Convolution operators supporting hypercyclic algebras☆ ,☆☆

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Abstract

We show that any convolution operator induced by a non-constant polynomial that vanishes at zero supports a hypercyclic algebra. This partially solves a question raised by R. Aron.

Keywords: Algebrability, hypercyclic algebras, convolution operators, hypercyclic subspaces, MacLane operator

1. Introduction

In 2001 Aron, Garcia and Maestre [1] called the attention to the wide range of examples supporting the following "principle": *In many different settings one encounters a problem which, at first glance, appears to have no solution at all. And, in fact, it frequently happens that there is a large linear subspace of solutions to the problem.* This originated growing interest in the study of *large* algebraic structures within nonlinear settings, giving rise to the notions of lineability, spaceability and algebrability, to name a few [3].

This has been the case, for instance, in the study of the set of hypercyclic vectors. It is well known that for any operator T on a topological vector space X, the set

$$HC(T) = \{f \in X : \quad \{f, Tf, T^2 f, \dots\} \text{ is dense in } X\}$$

of hypercyclic vectors for T is either empty or contains a dense infinite-dimensional linear subspace (but the origin), see [18]. In fact, when $HC(T)$ is non-empty it sometimes contains (but zero) a closed and infinite dimensional linear subspace, while other times the only closed subspaces it contains (but zero) are of finite dimension [8, 14]; see also [7, Ch. 8] and [15, Ch. 10].

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☆☆Dedicated to Professor Richard Aron on the occasion of his 70th birthday.

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When X is a topological algebra it is natural to ask whether $HC(T)$ can contain, or must always contain, a subalgebra (but the origin) whenever it is non-empty. Both questions have been answered by considering convolution operators on the space $X = \mathcal{H}(\mathbb{C})$ of entire functions on the complex plane $\mathbb{C}$, endowed with the compact-open topology; that convolution operators (other than scalar multiples of the identity) are hypercyclic was established by Godefroy and Shapiro [13], see also [10, 16, 2].

Aron et al [4, 5] showed that no translation operator τ_a

$$\tau_a(f)(z) = f(z+a) \quad f \in \mathcal{H}(\mathbb{C}), z \in \mathbb{C}$$

can support a hypercyclic algebra, in a very strong way: Indeed, for any positive integer p and any $f \in \mathcal{H}(\mathbb{C})$, the non-constant elements of the orbit of f^p under τ_a are those entire functions for which the multiplicities of their zeros are integer multiples of p. In stark contrast with this operator they also showed that the collection of entire functions f for which every power f^n $(n = 1, 2, \dots)$ is hypercyclic for the operator D of complex differentiation is residual in $\mathcal{H}(\mathbb{C})$.

Later Shkarin [17, Th. 4.1] showed that $HC(D)$ contained both a hypercyclic subspace and a hypercyclic algebra, and with a different approach Bayart and Matheron [7, Th. 8.26] also showed that the set of $f \in \mathcal{H}(\mathbb{C})$ that generate an algebra consisting entirely (but the origin) of hypercyclic vectors for D is residual in $\mathcal{H}(\mathbb{C})$. The abovementioned solutions by Aron et al, Bayart and Matheron, and Shkarin bear the question of which convolution operators on $\mathcal{H}(\mathbb{C})$ support a hypercyclic algebra. In this note we consider the following question:

Question 1. (Aron) Let Φ be a non-constant polynomial. Does $\Phi(D)$ support a hypercyclic algebra?

The purpose of this note is to show that the techniques by Bayart and Matheron provide an affirmative answer for the case when $\Phi(0) = 0$:

Theorem 1. *Let Ω be a simply connected planar domain and $\mathcal{H}(\Omega)$ the space of holomorphic functions on Ω endowed with the compact open topology. Let Φ be a non-constant polynomial with $\Phi(0) = 0$. Then the set of functions $f \in \mathcal{H}(\Omega)$ that generate a hypercyclic algebra for $\Phi(D)$ is residual in $\mathcal{H}(\Omega)$.*

2. Proof of Theorem 1

The proof of Theorem 1 follows that of [7, Th. 8.26]. We postpone the proof of Proposition 2 for later.

Proposition 2. *Let Φ be a polynomial with $\Phi(0) = 0$. Then for each pair (U, V) of non-empty open subsets of $\mathcal{H}(\Omega)$ and each $m \in \mathbb{N}$ there exists $P \in U$ and $q \in \mathbb{N}$ so that*

$$\begin{cases} \Phi(D)^q(P^j) = 0 & \text{for } 0 \le j < m, \\ \Phi(D)^q(P^m) \in V. \end{cases} \tag{2.1}$$

Proof of Theorem 1. For any $g \in \mathcal{H}(\Omega)$ and $\alpha \in \mathbb{C}^m$, we let $g_\alpha := \alpha_1 g + \cdots + \alpha_m g^m$. Let $(V_k)_k$ be a countable local basis of open sets of $\mathcal{H}(\Omega)$. For each $(k, s, m) \in \mathbb{N}^3$ we let $\mathcal{A}(k, s, m)$ denote the set of $f \in \mathcal{H}(\Omega)$ that satisfy the following property

$$\forall \alpha \in \mathbb{C}^m \text{with } \alpha_m = 1 \text{ and } \|\alpha\|_\infty \le s,\ \exists q \in \mathbb{N}: \ \Phi(D)^q(f_\alpha) \in V_k. \tag{2.2}$$

Each such $\mathcal{A}(k, s, m)$ is open and dense in $\mathcal{H}(\Omega)$, thanks to Proposition 2. By Baire's Theorem,

$$\mathcal{A} = \cap_{k,s,m \in \mathbb{N}} \mathcal{A}(k, s, m)$$

is residual in $\mathcal{H}(\Omega)$. Let $f \in \mathcal{A}$, and let g be in the algebra generated by f. Since a vector is hypercyclic if and only if any non-zero scalar multiple of it is hypercyclic, we may assume $g = f_\alpha = \alpha_1 f + \alpha_2 f^2 + \cdots +$

$\alpha_{m-1}f^{m-1}+f^m$. Then g is clearly hypercyclic for $\Phi(D)$. Indeed, given any non-empty open set U of $\mathcal{H}(\Omega)$, let $k\in\mathbb{N}$ so that $V_k\subset U$. Pick $s>\|\alpha\|_\infty$. Then since $f\in\mathcal{A}(k,s,m)$ we know that there exists q satisfying (2.2). Hence

$$\Phi(D)^q g=\Phi(D)^q(\alpha_1 f+\alpha_2 f^2+\cdots+\alpha_{m-1}f^{m-1}+f^m)\in V_k\subset U.$$

$\square$

Proof of Proposition 2. Let $\Phi(z)=z^r\sum_{j=0}^k a_j z^j$, with $a_0\neq 0$ and $r\in\mathbb{N}$, and let $(A,B)\in U\times V$ be polynomials. Enlarging the degree of B if necessary, we may assume that degree$(B)=p\in r\mathbb{N}$ and $p>m$. It suffices to show the following.

Claim 1. *For any large $n\in r\mathbb{N}$ there exist $(c_0,\dots,c_p)=(c_0(n),\dots,c_p(n))\in\mathbb{C}^{p+1}$ so that*

$$\begin{cases} R:=R_n=z^n\sum_{j=0}^p c_j z^j & \text{and}\\ q:=q_n=\frac{m}{r}n+(m-1)\frac{p}{r} \end{cases}$$

satisfy the following:

(i) $\Phi(D)^q((A+R)^j)=0$ *for* $0\le j<m$,
(ii) $\Phi(D)^q((A+R)^m)=\Phi(D)^q(R^m)=B$,
(iii) $R_n\underset{n\to\infty}{\to}0$ *in* $\mathcal{H}(\Omega)$.

To show the claim, notice that for each $0\le\ell\le m-1$ and each $s\in\mathbb{N}$ we have the inequality

$$\text{degree}(A^sR^\ell)\le\text{constant}+(m-1)n<(m-1)p+mn=rq$$

for all large n. Hence, since $\Phi(D)^q=(\sum_{\ell=0}^k a_\ell D^\ell)^q\ D^{rq}$, it follows that for large n we have

$$\begin{aligned}\Phi(D)^q\ ((A+R)^j)\ &=0\ \text{ for } 0\le j<m, \text{ and}\\ \Phi(D)^q((A+R)^m)\ &=\Phi(D)^q(R^m).\end{aligned}$$

So (i) holds, as well as the first equality in (ii), regardless of the selection $c=(c_0,\dots,c_p)$. Next, for each $s\in\mathbb{N}$ and multi-index $\gamma=(\gamma_0,\dots,\gamma_s)\in\mathbb{N}_0^{s+1}$ we let

$$|\gamma|=\sum_{j=0}^s\gamma_j \text{ and } \widehat{\gamma}=\sum_{j=0}^s j\gamma_j=\sum_{j=1}^s j\gamma_j.$$

Also, for $c=(c_0,\dots,c_s)\in(\mathbb{C}\setminus\{0\})^{s+1}$ and $|\gamma|=m$, we let

$$c^\gamma=\prod_{j=0}^s c_j^{\gamma_j} \text{ and } \binom{m}{\gamma}=\frac{m!}{\gamma_0!\gamma_1!\dots\gamma_s!}.$$

With this notation we have

$$\begin{aligned}\Phi(D)^q(R^m)&=\left(\sum_{\beta\in\mathbb{N}_0^{k+1}:|\beta|=q}\binom{q}{\beta}\ a^\beta\ D^{rq+\widehat{\beta}}\right)\left(\sum_{\alpha\in\mathbb{N}^{p+1}:\ |\alpha|=m}\binom{m}{\alpha}\ c^\alpha\ z^{nm+\widehat{\alpha}}\right)\\ &=\sum_{(\alpha,\beta)\in A}\binom{m}{\alpha}\ c^\alpha\ \binom{q}{\beta}\ a^\beta\ D^{rq+\widehat{\beta}}z^{nm+\widehat{\alpha}}\\ &=\sum_{(\alpha,\beta)\in A}\binom{m}{\alpha}\ c^\alpha\ \binom{q}{\beta}\ a^\beta\ \frac{(nm+\widehat{\alpha})!}{(\widehat{\alpha}-\widehat{\beta}-(m-1)p)!}z^{\widehat{\alpha}-\widehat{\beta}-(m-1)p}\end{aligned}$$

where

$$A = \Big\{(\alpha,\beta) \in \mathbb{N}_0^{p+1} \times N_0^{k+1} :\ |\alpha| = m,\ |\beta| = q,\ \text{and } rq + \widehat{\beta} \leq nm + \widehat{\alpha}\Big\}$$
$$= \Big\{(\alpha,\beta) \in \mathbb{N}_0^{p+1} \times N_0^{k+1} :\ |\alpha| = m,\ |\beta| = q,\ \text{and } mp - p \leq \widehat{\alpha} - \widehat{\beta}\Big\}.$$

Thus

$$\Phi(D)^q(R^m) = \sum_{i=0}^{p} \sum_{(\alpha,\beta)\in A_i} \binom{m}{\alpha} c^\alpha \binom{q}{\beta} a^\beta \frac{(nm+\widehat{\alpha})!}{i!} z^i$$

where for each $i = 0, \ldots p$

$$A_i = \{(\alpha,\beta) \in A :\ \widehat{\alpha} - \widehat{\beta} = i + (m-1)p\}.$$

In particular, $B = \sum_{i=0}^p b_i z^i = \Phi(D)^q(R^m)$ if and only if $c = (c_0, \ldots, c_p)$ is a solution of the system

$$b_i = \sum_{(\alpha,\beta)\in A_i} \binom{m}{\alpha} c^\alpha \binom{q}{\beta} a^\beta \frac{(nm+\widehat{\alpha})!}{i!} \qquad (0 \leq i \leq p). \tag{2.3}$$

We finish the proof of the claim using the following remark.

Remark 3. For each fixed $0 \leq \ell \leq p$, the following occurs:

(a) Each $(\alpha,\beta) \in A_\ell$ must satisfy $\alpha_0 = \cdots = \alpha_{\ell-1} = 0$. Otherwise, if $\alpha_s > 0$ with $0 \leq s \leq \ell - 1$ we'd have since $|\alpha| = m$ that $pm - (p-\ell) = \ell + (m-1)p \leq \widehat{\alpha} - \widehat{\beta} \leq \widehat{\alpha} \leq s\alpha_s + p(m-\alpha_s) = pm - (p-s)\alpha_s \leq pm - (p-s)$, a contradiction.

(b) If $(\alpha,\beta) \in A_\ell$ satisfies that $\alpha_\ell > 0$, then $\beta = (q, 0, \ldots, 0)$ and $\alpha_\ell = 1$, $\alpha_p = m-1$, and $\alpha_j = 0$ for $j \neq \ell, p$. This is forced from (a) and the inequalities $pm - (p-\ell) = \widehat{\alpha} - \widehat{\beta} \leq \widehat{\alpha} = \sum_{j=\ell}^p j\alpha_j \leq \ell\alpha_\ell + p(m-\alpha_\ell) = pm - (p-\ell)\alpha_\ell$.

(c) Let $A'_\ell := A_\ell \setminus \{((0, \ldots, 0, \underbrace{1}_{\ell^{th}}, 0, \ldots, 0, m-1), (q, 0, \ldots, 0))\}$. Then from (a) and (b) each $(\alpha,\beta) \in A'_\ell$ satisfies that $\alpha_0 = \cdots = \alpha_\ell = 0$.

(d) If $\beta \in \mathbb{N}_0^{k+1}$ satisfies $|\beta| = q$ and $\widehat{\beta} \in \{0, \ldots, \ell\}$, then $\binom{q}{\beta} \leq q^\ell$ and $|a^\beta| \leq (\max\{|a_0|, \ldots, |a_k|\})^\ell$.

Now, thanks to Remark 3 the system (2.3) is upper triangular and thus solvable, and any solution to (2.3) satisfies (ii) for sufficiently large n. To see (iii), it suffices to show that there exists $w > 1$ so that for each $\ell = 0, 1, \ldots, p$ we have

$$c_{p-\ell} = \mathrm{O}\left(\frac{w^n}{[(mn+mp)!]^{\frac{1}{m}}}\right) \qquad \text{as } n \to \infty. \tag{2.4}$$

Condition (2.4) ensures that $R_n \underset{n\to\infty}{\to} 0$ in $H(\Omega)$ as for each $M > 0$ we have $M^{n+i}|c_i| \underset{n\to\infty}{\to} 0$. Indeed, by (2.4) and Stirling's formula

$$\begin{aligned}\big(M^{n+i}|c_i|\big)^m &\leq \frac{M^{(n+i)m} w^{mn}}{(mn+mp)!} \\ &= o\left(\frac{(Mw)^{mn+mp}}{(\frac{mn+mp}{e})^{mn+mp}}\right) \\ &= O\left(\frac{eMw}{mn+mp}\right)^{mn+mp} \underset{n\to\infty}{\to} 0.\end{aligned}$$

So we finish by proving (2.4) by induction on ℓ. Taking $i = p$ in (2.3) we get -since in this case $(\alpha, \beta) \in A_p \Leftrightarrow \alpha = (0, \ldots, 0, m)$ and $\beta = (q, 0, \ldots, 0)$- that

$$\begin{aligned} p! b_p &= \sum_{(\alpha,\beta)\in A_p} \binom{m}{\alpha}\binom{q}{\beta} a^\alpha c^\beta (nm + \widehat{\alpha})! \\ &= \binom{m}{(0,\ldots,0,m)} \binom{q}{(q,0,\ldots,0)} \; a_0^q \; c_p^m (nm+mp)! \end{aligned}$$

Thus

$$c_p^m = \frac{p! b_p}{a_0^q \; (nm+mp)!} \tag{2.5}$$

and (2.4) holds for $\ell = 0$. Inductively, suppose there exists $w_{\ell-1} > 1$ so that

$$c_{p-j} = \mathrm{O}\left(\frac{w_{\ell-1}^n}{[(mn+mp)!]^{\frac{1}{m}}}\right) (n \to \infty)$$

for each $j = 0, \ldots, \ell - 1$. We want to show that for some $w > 1$

$$c_{p-\ell} = \mathrm{O}\left(\frac{w^n}{[(mn+mp)!]^{\frac{1}{m}}}\right) (n \to \infty). \tag{2.6}$$

Now, taking $i = p - \ell$ in (2.3) we have by Remark 3(b) and (c) that

$$\begin{aligned} (p-\ell)! \; b_{p-\ell} &= \sum_{(\alpha,\beta)\in A_{p-\ell}} \binom{m}{\alpha}\binom{q}{\beta} a^\beta c^\alpha (nm+\widehat{\alpha})! \\ &= m c_{p-\ell} c_p^{m-1} \; (nm+mp-\ell)! \; a_0^q + K_n, \end{aligned} \tag{2.7}$$

where $K_n = \sum_{(\alpha,\beta)\in A'_{p-\ell}} \binom{m}{\alpha}\binom{q}{\beta} a^\beta c^\alpha (nm+\widehat{\alpha})!$. Also, thanks to (2.5) we have that

$$\begin{aligned} c_p^{m-1} \; (nm+mp-\ell)! &= \frac{(p!b_p)^{1-\frac{1}{m}} \; [(nm+mp)!]^{\frac{1}{m}}}{(a_0^{1-\frac{1}{m}})^q \; \prod_{j=0}^{\ell-1}(nm+mp-j)} \\ &\sim \frac{[(nm+mp)!]^{\frac{1}{m}}}{(a_0^{1-\frac{1}{m}})^q \; n^\ell} \quad (n \to \infty). \end{aligned} \tag{2.8}$$

Now, let $(\alpha, \beta) \in A'_{p-\ell}$ be fixed. Notice that $(\widehat{\alpha}, \widehat{\beta}) \in \{(mp - j, j) : j = 0, \ldots, \ell\}$, and thus by Remark 3(d) that $\binom{q}{\beta} \leq q^\ell$ and $|a^\beta| \leq \|a\|_\infty^\ell$. Moreover, thanks to Remark 3(c) and our inductive hypothesis we also have

$$\begin{aligned} |c^\alpha| = |c_{p-\ell+1}^{\alpha_{p-\ell+1}} \cdots c_p^{\alpha_p}| &= \mathrm{O}\left(\left(\frac{w_{\ell-1}^n}{[(nm+mp)!]^{\frac{1}{m}}}\right)^{\alpha_{p-\ell+1}+\cdots+\alpha_p}\right) \\ &= \mathrm{O}\left(\frac{w_{\ell-1}^{nm}}{(nm+mp)!}\right). \end{aligned}$$

Hence

$$\left|\sum_{(\alpha,\beta)\in A'_{p-\ell}} \binom{m}{\alpha}\binom{q}{\beta} a^\beta c^\alpha (nm+\widehat{\alpha})!\right| = \mathrm{O}\left(n^\ell w_{\ell-1}^{mn}\right) \quad (n \to \infty). \tag{2.9}$$

So by (2.7), (2.8) and (2.9) we have

$$c_{p-\ell} = \mathrm{O}\left(\frac{n^{2\ell} w_{\ell-1}^{mn}}{[(nm+mp)!]^{\frac{1}{m}} a_0^{\frac{q}{m}}}\right) \quad (n \to \infty)$$

Thus any $w > w_{\ell-1}^{m} a_0^{-\frac{1}{r}}$ satisfies (2.6), and Claim 1 holds. The proof of Proposition 2 is now complete. $\square$

We conclude this note with the following natural questions:

Question 2. Can we eliminate the assumption $\Phi(0) = 0$ in Theorem 1?

It is also natural to ask for examples of algebras generated by bigger sets. In fact, the simplest case of the existence of algebras generated by two functions is still open.

Question 3. (Seoane-Sepúlveda) Does there exist a pair of algebraically independent hypercyclic functions $f, g \in \mathcal{H}(\mathbb{C})$ that altogether generate an algebra consisting entirely (but the origin) of hypercyclic vectors for D on $\mathcal{H}(\mathbb{C})$?

Finally, we may also ask about the existence of algebras of frequently hypercyclic vectors (respectively, of upper frequently hypercyclic vectors) for convolution operators on $\mathcal{H}(\mathbb{C})$. We note that every convolution operator $\Phi(D)$ that is not a scalar multiple of the identity is frequently hypercyclic [11]. Indeed, $\Phi(D)$ has a frequently hypercyclic subspace whenever $\Phi \in \mathcal{H}(\mathbb{C})$ is transcendental [12], but when Φ is a non-constant polynomial the operator $\Phi(D)$ has an upper frequently hypercycic subspace but has no frequently hypercyclic subspace [6, 9].

Question 4. Which convolution operators support an algebra of frequently hypercyclic (respectively, of upper frequently hypercyclic) vectors?

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