APPLIED GENERAL TOPOLOGY

Appl. Gen. Topol. 19, no. 2 (2018), 223-237
doi:10.4995/agt.2018.7955

Completely simple endomorphism rings of modules

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Communicated by F. Lin

Abstract

It is proved that if A_p is a countable elementary abelian p-group, then: (i) The ring $\mathrm{End}\,(A_p)$ does not admit a nondiscrete locally compact ring topology. (ii) Under (CH) the simple ring $\mathrm{End}\,(A_p)/I$, where I is the ideal of $\mathrm{End}\,(A_p)$ consisting of all endomorphisms with finite images, does not admit a nondiscrete locally compact ring topology. (iii) The finite topology on $\mathrm{End}\,(A_p)$ is the only second metrizable ring topology on it. Moreover, a characterization of completely simple endomorphism rings of modules over commutative rings is obtained.

2010 MSC: *16W80; 16N20; 16S50; 16N40.*

Keywords: *topological ring; endomorphism ring; Bohr topology; finite topology; locally compact ring.*

1. Introduction

The notion of associative simple ring can be extended for associative topological rings in several ways:

(i) simple abstract ring endowed with a nondiscrete ring topology (for instance, the classification of nondiscrete locally compact division rings, see [25, Chapter IV] and [4, 15, 16]; we refer to some historical notes about locally compact division rings to [29]);

Received 24 August 2017 – Accepted 09 July 2018

(ii) topological ring without nontrivial closed ideals (see [22, 31]).
(iii) topological ring R with the property that if $f : R \to S$ is a continuous homomorphism in a topological ring S, then either $f = 0$ or f is a topological embedding of R into S (see [24]).

In all cases it is assumed that multiplication is not trivial.

I. Kaplansky has mentioned (see [20], p. 56) that the classification of locally compact simple rings in positive characteristic p is difficult. He proved that every simple nondiscrete locally compact simple torsion-free ring is a matrix ring over a locally compact division ring. However in [26] (see also [30]) has been constructed a nondiscrete locally compact simple ring of positive characteristic which is not a matrix ring over a division ring. Thereby the program of classification of nondiscrete locally compact simple rings was finished. Nevertheless it is interesting to look for new examples of locally compact simple rings.

If A_p is a countable elementary abelian p-group and I is the ideal of the ring $\mathrm{End}\,(A_p)$ consisting of endomorphisms with finite images, then the factor ring $\mathrm{End}\,(A_p)/I$ is a simple von Neumann regular ring. We prove that under (CH) this ring does not admit a nondiscrete locally compact ring topology.

S. Ulam (see [23, Problem 96, p. 181]) posed the following problem: "*Can the group S_∞ of all permutations of integers so metrized that the group operation (composition of permutations) is a continuous function and the set S_∞ becomes, under this metric, a compact space? (locally compact?)*". E.D. Gaughan (see [10]) has solved this problem in the negative.

We study in §3 an analogous problem for the endomorphism ring of a countable elementary abelian p-group, namely: "*Does the endomorphism ring $\mathrm{End}\,(A_p)$ of a countable elementary p-group A_p admit a nondiscrete locally compact ring topology?*". Similarly to the Ulam's problem we obtain a negative answer to this question. Moreover, we prove that $\mathcal{T}_{fin}$ is the only ring topology $\mathcal{T}$ on $\mathrm{End}\,(A_p)$ such that $(\mathrm{End}\,(A_p), \mathcal{T})$ is complete and second metrizable.

We classify in §4 the completely simple rings $(\mathrm{End}\,(M), \mathcal{T}_{fin})$ of vector spaces M over division rings. Corollary 4.4 gives a characterization of semisimple left linearly compact minimal rings. It should be mentioned that Corollary 4.4 is related to a result from [3] stating that any semisimple ring admits at most one linearly compact topology.

Furthermore, we obtain in §5 a description of completely simple rings of the form $(\mathrm{End}\,(M_R), \mathcal{T}_{fin})$ of modules M over a commutative ring R. We extend the result of [28] to topological rings $(\mathrm{End}\,(M_R), \mathcal{T}_{fin})$.

2. Notation, Conventions and Preliminary Results

Rings are assumed to be associative, not necessarily with identity. Topological spaces are assumed to be completely regular. The *weight* (see [8], p.12) of the space X is denoted by $w(X)$. The *pseudocharacter* of a point $x \in X$ (see [8], p.135) is the smallest cardinal of the form $|\mathcal{U}|$, where $\mathcal{U}$ is a family of open subsets of X such that $\cap\mathcal{U} = \{x\}$. The closure of a subset A of the topological space X is denoted by $\overline{A}$ and the interior by $\mathrm{Int}(A)$ (see [8], p.14). A topological space X is called a *Baire space* (see [8], p.198) if for each sequence $\{X_1, X_2, \ldots\}$ of open dense subsets of X the intersection $\cap_{i=1}^{\infty} G_i$ is a dense set.

An abelian group A is called *elementary abelian p-group* (p prime) if $pa = 0$ for all $a \in A$. Such group is a direct sum of copies of the cyclic group $Z(p)$. The subring of a ring R generated by a subset S, is denoted by $\langle S\rangle$. A ring R is called *locally finite* if every its finite subset is contained in a finite subring. A topological ring $(R, \mathcal{T})$ is called *metrizable* if its underlying additive group satisfies the first axiom of countability. A ring R with 1 is called *Dedekind-finite* if each equality $xy = 1$ implies $yx = 1$. It is well-known that every finite ring with identity is Dedekind-finite. Since every compact ring with identity is a subdirect product of finite rings, it follows that every compact ring with identity is Dedekind-finite. If $A \subseteq R$, then $\mathrm{Ann}_l(A) := \{x \in R \mid xA = 0\}$. If X, Y are the subsets of R, then $X \cdot Y := \{xy \mid x \in X, y \in Y\}$. A topological ring R is called *compactly generated* (see [27, Chapter II]) if there exists a compact subset K such that $R = \langle K\rangle$. If $(R, \mathcal{T})$ is a topological ring and I is an ideal of R, then the quotient topology of the factor ring R/I is denoted by $\mathcal{T}/I$. If K is a subset of an abelian group A, then set

$$T(K) = \{\alpha \in \mathrm{End}\,(A) \mid \alpha(K) = 0\}.$$

When K runs over all finite subsets of A, the family $\{T(K)\}$ defines a ring topology $\mathcal{T}_{fin}$ on $\mathrm{End}\,(A)$. This topology is called the *finite topology.*

Lemma 2.1. *For any abelian group A the ring $(\mathrm{End}\,(A), \mathcal{T}_{fin})$ is complete.*

Proof. See [27, Theorem 19.2]. □

Lemma 2.2 (Cauchy's criterion)**.** *In a Hausdorff complete commutative group G, in order that a family $(x_\alpha)_{\alpha\in\Omega}$ should be summable it is necessary and sufficient that, for each neighborhood V of zero in G, there is a finite subset Ω_0 of Ω such that $\Sigma_{\alpha\in K} x_\alpha \in V$ for all finite subsets K of Ω which do not meet Ω.*

Proof. See [5], p.263. □

Lemma 2.3. *If $(x_\alpha)_{\alpha\in\Omega}$ is a summable subset in $(\mathrm{End}\,(A), \mathcal{T}_{fin})$ then every subset Δ of Ω the family $(x_\beta)_{\beta\in\Delta}$ is summable.*

Proof. Let V be a neighborhood of zero of $(\mathrm{End}\,(A), \mathcal{T}_{fin})$. We can consider without loss of generality that V is a left ideal of $\mathrm{End}\,(A)$. There exists a finite subset Ω_0 of Ω such that $\Sigma_{\alpha\in K} x_\alpha \in V$ for every finite subset K of Ω for which $K \cap \Omega_0 = \varnothing$. Let F be a finite subset of Δ such that $F \cap (\Omega_0 \cap \Delta) = \varnothing$. If $\alpha \in F$, then $\alpha \notin \Omega_0$, hence $\Sigma_{\alpha\in F} x_\alpha \in V$. By Cauchy's criterion the family $(x_\beta)_{\beta\in\Delta}$ is summable. □

A topological ring $(R, \mathcal{T})$ is called *minimal* (see, for instance, [7]) if there is no ring topology $\mathcal{U}$ such that $\mathcal{U} \leq \mathcal{T}$ and $\mathcal{U} \neq \mathcal{T}$. A topological ring $(R, \mathcal{T})$ is called *simple* if R is simple as a ring without topology. A topological ring $(R, \mathcal{T})$ is called *weakly simple* if $R^2 \neq 0$ and every its closed ideal is either 0 or R. A topological ring $(R, \mathcal{T})$ is called *completely simple* (see [24]) if $R^2 \neq 0$ and for every continuous homomorphism $f : (R, \mathcal{T}) \to (S, \mathcal{U})$ in a topological ring $(S, \mathcal{U})$ either $\ker(f) = R$ or f is a homeomorphism of $(R, \mathcal{T})$ on $\mathrm{Im}(f)$. Equivalently, $R^2 \neq 0$ and $(R, \mathcal{T})$ is weakly simple and minimal. Let M be a unitary right R-module over a commutative ring R with 1. The module M is called *divisible* if $Mr = M$ for every $0 \neq r \in R$. A right R-module M is called *faithful* if $Mr = 0$ implies $r = 0$ $(r \in R)$. A right R-module M is called *torsion-free* if $mr = 0$ implies that either $m = 0$ or $r = 0$, where $m \in M$ and $r \in R$. Recall that a submodule N of an R-module M is called *fully invariant* $\alpha(N) \subseteq N$ for every endomorphism α of M_R. We use in the sequel the notion and results from the books [8, 27].

Remark 2.4. If R is a von Neumann regular ring, then $R^2 = R$.

Lemma 2.5. *An ideal I of a von Neumann regular ring is von Neumann regular.*

Proof. Let $i \in I$. Thus there exists $x \in R$ such that $ixi = i$. It follows that $ixixi = i$ and $xix \in I$. □

Corollary 2.6. *If I an ideal of a von Neumann regular ring R, then any ideal H of I is an ideal of R, too.*

Proof. $RH = RH^2 \subseteq IH \subseteq H$. Similarly, $HR \subseteq H$. □

If A_p is a p-elementary countable group, then

$$I = \{\alpha \in \mathrm{End}\,(A_p) \mid |\mathrm{Im}(\alpha)| < \aleph_0\}.$$

Fix a linear basis $\{v_i \mid i \in \mathbb{N}\}$ of A_p over the field $\mathbb{F}_p$. Using this fixed basis, we define the map $e_i : A \to A$ such that

$$e_i(v_j) = \delta_{ij} v_j, \qquad (i, j \in \mathbb{N})$$

where δ_{ij} is the Kronecker delta.

Lemma 2.7. *We have for* $\mathrm{End}\,(A_p)$*:*

(i) *I is a von Neumann regular ring.*
(ii) *I is a simple ring.*
(iii) *The factor ring $\mathrm{End}(A_p)/I$ is simple von Neumann regular.*
(iv) *I is a locally finite ring.*

Proof. (i): The ring $\mathrm{End}\,(A_p)$ is regular (see [21, Theorem 4.27, p. 63]), so I is von Neumann regular by Lemma 2.5.

(ii), (iii): The ideal I is the only nontrivial ideal of the ring $\mathrm{End}\,(A_p)$ (see [17, §17, Theorem 1, p. 93]). This means that $\mathrm{End}\,(A)/I$ is simple. It is regular by the part (i).

(iv) Since I is simple (see [17, §12, Proposition 1]), it suffices to show that I contains a nonzero locally finite right ideal.

Let us show that the left ideal Ie_1 of I is locally finite as a ring (equivalently, as a $\mathbb{F}_p$-algebra). We have $0 \neq e_1 \in Ie_1$. If H is the left annihilator of Ie_1, then, obviously, H is a locally finite ring, hence it is locally finite as a $\mathbb{F}_p$-algebra. We claim that Ie_1/H is finite. Define $\beta_n \in H$ $(n \geq 2)$ in the following way

$$\beta_n(v_i) = \begin{cases} v_n, & \text{for} \quad i = 1; \\ 0, & \text{for} \quad i \neq 1. \end{cases}$$

Let us prove that $Ie_1 = \mathbb{F}_p e_1 + \Sigma_{n=2}^{\infty} \mathbb{F}_p \beta_n$.

If $\alpha \in I$, then $\alpha(v_1) = r_1 v_1 + \cdots + r_n v_n$, where $r_i \in \mathbb{F}_p$ and $n \in \mathbb{N}$, so

$$\begin{aligned} \alpha e_1(v_1) &= r_1 e_1(v_1) + r_2 \beta_2(v_1) + \cdots + r_n \beta_n(v_1) \\ &= (r_1 e_1 + r_2 \beta_2 + \cdots + r_n \beta_n)(v_1); \\ \alpha e_1(v_j) &= (r_1 e_1 + r_2 \beta_2 + \cdots + r_n \beta_n)(v_j) \qquad (j \neq 1). \end{aligned}$$

This yields

$$\alpha e_1 = r_1 e_1 + r_2 \beta_2 + \cdots + r_n \beta_n$$

and so $Ie_1 = \mathbb{F}_p e_1 + \Sigma_{n=2}^{\infty} \mathbb{F}_p \beta_n$.

In particular, $Ie_1 = \mathbb{F}_p e_1 + H$, and so H has a finite index in Ie_1. Clearly, Ie_1 is a locally finite $\mathbb{F}_p$-algebra (see [17, Proposition 1, p. 241]) and I is a locally finite $\mathbb{F}_p$-algebra (see [17, Proposition 2, p. 242]). □

The next result can be deduced from [27, Lemma 36.11].

Lemma 2.8. *Let A be a locally compact, compactly generated, and totally disconnected ring. If A contains a dense locally finite subring B, then A is compact.*

Proof. Let $A = \langle V \rangle$, where V is a compact symmetric neighborhood of zero. Since V is compact, the subset $V + V + V \cdot V$ also is compact. Since B is dense, $A = B + V$. By compactness of $V + V + V \cdot V$ there exists a finite subset $H \subseteq B$ such that $V + V + V \cdot V \subseteq H + V$. Since B is a locally finite ring, we can assume without loss of generality that H is a subring. Let $H \setminus \{0\} = \{h_1, \ldots, h_k\}$. The subset

$$H + h_1 V + \cdots + h_k V + V$$

is an open subgroup of $R(+)$. Indeed, this subset is symmetric and

$$\begin{aligned} (H + h_1 V + \cdots + h_k V + V) &+ (H + h_1 V + \cdots + h_k V + V) \\ &\subseteq H + h_1(V + V) + \cdots + h_k(V + V) + V + V \\ &\subseteq H + h_1 V + \cdots + h_k V + V. \end{aligned}$$

We prove by induction on m that

$$V^{[m]} \subseteq H + h_1 V + \cdots + h_k V + V, \qquad (m \in \mathbb{N})$$

where $V^{[1]} = V$ and $V^{[m]} = V^{[m-1]} \cdot V$ for all m.

The inclusion is obvious for $m = 1$.

Assume that the assertion has been proved for $m \geq 1$. Clearly,

$$\begin{aligned} V^{[m+1]} = V^{[m]} \cdot V \subseteq H \cdot V + h_1(V \cdot V) + \cdots + h_k(V \cdot V) + V \cdot V \subseteq \\ h_1 V + \cdots + h_k V + h_1(H + V) + \cdots + h_k(H + V) + H + V \subseteq \\ H + h_1 V + \cdots + h_k V + V. \end{aligned}$$

Consequently, $A = H + h_1V + \cdots + h_kV + V$, therefore A is compact. $\square$

An element x of a topological ring is called *discrete* if there exists a neighborhood V of zero such that $xV = 0$ (i.e., the right annihilator of x is open).

Lemma 2.9. *The set of all discrete elements of a topological ring is an ideal. A simple ring with identity does not contain nonzero discrete elements.*

3. Locally compact ring topologies on End (A) of a countable elementary abelian p-group A

Theorem 3.1. *Let R be a simple, nondiscrete and locally compact ring of* $\mathrm{char}(R) = p > 0$ *and* $1 \in R$. *If V is a compact open subring of R and* $\{e_\alpha \mid \alpha \in \Omega\}$ *is a set of orthogonal idempotents in R, then*

$$|\Omega| \leq w(V).$$

Proof. The ring R does not contain nonzero discrete elements by Lemma 2.9. Since R is locally compact and $\mathrm{char}(R) = p$, it is totally disconnected. Additionally, R has a fundamental system of neighborhoods of zero consisting of compact open subrings by [19, Lemma 9].

If V is a compact open subring of R, then by continuity of the ring operations for each $\alpha \in \Omega$ there exists an open ideal V_α of V such that $e_\alpha V_\alpha \subseteq V$. Clearly, there exists $y_\alpha \in V_\alpha$ for which $e_\alpha y_\alpha \neq 0$ since R has no nonzero discrete elements.
We claim that hold the following two properties:

(i) $e_\alpha y_\alpha \notin \overline{\{e_\beta y_\beta \mid \beta \neq \alpha\}}$ for each $\alpha \in \Omega$;

(ii) the set $X = \{e_\alpha y_\alpha \mid \alpha \in \Omega\}$ is a discrete subspace of V.

Indeed, if were $e_\alpha y_\alpha \in \overline{\{e_\beta y_\beta \mid \beta \neq \alpha\}}$ for some $\alpha \in \Omega$, then

$$\begin{aligned} e_\alpha y_\alpha = e_\alpha e_\alpha y_\alpha &\in e_\alpha \overline{\{e_\beta y_\beta \mid \beta \neq \alpha\}} \\ &\subseteq \overline{\{e_\alpha e_\beta y_\beta \mid \beta \neq \alpha\}} \\ &= \{0\}, \end{aligned}$$

so $e_\alpha y_\alpha = 0$, a contradiction. The part (i) is proved.

(ii) Now, for each $\alpha \in \Omega$ we have $V \setminus \overline{\{e_\beta y_\beta \mid \beta \neq \alpha\}}$ is open and, consequently,

$$(V \setminus \overline{\{e_\beta y_\beta \mid \beta \neq \alpha\}}) \cap X = \{e_\alpha y_\alpha\},$$

by (i). Therefore the point $e_\alpha y_\alpha (\alpha \in \Omega)$ of X is isolated. In other words, the subspace X of V is discrete.

Since X is discrete, $|\Omega| = |X| = w(X) \leq w(V)$ (see [1, Exercises 98-99, p. 72]). $\square$

Theorem 3.2. *Let A_p be a countable elementary abelian p-group. Then the ring*

$$I = \{\alpha \in \mathrm{End}\,(A_p) \mid |\mathrm{Im}(\alpha)| < \aleph_0\}$$

does not admit a nondiscrete ring topology $\mathcal{U}$ such that $(I, \mathcal{U})$ is a Baire space.

Proof. Put $S_n = \{\alpha \in I \mid \alpha(A) \subseteq \mathbb{F}_p v_1 + \cdots + \mathbb{F}_p v_n\}$, where $n \in \mathbb{N}$. Clearly, $I = \cup_{n\in\mathbb{N}} S_n$ and

$$S_n = \{\alpha \in I \mid e_i\alpha = 0 \quad \text{for} \quad i > n\} = \mathrm{Ann}_r\big(\{e_k \mid k > n\}\big).$$

This yields that the subset S_n is closed due the continuity of the ring operations.

Since I is a Baire space, there exists $n \in \mathbb{N}$ such that $\mathrm{Int}(S_n) \neq \varnothing$, hence S_n is an open subgroup.

Set $\beta \in I$ such that

$$\beta(v_i) = \begin{cases} v_{n+i}, & \text{for} \quad i = 1, \ldots, n; \\ 0 \quad , & \text{for} \quad i > n. \end{cases}$$

Let $W \subseteq S_n$ be a neighborhood of zero of $(I, \mathcal{U})$ such that $\beta W \subseteq S_n$. If $w \in W \setminus \{0\}$, then there exist $a \in A$ and $r_1, \ldots, r_n \in \mathbb{F}_p$ such that

$$0 \neq w(a) = \sum_{i=1}^{n} r_i v_i \qquad \text{and} \qquad \beta(w(a)) = \sum_{i=1}^{n} r_i v_{n+i}.$$

There exists $j \in 1, \ldots, n$ such that $r_j \neq 0$. Then

$$e_{n+j}\beta w(a) = r_j v_{n+j} \neq 0,$$

hence $e_{n+j}\beta w \neq 0$ and so $\beta w \notin S_n$, a contradiction. □

Corollary 3.3. *Under the notation of Theorem 3.2 the ring I does not admit a nondiscrete locally compact ring topology.*

Proof. This follows from the fact that each locally compact space is a Baire space (see [6, Theorem 1, p. 117]). □

Our main result is the following.

Theorem 3.4. *The endomorphism ring* $\mathrm{End}\,(A_p)$ *of a countable elementary abelian p-group A_p does not admit a nondiscrete locally compact ring topology.*

Proof. We use the notation and results from section 2. Denote $R = \mathrm{End}\,(A_p)$. Assume on the contrary that there exists on R a nondiscrete locally compact ring topology $\mathcal{T}$.

Fact 1. The ring $(R, \mathcal{T})$ has a fundamental system of neighborhoods of zero consisting of compact open subrings.

Since the additive group of the ring R has exponent p, it is totally disconnected (this follows from [12, Theorem 9.14, p. 95]). By I. Kaplansky's result (see [19, Lemma 9]), the ring $(R, \mathcal{T})$ has a fundamental system of neighborhoods of zero consisting of compact open subrings.

Fact 2. The group Re_i is countable for each $i \in \mathbb{N}$.

We claim that Re_i is infinite. Indeed, for each $j \in \mathbb{N}$ put $\beta_j \in R$ such that

$$\beta_j(v_k) = \begin{cases} v_j, & \text{for} \quad k = i; \\ 0, & \text{for} \quad k \neq i. \end{cases}$$

If $j \neq s$, then $\beta_j e_i(v_i) = \beta_j(v_i) = v_j$ and $\beta_s e_i(v_i) = \beta_s(v_i) = v_s$, hence $\beta_j e_i \neq \beta_s e_i$, so Re_i is infinite.

The ring Re_i is countable. Indeed, consider the mapping $\psi : Re_i \to A_p^{\mathbb{F}_p v_i}$, where

$$\psi(\alpha e_i)(rv_i) = \alpha(rv_i) \qquad \text{for all} \qquad r \in \mathbb{F}_p.$$

If $\alpha e_i \neq \beta e_i$ $(\alpha, \beta \in R)$, then there exists an element $x = \sum_j r_j v_j \in A_p$ such that $\alpha e_i(x) \neq \beta e_i(x)$, hence, $\alpha(r_i v_i) \neq \beta(r_i v_i)$. Thus

$$\psi(\alpha e_i)(r_i v_i) = \alpha(r_i v_i) \neq \beta(r_i v_i) = \psi(\beta e_i)(r_i v_i).$$

The latter means that ψ is an injective mapping of Re_i into $A^{\mathbb{F}_p v_i}$. Since $A^{\mathbb{F}_p v_i}$ is countable, Re_i is countable, too.
Fact 3. I is a closed ideal of R. We claim that I is not dense in the topological ring $(R, \mathcal{T})$. Assume the contrary. Since I is locally finite and is a maximal ideal, $(R, \mathcal{T})$ is topologically locally finite by Lemma 2.8. The ring R contains two elements x, y such that $xy = 1$ and $yx \neq 1$. The subring $\overline{\langle x, y\rangle}$ is compact, hence Dedekind-finite, a contradiction. We obtained that $(R/I, \mathcal{T}/I)$ is a nondiscrete metrizable locally compact ring.
Fact 4. I is a discrete ideal of R.
This follows from Theorem 3.2.
Fact 5. Re_i is a discrete left ideal of R for every $i \in \mathbb{N}$.
Indeed, $Re_i \subseteq I$ and I is discrete by Fact 4 for every $i \in \mathbb{N}$.
Fact 6. $\mathrm{Ann}_l(e_i)$ is open in R for every $i \in \mathbb{N}$.
Indeed, the group homomorphism $q : R \to Re_i, r \mapsto re_i$, is continuous. Since Re_i is discrete $q^{-1}(0) = \mathrm{Ann}_l(e_i)$ is open.
Fact 7. $\cap_i \mathrm{Ann}_l(e_i) = 0$.
Obvious.
Fact 8. $\mathcal{T} \geq \mathcal{T}_{fin}$.
We notice that $\mathrm{Ann}_l(e_i) = T(\{v_i\})$ for every $i \in \mathbb{N}$. For, if $\alpha e_i = 0$, then $\alpha(v_i) = \alpha e_i(v_i) = 0$, i.e., $\alpha \in T(\{v_i\})$. Conversely, if $\alpha \in T(\{v_i\})$, then $\alpha e_i(v_i) = \alpha(v_i) = 0$. If $j \neq i$ then $\alpha e_i(v_j) = 0$. Therefore $\alpha e_i = 0$. Moreover

$$T(\{v_1, \dots, v_n\}) = \cap_{i=1}^n T(\{v_i\}) = \cap_{i=1}^n \mathrm{Ann}_l(e_i) \in \mathcal{T} \qquad (\forall n \in \mathbb{N}).$$

Since the family $\{T(\{v_1, \dots, v_n\})\}$ forms a fundamental system of neighborhoods of zero of $(R, \mathcal{T}_{fin})$, we get that $\mathcal{T}_{fin} \leq \mathcal{T}$.
Fact 9. The ring $(R, \mathcal{T})$ is metrizable.
Since $\cap_{i\in\mathbb{N}} \mathrm{Ann}_l(e_i) = 0$, the pseudocharacter of $(R, \mathcal{T})$ is $\aleph_0$. If V is a compact open subring of $(R, \mathcal{T})$ (see Fact 1), then the pseudocharacter of V also is $\aleph_0$. However in every compact space the pseudocharacter of a point coincides with its character. Therefore $(R, \mathcal{T})$ is metrizable.
Fact 10. $(R/I, \mathcal{T}/I)$ has an open compact subring.
Indeed, it is well-known (see [19]) that every totally disconnected ring has a fundamental system of neighborhood of zero consisting of compact open subrings. Henceforth V is a fixed open compact subring of $(R/I, \mathcal{T}/I)$.
Fact 11. R/I contains a family of orthogonal idempotents of cardinality $2^{\aleph_0}$.
Indeed, the family $\{e_i\}_{i\in\mathbb{N}}$ of idempotents of the ring $(R, \mathcal{T}_{fin})$ is summable and $1_A = \Sigma_{n\in\mathbb{N}} e_n$, where 1_A is the identity of R.

The first ordinal number of cardinality $\mathfrak{c}$ of continuum is denoted by $\omega(\mathfrak{c})$. Let $\{\mathbb{N}(\alpha) \mid \alpha < \omega(\mathfrak{c})\}$ be a family of infinite almost disjoint subsets of $\mathbb{N}$ (see [8, Example 3.6.18, p. 175–176]). Put $f_{\mathbb{N}(\alpha)} = \Sigma_{i\in\mathbb{N}(\alpha)} e_i$ for each $\alpha < \omega(\mathfrak{c})$. The element $f_{\mathbb{N}(\alpha)}$ exists by Lemma 2.3. Then:

(i) $f_{\mathbb{N}(\alpha)} \notin I$ for every $\alpha < \omega(\mathfrak{c})$;

(ii) $f_{\mathbb{N}(\alpha)} f_{\mathbb{N}(\beta)} \in I$ for each $\alpha, \beta < \omega(\mathfrak{c})$ and $\alpha \neq \beta$.

If $g_\alpha = f_{\mathbb{N}(\alpha)} + I$ for each $\alpha < \omega(\mathfrak{c})$, then $\{g_\alpha \mid \alpha < \omega(\mathfrak{c})\}$ is the required system of orthogonal idempotents.

The subring V is metrizable (by Fact 9). Since V is compact and R/I is a simple von Neumann regular ring by Lemma 2.7 and $w(V) \leq \aleph_0$, we obtain a contradiction to Theorem 3.1. □

Theorem 3.5. *(CH) Under the notation of Theorem 3.4, the ring R/I does not admit a nondiscrete locally compact ring topology.*

Proof. Assume on the contrary that the factor ring R/I admits a nondiscrete locally compact ring topology $\mathcal{T}$, so $(R/I, \mathcal{T})$ contains an open compact subring V. Since the cardinality of R/I is continuum and V is infinite, the power of V is continuum. Since we have assumed (CH), the subring V is metrizable, hence second metrizable (see [14, 18]). However we have proved in Theorem 3.4 that the ring R/I contains a family of orthogonal idempotents of cardinality $\mathfrak{c}$, a contradiction with Theorem 3.1. □

Theorem 3.6. *The finite topology $\mathcal{T}_{fin}$ is the only second metrizable ring topology $\mathcal{T}$ on R for which $(R, \mathcal{T}_{fin})$ is complete.*

Proof. Let $K = \langle F\rangle$, where F is a finite subset of A. Clearly, there exists a subgroup A' of A such that $A = K \oplus A'$. Choose $\mathrm{e}_F \in R$ such that $\mathrm{e}_F \restriction_K = \mathrm{id}_K$ and $\mathrm{e}_F(A') = 0$. Clearly,

$$T(K) = R(1 - e_F)$$

and $\alpha K = 0$ if and only if $\alpha \in R(1 - e_F)$, so the family $\{R(1 - e_F)\}$, where F runs over all finite subset of A, forms a fundamental system of neighborhoods of zero for $(R, \mathcal{T}_{fin})$.

There exists an injective map of Re_F to $\mathrm{Hom}(K, A)$, so the left ideal Re_F is countable, due to countability $\mathrm{Hom}(K, A)$. Since $e_F^2 = e_F$, the Peirce decomposition

$$R = Re_F \oplus R(1 - e_F)$$

of R with respect to the idempotent e_F is a decomposition of the topological group $(R, +, \mathcal{T})$. It follows that Re_F is discrete, hence $R(1-e_F)$ is open (in the topology $\mathcal{T}$). Hence $\mathcal{T} \geq \mathcal{T}_{fin}$, so $\mathcal{T} = \mathcal{T}_{fin}$ (see [9, Theorem 30] or [11]). □

4. Completely simple topological endomorphism rings of vector spaces

Theorem 4.1. *Let A_F be a right vector space over a division ring F and $S = \mathrm{End}\,(A_F)$. The following conditions are equivalent:*

(i) *$(S, \mathcal{T}_{fin})$ is a completely simple topological ring.*

(ii) $\dim(A_F) = \infty$ *or* $\dim(A_F) < \infty$ *and* F *does not admit a nondiscrete ring topology.*

Proof. (i)⇒ (ii): If A_F is finite-dimensional, then S is discrete and isomorphic to the matrix ring $\mathrm{M(n, F)}$, where n is the dimension of A_F. Then, obviously, F does not admit a nondiscrete ring topology.

(ii)⇒ (i): If $\dim(A_F) = n < \infty$, then $S \cong \mathrm{M(n, F)}$. Since F does not admit nondiscrete ring topologies, the same holds for $\mathrm{M(n, F)}$.

Let A_F be infinite dimensional. Fix a basis $\{x_\alpha\}_{\alpha<\tau}$ over F, where τ is an infinite ordinal number. It is well-known that the topological ring $(S, \mathcal{T}_{fin})$ is weakly simple (see [22, Satz 12, p. 258]) and the family $\{T(x_\alpha)\}_{\alpha<\tau}$ is a prebase at zero for the finite topology $\mathcal{T}_{fin}$ of S.

Assume on the contrary that there exists a Hausdorff ring topology $\mathcal{T}$, coarser that $\mathcal{T}_{fin}$ and different from it. Let $e_\alpha \in S$ such that $e_\alpha^2 = e_\alpha$ and $e_\alpha(x_\beta) = \delta_{\alpha\beta}x_\alpha$ for each $\alpha < \tau$, where $\delta_{\alpha\beta}$ is the Kronecker delta.

Fact 1. $T(x_\alpha) = \mathrm{Ann}_l(e_\alpha)$ for each $\alpha < \tau$.

Indeed, if $p \in T(x_\alpha)$, then $pe_\alpha(x_\alpha) = p(x_\alpha) = 0$. If $\beta \neq \alpha$, then $e_\alpha(x_\beta) = 0$, hence $pe_\alpha = 0$, i.e. $p \in \mathrm{Ann}_l(e_\alpha)$. Conversely, if $pe_\alpha = 0$, then we have $p(x_\alpha) = pe_\alpha(x_\alpha) = 0$, i.e. $p \in T(x_\alpha)$.

Fact 2. There exists $\alpha_0 < \tau$ for which Se_{α_0} is nondiscrete in $(S, \mathcal{T})$.

Assume on the contrary that for every $\alpha < \tau$ there exists a neighborhood V_α of zero of $(S, \mathcal{T})$ such that $Se_\alpha \cap V_\alpha = 0$. If U_α is a neighborhood of zero of $(S, \mathcal{T})$ such that $U_\alpha e_\alpha \subseteq V_\alpha$, then $U_\alpha e_\alpha = 0$, hence $\mathrm{Ann}_l(e_\alpha) = T(x_\alpha)$ is open in $(S, \mathcal{T})$. Hence $\mathcal{T}_{fin} \leq \mathcal{T}$ and $\mathcal{T} = \mathcal{T}_{fin}$, a contradiction.

Fact 3. $(Se_{\alpha_0} \cap V)x_{\alpha_0} \nsubseteq \oplus_{\beta\in K} x_\beta F$

for any neighborhood V of of zero of $(S, \mathcal{T})$ and any finite subset K of the set $[0, \tau)$ of all ordinal numbers less than τ.

Assume on the contrary that there exists a finite subset K of $[0, \tau)$ and a neighborhood V of zero of $(S, \mathcal{T})$ such that

$$(Se_{\alpha_0} \cap V)x_{\alpha_0} \subseteq \oplus_{\beta\in K} x_\beta F. \tag{4.1}$$

Fix $\gamma \in [0, \tau) \setminus K$. For each $\beta \in K$ define $q_\beta \in S$ such that $q_\beta(x_\beta) = x_\gamma$ and $q(x_\delta) = 0$ for $\delta \neq \beta$.

Let V_0 be a neighborhood of zero of $(S, \mathcal{T})$ such that $V_0 \subseteq V$ and $q_\beta V_0 \subseteq V$ for all $\beta \in K$. There exists $0 \neq h \in Se_{\alpha_0} \cap V_0$ by Fact 2 and $hx_{\alpha_0} \neq 0$ by Fact 1. Since $Se_{\alpha_0} \cap V_0 \subseteq Se_{\alpha_0} \cap V$, we obtain that $hx_{\alpha_0} = \Sigma_{\beta\in K} x_\beta f_\beta$, $(f_\beta \in F)$ by (4.1). There exists $\beta_0 \in K$ such that $f_{\beta_0} \neq 0$ (because $hx_{\alpha_0} \neq 0$), so

$$q_{\beta_0}h = q_{\beta_0}(\Sigma_{\beta\in K} x_\beta f_\beta) = r_{\beta_0}x_\gamma \notin \oplus_{\beta\in K} x_\beta F,$$

a contradiction. Therefore Fact 3 is proved.

Now let V be a neighborhood of zero of $(S, \mathcal{T})$. Pick up a neighborhood V_0 of zero of $(S, \mathcal{T})$ such that $V_0 \cdot V_0 \subseteq V$. Since $\mathcal{T} \leq \mathcal{T}_{fin}$, there exists a finite subset K of $[0, \tau)$ such that

$$T(\{x_\beta \mid \beta \in K\}) \subseteq V_0.$$

We have $(Se_{\alpha_0} \cap V_0)x_{\alpha_0} \nsubseteq \oplus_{\beta \in K} x_\beta F$ by Fact 3. It follows that there exists $q \in Se_{\alpha_0} \cap V_0$ such that

$$q(x_{\alpha_0}) \notin \oplus_{\beta \in K} x_\beta F.$$

Clearly, $q(x_{\alpha_0}) \in A_F$, so it can be written as $q(x_{\alpha_0}) = \sum_{\alpha<\tau} x_\alpha f_\alpha$, where $f_\alpha \in F$ and there exists $\beta_0 \notin K$ such that $f_{\beta_0} \neq 0$.

Consider the element $s \in S$ such that $s(x_{\beta_0}) = x_{\alpha_0} f_{\beta_0}^{-1}$ and $s(x_\lambda) = 0$ for $\lambda \neq \beta_0$. Evidently, $s \in T(K)$, hence

$$sq \in T(K) \cdot V_0 \subseteq V_0 \cdot V_0 \subseteq V.$$

Moreover, $sq(x_{\alpha_0}) = s(x_{\beta_0} f_{\beta_0} + \cdots) = x_{\alpha_0}$. Since $q \in Se_{\alpha_0}$, we obtain that $sq(x_\beta) = 0$ for $\beta \neq \alpha_0$. Consequently, $e_{\alpha_0} = sq \in V$ for every neighborhood V of zero of $(S, \mathcal{T})$, a contradiction. □

Remark 4.2. The question of existence of a uncountable division ring which does not admit a nondiscrete Hausdorff ring topology is open. Several results on this topic can be found in Chapter 5 of [2].

Theorem 4.3. *Let $\prod_{\alpha\in\Omega} R_\alpha$ be a family of compact rings with identity. Then the product $(\prod_{\alpha\in\Omega} R_\alpha, \prod_{\alpha\in\Omega} \mathcal{T}_\alpha)$ is a minimal ring if and only if every $(R_\alpha, \mathcal{T}_\alpha)$ is a minimal topological ring. (Here $\prod_{\alpha\in\Omega} \mathcal{T}_\alpha$ is the product topology on the ring $\prod_{\alpha\in\Omega} R_\alpha$.)*

Proof. $\Rightarrow$: Assume on the contrary that there exists $\beta \in \Omega$ and a ring topology $\mathcal{T}'$ on R_β such that $\mathcal{T}' \leq \mathcal{T}_\beta$ and $\mathcal{T}' \neq \mathcal{T}_\beta$. Consider the product topology $\mathcal{U}$ on $\prod_{\alpha\in\Omega} R_\alpha$, where R_α is endowed with $\mathcal{T}_\alpha$ when $\alpha \neq \beta$ and R_β is endowed with $\mathcal{T}'$. Obviously, $\mathcal{U} \leq \prod_{\alpha\in\Omega} \mathcal{T}_\alpha$ and $\mathcal{U} \neq \prod_{\alpha\in\Omega} \mathcal{T}_\alpha$, a contradiction.
$\Leftarrow$: Denote by $\pi_\alpha(\alpha \in \Omega)$ the projection of $\prod_{\alpha\in\Omega} R_\alpha$ on R_α. By definition of the product topology, $\prod_{\alpha\in\Omega} \mathcal{T}_\alpha$ is the coarsest topology on $\prod_{\alpha\in\Omega} R_\alpha$ for which the projections $\pi_\alpha(\alpha \in \Omega)$ are continuous.

Let $\mathcal{U}$ be a ring topology on $\prod_{\alpha\in\Omega} R_\alpha$, $\mathcal{U} \leq \prod_{\alpha\in\Omega} \mathcal{T}_\alpha$ and $\beta \in \Omega$. Since

$$\mathcal{U} \restriction_{R_\beta \times \prod_{\gamma\neq\beta}\{0_\gamma\}} \leq \Big(\prod_{\alpha\in\Omega} \mathcal{T}_\alpha\Big) \restriction_{R_\beta \times \prod_{\gamma\neq\beta}\{0_\gamma\}},$$

it follows that $\mathcal{U} \restriction_{R_\beta \times \prod_{\gamma\neq\beta}\{0_\gamma\}} = (\prod_{\alpha\in\Omega} \mathcal{T}_\alpha) \restriction_{R_\beta \times \prod_{\gamma\neq\beta}\{0_\gamma\}}$ by minimality of $(R_\beta, \mathcal{T}_\beta)$.

Then the family $\{V \times \prod_{\gamma\neq\beta}\{0_\gamma\}\}$ when V runs all neighborhoods of zero of $(R_\beta, \mathcal{T}_\beta)$ is a fundamental system of neighborhoods of zero of

$$\Big(R_\beta \times \prod_{\gamma\neq\beta}\{0_\gamma\}, \quad \mathcal{U} \restriction_{R_\beta \times \prod_{\gamma\neq\beta}\{0_\gamma\}}\Big).$$

Since $R_\beta \times \prod_{\gamma\neq\beta}\{0_\gamma\}$ is an ideal with identity of $\prod_{\alpha\in\Omega} R_\alpha$, the topological ring $(\prod_{\alpha\in\Omega} R_\alpha, \mathcal{U})$ is a direct sum of ideals $R_\beta \times \prod_{\gamma\neq\beta}\{0_\gamma\}$ and $\{0_\beta\} \times \prod_{\gamma\neq\beta} R_\gamma$. Let V be a neighborhood of zero of $(R_\beta, \mathcal{T}_\beta)$. Then $V \times \prod_{\gamma\neq\beta} R_\gamma$ be a neighborhood of zero of $(\prod_{\alpha\in\Omega} R_\alpha, \mathcal{U})$ and $\pi_\beta(V \times \prod_{\gamma\neq\beta} R_\gamma) = V$.

We have proved that π_β is a continuous function from $(\prod_{\alpha\in\Omega} R_\alpha, \mathcal{U})$ to $(R_\beta, \mathcal{T}_\beta)$. It follows that $\prod_{\alpha\in\Omega} \mathcal{T}_\alpha \leq \mathcal{U}$ and so $\mathcal{U} = \prod_{\alpha\in\Omega} \mathcal{T}_\alpha$. □

Corollary 4.4. *A left linearly compact semisimple ring is minimal if and only if has no direct summands of the form $M(n, \Delta)$, where Δ is a division ring which does not admit a nondiscrete Hausdorff ring topology.*

Proof. This follows from Theorems 4.1, 4.3 and the Theorem of Leptin (see [22, Theorem 13, p. 258]) about the structure of left linearly compact semisimple rings. □

Corollary 4.5. *A semisimple linearly compact ring $(R, \mathcal{T})$ having no ideals isomorphic to matrix rings over infinite division rings is minimal.*

5. Completely simple endomorphism rings of modules

The endomorphism ring of a right R-module M is denoted by $\mathrm{End}\,(M_R)$.

Lemma 5.1. *Let M be a divisible, torsion-free module over a commutative domain R and K the field of fractions of R. The additive group of M has a structure of a vector K-space such that R-endomorphisms of M are exactly the K-linear transformations.*

Proof. We define a structure of a right vector K-space as follows: if $\frac{a}{b} \in K$ and $m \in M$, then there exists a unique $x \in M$ such that $ma = xb$; set $m \circ \frac{a}{b} = x$. Moreover, if $\frac{a}{b} = \frac{c}{d}$ and $0 \neq m \in M$, then $m \circ \frac{a}{b} = m \circ \frac{c}{d}$. Indeed, if $m \circ \frac{a}{b} = x$ and $m \circ \frac{c}{d} = y$, then $mad = xbd$ and $mbc = ybd$ which means that $xbd = ybd$, hence $x = y$.

Let $\alpha \in \mathrm{End}\,(M_R)$, $\frac{a}{b} \in K$, $m \in M$. By definition, $am = b(\frac{a}{b} \circ m)$, hence, $a\alpha(m) = b\alpha(\frac{a}{b} \circ m)$, which means that $\alpha(\frac{a}{b} \circ m) = \frac{a}{b} \circ \alpha(m)$, so α is a K-linear transformation. Note that, if $a \in R$ and $m \in M$, then $m \circ \frac{a}{1} = ma$.

Conversely, if α is a K-linear transformation, $a \in R$, $m \in M$, then

$$\alpha(\tfrac{a}{1} \circ m) = \tfrac{a}{1} \circ \alpha m,$$

i.e. $\alpha(am) = a\alpha(m)$. We have proved that every K-linear transformation is an right R-module homomorphism. □

Remark 5.2. The center $Z(R)$ of a weakly simple ring R is a domain.

Remark 5.3. For every right R-module M the underlying group $M(+)$ is a discrete left topological $(\mathrm{End}\,(M_R), \mathcal{T}_{fin})$-module.

Indeed, $T(m)(m) = 0$ for every $m \in M$. Moreover, $\mathrm{End}\,(M_R)\{0\} = \{0\}$, so M is a discrete left topological $(\mathrm{End}\,(M_R), \mathcal{T}_{fin})$-module.

Theorem 5.4. *Let M_R be a module over a commutative ring R.*

If the topological ring $(\mathrm{End}\,(M_R), \mathcal{T}_{fin})$ is weakly simple, then:

(i) *$P = \{r \in R \mid Mr = 0\}$ is a prime ideal of R.*

(ii) *M is a vector space over the field K of fractions of R/P and the R-endomorphisms of M are exactly the K-linear transformations.*

Conversely, if M_R is an R-module and are satisfied (i) and (ii), then the ring $(\mathrm{End}\,(M_R), \mathcal{T}_{fin})$ is a weakly simple topological ring.

Proof. $\Rightarrow$: If $(\mathrm{End}\,(M_R), \mathcal{T}_{fin})$ is weakly simple, then the mapping:

$$\alpha_r : M \to M, \qquad m \mapsto mr \qquad (r \in R) \tag{5.1}$$

is an R-module homomorphism and $\alpha_r \in Z(=$ the center of $\mathrm{End}\,(M_R))$.

First we show that the part (i) holds. Indeed, if $a, b \in R$ and $ab = 0$, then $\alpha_a \alpha_b = 0$ (see (5.1)). Thus $(\mathrm{End}\,(M_R)\alpha_a) \cdot (\mathrm{End}\,(M_R)\alpha_b) = 0$, so

$$(\overline{\mathrm{End}\,(M_R)\alpha_a}) \cdot (\overline{\mathrm{End}\,(M_R)\alpha_b}) = 0.$$

Since $\mathrm{End}\,(M_R)$ is weakly simple, one of them, say $\overline{\mathrm{End}\,(M_R)\alpha_a}$, is zero. This implies that $\alpha_a = 0$, hence $a \in P$.

(ii) The structure of R/P-module on M is defined as follows: if $r \in R$ and $m \in M$, then put $M(r + P) = mr$.

Note that M is a torsion-free right R/P-module. Assume that $m(r+P) = 0$, where $0 \neq r + P \in R/P$ and $0 \neq m \in M$. Then $mr = 0 = \alpha_r(m)$ (see (5.1)). Thus $\mathrm{End}\,(M_R)\alpha_r(m) = 0$. It follows that $\big(\overline{\mathrm{End}\,(M_R)\alpha_r}\big)(m) = 0$ by Remark 5.3. Since $\mathrm{End}\,(M_R)$ is weakly simple

$$\overline{\mathrm{End}\,(M_R)\alpha_r} = \mathrm{End}\,(M_R).$$

We obtained that $\mathrm{End}\,(M_R)(m) = 0$, so $m = 0$, a contradiction.

Under this convention R-submodules are exactly R/P-submodules and R-endomorphisms are exactly R/P-endomorphisms.

The module M is a divisible R/P-module. Indeed, if $0 \neq r+P \in R/P$, then $0 \neq M(r + P) = Mr$. Suppose that $Mr \neq M$. Consider

$$I = \{\alpha \in \mathrm{End}\,(M_R) \mid \alpha(M) \subseteq Mr\}.$$

Since Mr is a fully invariant submodule, I is a two-sided ideal of the ring $(\mathrm{End}\,(M_R), \mathcal{T}_{fin})$.

The ideal I is closed. Indeed, let $\alpha \in \overline{I}$. If $m \in M$, then there exists $\beta \in I$ such that $\alpha - \beta \in T(m)$. Clearly, $\alpha(m) = \beta(m) \in Mr$ and so $\alpha \in I$. We have proved that I is closed.

Since $1_M \notin I$, $I = 0$. It follows that $\alpha_r = 0$ (see (5.1)), a contradiction.

The module M has a structure of a right K-vector space and $\mathrm{End}\,(M_R)$ is exactly the ring of endomorphisms of M by Lemma 5.1.

The converse follows from Theorem 4.1. □

A characterization of completely simple topological ring $\mathrm{End}\,(M_R)$ is given by the following.

Theorem 5.5. *Let M_R be a module over a commutative ring R. The topological ring $(\mathrm{End}\,(M_R), \mathcal{T}_{fin})$ is completely simple if and only are satisfied the conditions (i) and (ii) of Theorem 5.4 and either*

(i) *M is finite or*
(ii) *M is infinite and the dimension of M over the field K is infinite.*

Proof. $\Rightarrow$: According to Theorem 5.4, the ideal P is prime and the topology of $\mathrm{End}\,(M_R)$ coincide with the finite topology of $\mathrm{End}\,(M_K)$, where K is the field of fractions of R/P. If M is finite, we have the part (i). Assume that

M is infinite. If R/P is finite, then the dimension of M over K is infinite. Suppose that R/P is infinite and $\dim_K(M) = n < \aleph_0$. Then M is isomorphic to $M(n, K)$. Since K is an infinite field, it admits a nondiscrete ring topology (see [13]) and we obtain a contradiction because $\mathrm{End}\,(M_R)$ is a discrete ring. Consequently $\dim_K(M)$ is infinite.
$\Leftarrow$ This follows from Theorems 4.1 and 5.4. □

Corollary 5.6. *The topological ring* $(\mathrm{End}\,(A), \mathcal{T}_{fin})$ *of an abelian group* A *is completely simple if and only one of the following conditions holds:*

(i) *A is a elementary abelian p-group.*
(ii) *A is a divisible torsion-free group of infinite rank.*

Acknowledgements. *Supported by UAEU UPAR (9) 2017 Grant G00002599.*

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