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The hull orthogonal of the unit interval [0,1]

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Abstract

In this paper, the full subcategory ***Hcomp*** *of* ***Top*** *whose objects are Hausdorff compact spaces is identified as the orthogonal hull of the unit interval* $\boldsymbol{I} = [0, 1]$*. The family of continuous maps rendered invertible by the reflector* $\beta \circ \rho$ *is deduced.*

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1. Introduction

In the literature, various approaches to the Stone-Čech Compactification βX of a topological space X are given, using constructions based on products of the interval unit $\mathbf{I}$, ultrafilters, and $C^\star$-algebras, respectively ([5], [7], [12] and [10]).

More than a compactification, the embedding of X into βX defines **Hcomp** as a reflective subcategory in the category **Tych** of Tychonoff spaces. Thus **Hcomp** is a reflective subcategory of **Top** with reflector $\beta \circ \rho$, where ρ is the Tychonoff reflector.

The year 1937 was an important one in establishing nice connections between topology and algebra. M. H. Stone and E. Čech published papers giving several fundamental properties of the compactification βX, which had been introduced by Tychonoff. For instance, Stone showed that any Tychonoff space X is $C^\star$-embedded in βX, and this can be interpreted algebraically as showing that the rings $C^\star(X)$ and $C^\star(\beta X)$ are isomorphic.

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Recall that if $\mathcal{D}$ is a reflective subcategory in a category $\mathcal{C}$, with reflector $\mathbf{F}$, then $\mathcal{D}^{\perp} = \{f \in hom_{\mathcal{C}} : \ F(f) \quad \text{is an isomorphism}\}$ and $\mathcal{D}^{\perp\perp} = \mathcal{D}$ (for more information see [1], [2] and [4]). In our case, we have $\mathbf{Hcomp}^{\perp} = \{f \in hom_{\mathbf{Top}} : \ \beta \circ \rho(f) \quad \text{is an isomorphism}\}$ and $\mathbf{Hcomp}^{\perp\perp} = \mathbf{Hcomp}$.

So on the one hand, if we consider the category **Sob** of sober spaces, it is not difficult to show that $\mathbf{Sob}^{\perp} = \{\delta\}^{\perp}$, where δ is the Sierpiński space, and thus $\mathbf{Sob} = \{\delta\}^{\perp\perp}$ which gives a characterization of sober spaces using only the space δ.

On the other hand, in [6], A. Haouati and S. Lazaar showed that the reflective subcategory **Hewitt** of **Top**, whose objects are real-compact spaces, is the orthogonal hull of the real line $\mathbb{R}$.

Analogous to $\mathbf{Sob}^{\perp} = \{\delta\}^{\perp}$ and $\mathbf{Hewitt}^{\perp} = \{\mathbb{R}\}^{\perp}$, we show in this paper that $\mathbf{Hcomp}^{\perp} = \{\mathbf{I}\}^{\perp}$ where $\mathbf{I}$ is the unit interval, and consequently the family of continuous maps rendered invertible by $\beta \circ \rho$ are those maps which are orthogonal to $\mathbf{I}$.

2. Some preliminary results

Let $\mathcal{C}$ be a category. An arrow f in $\mathcal{C}$ from A to B is said to be orthogonal to an object X in $\mathcal{C}$ if and only if for any arrow g from A to X, there exists a unique arrow $\widetilde{g}$ from B to X satisfying $\widetilde{g} \circ f = g$.

The orthogonal $\Sigma^{\perp}$ of a class of morphisms Σ is the class of objects orthogonal to every morphism in Σ [4]. The orthogonal of a class of objects is defined analogously.

Recall that a topological space is called completely regular (or Tychonoff) if it is T_1 and every closed subset F of the space is completely separated from any point x not in F. An other important characterization of completely regular spaces is given by the following theorem.

Theorem 2.1 ([12, Proposition 1.7])**.** *A space is completely regular if and only if the family of zero-sets of the space is a base for the closed sets (or equivalently, the family of cozero-sets is a base for the open sets).*

Notations 1. Lat X be a topological space. We denote by:

- $\mathbf{C}(X)$ the family off all continuous maps from X to $\mathbb{R}$.
- $\mathbf{C}^{\star}(X)$ the family off all bounded continuous maps from X to $\mathbb{R}$.
- $\mathbf{C}^{\star}_{\mathbf{I}}(X)$ the family off all continuous maps from X to $\mathbf{I}$.
- $\mathbf{C}_{[0,+\infty[}(X)$ the family off all positive continuous maps from X to $\mathbb{R}$.

Remark 2.2. Let f be a continuous map from a topological space X to $\mathbb{R}$. Consider the map $f_{\mathbf{I}}$ from X to I defined by $f_{\mathbf{I}} := \inf\{|f|, 1\}$.

Clearly $f(x) = 0$ if and only if $f_{\mathbf{I}}(x) = 0$ if and only if $|f|(x) = 0$. Therefore by Theorem 2.1 a topological space X is completely regular if and only if the family $\{h^{-1}(]0,1]) : h \in \mathbf{C}^{\star}_{\mathbf{I}}(X)\}$ (resp., $\{h^{-1}(]0,+\infty[) : h \in \mathbf{C}_{[0,+\infty[}(X)\}$ is a base for the open sets of X.

The following result is an easy observation from [2].

Proposition 2.3. *Let $\mathcal{C}$ be a category and $\mathcal{D}$ a reflective subcategory of $\mathcal{C}$, with reflector $\boldsymbol{F}$. An arrow in $\mathcal{C}$ is orthogonal to an object in $\mathcal{D}$ if and only if its $\boldsymbol{F}$-identification is also.*

Remark 2.4. In our case when we consider the reflective subcategory **Tych** of **Top**, with reflector ρ, where ρ is the Tychonoff reflector, a continuous map f between topological spaces is orthogonal to the unit interval $I = [0, 1]$ equipped with its usual topology if and only if its Tychonoff reflection $\rho(f)$ is orthogonal to $\mathbf{I}$.

Let us now give some properties of a continuous map orthogonal to I (resp., $[0, +\infty[$).

Proposition 2.5. *Let f be a continuous map from a functionally Hausdorff space to a topological space Y. If f is orthogonal to $\boldsymbol{I}$, then f is one-to-one.*

Proof. Let x and y be two points in X such that $f(x) = f(y)$. If x and y are distinct then there exists $g \in \mathbf{C}(X)$ such that $g(x) = 0$ and $g(y) = 1$ and thus $g_{\mathbf{I}}$ defined in Remark 2.2 satisfies also $g_{\mathbf{I}}(x) = 0$ and $g_{\mathbf{I}}(y) = 1$. The mapping $\widetilde{g}_{\mathbf{I}}$ in $\mathbf{C}^{\star}_{\mathbf{I}}(Y)$ obtained by orthogonality of f to I gives a contradiction. $\square$

Remark 2.6. By the same way as in Proposition 2.5, we can see easily that if we consider a continuous map from a functionally Hausdorff space to a topological space Y which is orthogonal to $[0, +\infty[$, then it is one to one. Indeed, it is enough to replace $g_{\mathbf{I}}$ in Proposition 2.5 by $|g|$.

Proposition 2.7. *Let f be a continuous map from a topological space X to a completely regular space Y. If f is orthogonal to $\boldsymbol{I}$, then, f is a dense mapping.*

Proof. Assume that $\overline{f(X)} \neq Y$ and let y be in Y and not in $\overline{f(X)}$. Since Y is completely regular, there exists a mapping h in $\mathbf{C}(Y)$ such that $h(y) = 0$ and $h(\overline{f(X)}) = \{1\}$. Then the mapping h_I from $\mathbf{C}^{\star}_{\mathbf{I}}(X)$ satisfies $h_I(y) = 0$ and $h_I(\overline{f(X)}) = \inf\{|f|, 1\}(\overline{f(X)}) = \inf\{1, 1\}(\overline{f(X)}) = \{1\}$. Now if we denote by 1_Y the constant map equal to 1 from Y to $\mathbf{I}$, we get:

$$\forall x \in X, (1_Y \circ f)(x) = (h_I \circ f)(x) = 1.$$

So,

$$1_Y \circ f = h_I \circ f.$$

This leads to a contradiction because f is orthogonal to $\mathbf{I}$ and the continuous maps 1_Y and h_I are not equal. $\square$

Remark 2.8. By the same way as in Proposition 2.7, we can see easily that if we consider a continuous map from a topological space X to a completely regular space Y which is orthogonal to $[0, +\infty[$, then it is a dense mapping. Indeed, it is enough to replace $h_{\mathbf{I}}$ in Proposition 2.7 by $|h|$.

Proposition 2.9. *Let f be a continuous map from a completely regular space X to a topological space Y. If f is orthogonal to $\boldsymbol{I}$, then $f(X)$ and X are homeomorphic.*

Proof. Let f_1 be the restriction of f to $f(X)$. Using Proposition 2.5, f_1 is a continuous bijective map, so it is sufficient to show that it is an open map.

Indeed, let $g^{-1}(]0,1])$ be an element of the base of open sets, cited in Remark 2.2, where $g \in \mathbf{C}^{\star}_{\mathbf{I}}(X)$. Since f is orthogonal to I, the unique map $\widetilde{g} \in \mathbf{C}^{\star}_{\mathbf{I}}(Y)$ such that $\widetilde{g} \circ f = g$ satisfies $f_1(g^{-1}(]0,1])) = \widetilde{g}^{-1}(]0,1]) \cap f(X)$, which is open in $f(X)$. $\square$

Remark 2.10. By the same way as in Proposition 2.9, any continuous map f from a completely regular space X to a topological space Y which is orthogonal to $[0,+\infty[$, then $f(X)$ and X are homeomorphic.

To conclude the three previous results, we can cite the following result.

Proposition 2.11. Every map $f : X \longrightarrow Y$ in the category **Tych** which is orthogonal to $\mathbf{I}$ (resp., $[0,+\infty[$) is a one-to-one dense mapping such that X and $f(X)$ are homeomorphic.

Proposition 2.12. *Let X be a Tychonoff space and $f : X \longrightarrow \boldsymbol{I}$ be a continuous map which is orthogonal to $\boldsymbol{I}$. Then f is an homeomorphism.*

Proof. By Proposition 2.11, it is enough to see that f is a surjective map. Suppose that $f(X) \neq \mathbf{I}$ and let y be in $\mathbf{I}$ not in $f(X)$.

We have two cases to discuss.

First case: $0 < y < 1$.

Let us denote by:

$$X^{<} = \{x \in X :\ 0 \leq f(x) < y\} \text{ and } X^{>} = \{x \in X :\ y < f(x) \leq 1\}.$$

So, one can check easily that $X^{<}$ and $X^{>}$ are a disjoint union of X. Then $f(X) = f(X^{<}) \cup f(X^{>})$ which implies that $\mathbf{I} = \overline{f(X)} = \overline{f(X^{>})} \cup \overline{f(X^{>})}$. Now since $\overline{f(X^{<})}$ (resp., $\overline{f(X^{>})}$) is closed containing $[0,y[$ (resp., $]y,1]$), then $\overline{f(X^{<})} = [0,y]$ (resp., $\overline{f(X^{>})} = [y,1]$).

Let consider the map g from X to $\mathbf{I}$ by $g(X^{<}) = \{\frac{y}{2}\}$ and $g(X^{>}) = \{\frac{y+1}{2}\}$. It is clear that g is continuous and thus by orthogonality of f to $\mathbf{I}$, let $\widetilde{g}$ be the continuous map from $\mathbf{I}$ to itself such that $\widetilde{g} \circ f = g$.

By density of $f(X^{<})$ (resp.,$f(X^{>})$) in $[0,y]$ (resp., $[y,1]$), consider a sequence (x_n) (resp., (z_n)) in $X^{<}$ (resp., $X^{>}$) with $(f(x_n))$ (resp., $(f(z_n))$) in $[0,y]$ (resp., $[y,1]$) converges to y. By preserving continuity under continuous maps, the constant sequences $(g(x_n) = \frac{y}{2})$ and $(g(z_n) = \frac{y+1}{2})$ must both converge to $\widetilde{g}(y)$ which is impossible.

Second case: $y \in \{0,1\}$.

In this case $f(X) \in \{]0,1],[0,1[,]0,1[\}$. Without loss in generality we can suppose that $f(X) =]0,1]$. Now, consider the map g from X to $\mathbf{I}$ defined by $g(x) = \mid \sin \frac{1}{f(x)} \mid$. Clearly g is a continuous map and since f is orthogonal to $\mathbf{I}$, there exists a unique map $\widetilde{g}$ from $\mathbf{I}$ to itself such that $\widetilde{g} \circ f = g$. So that for any $y \in]0,1]$, $\widetilde{g}(y) = \mid \sin \frac{1}{y} \mid$ which leads to a contradiction since $\widetilde{g}$ is continuous in 0. $\square$

3. Hcomp $=\{I\}^{\perp\perp}$

Before giving the main result of our paper, let us recall two important results introduced in chapter 6 in [5].

Theorem 3.1 ([5, Theorem 6.4])**.** *Let X be dense in T. The following statements are equivalent.*

1. *Every continuous mapping τ from X into any compact space Y has an extension to a continuous mapping from T into Y.*
2. *X is $C^\star$-embedded in T.*
3. *Any two disjoint zero-sets in X have disjoint closures in T.*
4. *For any two zero-sets Z_1 and Z_2 in X,*
$$cl_T(Z_1 \cap Z_2) = cl_T Z_1 \cap cl_T Z_2.$$
5. *Every point of T is the limit of a unique z-ultrafilter on X.*

Theorem 3.2 (Compactification Theorem, [5, Theorem 6.5])**.** *Every completely regular space X has a compactification βX, with the following equivalent properties.*

1. *(Stone) Every continuous mapping τ from X into any compact space Y has a continuous extension $\overline{\tau}$ from βX into Y.*
2. *(Stone-Cech) Every function f in $C^\star(X)$ has an extension to a function f^β in $C(\beta X)$.*
3. *(Cech) Any two disjoint zero-sets in X have disjoint closures in βX.*
4. *For any two zero-sets Z_1 and Z_2 in X,*
$$cl_{\beta X}(Z_1 \cap Z_2) = cl_{\beta X} Z_1 \cap cl_{\beta X} Z_2.$$
5. *Distinct z-ultrafilter on X have distinct limits in βX.*

Remark 3.3. The compactification βX in Theorem 3.2 is unique, in the following sense: if a compactification T of X satisfies anyone of the listed previous conditions, then there exists a homeomorphism from βX onto T that leaves X pointwise fixed.

Now, we are in a position to give our main result.

Theorem 3.4. ***Hcomp***$^\perp = \{\boldsymbol{I}\}^\perp$.

Proof. Clearly, $\mathbf{Hcomp}^\perp \subset \{\mathbf{I}\}^\perp$.

Conversely, let $f : X \longrightarrow T$ be a continuous map orthogonal to $\mathbf{I}$, Y a Hausdorff compact space and g a continuous map from X to Y.

By Remark 2.4, we may assume X and T are completely regular spaces. Now, using Proposition 2.11, we may assume X as a dense subset of the completely regular space T and replace f by the canonical injection from X to T. Now (2) $\Longrightarrow$ (1) of Theorem 3.1 applies, and thus g has a continuous extension $\widetilde{g}$ from T into Y. Furthermore, this extension is unique, since any two such continuous extensions must coincides on the dense subset X of the Hausdorff space T, and thus must be equal. □

The following corollaries are immediate.

Corollary 3.5. $\boldsymbol{Hcomp} = \{\boldsymbol{I}\}^{\perp\perp}$.

Corollary 3.6. *Let f be a continuous map. Then $\beta(\rho(f))$ is a homeomorphism if and only if f is orthogonal to $\boldsymbol{I}$. In particular, for a continuous map f between two Tychonoff spaces, $\beta(f)$ is an homeomorphism if and only if f is orthogonal to $\boldsymbol{I}$.*

Proof. Since the family of all morphisms rendered invertible by the reflector $\beta \circ \rho$ is exactly $\mathbf{Hcopm}^{\perp}$, an application of Theorem 3.4 gives the result. $\square$

Let us recall the definition introduced by Echi and Lazaar in [3].

Definition 3.7 ([3, Definition 3.2])**.** Let X be a topological space and H a subset of $\mathbf{C}(X)$. We say that H has the finite intersection property (FIP, for short) if for each finite subset J of H we have $\cap[f^{-1}(\{0\}, f \in J] \neq \varnothing$.

Theorem 3.8. *Let $f : X \longrightarrow Y$ be a continuous map which is orthogonal to $\boldsymbol{I}$. Then the following statements are equivalent.*

(1) *For each subset H of $C(Y)$ satisfying the FIP, $\cap[f^{-1}(\{0\}) : f \in H] \neq \varnothing$;*
(2) *$\beta(\rho(X)) = \rho(Y)$.*

Proof. (1) $\Longrightarrow$ (2) By [3, Proposition 3.6] $\rho(Y)$ is a completely regular compact space. Then $\rho(f)$ is a continuous map from the Tychonoff space $\rho(X)$ to the compact Tychonoff space $\rho(Y)$. Using the previous results, one can see that $\rho(f)(\rho(X))$ is a dense subset of the compact Hausdorff space $\rho(Y) = \beta(\rho(Y))$ which is $C^{\star}$-embedding. Hence by the Theorem 3.2 (2) and the Remark 3.3, $\beta(\rho(f)(\rho(X)) = \rho(Y)$. Finally, since $\rho(f)(\rho(X))$ and $\rho(X)$ are homeomorphic, (2) is satisfied.

(2) $\Longrightarrow$ (1) is an immediate consequence of [3, Proposition 3.6]. $\square$

Corollary 3.9. *Let $f : X \longrightarrow Y$ be a continuous map between Tychonoff spaces, with $f \perp \boldsymbol{I}$. Then the following statements are equivalent.*

(1) *Y is compact;*
(2) *$\beta(X) = Y$ (up to homeomorphism).*

Examples 3.10.

(1) Let $i :]0, 1] \longrightarrow [0, 1]$ be the canonical injection. Clearly i is a dense mapping between two Tychonoff spaces and the second space is compact. So, $\beta(]0, 1]) \neq [0, 1]$ because i is not orthogonal to $\mathbf{I}$. Indeed the continuous mapping $g :]0, 1] \to [0, 1]$ defined by $g(x) = |\sin(\dfrac{1}{x})|$ can not be extended to $x = 0$.
(2) Let $i :]0, 1] \longrightarrow]0, 1]$ the canonical injection. Clearly i is a dense mapping between two Tychonoff spaces. Since i is an isomorphism it is orthogonal to $\mathbf{I}$, but $\beta(]0, 1]) \neq]0, 1]$ because $]0, 1]$ is not compact.

Remark 3.11. Regarding [1], the authors in [1, Proposition 4.11], proved that, for any continuous map $f : X \longrightarrow Y$ between two Tychonoff spaces, $\beta(f)$ is a homeomorphism if and only if $\beta(f(X)) = \beta(Y)$, so in our case this result becomes trivial because $\beta(f) : \beta(X) \longrightarrow \beta(Y)$ is a homeomorphism if and only if f is orthogonal to $\mathbf{I}$ and in this situation, by Proposition 2.9, X is homoeomorphic to $f(X)$ and consequently $\beta(f(X))$ is homeomorphic to $\beta(X)$. Finally, $\beta(f(X)) = \beta(Y)$.

To finish this paper, we shield some light on the hull orthogonal of a given topological space. By [6], $\mathbf{Hewitt} = \{\mathbb{R}\}^{\perp\perp}$ and it is clear that any homeomorphic topological space to $\mathbb{R}$ satisfies also this property. The following example shows that the topological space $[0, +\infty[$, which is not homeomorphic to $\mathbb{R}$, satisfies also $\mathbf{Hewitt} = \{[0, +\infty[\}^{\perp\perp}$.

Proposition 3.12. ***Hewitt*** $= \{[0, +\infty[\}^{\perp\perp}$.

Proof. Since $\{\mathbb{R}\}^{\perp} = \mathbf{Hewitt}^{\perp}$ and $\{[0, +\infty[\} \subset \mathbf{Hewitt}$, then $\{\mathbb{R}\}^{\perp} \subset \{[0, +\infty[\}^{\perp}$.

Conversely, let $f : X \longrightarrow Y$ be a continuous map which is orthogonal to $[0, +\infty[$ and let us show that it is orthogonal to $\mathbb{R}$. By Proposition 2.3, we can suppose that X and Y are Tychonoff spaces. Now according to Proposition 2.11, we can suppose that X is a dense subset of a Tychonoff space Y and f is the canonical injection from X to Y.

For this, let g be a continuous map from X to $\mathbb{R}$. Then $g^+ = \max(g, 0)$ (resp., $-g^- = -\min\{g, 0\}$) is a continuous map from X to $[0, +\infty[$. By orthogonality of f to $[0, +\infty[$, there exists a continuous map $\widetilde{g^+}$ (resp.,$\widetilde{-g^-}$) from Y to $[0, +\infty[$ such that $\widetilde{g^+} \circ f = g^+$ (resp., $(\widetilde{-g^-}) \circ f = (-g^-)$). Hence

$$(\widetilde{g^+} - (\widetilde{-g^-})) \circ f = (\widetilde{g^+} \circ f) - ((\widetilde{-g^-}) \circ f) = (g^+) - (-g^-) = g^+ + g^- = g.$$

So the existence of a continuous map $\widetilde{g} = \widetilde{g^+} - \widetilde{-g^-}$ from Y to $\mathbb{R}$ such that $\widetilde{g} \circ g = f$. The uniqueness of a such function follows immediately from the density of X in Y and the fact that $\mathbb{R}$ is Hausdorff. $\square$

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