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Dynamics of real projective transformations

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Abstract

The dynamics of a projective transformation on a real projective space are studied in this paper. The two main aspects of these transformations that are studied here are the topological entropy and the zeta function. Topological entropy is an inherent property of a dynamical system whereas the zeta function is a useful tool for the study of periodic points. We find the zeta function for a general projective transformation but entropy only for certain transformations on the real projective line.

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1. Introduction

An $n-$dimensional real projective space, denoted by $\mathbb{P}_n(\mathbb{R})$ is the quotient space $S^n/_\sim$, where the antipodal points are identified under the relation, $\sim$. $\mathbb{P}_1(\mathbb{R})$ is also called the real projective line. Let $\pi : S^n \to \mathbb{P}_n(\mathbb{R})$ be the quotient map. A projective transformation on $\mathbb{P}_n(\mathbb{R})$ associated to a matrix $A \in GL_{n+1}(\mathbb{R})$, denoted by $\bar{A}$ is defined as $\bar{A}(\pi(x)) = \pi(Ax)$, for every $x \in S^n$.

A *discrete dynamical system* is, by definition, a pair (X, f), where X is a topological space and f is a self map on X i.e., $f : X \to X$. Though f can be any map in a general setting, we need it to be a continuous map in many cases. So, unless otherwise mentioned, we assume the map to be continuous. Since we consider only discrete dynamical systems in this paper,

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hereafter, we refer to them simply as dynamical systems. Given $x \in X$, the sequence $(x, f(x), f^2(x), f^3(x), ...)$ is called the trajectory of x, where $f^k(x) = f \circ f \circ ... \circ f(x)$ (k times) for $k \in \mathbb{N}$ and $f^0(x) = x$. The set $\{f^k(x) : k$ is a non-negative integer$\}$ is called the orbit of x. The study of dynamics is mainly about the eventual behavior of trajectories. A point $x \in X$ is said to be *periodic* if there is a $k \in \mathbb{N}$ such that $f^k(x) = x$; any such k is called a *period* of x and the least among them is called the *least period* of x. A periodic point x of period 1 is called a fixed point i.e., $f(x) = x$ and the set of fixed points of f is denoted by $Fix(f)$. We also use the notation $|Y|$ to denote the cardinality of any set Y.

In this paper, the typical dynamical system that we are going to consider is $(\mathbb{P}_n(\mathbb{R}), \bar{A})$. The periodic points of this system can be found very easily. If $v \in S^n$ is an eigenvector of A with eigenvalue λ, then $\bar{A}(\pi(v)) = \pi(Av) = \pi(\lambda v) = \pi(v)$; hence $\pi(v)$ is a fixed point. Conversely, if $\pi(v)$ is a periodic point with period k, then it is a fixed point of $\bar{A}^k$ and thus $\pi(A^k v) = \pi(v)$ i.e., $A^k v = \mu v$ for some scalar μ. Hence, v is an eigenvector of A^k. To sum up, we have shown that $\pi(v)$ is periodic if and only if v is an eigenvector of A^k for some $k \in \mathbb{N}$.

The dynamics of projective transformations are well studied in the literature. See for instance [4] and [6]. In this paper, we study the topological entropy and the zeta function of projective transformations.

One of the best ways of measuring the complexity of a dynamical system is finding its topological entropy. As stated in [3], topological entropy measures the exponential growth rate of the number of essentially different orbit segments of length n. On the other hand, the zeta function collects combinatorial information about the periodic points. In the next section, we calculate the entropy of certain projective transformations on the real projective line, followed by a section on finding the zeta function of a projective transformation on a projective space of any dimension.

2. Topological entropy

Topological entropy was introduced by Adler, Konheim and McAndrew [1] and here, we will use an equivalent definition for maps on compact metric spaces given by Bowen [2]. Most of the basic facts about entropy, that we mention here can be found in [3].

Given a compact metric space (X, d) and a continuous map $f : X \to X$, we define a new metric d_n, for every $n \in \mathbb{N}$ as $d_n(x, y) = max\ \{d(f^i(x), f^i(y)) : 0 \leq i < n\}$. It can be shown that each of these metrics induces the same topology on X as induced by d. A subset $E \subset X$ is called an $(n,\ \epsilon)-$separated set if for any two distinct points $x,\ y \in E$, $d_n(x, y) \geq \epsilon$. Since X is compact, every (n, ϵ)-separated set is a finite set; otherwise, there will be a sequence (x_k) in E with no convergent subsequence, as $d(x_k, x_{k+1}) \geq \epsilon$ for every $k \in \mathbb{N}$, thus contradicting the compactness of X. Now, let $sep(n,\ \epsilon,\ f)$ be the cardinality of an $(n,\ \epsilon)-$separated set with maximum cardinality.

Definition 2.1 (see [3]). The entropy $h(f)$ of a system (X, f) is defined as

$$h(f) = \lim_{\epsilon \to 0+} \limsup_{n \to \infty} \frac{1}{n} log(sep(n, \epsilon, f)). \tag{2.1}$$

Similar to $sep(n, \epsilon, f)$, two more numbers, namely $span(n, \epsilon, f)$ and $cov(n, \epsilon, f)$ can be defined for every $\epsilon > 0$ and $n \in \mathbb{N}$. Here, $cov(n, \epsilon, f)$ is the cardinality of a covering of X by least number of sets of d_n-diameter less than ϵ. It is well defined because, there does exist a finite cover of X by sets of d_n-diameter less than ϵ, as any cover of X with open sets of d_n-diameter less than ϵ will have a finite subcover for X. Finally, a subset $A \subset X$ is called an (n, ϵ)-spanning set in X, if for every $x \in X$, there is $y \in A$ such that $d_n(x, y) < \epsilon$. As X is compact, the open cover $\{B_{d_n}(x, \epsilon) : x \in X\}$ (where $B_{d_n}(x, \epsilon)$ is the open ball centered at x and has radius ϵ with respect to the d_n-diameter) has a finite subcover, say $\{B_{d_n}(x_1, \epsilon),\ B_{d_n}(x_2, \epsilon),\ \ldots,\ B_{d_n}(x_k, \epsilon)\}$. Then, the set $\{x_1,\ x_2,\ \ldots,\ x_k\}$ is an (n, ϵ)-spanning set. Since finite (n, ϵ)-spanning sets exist in a compact space, we can find an (n, ϵ)-spanning set with minimum cardinality. This minimum cardinality is called $span(n, \epsilon, f)$.

Lemma 2.2 (see [3]).

$$cov(n, 2\epsilon, f) \leq span(n, \epsilon, f) \leq sep(n, \epsilon, f) \leq cov(n, \epsilon, f). \tag{2.2}$$

Using this lemma, it follows easily that

$$h(f) = \lim_{\epsilon \to 0+} \limsup_{n \to \infty} \frac{1}{n} log(sep(n, \epsilon, f)) \tag{2.3}$$

$$= \lim_{\epsilon \to 0+} \limsup_{n \to \infty} \frac{1}{n} log(span(n, \epsilon, f)) \tag{2.4}$$

$$= \lim_{\epsilon \to 0+} \limsup_{n \to \infty} \frac{1}{n} log(cov(n, \epsilon, f)) \tag{2.5}$$

Proposition 2.3 (Proposition 2.5.3 in [3]). *The topological entropy of a continuous map $f : X \to X$ does not depend on the choice of a particular metric generating the topology of X.*

Proposition 2.4 ([3]). *The topological entropy of an isometry is zero.*

In the following proposition, $\mathcal{T}^2$ denotes the torus, $\mathbb{R}^2/\mathbb{Z}^2$. Any automorphism of this topological group, $\mathbb{R}^2/\mathbb{Z}^2$ which will be called a *toral automorphism*, is of the form $\pi'(x) \mapsto \pi'(Mx)$, where $\pi' : \mathbb{R}^2 \to \mathcal{T}^2$ is the canonical projection and $M \in GL_2(\mathbb{Z})$. We say that the automorphism is induced by the matrix M and denote it by T_M. If no eigenvalue of M has modulus 1, then T_M is called a hyperbolic toral automorphism.

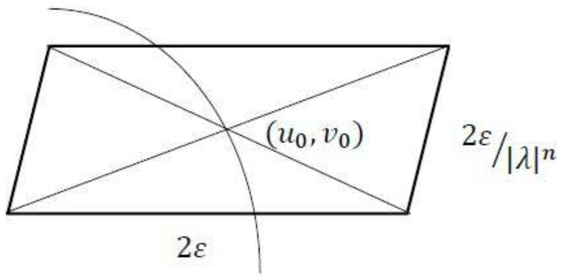
Proposition 2.5 (Proposition 2.6.1 in [3]). *The topological entropy of a hyperbolic toral automorphism $T_M : \mathcal{T}^2 \to \mathcal{T}^2$, with $det(M) = 1$ is equal to $log\ |\lambda|$, where λ is the eigenvalue of M such that $|\lambda| > 1$.*

All the above propositions can be found in [3]. Our proof of Theorem 2.6 relies mostly on the proof of Proposition 2.5, as given in [3].

Theorem 2.6. *Let A be a 2×2 matrix of determinant 1 with eigenvalues λ and $\frac{1}{\lambda}$, where $|\lambda| > 1$. Then the entropy of the corresponding projective transformation $\bar{A}$ is $log\ |\lambda|$.*

Proof. Let v_1 and v_2 be unit eigenvectors of A corresponding to λ and $\frac{1}{\lambda}$ respectively. Now, for $u,\ v \in S^1$, define $\tilde{d}(u,v) = max(|a_1|,\ |a_2|)$, where $u - v = a_1 v_1 + a_2 v_2$. $\tilde{d}$ is a metric on S^1 and it induces the metric d on $\mathbb{P}_1(\mathbb{R})$, where $d(x,y)$ is the $\tilde{d}$-distance between the sets $\pi^{-1}(x)$ and $\pi^{-1}(y)$.

Note that the definition of $\tilde{d}$ can be extended to a metric on $\mathbb{R}^2$ and an open ball of radius ϵ centered at $(u_0, v_0) \in \mathbb{R}^2$ under $\tilde{d}$ is a parallelogram centered at (u_0, v_0) with its sides parallel to v_1 and v_2 and each having a length 2ϵ. Then an open ball of radius ϵ in S^1 centered at $(u_0, v_0) \in S^1$ is an arc centered at (u_0, v_0), which is formed by the intersection of S^1 with the above parallelogram. Further, an ϵ $\tilde{d}_n$-ball in $\mathbb{R}^2$, with respect to the map induced by A is again a parallelogram with sides of lengths 2ϵ and $\frac{2\epsilon}{|\lambda|^n}$ which are parallel to v_1 and v_2 respectively. Now, an ϵ $\tilde{d}_n$-ball in S^1 is thus an arc passing through the center of a parallelogram with the above dimensions. So, its length is at least the smaller side of the parallelogram i.e., $\frac{2\epsilon}{|\lambda|^n}$. On the other hand, its length is at most the perimeter of the parallelogram, which is equal to $4\epsilon + \frac{4\epsilon}{|\lambda|^n}$ (See the figure). It follows from Archimedean property of real numbers that, if a and b are any two positive real numbers, then there is a positive integer k such that $(a + b) \leq kab$. Thus, we can find a positive integer depending on ϵ, say $k(\epsilon)$, such that $4\epsilon + \frac{4\epsilon}{|\lambda|^n} \leq \frac{k(\epsilon)\epsilon^2}{|\lambda|^n}$. Since π is a local isometry, for sufficiently small ϵ, we can assume that ϵ d_n-balls in $\mathbb{P}_1(\mathbb{R})$ have the same dimensions.



Since the diameter of an ϵ d_n-ball in $\mathbb{P}_1(\mathbb{R})$ is at most $\frac{k(\epsilon)\epsilon^2}{|\lambda|^n}$, the minimum number of such balls that are required to cover $\mathbb{P}_1(\mathbb{R})$ is $\frac{\pi}{\frac{k(\epsilon)\epsilon^2}{|\lambda|^n}}$, as the Euclidean length of $\mathbb{P}_1(\mathbb{R})$ is π. Since a set of diameter 2ϵ is contained in an open ball of radius ϵ, we have, $cov(n, 2\epsilon, \bar{A}) \geq \frac{\pi|\lambda|^n}{k(\epsilon)\epsilon^2}$. Thus, we have $h(\bar{A}) \geq log\ |\lambda|$.

Similarly, since the diameter of an ϵ d_n-ball in $\mathbb{P}_1(\mathbb{R})$ is at least $\frac{2\epsilon}{|\lambda|^n}$, $\mathbb{P}_1(\mathbb{R})$ can be covered by $\frac{\pi}{\frac{2\epsilon}{|\lambda|^n}}$ number of arcs. Hence, $cov(n, 2\epsilon, \bar{A}) \leq \frac{\pi|\lambda|^n}{2\epsilon}$. So, $h(\bar{A}) \leq log\ |\lambda|$.

Thus, we conclude that $h(\bar{A}) = log\ |\lambda|$. □

In [3], the authors have also given a proof of the fact that the entropy of a hyperbolic toral automorphism induced by a matrix A on an $n-$dimensional torus, $\mathcal{T}^n$ is equal to $\sum_{i=1}^{m} log|\alpha_i|$, where $\alpha_1,\ \alpha_2, \ldots,\ \alpha_m$ are those eigenvalues of A that have modulus strictly larger than 1 (See Proposition 2.6.4 in [3]). The proof relies on the idea of decomposing $\mathbb{R}^n$ in to generalised eigenspaces of A and is similar to the proof of the corresponding result on $2-$dimensional torus, mentioned above (Proposition 2.5). On the same lines, it is hoped that a result similar to Theorem 2.6 can be obtained for projective transformations on higher dimensional projective spaces also.

3. Zeta function

The zeta function collects combinatorial information about the periodic points. We follow [3] for the definition and other basic facts of the zeta function. For a dynamical system (X, f), if $|Fix(f^k)|$ is finite for every k, we define the zeta function $\zeta_f(z)$ of f to be the formal power series $\zeta_f(z) = exp(\sum_{k=1}^{\infty} \frac{1}{k}|Fix(f^k)|z^k)$. The zeta function can also be expressed by a product formula. Let $\mathcal{P}(f)$ denote the collection of all periodic orbits of f i.e., a typical element of $\mathcal{P}(f)$ will be $\{x_0,\ f(x_0),\ \ldots,\ f^{k-1}(x_0)\}$, where k is the least period of x_0. Now, the zeta function of f can be written as $\zeta_f(z) = \prod_{\gamma \in \mathcal{P}(f)} (1 - z^{|\gamma|})^{-1}$ where $|\gamma|$ is the number of elements in γ.

We use the following lemma in proving Theorem 3.2.

Lemma 3.1. *If μ is a non-zero eigenvalue of A^k for some $k \in \mathbb{N}$ such that there is a unique eigenvalue λ of A with $\lambda^k = \mu$, then the eigenspaces of A^k and A corresponding to μ and λ respectively, are same.*

The lemma follows easily from the facts that, under the assumptions of the hypothesis, the number of Jordan blocks in the Jordan normal form of A corresponding to λ is same as the number of Jordan blocks in the Jordan normal form of A^k corresponding to μ.

Theorem 3.2. *Let $\bar{A}$ be a projective transformation on $\mathbb{P}_n(\mathbb{R})$ induced by a matrix $A \in GL_{n+1}(\mathbb{R})$. $\bar{A}$ possesses zeta function if and only if each eigenspace of A is one-dimensional and no two eigenvalues have same absolute value. In such case, the zeta function is given by $\zeta_f(z) = \frac{1}{(1-z)^l}$.*

Proof. Suppose that each eigenspace of A is one-dimensional and no two eigenvalues of A have same absolute value. If $\pi(v)$ is a periodic point of $\bar{A}$ where $v \in S^1$, then v is an eigenvector of A^k for some $k \in \mathbb{N}$, say $A^k v = \mu v$, $\mu \in \mathbb{R}$. Then there is an eigenvalue λ of A such that $\lambda^k = \mu$. If λ_1 and λ_2 are two different eigenvalues of A such that $\lambda_1^k = \lambda_2^k = \mu$, then $|\lambda_1| = |\lambda_2|$, contrary to the hypothesis. Thus, λ is unique. Hence, by the above lemma, v lies in the eigenspace of A corresponding to λ. Thus $\bar{v}$ is a fixed point of $\bar{A}$ i.e., fixed points are the only periodic points. In other words, $Fix(\bar{A}^k) = Fix(\bar{A})$ for any k. Further, since each eigenspace of A is one-dimensional, there are as many fixed points as the eigenvalues. Thus, $\zeta_f(z) = exp(l \sum_{k=1}^{\infty} \frac{z^k}{k}) = \frac{1}{(1-z)^l}$, where l is the number of eigenvalues of A.

Conversely, suppose $\bar{A}$ possesses zeta function. Then there should be finitely many fixed points and thus each eigenspace should be one dimensional. If possible, suppose there are two different eigenvalues λ_1 and λ_2 such that $|\lambda_1| = |\lambda_2|$. Since λ_1 and λ_2 are real, $\lambda_1^2 = \lambda_2^2$; say $\mu = \lambda_1^2$. Then μ is an eigenvalue of A^2. If v_1 and v_2 are eigenvectors of A corresponding to λ_1 and λ_2 respectively, then v_1 and v_2 are eigenvectors of A^2 corresponding to the same eigenvalue μ. So, the dimension of eigenspace of A^2 corresponding to μ is greater than 1 and thus there are infinitely many periodic points of $\bar{A}$ with period 2, implying that the zeta function doesn't exist, contradicting the hypothesis. $\square$

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