Let us consider the monomial ideals of $\mathcal{O}_2$ given by $I = \langle x^{11}, x^8y^5, x^6y^9, y^{30} \rangle$ and $J = \langle x^{11}, x^8y^4, x^6y^{10}, y^{30} \rangle$. Then we observe that $\operatorname{ord}(I) = \operatorname{ord}(J) = 11$, $\mathcal{L}_0(I) = \mathcal{L}_0(J) = 30$ and $\operatorname{lct}(I) = \operatorname{lct}(J) = \frac{1}{7}$. However, by applying (4.1), we find that $\operatorname{lct}(\mathbf{m}_2 I) = \frac{3}{22}$ and $\operatorname{lct}(\mathbf{m}_2 J) = \frac{4}{29}$. Therefore I and J are not bi-Lipschitz equivalent, by Proposition 3.1 and Theorem 4.2.

Let us fix coordinates $x_1, \dots, x_n$ in $\mathbb{C}^n$. If I is an ideal of $\mathcal{O}_n$, then we denote by I^0 the ideal of $\mathcal{O}_n$ generated by those monomials x^k such that $k \in \Gamma_+(I)$. Since $I \subseteq I^0$, then $\operatorname{lct}(I) \leqslant \operatorname{lct}(I^0)$. Thus it is a natural question to ask when equality holds. Corollary 4.9

shows a sufficient condition for the equality $\mathrm{lct}(I) = \mathrm{lct}(I^0)$ and thus provides a useful tool for the computation of $\mathrm{lct}(I)$. We remark that the computation of $\mathrm{lct}(I)$ in general is a difficult problem (see for instance [40] and [41]).

If $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is an analytic map germ, then we denote by $D(f)$ the Jacobian matrix of f. We also define the matrix

$$N(f) = \begin{bmatrix} x_1 \frac{\partial f_1}{\partial x_1} & \cdots & x_n \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ x_1 \frac{\partial f_p}{\partial x_1} & \cdots & x_n \frac{\partial f_p}{\partial x_n} \end{bmatrix}. \tag{4.2}$$

Let us remark that, when $p = 1$, the ideal $I(f)$ associated to the function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is generated by the entries of $N(f)$.

Definition 4.6. [51] Let $f = (f_1, \dots, f_p) : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be an analytic map germ. Then f is called *strongly non-degenerate at the origin with respect to* $\Gamma_+(f)$ (or simply, *strongly non-degenerate*) if and only if, for any compact face Δ of $\Gamma_+(f)$ we have

$$f_\Delta^{-1}(0) \cap \big\{x \in \mathbb{C}^n : \mathrm{rank}(D(f_\Delta)(x)) < \min\{n, p\}\big\} \subseteq \{x \in \mathbb{C}^n : x_1 \cdots x_n = 0\},$$

where $f_\Delta = (f_{1,\Delta}, \dots, f_{p,\Delta})$, $f_{i,\Delta} = (f_i)_\Delta$, for all $i = 1, \dots, p$, and $D(f_\Delta)$ denotes the Jacobian matrix of f_Δ.

It is immediate to see that the case $p = 1$ of the above definition is equivalent to the condition of Newton non-degeneracy of functions. If A is a matrix of size $r \times s$ with entries in $\mathcal{O}_n$ and $1 \leqslant p \leqslant \min\{r, s\}$, then we denote by $\mathbf{I}_p(A)$ the ideal of $\mathcal{O}_n$ generated by the minors of size $p \times p$ of A.

Given a non-empty subset $X \subseteq \mathbb{R}^n$ and $\alpha \in \mathbb{R}_{>0}$, then we define $\alpha X = \{\alpha x : x \in X\}$. For the sake of completeness, in the following result we relate strongly non-degenerate maps with Newton non-degenerate ideals.

Lemma 4.7. *Let $f = (f_1, \dots, f_p) : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be a complex analytic map. Let $J = \langle x^k : k \in \Gamma_+(f) \rangle$. Then the following conditions are equivalent:*

(a) *f is strongly non-degenerate*
(b) *the ideal $K = \langle f_1, \dots, f_p \rangle \overline{J^{p-1}} + \mathbf{I}_p(N(f))$ is Newton non-degenerate.*

Proof. Let us observe first that $\Gamma_+(K) = \Gamma_+(J^p) = p\Gamma_+(J)$. Therefore, if $\Delta \subseteq \Gamma_+(K)$, then Δ is a compact face of $\Gamma_+(K)$ if and only if $\frac{1}{p}\Delta$ is a compact face of $\Gamma_+(J)$.

Let us fix a compact face Δ of $\Gamma_+(K)$. If m denotes the $p \times p$ minor of $N(f)$ formed by the first p columns of $N(f)$, then we observe that

$$m_\Delta = \det \begin{bmatrix} \left(x_1 \frac{\partial f_1}{\partial x_1}\right)_\Delta & \cdots & \left(x_p \frac{\partial f_1}{\partial x_p}\right)_\Delta \\ \vdots & & \vdots \\ \left(x_1 \frac{\partial f_p}{\partial x_1}\right)_\Delta & \cdots & \left(x_p \frac{\partial f_p}{\partial x_p}\right)_\Delta \end{bmatrix} = \det \begin{bmatrix} x_1 \frac{\partial f_{1,\Delta}}{\partial x_1} & \cdots & x_p \frac{\partial f_{1,\Delta}}{\partial x_p} \\ \vdots & & \vdots \\ x_1 \frac{\partial f_{p,\Delta}}{\partial x_1} & \cdots & x_p \frac{\partial f_{p,\Delta}}{\partial x_p} \end{bmatrix}$$

$$= x_1 \cdots x_p \det \begin{bmatrix} \frac{\partial f_{1,\Delta}}{\partial x_1} & \cdots & \frac{\partial f_{1,\Delta}}{\partial x_p} \\ \vdots & & \vdots \\ \frac{\partial f_{p,\Delta}}{\partial x_1} & \cdots & \frac{\partial f_{p,\Delta}}{\partial x_p} \end{bmatrix}. \tag{4.3}$$

The same conclusion analogously extends to any other $p \times p$ minor of $N(f)$.

If x^k is any monomial belonging to $\overline{J^{p-1}}$, that is, such that $k \in (p-1)\Gamma_+(J)$, then

$$\left(f_i x^k\right)_\Delta = (f_i)_{\frac{1}{p}\Delta} \left(x^k\right)_{\frac{p-1}{p}\Delta} \tag{4.4}$$

for all $i \in \{1, \ldots, p\}$. Therefore, by virtue of relations (4.3) and (4.4), the equivalence between (a) and (b) follows immediately. □

Here we recall a known result from [51].

Theorem 4.8. [51] *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be an analytic map germ and let us consider the ideal of $\mathcal{O}_n$ given by $J = \langle x^k : k \in \Gamma_+(f) \rangle$. If f is strongly non-degenerate and* $\mathrm{lct}(J) \leqslant p$, *then* $\mathrm{lct}(\langle f_1, \ldots, f_p \rangle) = \mathrm{lct}(J)$.

Proof. Let $I = \langle f_1, \ldots, f_p \rangle$. From [51, Theorem 2.7] we know that $-\mathrm{lct}(I)$ is equal to the real part of some pole of the Igusa zeta function associated to some representative of f. Therefore, by [51, Corollary 3.12] we obtain that $\mathrm{lct}(I) \geqslant \mathrm{lct}(J)$. But the inclusion $I \subseteq J$ implies that $\mathrm{lct}(I) \leqslant \mathrm{lct}(J)$. Hence the result follows. □

We refer to [25], [36, Example 1.10] or [41, Proposition 1.3] for the case $p = 1$ of the result above.

Corollary 4.9. *Let I be a proper ideal of $\mathcal{O}_n$ and let $p \in \mathbb{Z}_{\geqslant 1}$ such that* $\mathrm{lct}(I^0) \leqslant p$. *Let us suppose that there exists a map $f = (f_1, \ldots, f_p) : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ such that $f_1 \ldots, f_p \in I$, f is strongly non-degenerate and $P_{I^0} \in \Gamma_+(f)$. Then* $\mathrm{lct}(I) = \mathrm{lct}(I^0)$.

In particular, if $\mathrm{lct}(I^0) \leqslant 1$ *and there exists some $g \in I$ such that g is Newton non-degenerate and $\Gamma_+(g) = \Gamma_+(I)$, then* $\mathrm{lct}(I) = \mathrm{lct}(I^0)$.

Proof. Since $f_1 \ldots, f_p \in I$, then $\mathrm{lct}(\langle f_1, \ldots, f_p \rangle) \leqslant \mathrm{lct}(I) \leqslant \mathrm{lct}(I^0)$. Then, it suffices to show that $\mathrm{lct}(\langle f_1, \ldots, f_p \rangle) = \mathrm{lct}(I^0)$. Let $J = \langle x^k : k \in \Gamma_+(f) \rangle$. Since $f_1 \ldots, f_p \in I$, then $J \subseteq I^0$. We are assuming that $P_{I^0} \in \Gamma_+(J)$, hence $\mu(J) = \mu(I^0) \geqslant \frac{1}{p}$. That is, $\mathrm{lct}(J) \leqslant p$, by (4.1). Therefore we can apply Theorem 4.8 to deduce that $\mathrm{lct}(\langle f_1, \ldots, f_p \rangle) = \mathrm{lct}(J)$. Then the result follows. □

Let us observe that in the previous result, the assumption on $\mathrm{lct}(I^0)$ cannot be removed, as the following example shows.

Example 4.10. Let us consider the ideal of $\mathcal{O}_3$ given by $I = \langle y^2 - xz, x^3 - z^2 \rangle$. Then $I^0 = \overline{\langle x^3, y^2, z^2, xz \rangle}$. Therefore $\mathrm{lct}(I^0) = \frac{3}{2} > 1$, by (4.1). The function $g = y^2 - xz + x^3 - z^2$ verifies that $g \in I$, g is Newton non-degenerate and $\Gamma_+(g) = \Gamma_+(I)$. However, according to [40, Example 5.5], we have $\mathrm{lct}(I) = \frac{17}{12}$, which is different from $\mathrm{lct}(I^0) = \frac{3}{2}$.

Let $w \in \mathbb{Z}^n_{\geqslant 1}$ and let $h \in \mathcal{O}_n$. We define the *degree of* h *with respect to* w as $d_w(f) = \min\{\langle w, k\rangle : k \in \operatorname{supp}(h)\}$, where $\langle\,,\rangle$ stands for the standard scalar product in $\mathbb{R}^n$. We denote by $p_w(h)$ the polynomial obtained as the sum of all terms $a_k x^k$ such that $\langle w, k\rangle = d_w(h)$. We refer to $p_w(h)$ as the *principal part of* h *with respect to* w.

The function h is called *weighted homogeneous with respect to* w when $p_w(h) = h$. We say that h is *semi-weighted homogeneous with respect to* w when $p_w(h)$ has an isolated singularity at the origin.

In the following examples, most part of the computations about several combinatorial aspects of Newton polyhedra have been carried on with the help of the program *Gérmenes* developed by A. Montesinos-Amilibia [35].

Example 4.11. [22] Let $f_t : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be the analytic family of functions given by

$$f_t(x, y) = x^3 - 3t^2xy^4 + y^6$$

for all $(x, y) \in \mathbb{C}^2$, $t \in \mathbb{C}$. In this case, the support of x^4y belongs to the boundary of $\Gamma_+(f_0)$. In [22, Theorem 3.1] it is proven that if $t, t', 1 \pm 2t^3, 1 \pm 2t'^3 \in \mathbb{C} \smallsetminus \{0\}$ and if there exist a germ of bi-Lipschitz homeomorphism $\phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ such that $f_t = f_{t'} \circ \phi$, then $t^3 = \pm t'^3$. That is, this example proves the existence of moduli for bi-Lipschitz $\mathcal{R}$-equivalence of functions. We remark that the bi-Lipschitz equivalence of complex analytic set germs does not have moduli by [37].

Since $\Gamma_+(J(f_0)) = \Gamma_+(x^2, y^5)$ and $J(f_0)$ is Newton non-degenerate, we have $\operatorname{lct}(J(f_0)) = \frac{7}{10}$, by (4.1). If $t \neq 0$, then the function given by $g = \frac{\partial f_t}{\partial x}$ is Newton non-degenerate and $\Gamma_+(g) = \Gamma_+(J(f_t)) = \Gamma_+(x^2, y^4)$. Let us observe that $\operatorname{lct}(\langle x^2, y^4\rangle) = \frac{3}{4} < 1$. Thus $\operatorname{lct}(J(f_t)) = \frac{3}{4}$, by Corollary 4.9.

Example 4.12. Let us consider the function $f_c : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ given by

$$f_c(x, y, z) = c_1x^6 + c_2x^4y + c_3x^4z + c_4x^2y^2 + c_5x^2yz + c_6x^2z^2 + c_7y^3 + c_8y^2z + c_9yz^2 + c_{10}z^3$$

for all $(x, y, z) \in \mathbb{C}^3$ and all $c = (c_1, \dots, c_{10}) \in \mathbb{C}^{10}$. Then we observe that f_c is weighted homogeneous with respect to $w = (1, 1, 2)$ and $d_w(f) = 6$, for all $c \in \mathbb{C}^{10}$. It is straightforward to check that if $c_2 = c_3 = c_4 = c_5 = c_6 = 0$ and the other coefficients are chosen generically, then $J(f_c)$ is Newton non-degenerate and $\overline{J(f_c)} = \overline{\langle x^5, y^2, z^2\rangle}$. Therefore $\operatorname{lct}(J(f_c)) = \frac{6}{5}$ in this case, by (4.1).

Now, let us suppose that all the coefficients of f are chosen generically. Let $I = J(f_c)$. Then $I^0 = \overline{\langle x^4, y^2, z^2\rangle}$ and therefore $\operatorname{lct}(I^0) = \frac{5}{4}$. Let Δ denote the unique compact face of dimension 2 of $\Gamma_+(I)$. We observe that I is not Newton non-degenerate, since $(\frac{\partial f_c}{\partial x})_\Delta = 0$. Let us consider the map $g : (\mathbb{C}^3, 0) \to (\mathbb{C}^2, 0)$ given by $g = (\frac{\partial f_c}{\partial y}, \frac{\partial f_c}{\partial z})$. By a straightforward computation, using the fact that the coefficients of f are chosen generically, we obtain that g is strongly non-degenerate. Since $\operatorname{lct}(I^0) \leqslant 2$, we conclude that $\operatorname{lct}(J(f_c)) = \operatorname{lct}(I^0) = \frac{5}{4}$, by Corollary 4.9. Then, if we consider the function $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ given by $f(x, y, z) = x^6 + y^3 + z^3$, we have that f is not bi-Lipschitz $\mathcal{A}$-equivalent nor bi-Lipschitz $\mathcal{K}^*$-equivalent to f_c, for a generic choice of the vector of coefficients c, by Theorems 3.4 and 4.2(b).

Example 4.13. Let $f_t : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ be the analytic deformation given by the Briançon-Speder example [14]. That is

$$f_t(x, y, z) = x^5 + z^{15} + y^7 z + txy^6$$

for all $(x, y, z) \in \mathbb{C}^3$, $t \in \mathbb{C}$. We recall that, if $w = (3, 2, 1)$, then f_t is weighted homogeneous with respect to w, $d_w(f_t) = 15$ and f_t has an isolated singularity at the origin, for all $|t| \ll 1$. Let $J_1 = \langle x^k : k \in \Gamma_+(J(f_0))\rangle$ and let $J_2 = \langle x^k : k \in \Gamma_+(J(f_t))\rangle$, for $t \neq 0$. Then $J_1 \subseteq J_2$ and it is easy to check that

$$J_1 = \overline{\langle x^4, y^7, z^{14}, y^6 z\rangle} \qquad \text{and} \qquad J_2 = \overline{\langle x^4, y^6, z^{14}\rangle}.$$

The family f_t is not μ^*-constant and $\mathcal{L}_0^*(\nabla f_t)$ is not constant, since

$$\mu^*(f_t) = \begin{cases} (364, 28, 4) & \text{if } t = 0 \\ (364, 26, 4) & \text{if } t \neq 0. \end{cases} \qquad \mathcal{L}_0^*(\nabla f_t) = \begin{cases} (14, 7, 4) & \text{if } t = 0 \\ (14, 6.5, 4) & \text{if } t \neq 0. \end{cases}$$

Hence we observe that f_t is a Hickel singularity if and only if $t \neq 0$.

The ideal $J(f_0)$ is Newton non-degenerate (see [13]). Therefore, applying (4.1) we obtain that $\mathrm{lct}(J(f_0)) = \frac{10}{21}$, by [25]. Let us remark that $\Gamma_+(J(f_0))$ has only two compact faces of dimension 2.

By the lower semi-continuity of the log canonical threshold (see [36] or [32, Corollary 9.5.39]) we have that $\mathrm{lct}(J(f_0)) \leqslant \mathrm{lct}(J(f_t))$, for all $|t| \ll 1$. The inclusion $J(f_t) \subseteq J_2$ implies that $\mathrm{lct}(J(f_t)) \leqslant \mathrm{lct}(J_2) = \frac{41}{84}$.

Let $t \in \mathbb{C} \smallsetminus \{0\}$ such that $|t| < 1$. Let us define the function

$$g = \frac{\partial f_t}{\partial x} + \frac{\partial f_t}{\partial y} + \frac{\partial f_t}{\partial z}.$$

It is straightforward to see that $g \in J(f_t)$ and $\Gamma_+(g) = \Gamma_+(J_2)$. Moreover, g is Newton non-degenerate. Therefore, by Corollary 4.9, we obtain that

$$\mathrm{lct}(J(f_t)) = \mathrm{lct}(J_2) = \frac{41}{84}.$$

Then f_0 is not bi-Lipschitz $\mathcal{A}$-equivalent nor bi-Lipschitz $\mathcal{K}^*$-equivalent to f_t, if $t \neq 0$, $|t| \ll 1$, by Theorems 3.4 and 4.2(b).

Example 4.14. Let us consider the deformation $f_t : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ given by

$$f_t(x, y, z) = x^6 + y^4 + z^4 + tx^3yz + ty^3z.$$

for all $(x, y, z) \in \mathbb{C}^3$, $t \in \mathbb{C}$. This deformation is μ^*-constant: $\mu^*(f_t) = (45, 9, 3)$, for all $|t| \ll 1$. We remark that f_t is Newton non-degenerate, for all t, the Newton polyhedron $\Gamma_+(f_t)$ is also constant and f_t is weighted homogeneous with respect to $w = (2, 3, 3)$, $d_w(f_t) = 12$, for all t. However, as we will see, $\mathrm{lct}(J(f_t))$ is not constant for $|t| \ll 1$.

We observe that when $t = 0$, then $J(f_0)$ is Newton non-degenerate. Then $\mathrm{lct}(J(f_0)) = \frac{13}{15}$, by (4.1). Let $t \in \mathbb{C} \smallsetminus \{0\}$ such that $|t| < 1$. Let $J = \langle x^k : k \in \Gamma_+(J(f_t))\rangle$. Thus

$$J = \overline{\langle x^5, y^3, z^3, x^3 z, x^3 y\rangle}.$$

Therefore $\operatorname{lct}(J) = \frac{8}{9} \leqslant 1$, by (4.1). Let us define the function

$$g = \frac{\partial f_t}{\partial x} + \frac{\partial f_t}{\partial y} + \frac{\partial f_t}{\partial z}.$$

This function verifies that $g \in J(f_t) \subseteq J$, $\Gamma_+(g) = \Gamma_+(J)$ and g is Newton non-degenerate. By applying Corollary 4.9, we obtain that

$$\operatorname{lct}(J(f_t)) = \frac{8}{9}.$$

Then, by Theorems 3.4 and 4.2(b), it follows that f_0 is not bi-Lipschitz $\mathcal{A}$-equivalent nor bi-Lipschitz $\mathcal{K}^*$-equivalent to f_t, if $t \neq 0$, $|t| \ll 1$.

Example 4.15. Let us consider the deformation $f_t : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ given by

$$f_t(x, y, z) = x^6 + y^5 + z^{12} + xy^3z + tx^3y^2.$$

This deformation is not μ^*-constant, as is shown in [3]. Let $J_1 = \langle x^k : k \in \Gamma_+(J(f_0))\rangle$ and let $J_2 = \langle x^k : k \in \Gamma_+(J(f_t))\rangle$, for $t \neq 0$.

The ideal $J(f_0)$ is Newton non-degenerate, therefore $\operatorname{lct}(J(f_0)) = \operatorname{lct}(J_1) = \frac{71}{110}$. If $t \neq 0$, let us consider the function given by

$$g(x, y, z) = \frac{\partial f_t}{\partial x} + \frac{\partial f_t}{\partial y} + \frac{\partial f_t}{\partial z}.$$

This function verifies that $\Gamma_+(g) = \Gamma_+(J_2)$ and obviously $g \in J(f_t)$. Moreover, g is Newton non-degenerate and $\operatorname{lct}(J_2) = \frac{36}{55} \leqslant 1$. Thus, by Corollary 4.9, we obtain that $\operatorname{lct}(J(f_t)) = \operatorname{lct}(J_2) = \frac{36}{55}$, which is strictly bigger than $\operatorname{lct}(J(f_0))$. By Theorems 3.4 and 4.2(b), we obtain that f_0 is not bi-Lipschitz $\mathcal{A}$-equivalent nor bi-Lipschitz $\mathcal{K}^*$-equivalent to f_t, if $t \neq 0$, $|t| \ll 1$.

Example 4.16. Let $\alpha \in \mathbb{Z}_{\geqslant 3}$ such that α is odd and let $\beta \in \mathbb{Z}_{\geqslant 1}$ such that $3\alpha = 2\beta + 1$. Let us consider the deformation $f_t : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ given by

$$f_t(x, y, z) = x^3 + y^\beta z + z^{3\alpha} + y^{\beta+1} + txy^\alpha$$

for all $(x, y, z) \in \mathbb{C}^3$, $t \in \mathbb{C}$. Then we observe that f_t is semi-weighted homogeneous with respect to $(\alpha, 2, 1)$, $d_w(f) = 3\alpha$, $p_w(f_t) = x^3 + y^\beta z + z^{3\alpha} + txy^\alpha$ and $d_w(y^{\beta+1}) = d_w(f) + 1$. The deformation f_t is a slight modification of the example given in [14, p. 366]. That is, we have added the term $y^{\beta+1}$ to that example in order to have that f_t is a convenient function, for all t. The reason for this is to apply [8, Theorem 2.3] for obtaining the sequence $\mu^*(f_t)$ in terms of $\Gamma_+(f_t)$, since f_t contains parameters in the exponents.

Let us write $\alpha = 2k + 1$, for some $k \in \mathbb{Z}_{\geqslant 1}$. Then we can rewrite f_t as

$$f_t(x, y, z) = x^3 + y^{3k+1}z + z^{6k+3} + y^{3k+2} + txy^{2k+1}.$$

We first observe that $J(f_0)$ is Newton non-degenerate and $J(f_t)$ is not, if $t \neq 0$.

Let us define the ideals $K_1 = \langle x^k : k \in \Gamma_+(f_0)\rangle$ and $K_2 = \langle x^k : k \in \Gamma_+(f_t)\rangle$, for $t \neq 0$. An elementary combinatorial analysis shows that

$$K_1 = \overline{\left\langle x^3, y^{3k+1}z, z^{6k+3}, y^{3k+2}\right\rangle} \qquad K_2 = \overline{K_1 + \left\langle xy^{2k+1}\right\rangle}.$$

We remark that $xy^{2k+1} \notin K_1$, hence K_1 is strictly contained in K_2. We recall that if J is a monomial ideal of $\mathcal{O}_n$ of finite colength, then the multiplicity of J is expressed as $e(J) = n!\,\mathrm{V}_n(\mathbb{R}^n_{\geqslant 0} \smallsetminus \Gamma_+(J))$, where V_n denotes n-dimensional volume (see for instance [47]). Then we obtain the following multiplicities:

$$
\begin{aligned}
e(K_1) &= 54k^2 + 54k + 15 & e(K_2) &= 54k^2 + 54k + 14 \\
e(\mathbf{m}K_1) &= 54k^2 + 81k + 43 & e(\mathbf{m}K_2) &= 54k^2 + 81k + 39 \\
e(\mathbf{m}K_1^2) &= 432k^2 + 540k + 211 & e(\mathbf{m}K_2^2) &= 432k^2 + 540k + 191.
\end{aligned}
$$

Therefore, by the expression for mixed multiplicities of ideals given in [38, p. 409], and substituting the above relations, we obtain

$$
\begin{aligned}
e(K_1, K_1, \mathbf{m}) &= \frac{1}{3!}\left(2e(K_1) + e(\mathbf{m}) - e(K_1^2) - 2e(\mathbf{m}K_1) + e(\mathbf{m}K_1^2)\right) = 9k + 6 \\
e(K_2, K_2, \mathbf{m}) &= \frac{1}{3!}\left(2e(K_2) + e(\mathbf{m}) - e(K_2^2) - 2e(\mathbf{m}K_2) + e(\mathbf{m}K_2^2)\right) = 9k + 5.
\end{aligned}
$$

If J is a monomial ideal of $\mathcal{O}_n$ of finite colength, then we denote by $\nu^{(j)}(J)$ the value of $\mu^{(j)}(g)$, where g is any Newton non-degenerate function such that $\Gamma_+(g) = \Gamma_+(J)$, for all $j = 1, \ldots, n$ (see [8, Theorem 2.3]). Hence, the numbers $\nu^{(2)}(K_1)$ and $\nu^{(2)}(K_2)$ are given by

$$
\begin{aligned}
\nu^{(2)}(K_1) &= -\operatorname{ord}(K_1^{\{1,2\}}) - \operatorname{ord}(K_1^{\{1,3\}}) - \operatorname{ord}(K_1^{\{2,3\}}) \\
&\quad + \operatorname{ord}(K_1) + e(K_1, K_1, \mathbf{m}) + 1 = 6k + 2 \\
\nu^{(2)}(K_2) &= -\operatorname{ord}(K_2^{\{1,2\}}) - \operatorname{ord}(K_2^{\{1,3\}}) - \operatorname{ord}(K_2^{\{2,3\}}) \\
&\quad + \operatorname{ord}(K_2) + e(K_2, K_2, \mathbf{m}) + 1 = 6k + 1.
\end{aligned}
$$

Moreover

$$
\nu^{(3)}(K_1) = 36k^2 + 18k + 2 = \nu^{(3)}(K_2) \qquad \text{and} \qquad \nu^{(1)}(K_1) = 2 = \nu^{(1)}(K_2).
$$

Thus, if we fix an index $i \in \{1, 2, 3\}$ and $g_i : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ is any function with an isolated singularity at the origin such that $\Gamma_+(g_i) = \Gamma_+(K_i)$, then $\mu^{(j)}(g_i) \geqslant \nu^{(j)}(K_i)$, for all $j \in \{1, 2, 3\}$, and equality holds if g_i is Newton non-degenerate, by Theorem [8, Theorem 2.3].

Therefore, since f_t is Newton non-degenerate, for all t, we conclude that

$$
\mu^*(f_t) = \begin{cases} \left(36k^2 + 18k + 2,\ 6k + 2,\ 2\right) & \text{if } t = 0 \\ \left(36k^2 + 18k + 2,\ 6k + 1,\ 2\right) & \text{if } t \neq 0. \end{cases}
$$

Thus $\mu^{(2)}(f_t)$ is not constant. Moreover, following a procedure analogous to [12, Example 4.5], we obtain

$$
\mathcal{L}_0^*(\nabla f_t) = \begin{cases} \left(6k + 2,\ 3k + 1,\ 2\right) & \text{if } t = 0 \\ \left(6k + 2,\ 3k + \frac{1}{2},\ 2\right) & \text{if } t \neq 0. \end{cases}
$$

If $t \neq 0$, we observe that

$$e(J(f_t)) = 36k^2 + 18k + 2 = (6k+2)\left(3k + \frac{1}{2}\right)2$$

then f_t is a Hickel singularity, if $t \neq 0$, whereas f_0 is not Hickel. We also observe that

$$\mathrm{lct}(f_0) = \mathrm{lct}(K_1) = \frac{2k+4}{6k+3} = \mathrm{lct}(K_2) = \mathrm{lct}(f_t)$$

if $t \neq 0$. Moreover, since $J(f_0)$ is Newton non-degenerate, we deduce that

$$\mathrm{lct}(J(f_0)) = \mathrm{lct}\left(\langle x^2, y^{3k+1}, z^{6k+2}, y^{3k}z\rangle\right) = \frac{9k^2+12k+1}{18k^2+6k}.$$

If we fix $t \neq 0$, then the function

$$g = \frac{\partial f_t}{\partial x} + \frac{\partial f_t}{\partial y} + \frac{\partial f_t}{\partial z}$$

is Newton non-degenerate and $\Gamma_+(g) = \Gamma_+(J(f_t)) = \Gamma_+(x^2, y^{2k+1}, z^{6k+2})$. Thus, by Corollary 4.9 we obtain that

$$\mathrm{lct}(J(f_t)) = \mathrm{lct}\left(\langle x^2, y^{2k+1}, z^{6k+2}\rangle\right) = \frac{1}{2} + \frac{1}{2k+1} + \frac{1}{6k+2} = \frac{6k^2+13k+4}{12k^2+10k+2}.$$

Then $\mathrm{lct}(J(f_0)) = \mathrm{lct}(J(f_t)$ if and only if $k = \frac{1\pm\sqrt{7}}{6}$. That is $\mathrm{lct}(J(f_0)) \neq \mathrm{lct}(J(f_t))$, if $|t| \ll 1$, $t \neq 0$. This shows that the deformation f_t is not bi-Lipschitz $\mathcal{A}$-trivial nor bi-Lipschitz $\mathcal{K}^*$-trivial, by Theorems 3.4 and 4.2(b).

In view of Examples 4.13 and 4.16, we conjecture that $\mathcal{L}_0^*(I)$ is invariant in the bi-Lipschitz class of I. We give a result is this direction in Proposition 5.8. Moreover, we also expect that, if $f \in \mathcal{O}_n$ has an isolated singularity at the origin, then $\mu^*(f)$ is a bi-Lipschitz invariant of f.

5. Diagonal ideals, Hickel singularities and bi-Lipschitz equivalence

We say that an ideal I of $\mathcal{O}_n$ is *diagonal*, when there exist positive integers $a_1, \dots, a_n$ such that $\overline{I} = \overline{\langle x_1^{a_1}, \dots, x_n^{a_n}\rangle}$. We shall refer to $\{a_1, \dots, a_n\}$ as the set of exponents of I. If $a_n \geqslant \cdots \geqslant a_1$, then we recall that $\mathcal{L}_0^*(I) = (a_n, \dots, a_1)$, by [12, Corollary 4.2]. It is clear that any diagonal ideal is Hickel. The converse does not hold, as can be easily checked for the ideal $I = \langle x^3, xy, y^3\rangle \subseteq \mathcal{O}_2$.

If I is an ideal of $\mathcal{O}_n$ of finite colength, then we define the *Demailly-Pham number of* I, which we denote by $\mathrm{DP}(I)$, as

$$\mathrm{DP}(I) = \frac{1}{e_1(I)} + \frac{e_1(I)}{e_2(I)} \cdots + \frac{e_{n-1}(I)}{e_n(I)}.$$

Let I be an ideal of $\mathcal{O}_n$ of finite colength. By [18], we have $\mathrm{DP}(I) \leqslant \mathrm{lct}(I)$ (see also [10]). Then, applying inequality (2.2), we obtain that

$$\frac{1}{\mathcal{L}_0^{(1)}(I)} + \frac{1}{\mathcal{L}_0^{(2)}(I)} + \cdots + \frac{1}{\mathcal{L}_0^{(n)}(I)} \leqslant \frac{1}{e_1(I)} + \frac{e_1(I)}{e_2(I)} + \cdots + \frac{e_{n-1}(I)}{e_n(I)} = \mathrm{DP}(I) \leqslant \mathrm{lct}(I).$$

Moreover, by [10, Theorem 13], if $\mathrm{lct}(I) = \mathrm{lct}(I^0)$, then $\mathrm{DP}(I) = \mathrm{lct}(I)$ if and only if I is a diagonal ideal.

Proposition 5.1. *Let I and J be ideals of $\mathcal{O}_3$ of finite colength. If I and J are bi-Lipschitz equivalent and I is diagonal, then $\mathcal{L}_0^{(2)}(J) \geqslant \mathcal{L}_0^{(2)}(I)$.*

Proof. Since I is diagonal, we have the following equalities:

$$\frac{1}{\mathcal{L}_0^{(1)}(I)} + \frac{1}{\mathcal{L}_0^{(2)}(I)} + \frac{1}{\mathcal{L}_0^{(3)}(I)} = \frac{1}{e_1(I)} + \frac{e_1(I)}{e_2(I)} + \frac{e_2(I)}{e_3(I)} = \mathrm{DP}(I) = \mathrm{lct}(I). \tag{5.1}$$

By Theorem 4.2, we have that $\mathrm{lct}(I) = \mathrm{lct}(J)$. Then

$$\mathrm{lct}(I) = \mathrm{lct}(J) \geqslant \mathrm{DP}(J) \geqslant \frac{1}{\mathcal{L}_0^{(1)}(J)} + \frac{1}{\mathcal{L}_0^{(2)}(J)} + \frac{1}{\mathcal{L}_0^{(3)}(J)}. \tag{5.2}$$

We have $\mathrm{ord}(I) = \mathrm{ord}(J)$ and $\mathcal{L}_0^{(3)}(I) = \mathcal{L}_0^{(3)}(J)$, by Theorem 3.2. Then, by (5.1) and (5.2) it follows that $\mathcal{L}_0^{(2)}(J) \geqslant \mathcal{L}_0^{(2)}(I)$. □

Proposition 5.2. *Let us consider an analytic map $F : (\mathbb{C} \times \mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$. Let $f_t : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be the map given by $f_t(x) = F(t, x)$ and let I_t denote the ideal of $\mathcal{O}_n$ generated by the component functions of f_t, for all $|t| \ll 1$. Let us assume that I_t is an ideal of finite colength, for all $|t| \ll 1$, and I_0 is diagonal. If I_t is bi-Lipschitz trivial, then $e_i(I_t)$ is constant, for all $i = 1, \dots, n$ and all $|t| \ll 1$.*

Proof. The number $\mathrm{DP}(I_t)$ is lower semicontinuous (see [10, Corollary 12]), then $\mathrm{DP}(I_0) \leqslant \mathrm{DP}(I_t)$, for all $|t| \ll 1$. Moreover $\mathrm{lct}(I_t) = \mathrm{lct}(I_0)$, for all $|t| \ll 1$, by Theorem 4.2. Then if we fix some $t \in \mathbb{C}$ such that $|t| \ll 1$, we have the following inequalities:

$$\mathrm{DP}(I_0) \leqslant \mathrm{DP}(I_t) \leqslant \mathrm{lct}(I_t) = \mathrm{lct}(I_0) = \mathrm{DP}(I_0)$$

Hence $\mathrm{DP}(I_0) = \mathrm{DP}(I_t)$. This implies that $e_i(I_0) = e_i(I_t)$, for all $i = 1, \dots, n$ and all $|t| \ll 1$, by [10, Corollary 12]. □

Corollary 5.3. *Let $f_t : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be an analytic deformation such that f_t has an isolated singularity for all $|t| \ll 1$. Let us suppose that this deformation is bi-Lipschitz $\mathcal{A}$-trivial or bi-Lipschitz $\mathcal{K}^*$-trivial. If $J(f_0)$ is diagonal, then $\mu^*(f_t)$ is constant, for $|t| \ll 1$.*

Proof. This is a direct application of Proposition 5.2 to the family of gradient maps $\nabla f_t : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$. □

Corollary 5.4. *Let I and J be diagonal ideals of $\mathcal{O}_n$ such that I and J are bi-Lipschitz invariant. If $n \leqslant 3$ or if $n = 4$ and $e(I) = e(J)$, then the respective sets of exponents of I and J are equal.*

Proof. Let us write $I = \overline{\langle x_1^{a_1}, \dots, x_n^{a_n}\rangle}$ and $J = \overline{\langle x_1^{b_1}, \dots, x_n^{b_n}\rangle}$, for some positive integers a_i and b_i such that $a_1 \leqslant \cdots \leqslant a_n$ and $b_1 \leqslant \cdots \leqslant b_n$. Since $\mathcal{L}_0^*(I) = (a_n, \dots, a_1)$ and $\mathcal{L}_0^*(J) = (b_n, \dots, b_1)$, then the case where $n \leqslant 3$ follows by a direct application of Theorem 3.2 and Proposition 5.1.

Let us suppose that $n = 4$ and $e(I) = e(J)$. By Theorem 3.2 we have $a_1 = b_1$ and $a_4 = b_4$. The condition $e(I) = e(J)$ means that $a_1a_2a_3a_4 = b_1b_2b_3b_4$. Moreover, by Theorem 4.2 we have $\mathrm{lct}(I) = \mathrm{lct}(J)$. In particular, we deduce that a_2, a_3, b_2, b_3 are solutions of the system of equations formed by $\frac{1}{a_2} + \frac{1}{a_3} = \frac{1}{b_2} + \frac{1}{b_3}$ and $a_2a_3 = b_2b_3$. Since $a_2 \leqslant a_3$ and $b_2 \leqslant b_3$, then it follows that $a_2 = b_2$ and $a_3 = b_3$. Thus the result follows. □

It is worth remarking that, by the main result of [43] (see also [27]), if f and g are topologically equivalent Brieskorn-Pham singularities of $\mathcal{O}_n$, then the respective set of exponents of these functions are equal.

Lemma 5.5. *Let I be an ideal of $\mathcal{O}_n$ of finite colength. Then*

$$e_i(I) \leqslant \mathcal{L}_0^{(1)}(I) \cdots \mathcal{L}_0^{(i)}(I) \tag{5.3}$$

for all $i = 1, \dots, n$, and the following conditions are equivalent:

(a) *I is a Hickel ideal.*
(b) *$e_i(I) = \mathcal{L}_0^{(1)}(I) \cdots \mathcal{L}_0^{(i)}(I)$, for all $i = 1, \dots, n$.*
(c) *$\mathcal{L}_0^{(i)}(I) = \frac{e_i(I)}{e_{i-1}(I)}$, for all $i = 1, \dots, n$.*

Proof. Relation (5.3) follows as a direct consequence of (2.2). Let us prove (a) ⇒ (b). Let us assume that I is a Hickel ideal. By definition, we have $e(I) = \mathcal{L}_0^{(1)}(I) \cdots \mathcal{L}_0^{(n)}(I)$. In general, by (2.2) we know that $\frac{e(I)}{e_{n-1}(I)} \leqslant \mathcal{L}_0^{(n)}(I)$. Hence $\mathcal{L}_0^{(1)}(I) \cdots \mathcal{L}_0^{(n-1)}(I) \leqslant e_{n-1}(I)$. By (5.3), the opposite inequality also holds, then we obtain the equality $e_{n-1}(I) = \mathcal{L}_0^{(1)}(I) \cdots \mathcal{L}_0^{(n-1)}(I)$. By applying finite induction, then (b) follows.

The implication (b) ⇒ (c) is immediate. The implication (c) ⇒ (a) follows by observing that

$$e(I) = \frac{e_n(I)}{e_{n-1}(I)} \cdots \frac{e_2(I)}{e_1(I)} \frac{e_1(I)}{e_0(I)} = \mathcal{L}_0^{(n)}(I) \cdots \mathcal{L}_0^{(2)}(I)\mathcal{L}_0^{(1)}(I).$$

□

In the following result we show a relation between Hickel ideals and weighted homogeneous filtrations.

Proposition 5.6. *Let $w = (w_1, \dots, w_n) \in \mathbb{Z}^n_{\geqslant 1}$ such that $w_1 \geqslant \cdots \geqslant w_n$. Let $g = (g_1, \dots, g_n) : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ be a finite map and let I be the ideal of $\mathcal{O}_n$ generated by $g_1, \dots, g_n$. Let $d_i = d_w(g_i)$, for $i = 1, \dots, n$. Let us suppose that $d_1 \leqslant \cdots \leqslant d_n$. Then the following conditions are equivalent:*

(a) *$\mathcal{L}_0^{(i)}(I) = \frac{d_i}{w_i}$, for all $i = 1, \dots, n$.*
(b) *I is a Hickel ideal and $e_i(I) = \frac{d_1 \cdots d_i}{w_1 \cdots w_i}$, for all $i = 1, \dots, n$.*

Proof. We have that

$$\frac{d_1\cdots d_n}{w_1\cdots w_n}\leqslant e(g_1,\dots,g_n)=e(I)\leqslant \mathcal{L}_0^{(1)}(I)\cdots\mathcal{L}_0^{(n)}(I) \tag{5.4}$$

where the first inequality is well-known (see for instance [2, §12.3] or [17, §10.3]) and the second inequality comes from (2.3).

Let us see (a) ⇒ (b). If we suppose that $\mathcal{L}_0^{(i)}(I)=\frac{d_i}{w_i}$, for all $i=1,\dots,n$, then the inequalities of (5.4) become equalities. Hence $e(I)=\frac{d_1\cdots d_n}{w_1\cdots w_n}$, which means that g is semi-weighted homogeneous with respect to w by [13, Theorem 3.3] (see also [17, §10.3]) and $e(I)=\mathcal{L}_0^{(1)}(I)\cdots\mathcal{L}_0^{(n)}(I)$. Then I is a Hickel ideal.

By (2.2) we have that

$$\frac{e_i(I)}{e_{i-1}(I)}\leqslant \mathcal{L}_0^{(i)}(I)$$

for all $i=1,\dots,n$. In particular $e_{n-1}(I)\geqslant\frac{w_n}{d_n}e(I)=\frac{d_1\cdots d_{n-1}}{w_1\cdots w_{n-1}}$. Moreover, $e_{n-1}(I)\leqslant \mathcal{L}_0^{(1)}(I)\cdots\mathcal{L}_0^{(n-1)}(I)$, by (5.3). Thus $e_{n-1}(I)=\frac{d_1\cdots d_{n-1}}{w_1\cdots w_n}$. By the same argument, inductively we arrive to the relation $e_i(I)=\frac{d_1\cdots d_i}{w_1\cdots w_i}$, for all $i=1,\dots,n$.

The implication (b) ⇒ (a) is a direct application of Lemma 5.5. □

Let us observe that, in Proposition 5.6, we do not assume that g_i is weighted homogeneous with respect to w, for all $i=1,\dots,n$.

Remark 5.7. Let us fix a vector of weights $(w_1,\dots,w_n)\in\mathbb{Z}^n_{\geqslant 1}$ such that $w_1\geqslant\cdots\geqslant w_n$ and let $f:(\mathbb{C}^n,0)\to(\mathbb{C},0)$ be a semi-weighted homogeneous function germ. Let $d=d_w(f)$. In the article [16], Brzostowski showed that $\mathcal{L}_0^{(n)}(J(f))=\frac{d-w_n}{w_n}$, provided that $d\geqslant 2w_i$, for all $i=1,\dots,n$ (see [31] for the case $n=3$ of this result and [1]). If we apply Proposition 5.6 to ∇f, then we obtain a characterization of when $\mathcal{L}_0^{(i)}(J(f))=\frac{d-w_i}{w_i}$, for all $i=1,\dots,n$. If f is a function such that $J(f)$ satisfies conditions (a) or (b) of Proposition 5.6, then we will say that f is *w-optimal*. We remark that, if $f_t:(\mathbb{C}^3,0)\to(\mathbb{C},0)$ denotes the deformation of Example 4.13 or of Example 4.16, then f_t is *w-optimal* if and only if $t\neq 0$.

In the next result we will focus on bi-Lipschitz deformations of functions $f\in\mathcal{O}_n$ such that $J(f)$ is a diagonal ideal. This class of functions, which is included in the class of Hickel singularities, contains the class of homogeneous functions with an isolated singularity at the origin and Pham-Brieskorn singularities.

Proposition 5.8. *Let us fix an analytic family of functions $f_t:(\mathbb{C}^3,0)\to(\mathbb{C},0)$ such that f_t is Hickel, for all $|t|\ll 1$. If, in addition, $J(f_0)$ is diagonal and the family f_t is bi-Lipschitz trivial, then $\mathcal{L}_0^*(J(f_0))=\mathcal{L}_0^*(J(f_t))$, for all $|t|\ll 1$.*

Proof. By Theorem 3.4, the ideals $J(f_0)$ and $J(f_t)$ are bi-Lipschitz equivalent, then $\mathrm{ord}(J(f))=\mathrm{ord}(J(g))$ and $\mathcal{L}_0^{(3)}(J(f_0))=\mathcal{L}_0^{(3)}(J(f_t))$, for all $|t|\ll 1$. By Proposition 5.1 we also obtain that $\mathcal{L}_0^{(2)}(J(f_0))\leqslant\mathcal{L}_0^{(2)}(J(f_t))$, since we assume that $J(f_0)$ is diagonal.

We assume that the deformation (f_t) is bi-Lipschitz trivial, in particular, this is topologically trivial. Then $\mu(f_0) = \mu(f_t)$, for all $|t| \ll 1$. But we assume that f_t is Hickel, for all $t \neq 0$. Then

$$\mathcal{L}_0^{(1)}(J(f_0))\mathcal{L}_0^{(2)}(J(f_0))\mathcal{L}_0^{(3)}(J(f_0)) \geqslant \mu(f_0) = \mu(f_t) = \mathcal{L}_0^{(1)}(J(f_t))\mathcal{L}_0^{(2)}(J(f_t))\mathcal{L}_0^{(3)}(J(f_t)).$$

Then $\mathcal{L}_0^{(2)}(J(f_0)) \geqslant \mathcal{L}_0^{(2)}(J(f_t))$, for all $|t| \ll 1$. Hence we obtain the equality $\mathcal{L}_0^{(2)}(J(f_0)) = \mathcal{L}_0^{(2)}(J(f_t))$, for all $|t| \ll 1$. □

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