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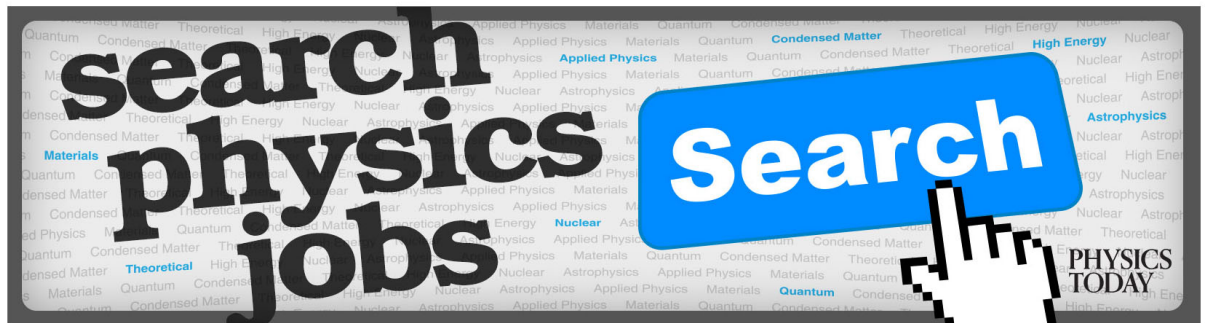
L^p-spectrum of the Schrödinger operator with inverted harmonic oscillator potential

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We determine the L^p-spectrum of the Schrödinger operator with the inverted harmonic oscillator potential $V(x) = -x^2$ for $1 \le p \le \infty$. *Published by AIP Publishing.*
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I. INTRODUCTION AND STATEMENT OF RESULTS

Schrödinger operators are usually considered on the physical Hilbert space L^2 of square–integrable functions. The mathematical interest in studying the L^p-spectrum goes back to Barry Simon,[16] who proposed to analyze localization properties of the binding in terms of L^p-estimates for the eigenfunctions. In many situations, the L^p-spectrum coincides with the usual L^2-spectrum (see, for example, Refs. 10, 17, and 3), but there are also examples where the spectrum does depend on p (see Refs. 11 and 5).

More recently, the interest in considering Schrödinger operators on the Banach spaces L^1 and L^∞ was revived from physics due to the connection between quantum mechanics and statistical mechanics.[7,8,13] In this context, one is interested in the harmonic potential having the opposite sign of the usual harmonic oscillator potential,

$$V(\mathbf{r}) = -\mathbf{r}^2, \qquad \mathbf{r} = (x, y, z) \in \mathbb{R}^3. \tag{1.1}$$

This repulsive potential might appear to be physically uninteresting. Yet it provides, among other examples, an approximate description of the expansion of the galaxies in our universe as governed by Hubble's law.[2,12] Although, at first sight, this is a purely classical–mechanical problem, Eddington pointed out long ago[6] that one should also consider the corresponding scattering problem in quantum mechanics. In this paper, we take up this suggestion and solve the nonrelativistic Schrödinger equation for the potential (1.1) (hereafter called *the Hubble potential* or also *the inverted harmonic oscillator potential*). We note that, in the L^2 case, the corresponding Hamiltonian was analyzed in detail in Ref. 1, with a focus on the dynamical picture and the sojourn time. Our analysis complements that of Ref. 1 in that we will step outside the Hilbert space $L^2(\mathbb{R}^3)$ and consider the Banach space $L^p(\mathbb{R}^3)$ for any $p \in [1, \infty]$. As explained in Refs. 7 and 8, the cases $p = 1$ and $p = \infty$ are of particular interest.

Separating variables in the standard way, we may restrict our attention to the 1-dimensional problem,

$$\mathcal{H}\psi(x) = \lambda\psi(x), \qquad \mathcal{H} = -\frac{d^2}{dx^2} - x^2. \tag{1.2}$$

As we will prove later, there exists no nontrivial solution to this eigenvalue equation that is square–integrable. Physically this is easily understood, since the repulsive potential $-x^2$ possesses scattering states only and no bound states.

Before embarking on the resolution of (1.2), it is instructive to put the *L^p-spectral theory* for the Hamiltonian (1.2) in the context of related harmonic potentials. As is well known, the Hamiltonian

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of the standard harmonic oscillator

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^2$$

on the Hilbert space $L^2(\mathbb{R})$ has a purely discrete spectrum and eigenfunctions given by

$$\psi_n(x) = H_n(x)\,\mathrm{e}^{-\frac{x^2}{2}}\,, \tag{1.3}$$

where $H_n(x)$ are the Hermite polynomials. Having an exponential decay at infinity, these eigenfunctions are also in L^p for all $p \in [1, \infty]$. Moreover, for any λ not in the L^2-spectrum, the Green's kernel $s_\lambda(u, u')$ can be constructed explicitly in terms of parabolic cylinder functions [for details, see (2.9) and (2.10) below]. Since this Green's kernel again decays exponentially at infinity, the corresponding integral operator defines the resolvent as a bounded operator on L^p for all $p \in [1, \infty]$. This shows that the spectrum of the harmonic oscillator is indeed independent of p.

The paper[4] is concerned with *complex deformations* of the harmonic oscillator of the form

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \mathrm{e}^{\mathrm{i}\varphi}\,x^2, \qquad \varphi \in (-\pi, \pi)\,. \tag{1.4}$$

The corresponding eigenfunctions are obtained by complex continuation of the eigensolutions

$$\psi(x) = H_n(z)\,\mathrm{e}^{-\frac{z^2}{2}}, \qquad z(x) = \mathrm{e}^{\frac{\mathrm{i}\varphi}{4}}\,x\,. \tag{1.5}$$

These solutions again decay exponentially at infinity and are thus in L^p for all $p \in [1, \infty]$. The point of interest in Ref. 4 is that these eigenfunctions do not form a Riesz basis in L^p. Constructing the resolvent via the Green's kernel, one can again show that the spectrum of the Hamiltonians (1.4) is independent of p.

The situation changes drastically when the angle φ in (1.4) is chosen as $\varphi = \pi$, which gives precisely the Hubble potential (1.2). Then the complex-deformed eigenfunctions (1.5) become

$$\psi(x) = H_n\big(\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}\,x\big)\,\mathrm{e}^{-\frac{\mathrm{i}x^2}{2}}\,.$$

Now the exponential is a phase factor which no longer gives decay at infinity. As we shall see in this paper, this indeed gives rise to an interesting p-dependence of the spectrum. Here is our main result:

Theorem 1.1. *The spectrum of the Hamiltonian $\mathcal{H}$ on $L^p(\mathbb{R})$ with the domain of definition $\mathcal{D}(\mathcal{H}) = C_0^\infty(\mathbb{R}) \subset L^2(\mathbb{R})$ is the strip,*

$$\begin{cases} |\mathrm{Im}\,\lambda| \leq \dfrac{2}{p} - 1 & \text{if } p \leq 2 \\[2mm] |\mathrm{Im}\,\lambda| \leq 1 - \dfrac{2}{p} & \text{if } p > 2 \end{cases}. \tag{1.6}$$

In the case $p > 2$, the interior of this strip is a point spectrum. In the case $p = \infty$, the whole strip is in the point spectrum.

II. DIAGONALIZATION OF THE HUBBLE HAMILTONIAN

We consider the Hubble Hamiltonian

$$\mathcal{H} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} - x^2 \tag{2.1}$$

on the Banach spaces $L^p(\mathbb{R})$, with $1 \leq p \leq \infty$, with the domain of definition

$$\mathcal{D}(\mathcal{H}) = C^\infty(\mathbb{R}) \cap L^p(\mathbb{R})\,.$$

In the case $p = 2$, Nelson's commutator theorem and the Faris–Lavine theorem imply that the Hubble Hamiltonian (2.1) is essentially self-adjoint on $C_0^\infty(\mathbb{R}) \subset L^2(\mathbb{R})$ (see Ref. 15, Theorem X36, Theorem X38, and the corollaries thereafter).

A. Eigenfunctions

Our starting point is the eigenvalue equation

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^2\right)\psi(x) = -\lambda\psi(x), \quad x \in \mathbb{R}, \quad \lambda \in \mathbb{C}. \tag{2.2}$$

The eigenvalue λ need not be real. Indeed, in the case $p \neq 2$, the notion of self-adjointness is not defined, so that the eigenvalues are not guaranteed to be real.

A fundamental set of solutions to (2.2), known as *parabolic cylinder functions*, can be found in the literature.[9,14] However, for later purposes, it will be illustrative to recall the actual resolution of the differential equation (2.2). Thus we first look for a factorization of $\psi(x)$ in the form

$$\psi(x) = v(x)\exp(\beta x^2), \qquad \beta \in \mathbb{C}, \tag{2.3}$$

where β is some constant to be chosen later. With (2.3) in (2.2), one finds

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}v(x) + 4\beta x\frac{\mathrm{d}}{\mathrm{d}x}v(x) + \left[(2\beta + \lambda) + (4\beta^2 + 1)x^2\right]v(x) = 0. \tag{2.4}$$

The choice $\beta = \mathrm{i}/2$ simplifies (2.4) to

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}v(x) + 2\mathrm{i}x\frac{\mathrm{d}}{\mathrm{d}x}v(x) + (\mathrm{i} + \lambda)v(x) = 0. \tag{2.5}$$

Finally, the change of variables $z = \mathrm{e}^{\mathrm{i}\frac{3\pi}{4}}x$ reduces (2.5) to

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}\bar{v}(z) - 2z\frac{\mathrm{d}}{\mathrm{d}z}\bar{v}(z) - (1 - \mathrm{i}\lambda)\bar{v}(z) = 0, \tag{2.6}$$

where we have defined

$$\bar{v}(z) := v(\mathrm{e}^{-\mathrm{i}\frac{3\pi}{4}}z) = v(x).$$

Now (2.6) is a particular case of the Hermite differential equation on the complex plane (Ref. 14, Chap. 10),

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}f(z) - 2z\frac{\mathrm{d}}{\mathrm{d}z}f(z) + 2\nu f(z) = 0, \qquad \nu \in \mathbb{C}, \tag{2.7}$$

where the relation between the eigenvalue λ and the index ν is

$$\lambda = -\mathrm{i}(2\nu + 1). \tag{2.8}$$

It turns out that for $\nu \in \mathbb{C} \setminus \mathbb{N}$, the Hermite functions [see Ref. 14, Eq. (10.4.3)],

$$H_\nu(z) = \frac{1}{2\Gamma(-\nu)}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\Gamma\left(\frac{n-\nu}{2}\right)(2z)^n,$$

solve (2.7). Now $H_\nu(z)$ defines an entire function of $z \in \mathbb{C}$. In the limit when ν tends to an integer, the above series reduces to the corresponding Hermite polynomials in (1.3). Moreover, the general solution to the Hermite equation (2.7) can be expressed as a linear combination of the two functions $H_\nu(z)$ and $\exp(z^2)H_{-(\nu+1)}(\mathrm{i}z)$, since both satisfy (2.7) and have a nonvanishing Wronskian. Thus the general solution to (2.6) is a linear combination of the two linearly independent functions

$$\bar{v}_\nu(z) = H_\nu(z), \qquad \bar{v}_{-(\nu+1)}(z) = \exp(z^2)H_{-(\nu+1)}(\mathrm{i}z),$$

and the general eigenfunction (2.3) becomes a linear combination of the two linearly independent eigenfunctions

$$\psi_\nu(x) := \exp\left(\frac{\mathrm{i}}{2}x^2\right)H_\nu\left(\mathrm{e}^{\mathrm{i}\frac{3\pi}{4}}x\right), \tag{2.9}$$

$$\psi_{-(\nu+1)}(x) := \exp\left(-\frac{\mathrm{i}}{2}x^2\right)H_{-(\nu+1)}\left(\mathrm{e}^{\mathrm{i}\frac{5\pi}{4}}x\right), \tag{2.10}$$

both corresponding to the same eigenvalue λ in (2.8).

B. Asymptotics of eigenfunctions

We now determine the asymptotics of the solutions (2.9) and (2.10) as $x \to \pm\infty$. We begin with the function ψ_ν. Asymptotically as $x \to \infty$, the argument of the Hermite functions $H_\nu(z)$ lies on the ray $z = e^{i\frac{3\pi}{4}} x$ with $x > 0$. The asymptotics of the Hermite functions is given by [Ref. 14, Eq. (10.6.7)]

$$H_\nu(z) \sim (2z)^\nu - \frac{\sqrt{\pi} e^{i\pi\nu}}{\Gamma(-\nu)} z^{-\nu-1} e^{z^2}, \quad |z| \to \infty, \quad \frac{\pi}{4} < \arg(z) < \frac{5\pi}{4} .$$

We thus obtain

$$H_\nu(e^{i\frac{3\pi}{4}} x) \sim a_\nu x^\nu + b_\nu x^{-(\nu+1)} e^{-ix^2} \qquad \text{as } x \to \infty ,$$

where the coefficients are given by

$$a_\nu = 2^\nu e^{i\frac{3\nu\pi}{4}}, \quad b_\nu = -\frac{\sqrt{\pi} e^{i\pi\nu}}{\Gamma(-\nu)} e^{-i(\nu+1)\frac{3\pi}{4}} .$$

As a consequence, the wavefunction (2.9) has the asymptotic behavior

$$\psi_\nu(x) \sim a_\nu x^\nu e^{ix^2/2} + b_\nu x^{-(\nu+1)} e^{-ix^2/2} \qquad \text{as } x \to \infty . \tag{2.11}$$

Asymptotically as $x \to -\infty$, the argument of the Hermite functions in (2.9) is on the ray $z = e^{i\frac{7\pi}{4}} |x|$. The corresponding asymptotics is [Ref. 14, Eq. (10.6.6)]

$$H_\nu(z) \sim (2z)^\nu, \quad |z| \to \infty, \quad |\arg(z)| < \frac{3\pi}{4} . \tag{2.12}$$

Then

$$H_\nu(e^{i\frac{7\pi}{4}} |x|) \sim c_\nu |x|^\nu, \quad x \to -\infty, \quad c_\nu := 2^\nu e^{i\frac{7\nu\pi}{4}} ,$$

and the wavefunction (2.9) becomes

$$\psi_\nu(x) \sim c_\nu |x|^\nu e^{ix^2/2} \qquad \text{as } x \to -\infty . \tag{2.13}$$

For the asymptotics of the solution $\psi_{-(\nu+1)}$ as $x \to \infty$, we make use of the asymptotic expansion of the Hermite function [Ref. 14, Eq. (10.6.8)]

$$H_\nu(z) \sim (2z)^\nu - \frac{\sqrt{\pi} e^{-i\pi\nu}}{\Gamma(-\nu)} z^{-\nu-1} e^{z^2}, \quad |z| \to \infty, \quad -\frac{5\pi}{4} < \arg(z) < -\frac{\pi}{4} ,$$

and the replacement $\nu \to -(\nu+1)$ leads to

$$H_{-(\nu+1)}(z) \sim (2z)^{-(\nu+1)} - \frac{\sqrt{\pi} e^{i\pi(\nu+1)}}{\Gamma(\nu+1)} z^\nu e^{z^2}, \quad -\frac{5\pi}{4} < \arg(z) < -\frac{\pi}{4} .$$

Hence

$$H_{-(\nu+1)}\left(e^{i\frac{5\pi}{4}} x\right) \sim d_\nu x^{-(\nu+1)} + e_\nu x^\nu e^{ix^2}, \quad x \to \infty ,$$

where the x-independent coefficients d_ν and e_ν are

$$d_\nu = 2^{-(\nu+1)} e^{-i\frac{5\pi}{4}(\nu+1)}, \quad e_\nu = -\frac{\sqrt{\pi} e^{i\pi(\nu+1)}}{\Gamma(\nu+1)} e^{i\frac{5\pi}{4}\nu} .$$

Thus the wavefunction (2.10) reads asymptotically,

$$\psi_{-(\nu+1)}(x) \sim d_\nu x^{-(\nu+1)} e^{-ix^2/2} + e_\nu x^\nu e^{ix^2/2} \qquad \text{as } x \to \infty . \tag{2.14}$$

Finally, for the asymptotics $x \to -\infty$, we again use (2.12) for ν replaced by $-(\nu+1)$,

$$H_{-(\nu+1)}(z) \sim (2z)^{-(\nu+1)}, \quad |z| \to \infty, \quad |\arg(z)| < \frac{3\pi}{4} .$$

Hence

$$H_{-(\nu+1)}\left(e^{i\frac{\pi}{4}} |x|\right) \sim f_\nu |x|^{-(\nu+1)}, \quad f_\nu := 2^{-(\nu+1)} e^{-i\frac{(\nu+1)\pi}{4}}, \quad x \to -\infty .$$

We conclude that the wavefunction (2.10) has the asymptotics

$$\psi_{-(\nu+1)}(x) \sim f_\nu |x|^{-(\nu+1)} e^{-ix^2/2} \qquad \text{as } x \to -\infty . \tag{2.15}$$

III. RESOLVENT ESTIMATES

We now write the eigenvalue equation (2.2) as the one-dimensional Schrödinger equation

$$\mathcal{H}\phi(x) = \lambda\,\phi(x) \qquad \text{with} \quad \mathcal{H} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} - x^2\,. \tag{3.1}$$

We may assume that λ is in the upper half plane,

$$\mathrm{Im}\,\lambda \geq 0\,, \tag{3.2}$$

because otherwise, we may analyze the complex conjugate of (3.1). As $x \to \pm\infty$, the solutions have the asymptotics

$$\phi(x) = c_1^{\pm}\, e^{\frac{ix^2}{2} + \frac{i\lambda-1}{2}\,\log|x|}\left(1 + \mathcal{O}(|x|^{-\frac{1}{2}})\right) + c_2^{\pm}\, e^{-\frac{ix^2}{2} - \frac{i\lambda+1}{2}\,\log|x|}\left(1 + \mathcal{O}(|x|^{-\frac{1}{2}})\right)\,. \tag{3.3}$$

This asymptotics can be understood immediately by computing the second derivatives of the approximate solution,

$$\phi_{\mathrm{approx}}(x) := c_1^{\pm}\, e^{\frac{ix^2}{2} + \frac{i\lambda-1}{2}\,\log|x|} + c_2^{\pm}\, e^{-\frac{ix^2}{2} - \frac{i\lambda+1}{2}\,\log|x|},$$

$$\phi'_{\mathrm{approx}}(x) = c_1^{\pm}\left(ix + \frac{i\lambda-1}{2x}\right) e^{\frac{ix^2}{2} + \frac{i\lambda-1}{2}\,\log|x|}$$

$$\qquad\qquad + c_2^{\pm}\left(-ix - \frac{i\lambda+1}{2x}\right) e^{-\frac{ix^2}{2} - \frac{i\lambda+1}{2}\,\log|x|},$$

$$\phi''_{\mathrm{approx}}(x) = c_1^{\pm}\left\{i + \left(ix + \frac{i\lambda-1}{2x}\right)^2 + \mathcal{O}(x^{-2})\right\} e^{\frac{ix^2}{2} + \frac{i\lambda-1}{2}\,\log|x|}$$

$$\qquad\qquad + c_2^{\pm}\left\{-i + \left(-ix - \frac{i\lambda+1}{2x}\right)^2 + \mathcal{O}(x^{-2})\right\} e^{-\frac{ix^2}{2} - \frac{i\lambda+1}{2}\,\log|x|}$$

$$\qquad = c_1^{\pm}\left\{-x^2 - \lambda + \mathcal{O}(x^{-2})\right\} e^{\frac{ix^2}{2} + \frac{i\lambda-1}{2}\,\log|x|}$$

$$\qquad\qquad + c_2^{\pm}\left\{-x^2 - \lambda + \mathcal{O}(x^{-2})\right\} e^{-\frac{ix^2}{2} - \frac{i\lambda+1}{2}\,\log|x|}$$

$$\qquad = (-x^2 - \lambda)\,\phi_{\mathrm{approx}}(x)\left(1 + \mathcal{O}(x^{-2})\right)\,.$$

We introduce the functions $\acute{\phi}$ and $\grave{\phi}$ as solutions with the fastest possible asymptotic decay, i.e.,

$$\acute{\phi}(x) = e^{\frac{ix^2}{2} + \frac{i\lambda-1}{2}\,\log|x|}\left(1 + \mathcal{O}((-x)^{-\frac{1}{2}})\right) \qquad \text{as } x \to -\infty, \tag{3.4}$$

$$\grave{\phi}(x) = e^{\frac{ix^2}{2} + \frac{i\lambda-1}{2}\,\log|x|}\left(1 + \mathcal{O}(x^{-\frac{1}{2}})\right) \qquad \text{as } x \to +\infty\,. \tag{3.5}$$

Lemma 3.1. The solutions $\acute{\phi}$ and $\grave{\phi}$ form a fundamental system.

Proof. Comparing (3.4) with (2.13) and using (2.8), one concludes that $\acute{\phi}$ is a multiple of ψ_ν. According to (2.11) and noting that $b_\nu \neq 0$ (here we make use of the fact that $\mathrm{Im}\,\lambda \geq 0$, implying that $\nu \notin \mathbb{N}_0$), one observes the asymptotics $\acute{\phi}$ as $x \to \infty$ is different from the asymptotics of $\grave{\phi}$ in (3.5). Hence $\acute{\phi}$ and $\grave{\phi}$ are linearly independent. $\qquad\square$

The fundamental solutions $\acute{\phi}$ and $\grave{\phi}$ have the asymptotics

$$\acute{\phi}(x) = \acute{c}_1\, e^{\frac{ix^2}{2} + \frac{i\lambda-1}{2}\,\log|x|}\left(1 + \mathcal{O}(x^{-\frac{1}{2}})\right)$$

$$\qquad + \acute{c}_2\, e^{-\frac{ix^2}{2} - \frac{i\lambda+1}{2}\,\log|x|}\left(1 + \mathcal{O}(x^{-\frac{1}{2}})\right) \qquad \text{as } x \to +\infty, \tag{3.6}$$

$$\grave{\phi}(x) = \grave{c}_1\, e^{\frac{ix^2}{2} + \frac{i\lambda-1}{2}\,\log|x|}\left(1 + \mathcal{O}((-x)^{-\frac{1}{2}})\right)$$

$$\qquad + \grave{c}_2\, e^{-\frac{ix^2}{2} - \frac{i\lambda+1}{2}\,\log|x|}\left(1 + \mathcal{O}((-x)^{-\frac{1}{2}})\right) \qquad \text{as } x \to -\infty\,. \tag{3.7}$$

Here the parameters $\acute{c}_2$ and $\grave{c}_2$ are both non-zero because otherwise the solutions $\acute{\phi}$ and $\grave{\phi}$ would be linearly dependent in view of (3.4) and (3.5). Computing their Wronskian at, for example, $x = +\infty$ gives

$$
\begin{aligned}
w(\acute{\phi}, \grave{\phi}) &= \lim_{x \to \infty} (\acute{\phi}'(x)\, \grave{\phi}(x) - \acute{\phi}(x)\, \grave{\phi}'(x)) \\
&= \acute{c}_2 \lim_{x \to \infty} w\left(e^{-\frac{ix^2}{2} - \frac{i\lambda+1}{2}\, \log x}, \, e^{\frac{ix^2}{2} + \frac{i\lambda-1}{2}\, \log x}\right) \\
&= \acute{c}_2 \lim_{x \to \infty} \left\{\left(-ix - \frac{i\lambda+1}{2x}\right) - \left(ix + \frac{i\lambda-1}{2x}\right)\right\} e^{-\frac{ix^2}{2} - \frac{i\lambda+1}{2}\, \log x}\, e^{\frac{ix^2}{2} + \frac{i\lambda-1}{2}\, \log x} \\
&= \acute{c}_2 \lim_{x \to \infty} \left\{-2ix - \frac{i\lambda}{x}\right\} e^{-\log x} = -2i\acute{c}_2 \neq 0 .
\end{aligned}
\tag{3.8}
$$

Since this Wronskian is non-zero, we may introduce *Green's kernel* by

$$
s_\lambda(x, x') = \frac{1}{w(\acute{\phi}, \grave{\phi})} \times \begin{cases} \acute{\phi}(x)\, \grave{\phi}(x') & \text{if } x \leq x' \\ \acute{\phi}(x')\, \grave{\phi}(x) & \text{if } x' < x \end{cases}.
\tag{3.9}
$$

We also consider the Green's kernel as the integral kernel of a corresponding operator s_λ, i.e.,

$$
(s_\lambda \psi)(x) := \int_{-\infty}^{\infty} s_\lambda(x, x')\, \psi(x')\, dx' .
$$

The distributional relation

$$
(\mathcal{H}_x - \lambda)\, s_\lambda(x, x') = \delta(x - x')
$$

implies that, formally, the operator s_λ coincides with the resolvent,

$$
s_\lambda \overset{\text{formally}}{=} (\mathcal{H} - \lambda)^{-1} .
$$

In order to establish the existence of the resolvent, it suffices to show that s_λ is a bounded operator on L^p. We begin with a preparatory lemma.

Lemma 3.2. For every $\lambda \in \mathbb{C}$ with $\text{Im}\,\lambda \geq 0$, there is a constant $c = c(\lambda)$ such that

$$
|s_\lambda(x, x')| \leq \frac{c}{\sqrt{1+|x|}\,\sqrt{1+|x'|}}\, \min\left(\left|\frac{x}{x'}\right|, \left|\frac{x'}{x}\right|\right)^{\frac{\text{Im}\,\lambda}{2}} .
$$

Proof. By symmetry, it clearly suffices to consider the case $x' > x$, so that

$$
s_\lambda(x, x') = -\frac{\acute{\phi}(x)\, \grave{\phi}(x')}{2i\acute{c}_2} .
$$

According to the asymptotics (3.4)–(3.7),

$$
\begin{aligned}
|\acute{\phi}(x)| &\approx \frac{1}{\sqrt{|x|}}\, e^{-\frac{\text{Im}\,\lambda}{2}\, \log|x|}, \quad |\grave{\phi}(x')| \approx \frac{1}{\sqrt{|x'|}}\, e^{\frac{\text{Im}\,\lambda}{2}\, \log|x'|} & \text{as } x, x' \to -\infty, \\
|\acute{\phi}(x)| &\approx \frac{1}{\sqrt{|x|}}\, e^{\frac{\text{Im}\,\lambda}{2}\, \log|x|}, \quad |\grave{\phi}(x')| \approx \frac{1}{\sqrt{|x'|}}\, e^{-\frac{\text{Im}\,\lambda}{2}\, \log|x'|} & \text{as } x, x' \to +\infty .
\end{aligned}
$$

Moreover, the solutions $\acute{\phi}$ and $\grave{\phi}$ are clearly bounded on any compact set. This gives the result. □

Proposition 3.3. Assume that for $p \in [1, \infty]$, the following inequality holds:

$$
\frac{\text{Im}\,\lambda}{2} > \left|\frac{1}{2} - \frac{1}{p}\right| .
$$

Then the resolvent exists as a bounded operator on $L^p(\mathbb{R})$.

Proof. Let $p, q \in [1, \infty]$ be conjugate Hölder exponents, i.e.,

$$
\frac{1}{p} + \frac{1}{q} = 1 .
\tag{3.10}
$$

Our goal is to show that there is a constant $c = c(\lambda)$ such that for all $\psi \in L^p(\mathbb{R})$ and $\phi \in L^q(\mathbb{R})$, the inequality

$$\int_{\mathbb{R}^2} |\overline{\phi(x)} \, s_\lambda(x, y) \, \psi(y)| \, \mathrm{d}x \, \mathrm{d}y \leq c \, \|\phi\|_{L^q} \, \|\psi\|_{L^p} \tag{3.11}$$

holds. Indeed, in the case $1 \leq p < \infty$, we can then use the fact that $(L^p)^* = L^q$ to conclude that s_λ is a bounded operator on L^p. In the case $p = \infty$, we set $\eta = s_\lambda \psi$. For any given $\varepsilon > 0$, we can choose a set Ω with non-zero Lebesgue measure such that

$$\eta|_\Omega \geq \|\eta\|_{L^\infty} - \varepsilon \qquad \text{or} \qquad \eta|_\Omega \leq -\|\eta\|_{L^\infty} + \varepsilon \, .$$

Choosing ϕ as the characteristic function $\phi = \chi_\Omega$, it follows that

$$\int_{\mathbb{R}^2} |\overline{\phi(x)} \, s_\lambda(x, y) \, \psi(y)| \, \mathrm{d}x \, \mathrm{d}y \geq (\|\eta\|_{L^\infty} - \varepsilon) \, \|\phi\|_{L^1} \, .$$

Therefore, the inequality (3.11) implies that

$$(\|\eta\|_{L^\infty} - \varepsilon) \leq c \, \|\psi\|_{L^\infty} \, .$$

Since ε is arbitrary, we conclude that s_λ is a bounded operator on L^∞.

It remains to derive the inequality (3.11). Employing the estimate of Lemma 3.2, we obtain

$$\left| \int_{R^2} \overline{\phi(x)} \, s_\lambda(x, y) \, \psi(y) \, \mathrm{d}x \, \mathrm{d}y \right|$$

$$\leq c \int_{\mathbb{R}^2} \frac{|\phi(x)| \, |\psi(y)|}{\sqrt{1 + |x|} \, \sqrt{1 + |y|}} \, \min\left(\left|\frac{x}{y}\right|, \left|\frac{y}{x}\right|\right)^{\frac{|\mathrm{Im}\,\lambda|}{2}} \, \mathrm{d}x \, \mathrm{d}y \, .$$

Now it is useful to choose polar coordinates $r \in \mathbb{R}^+$, $\varphi \in [0, 2\pi)$, i.e.,

$$x = r \cos \varphi \, , \qquad y = r \sin \varphi \, .$$

We thus obtain the estimate

$$\left| \int_{\mathbb{R}^2} \overline{\phi(x)} \, s_\lambda(x, y) \, \psi(y) \, \mathrm{d}x \, \mathrm{d}y \right|$$

$$\leq c \int_0^{2\pi} \mathrm{d}\varphi \int_0^\infty r \, \mathrm{d}r \, \frac{|\phi(r \cos \varphi)| \, |\psi(r \sin \varphi)|}{\sqrt{1 + r \, |\cos \varphi|} \, \sqrt{1 + r \, |\sin \varphi|}} \, \min\left(|\cot \varphi|, |\tan \varphi|\right)^{\frac{|\mathrm{Im}\,\lambda|}{2}}$$

$$\leq c \int_0^{2\pi} \mathrm{d}\varphi \int_0^\infty \mathrm{d}r \, \frac{|\phi(r \cos \varphi)| \, |\psi(r \sin \varphi)|}{\sqrt{|\cos \varphi \, \sin \varphi|}} \, \min\left(|\cot \varphi|, |\tan \varphi|\right)^{\frac{|\mathrm{Im}\,\lambda|}{2}} \, . \tag{3.12}$$

Now the r-integral can be estimated with Hölder's inequality,

$$\int_0^\infty |\phi(r \cos \varphi)| \, |\psi(r \sin \varphi)| \, \mathrm{d}r \leq \|\phi(.\,\cos \varphi)\|_{L^q} \, \|\psi(.\,\sin \varphi)\|_{L^p} \, .$$

The obtained norms can be simplified by rescaling. Namely, in the case $1 \leq p < \infty$, the transformation

$$\|\psi(.\,\sin \varphi)\|_{L^p} \leq \left(\int_{-\infty}^\infty |\psi(x \sin \varphi)|^p \, \mathrm{d}x \right)^{\frac{1}{p}} = \left(\int_{-\infty}^\infty |\psi(u)|^p \, \frac{\mathrm{d}u}{|\sin \varphi|} \right)^{\frac{1}{p}}$$

implies that

$$\|\psi(.\,\sin \varphi)\|_{L^p} \leq |\sin \varphi|^{-\frac{1}{p}} \, \|\psi\|_{L^p} \, .$$

In the case $p = \infty$, this inequality is again satisfied trivially. Rescaling the q-norm similarly, we obtain the inequality

$$\int_0^\infty |\phi(r \cos \varphi)| \, |\psi(r \sin \varphi)| \, \mathrm{d}r \leq |\cos \varphi|^{-\frac{1}{q}} \, |\sin \varphi|^{-\frac{1}{p}} \, \|\phi\|_{L^q} \, \|\psi\|_{L^p} \, .$$

Using this inequality in (3.12), we get the estimate

$$\left| \int_{\mathbb{R}^2} \overline{\phi(x)} \, s_\lambda(x, y) \, \psi(y) \, \mathrm{d}x \, \mathrm{d}y \right|$$

$$\leq \|\phi\|_{L^q} \, \|\psi\|_{L^p} \int_0^{2\pi} \frac{1}{|\cos\varphi|^{\frac{1}{2}+\frac{1}{q}} \, |\sin\varphi|^{\frac{1}{2}+\frac{1}{p}}} \, \min\left(|\cot\varphi|, |\tan\varphi|\right)^{\frac{|\mathrm{Im}\lambda|}{2}} \, \mathrm{d}\varphi.$$

It remains to analyze whether the φ-integral is finite. To this end, we need to consider the poles of the integrand,

$$\frac{1}{|\cos\varphi|^{\frac{1}{2}+\frac{1}{q}} \, |\sin\varphi|^{\frac{1}{2}+\frac{1}{p}}} \, \min\left(|\cot\varphi|, |\tan\varphi|\right)^{\frac{|\mathrm{Im}\lambda|}{2}}$$

$$\sim \begin{cases} |\sin\varphi|^{\frac{|\mathrm{Im}\lambda|}{2}-\frac{1}{2}-\frac{1}{p}} & \text{near } \varphi = 0, \pi \\ |\cos\varphi|^{\frac{|\mathrm{Im}\lambda|}{2}-\frac{1}{2}-\frac{1}{q}} & \text{near } \varphi = \frac{\pi}{2}, \frac{3\pi}{2} \end{cases}.$$

These poles are integrable if and only if

$$\frac{|\mathrm{Im}\lambda|}{2} - \frac{1}{2} - \frac{1}{p} > -1 \qquad \text{and} \qquad \frac{|\mathrm{Im}\lambda|}{2} - \frac{1}{2} - \frac{1}{q} > -1 \, .$$

This gives the result. □

IV. LOCALIZING THE POINT SPECTRUM IN THE CASE $p > 2$

Lemma 4.1. In the case $p > 2$, the strip

$$0 < \mathrm{Im}\lambda < 1 - \frac{2}{p} \tag{4.1}$$

is in the point spectrum. In the case $p = \infty$, the point spectrum is given by the strip

$$0 \leq \mathrm{Im}\lambda \leq 1 \, . \tag{4.2}$$

Proof. According to the asymptotic expansions (3.4)–(3.7), the fundamental solutions are bounded by

$$|\phi(x)| \lesssim c(\lambda) \, (1 + |x|)^{\frac{|\mathrm{Im}\lambda|}{2}-\frac{1}{2}} \, . \tag{4.3}$$

In the case $1 \leq p < \infty$, this function is in the Banach space L^p if and only if

$$p\left(\frac{|\mathrm{Im}\lambda|}{2} - \frac{1}{2}\right) < -1 \, .$$

In particular, the function ϕ is in the point spectrum if (4.1) holds.

In the case $p = \infty$, the estimate (4.3) shows that under the assumption (4.2) the function ϕ is essentially bounded and is thus an eigenvector. This concludes the proof. □

V. LOCALIZING THE SPECTRUM IN THE CASE $1 \leq p < 2$

Lemma 5.1. In the case $1 \leq p < 2$, the spectrum of the Hamiltonian contains the strip

$$0 < \mathrm{Im}\lambda < \frac{2}{p} - 1 \, . \tag{5.1}$$

Proof. We proceed indirectly. Assume that λ is not in the spectrum. Then the resolvent $(\mathcal{H} - \lambda)^{-1} : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ exists and is continuous. Given a test function $\eta \in C_0^\infty(\mathbb{R})$, the function $\phi := (\mathcal{H} - \lambda)^{-1})\eta \in L^p(\mathbb{R})$ is a weak solution of the differential equation $(\mathcal{H} - \lambda)\phi = \eta$. As a consequence, outside the support of η, the function ϕ is a solution of the ODE (3.1). Combining the asymptotics (3.3) with the fact that $\phi \in L^p(\mathbb{R})$ implies that ϕ must vanish identically outside the support of η [note that *both* summands in (3.3) are *not* in L^p]. Considering the limiting case where η approaches a Dirac distribution, we obtain a contradiction. □

VI. PROOF OF THE MAIN THEOREM

Proof of Theorem 1.1. We may assume that $\text{Im}\,\lambda \geq 0$ because otherwise we may take the complex conjugate of Eq. (3.1). From Proposition 3.3, we know that the spectrum lies inside the strip

$$\frac{\text{Im}\,\lambda}{2} \leq \left| \frac{1}{2} - \frac{1}{p} \right|.$$

According to Lemmas 4.1 and 5.1, the interior of this strip belongs to the spectrum. Since the spectrum is a closed set, we obtain (1.6). The statement on the point spectrum is proved in Lemma 4.1.   □

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