Summary

An important area of Applied Mathematics is Matrix Analysis due to the fact that many problems can be reformulated in terms of matrices and, in this way, their resolution is facilitated. The inverse eigenvalue problem consists of the reconstruction of a matrix from given spectral data. This type of problems occurs in different engineering areas and arises in numerous applications where the parameters of a particular physical system are determined from previous knowledge or expected dynamic behavior. In this thesis the inverse eigenvalue problem for three specific sets of matrices is solved.

Inverse eigenvalue problems have been studied from theoretical and numerical points of view as well as from their applications. The list of applications is vast. For instance, we can mention control theory, identification of systems, analysis and design of structures, geophysical studies, molecular spectroscopy, and circuit theory, among others. Some of these applications will be described in Chapter 1 of this thesis.

In several cases, in order to make the inverse eigenvalue problem reasonable, it is necessary to impose some additional conditions on the solution matrices, that is, those matrices must have a specific structure. In summary, an inverse

eigenvalue problem properly posed must satisfy two constraints: one referring to the spectral data and the other to the desirable structure.

Given a matrix X and a diagonal matrix D, solutions of the equation $AX = XD$ are searched, where A is a matrix with a prescribed structure and a predefined spectrum. Based on these restrictions on matrix A, a variety of inverse eigenvalue problems arise.

For example, the inverse eigenvalue problem for centrosymmetric matrices was addressed by F. Zhou, X. Hu, and L. Zhang in [49]. Using the singular value decomposition and the Moore-Penrose inverse, they found conditions to guarantee the existence of solution. The centrosymmetric matrices have applications in information theory and in theory of linear systems, among others.

In the article [38] appeared in 2005, Z. Y. Peng considered the inverse eigenvalue problem for the case where A is a hermitian and antireflexive matrix with respect to a generalized reflexion matrix. Five years later, M. Liang and L. Dai stated in [32] the solvability conditions for the left and right inverse eigenvalue problem for generalized reflexive and antireflexive matrices. The general expression of the solution was also given. In the same year, L. Lebtahi and N. Thome solved in [28] the problem for the case of a matrix A that is hermitian and reflexive or antireflexive with respect to a matrix J that is tripotent and hermitian.

In Chapter 2 of this work the results of [28] are extended to the case of a matrix A that is hermitian and reflexive with respect to a matrix J which is $\{k+1\}$-potent and normal. Theorem 2.2.1 provides conditions under which the problem has a solution and the explicit form of the general solution is given. In addition, in case of the set of solutions of the inverse eigenvalue problem is not empty, the associated Procrustes problem is solved.

The Procrustes problem, or the best approximation problem, associated to the inverse eigenvalue one can be described synthetically as follows: given an experimentally obtained matrix, the problem consists on finding a matrix from the problem solution set (and, therefore, with the desired structure), such that it is the best approximation to the data matrix. For simplicity, the Frobenius norm is generally used.

On the other hand, Hamiltonian and skewHamiltonian matrices appear in the resolution of important problems of Systems and Control Theory. They arise, for example, in optimal linear quadratic control [34, 42], in the calculation of the norm H_∞ of a stable system [50], and in the resolution of the algebraic Riccati equations [27], among others. The inverse eigenvalue problem for hermitian and generalized Hamiltonian matrices was analyzed by Z. Zhang, X. Hu and L. Zang in [48] and, afterwards, the case of hermitian and skewHamiltonian generalized matrices by Z. Bai was considered. In both cases, not only the inverse eigenvalue problem was studied but also uniqueness of solution for the best approximation problem was proved and the solution was presented.

An extension of the Hamiltonian matrices are the J-Hamiltonian matrices defined for the first time in [14], and it is one of the original contributions of this work. In Chapters 3 and Chapter 4 of this thesis the inverse eigenvalue the respective problems for normal J-Hamiltonian matrices and for normal J-skewHamiltonian matrices are studied. For the resolution of the normal J-Hamiltonian matrices case, the structure of this type of matrices is fisrtly analyzed and, then, four methods are presented. The first two methods are general, they give conditions under which the problem is solvable and, among the solutions normal J-Hamiltonian matrices are found. The third method is formalized in the Theorem 3.2.2. It provides the conditions under which the problem has a solution and the infinite solutions are presented, but with this method we are not able to obtain all of them. Finally, the last method states the form of all the solutions. The main result is established in the Theorem 3.2.3. A complete section is dedicated to solve the associated optimization Procrustes problem in case of the problem admits solution. The main result is presented in Theorem 3.3.1.

Below, a summary of the organization of this thesis and a brief description of its four chapters are presented.

Chapter 1 contains an introduction to the inverse eigenvalue problem, the Procrustes problem, and some other ones studied in the literature. Also, definitions, properties, lemmas, and theorems used throughout this work are presented.

In Chapter 2, the inverse eigenvalue problem for a hermitian reflexive matrix with respect to a normal $\{k+1\}$-potent matrix is studied, as well as the associated optimization Procrustes problem. In addition, an algorithm that solves the Procrustes problem is designed and an example that shows the performance of the algorithm is given.

The inverse eigenvalue problem for a normal J-Hamiltonian matrix is investigated in Chapter 3 by using several methods. The associated optimization Procrustes problem is also considered. As in Chapter 2, an algorithm that allows us to calculate the solution of the optimization problem is proposed. Some examples where its performance is showed are provided.

Finally, in Chapter 4, based on the results obtained in Chapter 3, the inverse eigenvalue problem for normal J-skewHamiltonian matrices is addressed. Following the line of Chapters 2 and Chapter 3, an algorithm that solves the Procrustes problem is presented and some illustrative examples of application of the results are presented.

The main contributions obtained in this thesis were published in scientific journals and presented at congresses. They can be seen in [13, 14, 15, 16, 17, 18].

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