Some embedding theorems for Hörmander-Beurling spaces

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Dedicated to Professor Manuel Valdivia on the occasion of his 80th birthday

Abstract

In this paper we prove a number of results on sequence space representations and embedding theorems of Hörmander-Beurling spaces. As a consequence and using sharp results of Meise, Taylor and Vogt, a result of Kaballo on short sequences and hypoelliptic operators is extended to ω-hypoelliptic differential operators and to the vector-valued setting.

Key words: Beurling ultradistributions, Hörmander spaces, Hörmander-Beurling spaces, $\omega-$hypoelliptic differential operators.

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1 Introduction and notations

It is well known that the Hörmander spaces $\mathcal{B}_{p,k}$, $\mathcal{B}_{p,k}^{\text{loc}}(\Omega)$ and $\mathcal{B}_{p,k}^{c}(\Omega)$ play a crucial role in the theory of linear partial differential operators (see [2,15,16]). Our research pursues the study on Hörmander spaces and Hörmander spaces in the sense of Beurling and Björck [2] (=Hörmander-Beurling spaces) carried out in [2,8,14–16,19,40,45] and [5,29–31,36,37,44] (see also [18]). In this paper we prove a number of results on sequence space representations and

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embedding theorems of Hörmander-Beurling spaces (extending corresponding results of [29–31]) and as a consequence, and using results of Meise, Taylor and Vogt [24], a result of Kaballo [19] on short sequences and hypoelliptic differential operators is extended to $\omega-$hypoelliptic differential operators and to the vector-valued setting.

The organization of the paper is as follows. Section 2 contains some basic facts about scalar and vector-valued Beurling ultradistributions and the definitions of the spaces which are considered in the paper. In Section 3 we generalize to UMD spaces the Theorem 4.6 of [31], we prove an embedding (and sequence spaces representation) theorem for vector-valued Hörmander-Beurling spaces, we give a result of Rosenthal type [38] (every weakly compact subset of $\mathcal{B}^{\mathrm{loc}}_{\infty,k}(\Omega, E)$ is separable when E is a closed subspace of $l^{\mathbb{N}}_{\infty}$)(see Remark 3.1.1), we prove an embedding theorem when E is non-separable Fréchet space and we pose the following question: Is $\mathcal{B}^{\mathrm{loc}}_{\infty,k}(\Omega, l_{\infty})$ isomorphic to a complemented subspace of $l^{\mathbb{N}}_{\infty}$? (see Remark 3.1.3). In Section 4 we show that, in general, the topology induced by $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E)$ on $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega) \bigotimes E$ is strictly finer than the ε topology and strictly coarser than the π topology (our example extends to $1 < p < \infty$, by using a different technique, the example studied in [31, Remark 4.7.2]) and we pose another question: Are the spaces $\mathcal{B}^{\mathrm{loc}}_{\infty,k}(\Omega, l_{\infty})$ and $\mathcal{B}^{\mathrm{loc}}_{\infty,k}(\Omega)\hat{\otimes}_{\varepsilon} l_{\infty}$ isomorphic? We also give a sequence space representation theorem when E is a nuclear Fréchet space (for example it is shown that if $E \simeq s$ or $s^{\mathbb{N}}$ then $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E)$ is isomorphic to $(\mathcal{D}_{L^p})^{\mathbb{N}}$). Then, using results of Meise, Taylor and Vogt [24], we extend a result of Kaballo [19] to $\omega-$hypoelliptic differential operators.

Notations. The linear spaces we use are defined over $\mathbb{C}$. Let E and F be locally convex spaces. Then $\mathcal{L}_b(E, F)$ is the locally convex space of all continuous linear operators equipped with the bounded convergence topology. The (topological) dual of E is denoted by E' and is given the strong topology so that $E' = \mathcal{L}_b(E, \mathbb{C})$. $E\hat{\otimes}_{\varepsilon}F$ (resp. $E\hat{\otimes}_{\pi}F$) is the completion of the injective (resp. projective) tensor product of E and F. If E and F are (topologically) isomorphic we put $E \simeq F$. If E is isomorphic to a subspace (resp. complemented subspace) of F we write $E \subset F$ (resp. $E < F$). We put $E \hookrightarrow F$ if E is a linear subspace of F and the canonical injection is continuous (we replace $\hookrightarrow$ by $\overset{d}{\hookrightarrow}$ if E is also dense in F). If $(E_n)^{\infty}_{n=1}$ is a sequence of locally convex spaces, $\prod^{\infty}_{n=1} E_n$ ($E^{\mathbb{N}}$ if $E_n = E$ for all n) is the topological product of the spaces E_n; $\bigoplus^{\infty}_{n=1} E_n$ ($E^{(N)}$ if $E_n = E$ for all n) is the locally convex direct sum of the spaces E_n. The Fréchet space defined by the projective sequence of Fréchet spaces E_n and linking maps A_n will be denoted by $\mathrm{proj}(E_n, A_n)$ (or $\mathrm{proj}E_n$, for short). This projective limit is said to be reduced if $\overline{\mathrm{Im}P_j} = E_j$ for $j = 1, 2, ...,$ being $P_j : \mathrm{proj}(E_n, A_n) \to E_j : (e_n)^{\infty}_1 \to e_j$. If the E_n are Banach spaces and the maps A_n are surjective then $\mathrm{proj}(E_n, A_n)$ is said to be a quojection (see e.g. [28]).

Let $1 \le p \le \infty$, $k : \mathbb{R}^n \to (0, \infty)$ a Lebesgue measurable function, and E a Fréchet space. Then $L_p(E)$ is the set of all (equivalence classes of) Bochner measurable functions $f : \mathbb{R}^n \to E$ for which $\|f\|_p = \left(\int_{\mathbb{R}^n} \|f(x)\|^p dx\right)^{1/p}$ is finite (with the usual modification when $p = \infty$) for all $\|\cdot\| \in \mathrm{cs}(E)$ (see, e.g. [10]). $L_{p,k}(E)$ denotes the set of all Bochner measurable functions $f : \mathbb{R}^n \to E$ such that $kf \in L_p(E)$. Putting $\|f\|_{L_{p,k}(E)} = \|kf\|_p$ for all $f \in L_{p,k}(E)$ and for all $\|\cdot\| \in \mathrm{cs}(E)$, $L_{p,k}(E)$ becomes a Fréchet space isomorphic to $L_p(E)$. When E is the field $\mathbb{C}$, we simply write L_p and $L_{p,k}$. If $f \in L_1(E)$ the Fourier transform of f , $\hat{f}$ or $\mathcal{F}f$, is defined by $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-i\xi x}dx$. If f is a function on $\mathbb{R}^n$ then $\tilde{f}(x) = f(-x)$ for $x \in \mathbb{R}^n$.

Finally we recall the definition of A_p^* functions. A positive, locally integrable function ω on $\mathbb{R}^n$ is in A_p^* provided, for $1 < p < \infty$,

$$\sup_R \left(\frac{1}{|R|}\int_R \omega dx\right)\left(\frac{1}{|R|}\int_R \omega^{-p'/p} dx\right)^{p/p'} < \infty,$$

where R runs over all bounded $n-$dimensional intervals. The basic properties of these functions can be found in [9].

2 Spaces of Beurling ultradistributions. Hörmander-Beurling spaces

In this section we collect some basic facts about vector-valued (Beurling) ultradistributions and we recall the definitions of the vector-valued Hörmander-Beurling spaces. Comprehensive treatments of the theory of (scalar or vector-valued) ultradistributions can be found in [2,13,20,21]. Our notations are based on [2] and [41].

Let $\mathcal{M}$ (or $\mathcal{M}_n$) be the set of all functions ω on $\mathbb{R}^n$ such that $\omega(x) = \sigma(|x|)$ where $\sigma(t)$ is an increasing continuous concave function on $[0, \infty[$ with the following properties:
(i) $\sigma(0) = 0$,
(ii)$\int_0^\infty \frac{\sigma(t)}{1+t^2} dt < \infty$ (Beurling's condition),
(iii) there exist a real number a and a positive number b such that

$$\sigma(t) \geq a + b\log(1 + t) \quad \text{for all} \quad t \geq 0.$$

The assumption (ii) is essentially the Denjoy-Carleman non-quasianalyticity condition (see [2]). The two most prominent examples of functions $\omega \in \mathcal{M}$ are given by $\omega(x) = \log(1 + |x|)^d$, $d > 0$, and $\omega(x) = |x|^\beta$, $0 < \beta < 1$.

If $\omega \in \mathcal{M}$ and E is a Fréchet space, we denote by $\mathcal{D}_\omega(E)$ the set of all functions $f \in L_1(E)$ with compact support, such that $\|f\|_\lambda = \int_{\mathbb{R}^n} \|\hat{f}(\xi)\|e^{\lambda\omega(\xi)}d\xi < \infty$, for all $\lambda > 0$ and for all $\|\cdot\| \in \mathrm{cs}(E)$. For each compact subset K of $\mathbb{R}^n$, $\mathcal{D}_\omega(K, E) = \{f \in \mathcal{D}_\omega(E) : \mathrm{supp} f \subset K\}$, equipped with the topology induced by the family of seminorms $\{\|\cdot\|_\lambda : \|\cdot\| \in \mathrm{cs}(E), \lambda > 0\}$, is a Fréchet space and $\mathcal{D}_\omega(E) = \underset{\overrightarrow{K}}{\mathrm{ind}}\mathcal{D}_\omega(K, E)$ becomes a strict (LF)-space. If Ω is any open set in $\mathbb{R}^n$, $\mathcal{D}_\omega(\Omega, E)$ is the subspace of $\mathcal{D}_\omega(E)$ consisting of all functions f with $\mathrm{supp} f \subset \Omega$. $\mathcal{D}_\omega(\Omega, E)$ is endowed with the corresponding inductive limit topology: $\mathcal{D}_\omega(\Omega, E) = \underset{\overrightarrow{K}}{\text{ind}}\mathcal{D}_\omega(K, E)$. Let $\mathcal{S}_\omega(E)$ be the set of all functions $f \in L_1(E)$ such that both f and $\hat{f}$ are infinitely differentiable functions on $\mathbb{R}^n$ with $\sup_{x\in\mathbb{R}^n} e^{\lambda\omega(x)} \|\partial^\alpha f(x)\| < \infty$ and $\sup_{x\in\mathbb{R}^n} e^{\lambda\omega(x)} \|\partial^\alpha \hat{f}(x)\| < \infty$ for all multi-indices α and all positive numbers λ and all $\|\cdot\| \in \text{cs}(E)$. $\mathcal{S}_\omega(E)$ with the topology induced by the above family of seminorms is a Fréchet space and the Fourier transformation $\mathcal{F}$ is an automorphism of $\mathcal{S}_\omega(E)$. If $E = \mathbb{C}$ then $\mathcal{D}_\omega(E)$ and $\mathcal{S}_\omega(E)$ coincide with the spaces $\mathcal{D}_\omega$ and $\mathcal{S}_\omega$ (see [2]). Let us recall that, by Beurling's condition, the space $\mathcal{D}_\omega$ is non-trivial and the usual procedure of the resolution of unity can be established with $\mathcal{D}_\omega-$functions (see [2]). Furthermore $\mathcal{D}_\omega \overset{d}{\hookrightarrow} \mathcal{D}$ (see [2]) and $\mathcal{D}_\omega$ is nuclear [45]. On the other hand, $\mathcal{D}_\omega = \mathcal{D} \cap \mathcal{S}_\omega$, $\mathcal{D}_\omega \overset{d}{\hookrightarrow} \mathcal{S}_\omega \overset{d}{\hookrightarrow} \mathcal{S}$ (see [2]) and $\mathcal{S}_\omega$ is nuclear too (see [13]). If $\mathcal{E}_\omega$ is the set of multipliers on $\mathcal{D}_\omega$, i.e., the set of all functions $f : \mathbb{R}^n \to \mathbb{C}$ such that $\varphi f \in \mathcal{D}_\omega$, for all $\varphi \in \mathcal{D}_\omega$, then $\mathcal{E}_\omega$ with the topology generated by the seminorms $\{f \to \|\varphi f\|_\lambda = \int_{\mathbb{R}^n} |\widehat{\varphi f}(\xi)| e^{\lambda\omega(\xi)} d\xi : \lambda > 0, \varphi \in \mathcal{D}_\omega\}$ becomes a nuclear Fréchet space (see [45]) and $\mathcal{D}_\omega \overset{d}{\hookrightarrow} \mathcal{E}_\omega$. Using the above results and [21, Theorem 1.12] we can identify $\mathcal{S}_\omega(E)$ with $\mathcal{S}_\omega \hat{\otimes}_\varepsilon E$. However, though $\mathcal{D}_\omega \otimes E$ is dense in $\mathcal{D}_\omega(E)$, in general $\mathcal{D}_\omega(E)$ is not isomorphic to $\mathcal{D}_\omega \hat{\otimes}_\varepsilon E$ (cf., e.g. [12]). A continuous linear operator from $\mathcal{D}_\omega$ into E is said to be a (Beurling) ultradistribution with values in E. We write $\mathcal{D}'_\omega(E)$ for the space of all E-valued (Beurling) ultradistributions endowed with the bounded convergence topology, thus $\mathcal{D}'_\omega(E) = \mathcal{L}_b(\mathcal{D}_\omega, E)$. $\mathcal{D}'_\omega(\Omega, E) = \mathcal{L}_b(\mathcal{D}'_\omega(\Omega), E)$ is the space of all (Beurling) ultradistributions on Ω with values in E. A continuous linear operator from $\mathcal{S}_\omega$ into E is said to be an $E-$valued tempered ultradistribution. $\mathcal{S}'_\omega(E)$ is the space of all E-valued tempered ultradistributions equipped with the bounded convergence topology, i.e., $\mathcal{S}'_\omega(E) = \mathcal{L}_b(\mathcal{S}_\omega, E)$. The Fourier transformation $\mathcal{F}$ is an automorphism of $\mathcal{S}'_\omega(E)$.

If $\omega \in \mathcal{M}$, then $\mathcal{K}_\omega$ is the set of all positive functions k on $\mathbb{R}^n$ for which there exists a positive constant N such that $k(x+y) \le e^{N\omega(x)} k(y)$ for all x and y in $\mathbb{R}^n$, cf. [2] (when $\omega(x) = \log(1+|x|)$ the functions k of the corresponding class $\mathcal{K}_\omega$ are called temperate weight functions, see [16]). If $k, k_1, k_2 \in \mathcal{K}_\omega$ and s is a real number then $\log k$ is uniformly continuous, $k^s \in \mathcal{K}_\omega$, $k_1k_2 \in \mathcal{K}_\omega$ and $M_k(x) = \sup_{y\in\mathbb{R}^n} \frac{k(x+y)}{k(y)} \in \mathcal{K}_\omega$ (see [2]). If $u \in L_1^{\text{loc}}$ and $\int_{\mathbb{R}^n} \varphi(x)u(x)dx = 0$ for all $\varphi \in \mathcal{D}_\omega$, then $u = 0$ a.e. (see [2]). This result, the Hahn-Banach theorem and [7, Chapter II, Corollary 7] prove that if $k \in \mathcal{K}_\omega$, $p \in [1, \infty]$ and E is a Fréchet space, we can identify $f \in L_{p,k}(E)$ with the $E-$valued tempered ultradistribution $\varphi \to \langle\varphi, f\rangle = \int_{\mathbb{R}^n} \varphi(x)f(x)dx$, $\varphi \in \mathcal{S}_\omega$, and $L_{p,k}(E) \hookrightarrow \mathcal{S}'_\omega(E)$. If $\omega \in \mathcal{M}$, $k \in \mathcal{K}_\omega$, $p \in [1, \infty]$ and E is a Fréchet space, we denote by $\mathcal{B}_{p,k}(E)$ the set of all $E-$valued tempered ultradistributions T for which there exists a function $f \in L_{p,k}(E)$ such that $\langle\varphi, \hat{T}\rangle = \int_{\mathbb{R}^n} \varphi(x)f(x)dx$, $\varphi \in \mathcal{S}_\omega$. $\mathcal{B}_{p,k}(E)$ with the seminorms $\{\|T\|_{p,k} = ((2\pi)^{-n} \int_{\mathbb{R}^n} \|k(x)\hat{T}(x)\|^p dx)^{1/p} : \|\cdot\| \in \text{cs}(E)\}$ (usual modification if $p = \infty$), becomes a Fréchet space isomorphic to $L_{p,k}(E)$. Spaces $\mathcal{B}_{p,k}(E)$ are called Hörmander-Beurling spaces with values in E (see [2] for the scalar case and [44] for the vector-valued case). We denote by

$\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E)$ (see [30]) the space of all $E-$valued ultradistributions $T \in \mathcal{D}'_\omega(\Omega, E)$ such that, for every $\varphi \in \mathcal{D}_\omega(\Omega)$, the map $\varphi T : \mathcal{S}_\omega \to E$ defined by $\langle u, \varphi T\rangle = \langle u\varphi, T\rangle$, $u \in \mathcal{S}_\omega$, belongs to $\mathcal{B}_{p,k}(E)$. The space $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E)$ is a Fréchet space with the topology generated by the seminorms $\{\|\cdot\|_{p,k,\varphi} : \varphi \in \mathcal{D}_\omega(\Omega), \|\cdot\| \in \mathrm{cs}(E)\}$, where $\|T\|_{p,k,\varphi} = \|\varphi T\|_{p,k}$ for $T \in \mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E)$, and $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E) \hookrightarrow \mathcal{D}'_\omega(\Omega, E)$. We shall also use the spaces $\mathcal{B}^{\mathrm{c}}_{p,k}(\Omega, E)$ which generalize the scalar spaces $\mathcal{B}^{\mathrm{c}}_{p,k}(\Omega)$ considered by Hörmander in [16], by Vogt in [45] and by Björck in [2]. If ω, k, p, Ω and E are as above, then $\mathcal{B}^{\mathrm{c}}_{p,k}(\Omega, E) = \bigcup_{j=1}^{\infty}[\mathcal{B}_{p,k}(E) \cap \mathcal{E}'_\omega(K_j, E)]$ (here (Kj) is any fundamental sequence of compact subsets of Ω and $\mathcal{E}'_\omega(K_j, E)$ denotes the set of all $T \in \mathcal{D}_\omega(E)$ such that supp$T \subset K_j$). Since for every compact $K \subset \Omega$, $\mathcal{B}_{p,k}(E) \cap \mathcal{E}'_\omega(K_j, E)$ is a Fréchet space with the topology induced by $\mathcal{B}_{p,k}(E)$, it follows that $\mathcal{B}^{\mathrm{c}}_{p,k}(\Omega, E)$ becomes a strict (LF)-space (strict (LB)-space if E is a Banach space): $\mathcal{B}^{\mathrm{c}}_{p,k}(\Omega, E) = \underset{\overrightarrow{j}}{\mathrm{ind}}[\mathcal{B}_{p,k}(E) \cap \mathcal{E}'_\omega(K_j, E)]$. These spaces are studied in [36] and [31].

3 An embedding theorem

In this section we generalize to UMD spaces the Theorem 4.6 of [31], we prove an embedding theorem for vector-valued Hörmander-Beurling spaces (Theorem 3.1, see also Remark 3.1.2) and we give a result of Rosenthal type [38] (every weakly compact subset of $\mathcal{B}^{\mathrm{loc}}_{\infty,k}(\Omega, E)$ is separable when E is a closed subspace of $l_\infty^{\mathbb{N}}$; see Remark 3.1.1).

We shall need the following technical result.

Lemma 3.1 *Let Ω be an open set in $\mathbb{R}^n$, $\omega \in \mathcal{M}$, $k \in \mathcal{K}_\omega$ and $1 \le p \le \infty$. Let $E = \mathrm{proj}(E_j, A_j)$ be the reduced projective limit of the projective sequence of Fréchet spaces E_j and linking maps A_j. Then the map*

$$P : \mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E) \longrightarrow \mathrm{proj}\Big(\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E_j), \overline{A}_j\Big) : T \to \Big(P_j \circ T\Big)_1^\infty$$

is an isomorphism ($\overline{A}_j$ is the map $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E_{j+1}) \to \mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E_j) : T \to A_j \circ T$) and this projective limit is reduced if $p < \infty$. If $E = \prod_{j=1}^\infty E_j$ then the space $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E)$ is isomorphic to $\prod_{j=1}^\infty \mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E_j)$.

Proof. Although the proof of the lemma is straightforward, for the sake of completeness we give here the proof of the surjectivity of P: Let $\Big(T_j\Big)_1^\infty$ be any element in $\mathrm{proj}\Big(\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E_j), \overline{A}_j\Big)$. For each $\varphi \in \mathcal{D}_\omega(\Omega)$ and each $j \ge 1$, we have $A_j\Big(\langle\varphi, T_{j+1}\rangle\Big) = \langle\varphi, A_j \circ T_{j+1}\rangle = \langle\varphi, T_j\rangle$ and so $\Big(\langle\varphi, T_j\rangle\Big)_1^\infty \in \mathrm{proj}(E_j, A_j)$. Let $T : \mathcal{D}_\omega \to E$ be defined by $\langle\varphi, T\rangle := \Big(\langle\varphi, T_j\rangle\Big)_1^\infty$ for $\varphi \in \mathcal{D}_\omega(\Omega)$. Let us prove that $T \in \mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E)$, i.e., that for every $\varphi \in \mathcal{D}_\omega(\Omega)$ there is an $f \in L_{p,k}(E)$ such that $\Big\langle\theta, \widehat{\varphi T}\Big\rangle = \int_{\mathbb{R}^n} \theta(x) f(x) dx$ for all $\theta \in \mathcal{S}_\omega$. Given such a φ let $f_j \in L_{p,k}(E_j)$, $j = 1, 2, \ldots$, such that $\Big\langle\theta, \widehat{\varphi T_j}\Big\rangle = \int_{\mathbb{R}^n} \theta(x) f_j(x) dx$ for all $\theta \in \mathcal{S}_\omega$. Then, for every $\theta \in \mathcal{S}_\omega$, we have $\int_{\mathbb{R}^n} \theta(x) A_j \circ f_{j+1}(x) dx = A_j\Big(\int_{\mathbb{R}^n} \theta(x) f_{j+1}(x) dx\Big) = A_j\Big(\langle \theta, (\varphi T_{j+1})^\wedge \rangle\Big) = \langle \theta, A_j \circ (\varphi T_{j+1})^\wedge \rangle = \langle \theta, [\varphi(A_j \circ T_{j+1})]^\wedge \rangle = \langle \theta, (\varphi T_j)^\wedge \rangle = \int_{\mathbb{R}^n} \theta(x) f_j(x) dx$, that is, $\int_{\mathbb{R}^n} \theta(x)[A_j \circ f_{j+1}(x) - f_j(x)] dx = 0$. Hence it follows (see Section 2) that $A_j \circ f_{j+1}(x) = f_j(x)$ for almost all $x \in \mathbb{R}^n$. Then, modifying the functions f_j in a nullset if necessary, we get $\big(f_j(x)\big)_1^\infty \in \mathrm{proj}(E_j, A_j)$ for all $x \in \mathbb{R}^n$. It is easy to verify that the function $f(x) = \big(f_j(x)\big)_1^\infty$ is Bochner measurable. In fact, if $\phi \in E'$ we can find $N \geq 1$ and $(e'_1, \ldots, e'_N) \in E'_1 \times \cdots \times E'_N$ (see, e.g. [25]) such that $\langle (e_j)_1^\infty, \phi \rangle = \sum_{j=1}^N \langle e_j, e'_j \rangle$, $(e_j)_1^\infty \in E$. Thus $\phi \circ f = \sum_{j=1}^N e'_j \circ f_j$ is measurable. Moreover, if N_j is a nullset such that $f_j(\mathbb{R}^n \setminus N_j)$ is separable, then $f(\mathbb{R}^n \setminus \bigcup N_j)$ is also separable. Hence by the Pettis's measurability theorem (in Fréchet spaces, see e.g. [10]) it follows that f is Bochner measurable. Then, by using the properties of the f_j, $j = 1, 2, \ldots$, we conclude that $f \in L_{p,k}(E)$. Finally, since $\int_{\mathbb{R}^n} \theta(x) f(x) dx = \Big(\int_{\mathbb{R}^n} \theta(x) f_j(x) dx\Big)_1^\infty = \Big(\langle \theta, \widehat{\varphi T_j} \rangle\Big)_1^\infty = \Big(\langle \hat{\theta}\varphi, T_j \rangle\Big)_1^\infty = \langle \hat{\theta}\varphi, T \rangle = \langle \theta, \widehat{\varphi T} \rangle$ for all $\theta \in \mathcal{S}_\omega$, it follows that $T \in \mathcal{B}_{p,k}^{\mathrm{loc}}(\Omega, E)$. Thus P is surjective. ■

The next lemma generalizes to UMD spaces the Theorem 4.6 of [31]. We will reason as we did in [31] but we will use Theorem 4.2 of [29] instead of Corollary 4.2 of [29]. For convenience of the reader we will give a complete proof. The following elementary fact will be used: "Let $F = \underset{\rightarrow j}{\mathrm{ind}} F_j$ be the strict inductive limit of a properly increasing sequence $F_1 \subset F_2 \subset \ldots$ of Banach spaces. Assume that every F_j is a complemented subspace of F_{j+1} and that G_j is a topological complement of F_j in F_{j+1}. Then the mapping $F_1 \oplus G_1 \oplus G_2 \oplus \cdots \to F : (f_1, g_1, g_2, ...) \to f_1 + g_1 + g_2 + \ldots$ is an isomorphism". We will also need the weighted L_p-spaces of vector-valued entire analytic functions $L_{p,k}^K(E)$ and the operators $S_K(f) = \mathcal{F}^{-1}(\chi_K \hat{f})$ (see [29] and [41]).

Lemma 3.2 *Let Ω be an open set in $\mathbb{R}^n$, $p \in (1, \infty)$ and k a temperate weight function on $\mathbb{R}^n$ with $k^p \in A_p^*$. Let E be a Banach space with the UMD-property. Then the space $\mathcal{B}_{p,k}^{\mathrm{loc}}(\Omega, E)$ is isomorphic to $\prod_{j=0}^\infty H_j$ where H_0 is isomorphic to $l_p(E)$ and H_j is isomorphic to a complemented subspace of $l_p(E)$ for $j = 1, 2, \ldots$.*

Proof. Let (K_j) be a covering of Ω consisting of compact sets such that $K_j \subset \overset{\circ}{K}_{j+1}$, $K_j = \overline{\overset{\circ}{K}_j}$ and $\overset{\circ}{K}_j$ has the segment property (we may also assume, without loss of generality, that each K_j is a finite union of $n-$dimensional compact intervals). Then $\mathcal{B}_{p,k}^{\mathrm{c}}(\Omega, E) = \underset{\rightarrow j}{\mathrm{ind}}\Big[\mathcal{B}_{p,k}(E) \cap \mathcal{E}'(K_j, E)\Big]$. In this inductive limit, the step $\mathcal{B}_{p,k}(E) \cap \mathcal{E}'(K_j, E)$ is isomorphic (via Fourier transform) to $L_{p,k}^{-K_j}(E)$ and this space is isomorphic, by Theorem 4.2 and Corollary 5.1 of [29], to $l_p(E)$. Furthermore, $L_{p,k}^{-K_j}(E)$ is a complemented subspace of $L_{p,k}^{-K_{j+1}}(E) : L_{p,k}^{-K_{j+1}}(E) = L_{p,k}^{-K_j}(E) \oplus \Big[\ker S_{-K_j} \cap L_{p,k}^{-K_{j+1}}(E)\Big]$. Thus, the space $G_j = \ker S_{-K_j} \cap L_{p,k}^{-K_{j+1}}(E)$ is isomorphic to an infinite-dimensional complemented subspace of $l_p(E)$. Then, by using the former result, we obtain $\mathcal{B}^{c}_{p,k}(\Omega,E) \simeq L^{-K_1}_{p,k}(E)\oplus G_1\oplus G_2\oplus\cdots \simeq l_p(E)\oplus G_1\oplus G_2\oplus\ldots$. Next, since $1/\tilde{k}$ is a temperate weight function on $\mathbb{R}^n$ such that $1/\tilde{k}^{p'} \in A^*_{p'}$ and $E' \in UMD$ (see [39]), we see that $\mathcal{B}^{c}_{p',1/\tilde{k}}(\Omega,E') \simeq \bigoplus_{j=0}^{\infty} B_j$ where $B_0 \simeq l_{p'}(E')$ and $B_j < l_{p'}(E')$ for $j = 1,2,\ldots$. Therefore, by Theorem 3.2 of [31] (see [16] also), we get $\mathcal{B}^{loc}_{p,k}(\Omega,E) \simeq \left(\mathcal{B}^{c}_{p',1/\tilde{k}}(\Omega,E')\right)' \simeq \left(\bigoplus_{j=0}^{\infty} B_j\right)' \simeq \prod_{j=0}^{\infty} B'_j = \prod_{j=0}^{\infty} H_j$ (here $H_j = B'_j$) where $H_0 \simeq l_p(E)$ and $H_j < l_p(E)$ for $j = 1,2,\ldots$, and the proof is complete. ■

Remark. One can improve Lemma 3.2 by using [45]. Indeed, using the arguments of [45] it can be shown that $\mathcal{B}^{loc}_{p,k}(\Omega,E) \simeq (\mathcal{B}_{p,k}(E)\cap\mathcal{E}'(Q,E))^{\mathbb{N}}$ where $Q = [0,1]^n$. Then, reasoning as in the lemma, we obtain the isomorphism $\mathcal{B}^{loc}_{p,k}(\Omega,E) \simeq (l_p(E))^{\mathbb{N}}$.

We now present the main result of this section, an embedding (and sequence space representation) theorem for vector-valued Hörmander-Beurling spaces (see also Remark 3.1). We also pose a related question (Remark 3.1.3): Is $\mathcal{B}^{loc}_{\infty,k}(\Omega,l_\infty)$ isomorphic to a complemented subspace of $l^{\mathbb{N}}_\infty$? We will use the Fréchet spaces $l_{q^+} = \bigcap_{p>q} l_p$ and $L_{q^-} = \bigcap_{p<q} L_p([0,1])$ (these spaces have an interest in the structure theory of Fréchet spaces and are primary and have all nuclear $\Lambda_1(\alpha)-$spaces as complemented subspaces, see [27] and [3]).

Theorem 3.1 *Let Ω be an open set in $\mathbb{R}^n$, $\omega \in \mathcal{M}$, $k \in \mathcal{K}_\omega$ and $1 \le p,q \le \infty$, and let E be a Fréchet space.*

(1) If $p < \infty$ and E is separable then $\mathcal{B}^{loc}_{p,k}(\Omega,E)$ is isomorphic to a subspace of $\left(C([0,1])\right)^{\mathbb{N}}$ and this space does not contain any complemented copy of $\mathcal{B}^{loc}_{p,k}(\Omega,E)$.

(2) If E is separable and infinite-dimensional and $E \not\simeq \mathbb{C}^{\mathbb{N}}$ then $\mathcal{B}^{loc}_{\infty,k}(\Omega,E)$ is isomorphic to a subspace of $l^{\mathbb{N}}_\infty$ but this space does not contain any complemented copy of $\mathcal{B}^{loc}_{\infty,k}(\Omega,E)$. If $E \simeq \mathbb{C}^{\mathbb{N}}$ then $\mathcal{B}^{loc}_{\infty,k}(\Omega,E)$ is isomorphic to $l^{\mathbb{N}}_\infty$.

(3) Suppose $E \subset F^{\mathbb{N}}$ (resp. $< F^{\mathbb{N}}$) where F is a Banach space. Then $l^{\mathbb{N}}_1 < \mathcal{B}^{loc}_{1,k}(\Omega,E) \subset \left(l_1(F)\right)^{\mathbb{N}}$ (resp. $< \left(l_1(F)\right)^{\mathbb{N}}$). If F is a dual space and has the Radon-Nikodým property, then $l^{\mathbb{N}}_\infty < \mathcal{B}^{loc}_{\infty,k}(\Omega,E) \subset \left(l_\infty(F)\right)^{\mathbb{N}}$ (resp. $< \left(l_\infty(F)\right)^{\mathbb{N}}$). If F has the UMD-property then $l^{\mathbb{N}}_p < \mathcal{B}^{loc}_{p,k}(\Omega,E) \subset \left(l_p(F)\right)^{\mathbb{N}}$ (resp. $< \left(l_p(F)\right)^{\mathbb{N}}$) provided that $1 < p < \infty$ and k is a temperate weight with $k^p \in A^*_p$; in particular, $\mathcal{B}^{loc}_{p,k}(\Omega,l^{\mathbb{N}}_p)$ is isomorphic to $l^{\mathbb{N}}_p$.

(4) Suppose $1 < p < \infty$ and that k is a temperate weight with $k^p \in A^*_p$, and let $E = l_{q^+}$ with $q < \infty$ (resp. $L_{q^-}([0,1])$ with $1 < q$). Let $(q_j)_1^\infty$ be any sequence such that $q_j \searrow q$ (resp. $q_j \nearrow q$). Then $\mathcal{B}^{loc}_{p,k}(\Omega,E)$ is isomorphic to a subspace of $G := \left(\prod_{j=1}^{\infty} l_p(l_{q_j})\right)^{\mathbb{N}}$ (resp. $H := \left(\prod_{j=1}^{\infty} l_p(L_{q_j}([0,1]))\right)^{\mathbb{N}}$) but G (resp. H) does not contain any complemented copy of $\mathcal{B}^{loc}_{p,k}(\Omega,E)$.

(5) Let p, k, q and $(q_j)_1^\infty$ be as in 4. Let X be a Banach subspace of $\mathcal{B}^{loc}_{p,k}(\Omega,l_{q^+})$

(resp. $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, L_{q^-}([0,1]))$). Then X is isomorphic to a subspace of $l_p(l_{q_1} \oplus \cdots \oplus l_{q_m})$ (resp. $l_p(L_{q_1}([0,1]) \oplus \cdots \oplus L_{q_m}([0,1]))$) for some integer m.

Proof. 1. The first claim is a consequence from the fact that every separable Fréchet space is isomorphic to a subspace of $\Big(C([0,1])\Big)^{\mathbb{N}}$ (see e.g. [1, p.51]). Now suppose that $\Big(C([0,1])\Big)^{\mathbb{N}}$ contains a complemented copy of $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E)$. Then $\Big(C([0,1])\Big)^{\mathbb{N}}$ also contains a complemented copy of $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega)$ since this space is clearly isomorphic to a complemented subspace of $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E)$. Hence it follows, if $p = 1$, that $\Big(C([0,1])\Big)^{\mathbb{N}}$ contains a complemented copy of $l_1^{\mathbb{N}}$ (the proof given in [45] of the isomorphism $\mathcal{B}^{\mathrm{loc}}_{1,k}(\Omega) \simeq l_1^{\mathbb{N}}$ is also valid for weights $k \in \mathcal{K}_\omega$). Then l_1 becomes isomorphic to a complemented subspace of $C([0,1])$ (see e.g. [6])which contradicts Corollary 2 in [33]. In case $p > 1$ we can apply Proposition 3.7 in [26] and obtain the isomorphism $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega) \simeq \mathbb{C}^{\mathbb{N}}$. This contradicts the fact that $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega)$ is a non-Montel Fréchet space (see [15, Theorem 2.3.9] and [16]). Consequently,$\Big(C([0,1])\Big)^{\mathbb{N}}$ does not contain any complemented copy of $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E)$.

2. We know that $E \subset l_\infty^{\mathbb{N}}$ ([1, p.51]), that $L_\infty \simeq l_\infty$ ([23]) and that $L_\infty(L_\infty) \subset \Big(L_1(L_1)\Big)' \simeq L_1' \simeq L_\infty$ (but $L_\infty(L_\infty) \not\simeq L_\infty$, see [4]). Hence and from Lemma 3.1 it follows that $\mathcal{B}^{\mathrm{loc}}_{\infty,k}(\Omega, E) \subset \mathcal{B}^{\mathrm{loc}}_{\infty,k}\Big(\Omega, L_\infty^{\mathbb{N}}\Big) \simeq \Big(\mathcal{B}^{\mathrm{loc}}_{\infty,k}\Big(\Omega, L_\infty\Big)\Big)^{\mathbb{N}} \subset \Big(\Big(L_\infty(L_\infty)\Big)^{\mathbb{N}}\Big)^{\mathbb{N}} \simeq \Big(L_\infty(L_\infty)\Big)^{\mathbb{N}} \subset L_\infty^{\mathbb{N}} \simeq l_\infty^{\mathbb{N}}$. However, if $E \not\simeq \mathbb{C}^{\mathbb{N}}$, the space $l_\infty^{\mathbb{N}}$ can not contain any complemented copy of $\mathcal{B}^{\mathrm{loc}}_{\infty,k}(\Omega, E)$ by virtue of Proposition 3.12 in [26] (recall that $E < \mathcal{B}^{\mathrm{loc}}_{\infty,k}(\Omega, E)$). On the other hand, if $E \simeq \mathbb{C}^{\mathbb{N}}$ then $\mathcal{B}^{\mathrm{loc}}_{\infty,k}(\Omega, E) \simeq \Big(\mathcal{B}^{\mathrm{loc}}_{\infty,k}(\Omega)\Big)^{\mathbb{N}} \simeq \Big(l_\infty^{\mathbb{N}}\Big)^{\mathbb{N}} \simeq l_\infty^{\mathbb{N}}$ by Lemma 3.1 and [31, Theorem 4.2(3)].

3. By Lemma 3.1 and by [45] and [31, Theorem 4.2(2)], we have $l_1^{\mathbb{N}} \simeq \mathcal{B}^{\mathrm{loc}}_{1,k}(\Omega) < \mathcal{B}^{\mathrm{loc}}_{1,k}(\Omega, E) \subset$ (resp. $<$)$\mathcal{B}^{\mathrm{loc}}_{1,k}\Big(\Omega, F^{\mathbb{N}}\Big) \simeq \Big(\mathcal{B}^{\mathrm{loc}}_{1,k}\Big(\Omega, F\Big)\Big)^{\mathbb{N}} \simeq \Big((l_1(F)^{\mathbb{N}}\Big)^{\mathbb{N}} \simeq \Big(l_1(F)\Big)^{\mathbb{N}}$.
If F is a dual space and has the Radon-Nikodým property then $l_\infty^{\mathbb{N}} \simeq \mathcal{B}^{\mathrm{loc}}_{\infty,k}(\Omega) < \mathcal{B}^{\mathrm{loc}}_{\infty,k}(\Omega, E) \subset$ (resp. $<$)$\mathcal{B}^{\mathrm{loc}}_{\infty,k}\Big(\Omega, F^{\mathbb{N}}\Big) \simeq \Big(\mathcal{B}^{\mathrm{loc}}_{\infty,k}\Big(\Omega, F\Big)\Big)^{\mathbb{N}} \simeq \Big((l_\infty(F)^{\mathbb{N}}\Big)^{\mathbb{N}} \simeq \Big(l_\infty(F)\Big)^{\mathbb{N}}$ by virtue of Lemma 3.1 and [31, Theorem 4.2(3)].
Suppose now that F has the UMD-property, $1 < p < \infty$ and $k^p \in A_p^*$. By using [31, Remark 4.7(1)] (see also [14]), Lemma 3.1 and Lemma 3.2, we get $l_p^{\mathbb{N}} \simeq \mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega) < \mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E) \subset$ (resp. $<$)$\mathcal{B}^{\mathrm{loc}}_{p,k}\Big(\Omega, F^{\mathbb{N}}\Big) \simeq \Big(\mathcal{B}^{\mathrm{loc}}_{p,k}\Big(\Omega, F\Big)\Big)^{\mathbb{N}} < \Big((l_p(F)^{\mathbb{N}}\Big)^{\mathbb{N}} \simeq \Big(l_p(F)\Big)^{\mathbb{N}}$. Hence and from [42, (1)p.331] it follows that $\mathcal{B}^{\mathrm{loc}}_{p,k}\Big(\Omega, l_p^{\mathbb{N}}\Big) \simeq l_p^{\mathbb{N}}$ (see also [31, Remark 4.7(1)] or [14]).

4. Since the proofs of both claims are similar, we shall only proceed with the proof of the second one.
Put $E = L_{q^-}([0,1])$ and let (q_j) be a sequence such that $q_j \nearrow q$. Then, tak-

ing into account Lemma 3.1 and Lemma 3.2 (the spaces $L_{q_j}([0,1])$ have the UMD-property, see e.g. [39]), we have $\mathcal{B}^{\rm loc}_{p,k}(\Omega,E)\subset\mathcal{B}^{\rm loc}_{p,k}\Big(\Omega,\prod_{j=1}^{\infty}L_{q_j}([0,1])\Big)\simeq\prod_{j=1}^{\infty}\mathcal{B}^{\rm loc}_{p,k}\Big(\Omega,L_{q_j}([0,1])\Big)<\prod_{j=1}^{\infty}\Big(l_p\Big(L_{q_j}([0,1])\Big)\Big)^{\mathbb{N}}\simeq\Bigg(\prod_{j=1}^{\infty}l_p\Big(L_{q_j}([0,1])\Big)\Bigg)^{\mathbb{N}}=H$. Furthermore, since all complemented subspace of a quojection is a quojection (see [28]), H is a quojection (actually $H\simeq\prod_{r=1}^{\infty}X_r$ where each X_r coincides with some $l_p(L_{q_j}([0,1]))$), $E<\mathcal{B}^{\rm loc}_{p,k}(\Omega,E)$ and E is not a quojection (see [3]), it follows that H does not contain any complemented copy of $\mathcal{B}^{\rm loc}_{p,k}(\Omega,E)$.
5. Let X be a Banach subspace of $\mathcal{B}^{\rm loc}_{p,k}(\Omega,l_{q^+})$ (resp. $\mathcal{B}^{\rm loc}_{p,k}(\Omega,L_{q^-}([0,1]))$). By using 4 we see that X is isomorphic to a subspace of $\prod_{r=1}^{\infty}Y_r$ (resp. $\prod_{r=1}^{\infty}X_r$) where each Y_r (resp. X_r) coincides with some $l_p(l_{q_j})$ (resp. $l_p(L_{q_j}([0,1]))$), thus ([6]) X becomes isomorphic to a subspace of $l_p(l_{q_1}\oplus\cdots\oplus l_{q_m})$ (resp. $l_p(L_{q_1}([0,1])\oplus\cdots\oplus L_{q_m}([0,1])))$ for some integer m. ∎

Remark 3.1 1. In [38] Rosenthal showed that if (Ω,Σ,μ) is a finite measure space then every weakly compact subset of $L_\infty(\mu)$ is norm separable. By using this result it is easy to show that if $E\subset l_\infty^{\mathbb{N}}$ then every weakly compact subset of $\mathcal{B}^{\rm loc}_{\infty,k}(\Omega,E)$ (and hence every WCG subspace of $\mathcal{B}^{\rm loc}_{\infty,k}(\Omega,E)$) is separable. In fact, let K be a weakly compact subset of $\mathcal{B}^{\rm loc}_{\infty,k}(\Omega,E)$. Then K becomes a weakly compact subset of $\Big(L_\infty([0,1])\Big)^{\mathbb{N}}$ (see the proof of Theorem 3.1(2) and recall that $l_\infty\simeq L_\infty([0,1])$). Now the weak topology

$$\sigma\Big((L_\infty([0,1]))^{\mathbb{N}},((L_\infty([0,1]))^{\mathbb{N}})'\Big)$$

is the product of the weak topologies (see, e.g. [17, p.167]). Consequently the projection of K on every factor $L_\infty([0,1])$ is weakly compact and, by the Rosenthal's result, is norm separable. Hence it follows that K is separable in $(L_\infty([0,1]))^{\mathbb{N}}$ and so is separable in $\mathcal{B}^{\rm loc}_{\infty,k}(\Omega,E)$.
2. Evidently it is possible to replace $C([0,1])$ by l_∞ in Theorem 3.1(1). In the non-separable case we have the following extension: "Let $p<\infty$ be. Let E be a non-separable Fréchet space and let I be a set such that card$I$ = dens$E$. Then $\mathcal{B}^{\rm loc}_{p,k}(\Omega,E)\subset(l_\infty(I))^{\mathbb{N}}$ and this space does not contain any complemented copy of $\mathcal{B}^{\rm loc}_{p,k}(\Omega,E)$." In fact, let $(E_j)_{j=1}^{\infty}$ be a sequence of Banach spaces, with dens$E_j\le$ dens$E$ for all j, such that E is isomorphic to a subspace of $\prod_{j=1}^{\infty}E_j$ (see, e.g. [1, p.34]). Since dens$L_p(E_j)\le$ card$I$, we get $L_p(E_j)\subset l_\infty(I)$ ([1, p.50]) and

$$\mathcal{B}^{\rm loc}_{p,k}(\Omega,E)\subset\mathcal{B}^{\rm loc}_{p,k}(\Omega,\prod_{j=1}^{\infty}E_j)\simeq\prod_{j=1}^{\infty}\mathcal{B}^{\rm loc}_{p,k}(\Omega,E_j)\subset\prod_{j=1}^{\infty}(L_p(E_j))^{\mathbb{N}}$$
$$\subset\prod_{j=1}^{\infty}(l_\infty(I))^{\mathbb{N}}\simeq(l_\infty(I))^{\mathbb{N}}.$$

Finally, since $l_\infty(I)=C(\beta I)$ (βI is the Stone-Čech compactification of I re-

garded in its discrete topology) and βI is extremally disconnected, we apply [26, Proposition 3.12].
3. We finish this note by posing the following question: Let Ω be an open set in $\mathbb{R}^n$, $\omega \in \mathcal{M}$ and $k \in \mathcal{K}_\omega$. Is $\mathcal{B}^{\mathrm{loc}}_{\infty,k}(\Omega, l_\infty)$ isomorphic to a complemented subspace of $l_\infty^{\mathbb{N}}$? (If the answer to this question were yes, $\mathcal{B}^{\mathrm{loc}}_{\infty,k}(\Omega, l_\infty)$ would be isomorphic to $l_\infty^{\mathbb{N}}$ since $\mathcal{B}^{\mathrm{loc}}_{\infty,k}(\Omega) \simeq l_\infty^{\mathbb{N}} < \mathcal{B}^{\mathrm{loc}}_{\infty,k}(\Omega, l_\infty) < l_\infty^{\mathbb{N}}$ implies $\mathcal{B}^{\mathrm{loc}}_{\infty,k}(\Omega, l_\infty) \simeq l_\infty^{\mathbb{N}}$ in virtue of [42, (1) p.331]).

4 On sequence space representations of Hörmander-Beurling spaces and applications

In this section a number of results on sequence space representations of vector-valued Hörmander-Beurling spaces are given (Theorem 4.1; see also Lemma 3.2, [30] and [31]). As a consequence, and using sharp results of Meise, Taylor and Vogt [24], a result of Kaballo (see [19]) on short sequences and hypoelliptic differential operators is extended to ω-hypoelliptic differential operators and to the vector-valued setting.

Lemma 4.1 *Let Ω be an open set in $\mathbb{R}^n$, $\omega \in \mathcal{M}$, $k \in \mathcal{K}_\omega$ and $1 \le p < \infty$. Let E be a Fréchet space. Then the topology induced by $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E)$ on $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega) \otimes E$ is intercalated between the ε and π topologies.*

Proof. Taking into account the corresponding fundamental systems of seminorms the proof is immediate since, for every $\varphi \in D_\omega(\Omega)$ and every $\|\cdot\| \in \mathrm{cs}(E)$, we have

$$\|T\|_{p,k,\varphi} \le \inf \left\{ \sum_1^m \|u_j\|_{p,k,\varphi} \|e_j\| : T = \sum_1^m u_j \otimes e_j \right\}$$

for all $T \in \mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega) \otimes E$, and, for every neighborhood U of 0 in $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega)$ and every $\|\cdot\| \in \mathrm{cs}(E)$, we have

$$\sup_{(\xi,e')\in U^0\times V^0} \left| \sum_1^m \langle u_j, \xi\rangle \langle e_j, e'\rangle \right| \le \max_{1\le i\le r} \|T\|_{p,k,\varphi_i}$$

(here $\varphi_1, \dots, \varphi_r \in D_\omega(\Omega)$ generate U and $V = \{e \in E : \|e\| \le 1\}$) for all $T = \sum_1^m u_j \otimes e_j \in \mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega) \otimes E$. ■

Remark 4.1 1. Note that, in general, the topology induced by $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E)$ on $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega) \otimes E$ is strictly finer than the ε topology and strictly coarser than the π topology: In fact let $1 < p < \infty$, let k a temperate weight function on $\mathbb{R}^n$ with $k^p \in A_p^*$ and assume that $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, l_p)$ contains a complemented copy of $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega)\hat{\otimes}_\varepsilon l_p$. Then, by [31, Remark 4.7(1)] (see also Theorem 3.1(3)) and [22, (5) p.282], we get $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega)\hat{\otimes}_\varepsilon l_p \simeq l_p^{\mathbb{N}}\hat{\otimes}_\varepsilon l_p \simeq \left(l_p\hat{\otimes}_\varepsilon l_p\right)^{\mathbb{N}} < \mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, l_p) \simeq l_p^{\mathbb{N}}$. Hence and from [6] it follows that $l_p\hat{\otimes}_\varepsilon l_p < l_p$, that is to say (since l_p is prime [23, Theorem 2.4.3]), that $l_p\hat{\otimes}_\varepsilon l_p \simeq l_p$. But this is false since $l_p\hat{\otimes}_\varepsilon l_p$ fails to have the

uniform approximation property (UAP, for short; see [34, p.350]) whereas $l_p \in$ UAP by [35]. Therefore, $\mathcal{B}^{\text{loc}}_{p,k}(\Omega)\hat{\otimes}_\varepsilon l_p$ can not be isomorphic to a complemented subspace of $\mathcal{B}^{\text{loc}}_{p,k}(\Omega, l_p)$. In particular, since $\mathcal{B}^{\text{loc}}_{p,k}(\Omega) \otimes l_p$ is dense in $\mathcal{B}^{\text{loc}}_{p,k}(\Omega, l_p)$, the ε topology is strictly coarser than the topology induced by $\mathcal{B}^{\text{loc}}_{p,k}(\Omega, l_p)$. (A different proof, for the case $2 \leq p < \infty$, is given in [31, Remark 4.7(2)]). In a similar way it can be shown that the topology induced by $\mathcal{B}^{\text{loc}}_{p,k}(\Omega, l_p)$ on $\mathcal{B}^{\text{loc}}_{p,k}(\Omega) \otimes l_p$ is strictly coarser than the π topology (recall that $l_p\hat{\otimes}_\pi l_p \notin$ UAP [34, p.350]).
2. If $p = 1$ and k is any weight in $\mathcal{K}_\omega$ one can argue as in 1 (by using [31, Theorem 4.2(2)] and the well known fact that $l_1\hat{\otimes}_\varepsilon l_1$ is not isomorphic to l_1 [7, Chapter VIII]) and show that the topology induced by $\mathcal{B}^{\text{loc}}_{1,k}(\Omega, l_1)$ on $\mathcal{B}^{\text{loc}}_{1,k}(\Omega) \otimes l_1$ is strictly finer than the ε topology.
3. The assertions in the above notes continue to hold when one replaces l_p by $l_p^{\mathbb{N}}$ in 1 and l_1 by $l_1^{\mathbb{N}}$ in 2.
4. Notice also that if the answer to the posed question in Remark 3.1.3 were affirmative, then $\mathcal{B}^{\text{loc}}_{\infty,k}(\Omega)\hat{\otimes}_\varepsilon l_\infty$ would not be isomorphic to $\mathcal{B}^{\text{loc}}_{\infty,k}(\Omega, l_\infty)$ for any $k \in \mathcal{K}_\omega$. In fact, if these spaces were isomorphic then, by [31, Theorem 4.2(3)], [22, (5) p.282], [22, (2) p.287] and a result of Cembranos and Freniche [4, Theorem 3.2.1], we would have $l_\infty^{\mathbb{N}} \simeq l_\infty^{\mathbb{N}}\hat{\otimes}_\varepsilon l_\infty \simeq \left(l_\infty\hat{\otimes}_\varepsilon l_\infty\right)^{\mathbb{N}} \simeq \left(C(\beta\mathbb{N})\hat{\otimes}_\varepsilon l_\infty\right)^{\mathbb{N}} \simeq \left(C(\beta\mathbb{N}, l_\infty)\right)^{\mathbb{N}} > c_0^{\mathbb{N}}$. Therefore c_0 would become a complemented subspace of l_∞ which contradicts a classical result of Phillips (see e.g. [4, Corollary 1.3.2]).

Theorem 4.1 *Let Ω be an open set in $\mathbb{R}^n$, $\omega \in \mathcal{M}$, $k \in \mathcal{K}_\omega$ and $1 \leq p < \infty$. Let E be a nuclear Fréchet space. Then*

(a) $\mathcal{B}^{\text{loc}}_{p,k}(\Omega, E) = \mathcal{B}^{\text{loc}}_{p,k}(\Omega)\hat{\otimes}_\varepsilon E$

(b) if $p = 1$, or, $1 < p < \infty$ and k is a temperate weight with $k^p \in A_p^*$, then $\mathcal{B}^{\text{loc}}_{p,k}(\Omega, E) \simeq \left(l_p(E)\right)^{\mathbb{N}}$

(c) if $p = 1$, or, $1 < p < \infty$ and k is a temperate weight with $k^p \in A_p^*$, and $E \simeq s$ or $s^{\mathbb{N}}$, then $\mathcal{B}^{\text{loc}}_{p,k}(\Omega, E) \simeq (\mathcal{D}_{L^p})^{\mathbb{N}}$

(d) if E is infinite dimensional and $E \not\simeq \mathbb{C}^{\mathbb{N}}$, then $\mathcal{B}^{\text{loc}}_{p,k}(\Omega, E)$ is isomorphic to a (non complemented) subspace of $(L_p([0, 1]))^{\mathbb{N}}$

(e) if E is a power series space of finite type, then $\mathcal{B}^{\text{loc}}_{p,k}(\Omega, E)$ is isomorphic to a complemented subspace of $\mathcal{B}^{\text{loc}}_{p,k}(\Omega, l_{q^+})$ (resp. $\mathcal{B}^{\text{loc}}_{p,k}(\Omega, L_{q^-}([0, 1]))$) for any $q \in [1, \infty[$ (resp. $q \in]1, \infty]$)

(f) if X is a Banach subspace of $\mathcal{B}^{\text{loc}}_{p,k}(\Omega, E)$, then X is isomorphic to a subspace of $L_p([0, 1])$

(g) if $p = 1$, or, $1 < p < \infty$ and k is a temperate weight with $k^p \in A_p^*$, and X is a Banach subspace of $\mathcal{B}^{\text{loc}}_{p,k}(\Omega, E)$, then X is isomorphic to a subspace of l_p

(h) if $1 < p_1, p_2 < \infty$, and k_1, k_2 are temperate weights such that $k_1^{p_1} \in A_{p_1}^*$, $k_2^{p_2} \in A_{p_2}^*$, then $\mathcal{B}^{\text{loc}}_{p_1,k_1}(\Omega, E) \simeq \mathcal{B}^{\text{loc}}_{p_2,k_2}(\Omega, E)$ if and only if $p_1 = p_2$

(i) $\mathcal{B}^{\text{loc}}_{p,k}(\Omega, E)$ is quasinormable, and if $p > 1$ every quotient of $\mathcal{B}^{\text{loc}}_{p,k}(\Omega, E)$ by a closed subspace is reflexive

(j) every exact sequence $0 \longrightarrow \mathcal{B}_{p,k}^{\mathrm{loc}}(\Omega) \longrightarrow G \longrightarrow E \longrightarrow 0$ *where* G *is a Fréchet space,* $1 < p < \infty$ *and* k *is a temperate weight with* $k^p \in A_p^*$*, splits.*

Proof. (a) This is an immediate consequence of Lemma 4.1, the nuclearity of E, the denseness of $\mathcal{D}_\omega(\Omega) \otimes E$ in $\mathcal{B}_{p,k}^{\mathrm{loc}}(\Omega, E)$ (use [36, Proposition 3.4]) and the completeness of $\mathcal{B}_{p,k}^{\mathrm{loc}}(\Omega, E)$.

(b) By using (a), [31, Theorem 4.2], [31, Remark 4.7(1)], [22, (5) p.282], [22, (5) p.198] and [22, (5) p.291], we get $\mathcal{B}_{p,k}^{\mathrm{loc}}(\Omega, E) = \mathcal{B}_{p,k}^{\mathrm{loc}}(\Omega)\hat{\otimes}_\varepsilon E \simeq l_p^{\mathbb{N}}\hat{\otimes}_\varepsilon E \simeq \left(l_p\hat{\otimes}_\varepsilon E\right)^{\mathbb{N}} \simeq \left(l_p(E)\right)^{\mathbb{N}}$.

(c) By Valdivia [43] and Vogt [45], we know that $\mathcal{D}_{L^p}$ is isomorphic to $l_p\hat{\otimes}_\varepsilon s$. Hence and from (b) and [22, (5) p.282] it follows that $\mathcal{B}_{p,k}^{\mathrm{loc}}(\Omega, s) \simeq \left(l_p\hat{\otimes}_\varepsilon s\right)^{\mathbb{N}} \simeq (\mathcal{D}_{L^p})^{\mathbb{N}}$ and $\mathcal{B}_{p,k}^{\mathrm{loc}}(\Omega, s^{\mathbb{N}}) \simeq \left(l_p\hat{\otimes}_\varepsilon s^{\mathbb{N}}\right)^{\mathbb{N}} \simeq \left((l_p\hat{\otimes}_\varepsilon s)^{\mathbb{N}}\right)^{\mathbb{N}} \simeq \left(l_p\hat{\otimes}_\varepsilon s\right)^{\mathbb{N}} \simeq (\mathcal{D}_{L^p})^{\mathbb{N}}$.

(d) The space E is isomorphic to a subspace of $\left(L_p([0,1])\right)^{\mathbb{N}}$ (see e.g. [17, p.483]). Hence and from Lemma 3.1 it follows that

$$\begin{aligned}\mathcal{B}_{p,k}^{\mathrm{loc}}(\Omega, E) \subset \mathcal{B}_{p,k}^{\mathrm{loc}}\Big(\Omega, \left(L_p([0,1])\right)^{\mathbb{N}}\Big) &\simeq \Big(\mathcal{B}_{p,k}^{\mathrm{loc}}\big(\Omega, L_p([0,1])\big)\Big)^{\mathbb{N}} \\ \subset \Big(\big(L_p(L_p([0,1]))\big)^{\mathbb{N}}\Big)^{\mathbb{N}} &\simeq \Big(\big(L_p([0,1])\big)^{\mathbb{N}}\Big)^{\mathbb{N}} \simeq \big(L_p([0,1])\big)^{\mathbb{N}}.\end{aligned}$$

Now we prove that $\mathcal{B}_{p,k}^{\mathrm{loc}}(\Omega, E)$ can not be isomorphic to a complemented subspace of $\left(L_p([0,1])\right)^{\mathbb{N}}$. If this were not the case, E would also be isomorphic to a complemented subspace of $\left(L_p([0,1])\right)^{\mathbb{N}}$. Then E would become a quojection (see e.g. [26]) and thus $E \simeq \mathbb{C}^{\mathbb{N}}$ (see again [26]), a contradiction.

(e) We know that all nuclear $\Lambda_1(\alpha)-$spaces are complemented subspaces of l_{q^+} when $1 \leq q < \infty$ [27] and of $L_{q^-}([0,1])$ when $1 < q \leq \infty$ [3]. Thus, if $E = \Lambda_1(\alpha)$, we have $\mathcal{B}_{p,k}^{\mathrm{loc}}(\Omega, \Lambda_1(\alpha)) < \mathcal{B}_{p,k}^{\mathrm{loc}}(\Omega, l_{q^+})$ (resp. $< \mathcal{B}_{p,k}^{\mathrm{loc}}(\Omega, L_{q^-}([0,1]))$).

(f) By (d) X is isomorphic to a subspace of $\left(L_p([0,1])\right)^{\mathbb{N}}$ and thus (see [6]) isomorphic to a subspace of $L_p([0,1])$.

(g) Since E is isomorphic to a subspace of $l_p^{\mathbb{N}}$ [17, p.483], we may apply Theorem 3.1(3) and conclude that X is also isomorphic to a subspace of $l_p^{\mathbb{N}}$. Thus [6] X becomes isomorphic to a subspace of l_p.

(h) ($\Rightarrow$) From [31, Remark 4.7(1)], the hypothesis and (g) it follows that $l_{p_1} \subset l_{p_2}$ (and $l_{p_2} \subset l_{p_1}$). As is well known this implies $p_1 = p_2$. ($\Leftarrow$) It suffices to apply (b).

(i) Taking into account (b) and recalling that the product of a family of quasinormable spaces is quasinormable [11, p.107] and that the tensor product $\hat{\otimes}_\varepsilon$ of a Banach space and a nuclear space is also quasinormable [12, Ch. II, Proposition 13 p.76], we see that $\mathcal{B}_{p,k}^{\mathrm{loc}}(\Omega, E)$ becomes a quasinormable space. Finally, since $\mathcal{B}_{p,k}^{\mathrm{loc}}(\Omega, E) \subset (L_p([0,1]))^{\mathbb{N}}$ (see the proof of (d)), we conclude the proof by virtue of [11, Corollary p.101].

(j) Since the Fréchet space $\mathcal{B}_{p,k}^{\mathrm{loc}}(\Omega)$ is a quojection (we know that this space is isomorphic to $l_p^{\mathbb{N}}$, see [31] or [14]) it suffices to apply [46, Theorems 5.2 and

1.8]. ■

Remark 4.2 1. Concerning Theorem 4.1 (c) let us recall that a large number of standard spaces of test functions are isomorphic to s or $s^{\mathbb{N}}$. For example, $\mathcal{S}(\mathbb{R}^n) \simeq s$ [42,25], $\mathcal{D}(K) \simeq s$ (K is a compact set in $\mathbb{R}^n$ such that $\overset{\circ}{K} \neq \emptyset$; see [42] and [45]), $C^\infty(\Omega) \simeq s^{\mathbb{N}}$ (Ω is an open set in $\mathbb{R}^n$; see [42] and [45]), $C^\infty(V) \simeq s$ (V is an $n-$dimensional compact $C^\infty-$differentiable manifold; see [42]), $C^\infty(W) \simeq s^{\mathbb{N}}$ (W is an $n-$dimensional $C^\infty-$differentiable manifold not compact and countable at infinity; see [42]).
2. It is well known (see [25]) that the space $A(\mathbb{C}^d)$ of all entire analytic functions can not be isomorphic to either s or $s^{\mathbb{N}}$ but it is isomorphic to a complemented subspace of s. However, if p and k are as in Theorem 4.1 (c), $\mathcal{B}^{\text{loc}}_{p,k}(\Omega, A(\mathbb{C}^d))$ and $(\mathcal{D}_{L^p})^{\mathbb{N}}$ are isomorphic. In fact, we know that

$$\mathcal{B}^{\text{loc}}_{p,k}(\Omega, A(\mathbb{C}^d)) \simeq \mathcal{B}^{\text{loc}}_{p,k}(\Omega)\widehat{\otimes}_\varepsilon A(\mathbb{C}^d)) \simeq l_p^{\mathbb{N}}\widehat{\otimes}_\varepsilon A(\mathbb{C}^d) \simeq \left(l_p\widehat{\otimes}_\varepsilon A(\mathbb{C}^d)\right)^{\mathbb{N}}$$

and that $A(\mathbb{C}^d) \simeq \Lambda_\infty(\alpha)$ with $\alpha_n = n^{1/\alpha}$. But, by [47, 1.1 Proposition] (the proof given there works for any $p \geq 1$) we have $l_p\widehat{\otimes}_\varepsilon A(\mathbb{C}^d) \simeq l_p\widehat{\otimes}_\varepsilon s$, therefore $\mathcal{B}^{\text{loc}}_{p,k}(\Omega, A(\mathbb{C}^d) \simeq (\mathcal{D}_{L^p})^{\mathbb{N}}$.

In [19] Kaballo showed that the short sequence $0 \longrightarrow N(P(D)) \longrightarrow \mathcal{B}^{\text{loc}}_{p,kP'}(\Omega) \longrightarrow \mathcal{B}^{\text{loc}}_{p,k}(\Omega) \longrightarrow 0$ is an $(\epsilon L)-$triple when the differential operator $P(D)$ is hypoelliptic and it does not split when $P(D)$ is elliptic (recall that a short exact sequence of locally convex spaces $0 \longrightarrow E \longrightarrow F \overset{q}{\longrightarrow} G \longrightarrow 0$ is called an $(\epsilon L)-$triple, if for every Banach space X the mapping $q\hat{\otimes}_\epsilon \text{id} : F\hat{\otimes}_\epsilon X \to G\hat{\otimes}_\epsilon X$ is surjective). In the next theorem this result is extended to ω-hypoelliptic differential operators and to the vector-valued setting. The extension is essentially a consequence of results of Meise, Taylor and Vogt [24, Theorem 2.10, Corollary 2.16] (see also Vogt [46]) and Theorem 4.1. We will consider weights in the class $\mathcal{M}^*$ ($\omega \in \mathcal{M}^*$ if $\omega(x) = \sigma(|x|) \in \mathcal{M}$ and σ is as in [24, Definition 1.1]). For example, the weight $\omega(x) = |x|^\beta$ belongs to $\mathcal{M}^*$ when $0 < \beta < 1$. On the other hand, if $P(x) = \sum_{|\alpha|\leq m} a_\alpha x^\alpha$ is a complex polynomial in n variables then $P'(x)$ denotes the function $x \to \left(\sum_{|\alpha|\geq 0} |\partial^\alpha P(x)|^2\right)^{1/2}$. An open set $\Omega \subset \mathbb{R}^n$ is called P–convex (P–convex for supports in [16, Definition 10.6.1]) if to every compact set $K \subset \Omega$ there exists another compact set $K' \subset \Omega$ such that $\phi \in \mathcal{D}(\Omega)$ and $\text{supp}\, P(-D)\phi \subset K$ implies $\text{supp}\, \phi \subset K'$. Finally we refer the reader to [2,15,16] for the theory of linear partial differential operators.
Theorem 4.2 *Let $P(D)$ be a linear partial differential operator with constant coefficients in $\mathbb{R}^n$ ($n \geq 2$), Ω an open subset of $\mathbb{R}^n$, $\omega \in \mathcal{M}^*$, $k \in \mathcal{K}_\omega$ and $1 \leq p < \infty$.*
(1) If $P(D)$ is $\omega-$hypoelliptic and Ω is $P-$convex, then the short sequence

$$0 \longrightarrow N(P(D)) \longrightarrow \mathcal{B}^{\text{loc}}_{p,kP'}(\Omega) \overset{P(D)}{\longrightarrow} \mathcal{B}^{\text{loc}}_{p,k}(\Omega) \longrightarrow 0$$

is exact, it does not split and it is an $(\epsilon L)-$triple (here $N(D)$ is the kernel of $P(D)$). The dual sequence

$$0 \longrightarrow \Big(\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega)\Big)' \overset{{}^tP(D)}{\longrightarrow} \Big(\mathcal{B}^{\mathrm{loc}}_{p,kP'}(\Omega)\Big)' \longrightarrow \Big(N(P(D))\Big)' \longrightarrow 0$$

is topologically exact and it does not split either.

(2) *If $P(D)$ is $\omega-$hypoelliptic, Ω is $\widetilde{P}-$convex and $1 < p < \infty$, there exist a short sequence*

$$0 \longrightarrow \mathcal{B}^{\mathrm{c}}_{p,k}(\Omega) \longrightarrow \mathcal{B}^{\mathrm{c}}_{p,k/P'}(\Omega) \longrightarrow \Big(N(P(-D))\Big)' \longrightarrow 0$$

which is topologically exact and it does not split.

(3) *If $P(D)$ is $\omega-$hypoelliptic, Ω is $P-$convex and E is a nuclear Fréchet space, the short sequence*

$$0 \longrightarrow N(P_E(D)) \longrightarrow \mathcal{B}^{\mathrm{loc}}_{p,kP'}(\Omega, E) \overset{P_E(D)}{\longrightarrow} \mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega, E) \longrightarrow 0$$

is exact and an $(\epsilon L)-$triple (here $P_E(D) : \mathcal{D}'_\omega(\Omega, E) \to \mathcal{D}'_\omega(\Omega, E)$ is defined by $\langle \varphi, P_E(D)T\rangle = \langle P(-D)\varphi, T\rangle$ for all $\varphi \in \mathcal{D}_\omega(\Omega)$ and all $T \in \mathcal{D}'_\omega(\Omega, E)$).

Proof. 1. It follows from the hypothesis and [2, Theorem 3.3.3] that $P(D)$ is a continuous linear operator of $\mathcal{B}^{\mathrm{loc}}_{p,kP'}(\Omega)$ (resp. $\mathcal{E}_\omega(\Omega)$) onto $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega)$ (resp. $\mathcal{E}_\omega(\Omega)$). Furthermore $N(P(D))$ coincides, algebraic and topologically, with the subspace $\{f \in \mathcal{E}_\omega(\Omega) : P(D)f = 0\}$ of $\mathcal{E}_\omega(\Omega)$ in virtue of [2, Theorem 4.1.1], the embedding $\mathcal{E}_\omega(\Omega) \hookrightarrow \mathcal{B}^{\mathrm{loc}}_{p,kP'}(\Omega)$ [2, Theorem 2.3.5] and the closed graph theorem; thus $N(P(D))$ is a nuclear Fréchet space ($\mathcal{E}_\omega(\Omega)$ is nuclear by [45]). It is then clear that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N(P(D)) & \longrightarrow & \mathcal{B}^{\mathrm{loc}}_{p,kP'}(\Omega) & \overset{P(D)}{\longrightarrow} & \mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega) & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & N(P(D)) & \longrightarrow & \mathcal{E}_\omega(\Omega) & \overset{P(D)}{\longrightarrow} & \mathcal{E}_\omega(\Omega) & \longrightarrow & 0 \end{array}$$

is commutative. Since, by the Meise-Taylor-Vogt theorem [24, Theorem 2.10, Corollary 2.16], the second row of this diagram does not split, it follows that the first row does not split either (see [32]). The first row is an $(\epsilon L)-$triple by the nuclearity of $N(P(D))$ and [19, Theorem 2.9]. Next consider the dual diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \left(\mathcal{B}^{loc}_{p,k}(\Omega)\right)' & \xrightarrow{{}^tP(D)} & \left(\mathcal{B}^{loc}_{p,kP'}(\Omega)\right)' & \longrightarrow & (N(P(D)))' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \text{id} & & \\
0 & \longrightarrow & \mathcal{E}'_\omega(\Omega) & \xrightarrow{{}^tP(D)} & \mathcal{E}'_\omega(\Omega) & \longrightarrow & (N(P(D)))' & \longrightarrow & 0
\end{array}$$

This diagram is also commutative and since $N(P(D))$ is quasinormable (see e.g. [25, Corollary 28.5]) its rows are topologically exact sequences (use [25, Proposition 26.18]). Its second row does not split because the second row of the previous diagram does not split either and the space $\mathcal{E}_\omega(\Omega)$ is reflexive (see [32]). Hence it follows that the first row does not split either.

2. Since $\widetilde{P}(D) = P(-D)$ and Ω is $\widetilde{P}-$convex, it follows from 1 that the short sequence $0 \longrightarrow \left(\mathcal{B}^{\rm loc}_{p',1/\tilde{k}}(\Omega)\right)' \xrightarrow{{}^tP(D)} \left(\mathcal{B}^{\rm loc}_{p',\frac{1}{\tilde{k}}\widetilde{P}'}(\Omega)\right)' \longrightarrow \left(N(P(-D))\right)' \longrightarrow 0$ is topologically exact and it does not split. Using the isomorphisms [31, Theorem 3.2] $\left(\mathcal{B}^{\rm loc}_{p',1/\tilde{k}}(\Omega)\right)' \simeq \mathcal{B}^{\rm c}_{p,k}(\Omega)$, $\left(\mathcal{B}^{\rm loc}_{p',\frac{1}{\tilde{k}}\widetilde{P}'}(\Omega)\right)' \simeq \mathcal{B}^{\rm c}_{p,k/P'}(\Omega)$ one easily concludes the proof.

3. According to 1 we have the exact sequence $0 \longrightarrow N(P(D)) \longrightarrow \mathcal{B}^{\rm loc}_{p,kP'}(\Omega) \xrightarrow{P(D)} \mathcal{B}^{\rm loc}_{p,k}(\Omega) \longrightarrow 0$ then also $0 \longrightarrow N(P(D))\hat{\otimes}_\varepsilon E \longrightarrow \mathcal{B}^{\rm loc}_{p,kP'}(\Omega)\hat{\otimes}_\varepsilon E \xrightarrow{P(D)\hat{\otimes}_\varepsilon{\rm id}} \mathcal{B}^{\rm loc}_{p,k}(\Omega)\hat{\otimes}_\varepsilon E \longrightarrow 0$ is exact (the second arrow is injective by [22, Proposition 5 p.277] and $P(D)\hat{\otimes}_\varepsilon{\rm id}$ is surjective by the nuclearity of E and [22, Proposition 7 p.189]). On the other hand from [22, Proposition 7 p.189] and [22, Proposition 7 p.174] it follows that $N(P_E(D)) = N(P(D)\hat{\otimes}_\varepsilon{\rm id}) = \overline{N(P(D)) \otimes E}^{\mathcal{B}^{\rm loc}_{p,kP'}(\Omega)\hat{\otimes}_\varepsilon E} = N(P(D))\hat{\otimes}_\varepsilon E$. Furthermore, by virtue of Theorem 4.1(a), we have $\mathcal{B}^{\rm loc}_{p,kP'}(\Omega)\hat{\otimes}_\varepsilon E = \mathcal{B}^{\rm loc}_{p,kP'}(\Omega, E)$ and $\mathcal{B}^{\rm loc}_{p,k}(\Omega)\hat{\otimes}_\varepsilon E = \mathcal{B}^{\rm loc}_{p,k}(\Omega, E)$. Therefore we have the exact sequence $0 \longrightarrow N(P_E(D)) \longrightarrow \mathcal{B}^{\rm loc}_{p,kP'}(\Omega, E) \xrightarrow{P_E(D)} \mathcal{B}^{\rm loc}_{p,k}(\Omega, E) \longrightarrow 0$. Finally the nuclearity of $N(P_E(D))$ and Theorem 2.9 in [19] show that this sequence is also an $(\epsilon L)-$triple. ■

Remark. For results on the splitting of partial differential operators between $\mathcal{B}^{\rm loc}_{p,k}$–spaces in the temperate case see also [14].

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