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A viscosity iterative technique for equilibrium and fixed point problems in a Hadamard space

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Abstract

The main purpose of this paper is to introduce a viscosity-type proximal point algorithm, comprising of a nonexpansive mapping and a finite sum of resolvent operators associated with monotone bifunctions. A strong convergence of the proposed algorithm to a common solution of a finite family of equilibrium problems and fixed point problem for a nonexpansive mapping is established in a Hadamard space. We further applied our results to solve some optimization problems in Hadamard spaces.

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1. Introduction

Optimization theory is one of the most flourishing areas of research in mathematics that has received a lot of attention in recent time. One of the most important problems in optimization theory is the Equilibrium Problem (EP) since it includes many other optimization and mathematical problems as special cases; namely, minimization problems, variational inequality problems, complementarity problems, fixed point problems, convex feasibility problems, among

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others (see Section 4, for details). Thus, EPs are of central importance in optimization theory as well as in nonlinear and convex analysis. Given a nonempty set C and $f : C \times C \to \mathbb{R}$, the EP is defined as follows:

$$\text{Find } x^* \in C \text{ such that } f(x^*, y) \geq 0, \quad \forall y \in C. \tag{1.1}$$

The point x^* for which (1.1) is satisfied is called an equilibrium point of f. Throughout this paper, we shall denote the solution set of problem (1.1) by $\mathrm{EP}(f, C)$. EPs have been widely studied in Hilbert, Banach and topological vector spaces by many authors (see [5, 10, 16, 28]), as well as in Hadamard manifolds (see [9, 26]). One of the most popular and effective method used for solving problem (1.1) and other related optimization problems is the Proximal Point Algorithm (PPA) which was introduced in Hilbert space by Martinet [25] in 1970 and was further extensively studied in the same space by Rockafellar [30] in 1976. The PPA and its generalizations have also been studied extensively in Banach spaces and Hadamard manifolds (see [9, 14, 22, 28] and the references therein).

Recently, researchers are beginning to extend the study of the PPA and its generalizations to Hadamard spaces. For instance, Bačák [2] studied the following PPA for finding minimizers of proper convex and lower semicontinuous functionals in Hadamard spaces: Let X be a Hadamard space, then for arbitrary point $x_1 \in X$, define the sequence $\{x_n\}$ iteratively by

$$x_{n+1} = \mathrm{prox}^f_{\mu_n}(x_n), \tag{1.2}$$

where $\mu_n > 0$ for all $n \geq 1$, and $\mathrm{prox}^f_\mu : X \to X$ is the Moreau-Yosida resolvent of a proper convex and lower semicontinuous functional f defined by

$$\mathrm{prox}^f_\mu(x) = \arg\min_{v \in X} \left(f(v) + \frac{1}{2\mu} d^2(v, x) \right). \tag{1.3}$$

Bačák [2] proved that (1.2) Δ-convergence to a minimizer of f. In 2016, Suparatulatorn *et al* [33] extended the results of Bačák [2] by proposing the following Halpern-type PPA for approximating a minimizer of a proper convex and lower semicontinuous functional which is also a fixed point of a nonexpansive mapping in Hadamard spaces:

$$\begin{cases} u, x_1 \in X, \\ y_n = \mathrm{prox}^f_{\mu_n}(x_n), \\ x_{n+1} = \alpha_n u \oplus (1 - \alpha_n) T y_n \ n \geq 1, \end{cases} \tag{1.4}$$

where $\{\alpha_n\} \subset (0, 1)$ and $\mu_n \geq \lambda > 0$. They obtained a strong convergence result under some mild conditions. The PPA was also studied by Khatibzadeh and Ranjbar in [19] for finding zeroes of monotone operators and in [20] for solving variational inequality problems in Hadamard spaces. Based on the results of Suparatulatorn *et al* [33], Khatibzadeh and Ranjbar [19], Okeke and Izuchukwu [27] studied the Halpern-type PPA and obtained a strong convergence results for finding a minimizer of a proper convex and lower semicontinuous functional which is also a zero of a monotone operator and a fixed point of a nonexpansive

mapping. For more recent important results on PPA in Hadamard spaces and other general metric spaces, see [1, 17, 36] and the references therein.
Very recently, Kumam and Chaipunya [22] studied EP (1.1) in Hadamard spaces. First, they established the existence of an equilibrium point of a bifunction satisfying some convexity, continuity and coercivity assumptions, and they also established some fundamental properties of the resolvent of the bifunction. Furthermore, they studied the PPA for finding an equilibrium point of a monotone bifunction in a Hadamard space. More precisely, they proved the following theorem.

Theorem 1.1. *Let C be a nonempty closed and convex subset of a Hadamard space X and $f : C \times C \to \mathbb{R}$ be monotone, Δ-upper semicontinuous in the first variable such that $D(J_\lambda^f) \supset C$ for all $\lambda > 0$ (where $D(J_\lambda^f)$ means the domain of J_λ^f). Suppose that $EP(C, f) \neq \varnothing$ and for an initial guess $x_0 \in C$, the sequence $\{x_n\} \subset C$ is generated by*

$$x_n := J_{\lambda_n}^f(x_{n-1}),\ n \in \mathbb{N},$$

where $\{\lambda_n\}$ is a sequence of positive real numbers bounded away from 0. Then, $\{x_n\}$ Δ-converges to an element of $EP(C, f)$.

Motivated by the above results of Kumam and Chaipunya [22], we study some other important properties of the resolvent of monotone bifunctions. We then introduce a viscosity-type PPA comprising of a nonexpansive mapping and a finite sum of resolvent operators associated with these bifunctions. We prove that the sequence generated by our proposed algorithm converges strongly to a common solution of a finite family of equilibrium problems which is also a fixed point of a nonexpansive mapping. Furthermore, we applied our results to solve some optimization problems in Hadamard spaces. Our results serve as a continuation of the work of Kumam and Chaipunya [22]. They also extend related results from Hilbert spaces and Hadamard manifolds to Hadamard spaces, and they complement some recent important results in Hadamard spaces.

2. Preliminaries

In this section, we recall some basic and useful results that will be needed in establishing our main results. We categorize our study into brief-detailed subsections.

2.1. **Geometry of Hadamard spaces.**

Definition 2.1. Let (X, d) be a metric space, $x, y \in X$ and $I = [0, d(x, y)]$ be an interval. A curve c (or simply a geodesic path) joining x to y is an isometry $c : I \to X$ such that $c(0) = x$, $c(d(x, y)) = y$ and $d(c(t), c(t') = |t - t'|$ for all $t, t' \in I$. The image of a geodesic path is called the geodesic segment, which is denoted by $[x, y]$ whenever it is unique.

Definition 2.2 ([13])**.** A metric space (X, d) is called a geodesic space if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if every two points of X are joined by exactly one geodesic. A subset C of X is said to be convex if C includes every geodesic segments joining two of its points. Let $x, y \in X$ and $t \in [0,1]$, we write $tx \oplus (1-t)y$ for the unique point z in the geodesic segment joining from x to y such that

$$d(x,z) = (1-t)d(x,y) \quad \text{and} \quad d(z,y) = td(x,y). \tag{2.1}$$

A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (X,d) consists of three vertices (points in X) with unparameterized geodesic segment between each pair of vertices. For any geodesic triangle there is comparison (Alexandrov) triangle $\bar{\Delta} \subset \mathbb{R}^2$ such that $d(x_i, x_j) = d_{\mathbb{R}^2}(\bar{x}_i, \bar{x}_j)$ for $i, j \in \{1,2,3\}$. Let Δ be a geodesic triangle in X and $\bar{\Delta}$ be a comparison triangle for $\bar{\Delta}$, then Δ is said to satisfy the CAT(0) inequality if for all points $x, y \in \Delta$ and $\bar{x}, \bar{y} \in \bar{\Delta}$,

$$d(x,y) \leq d_{\mathbb{R}^2}(\bar{x}, \bar{y}). \tag{2.2}$$

Let x, y, z be points in X and y_0 be the midpoint of the segment $[y,z]$, then the CAT(0) inequality implies

$$d^2(x, y_0) \leq \frac{1}{2}d^2(x,y) + \frac{1}{2}d^2(x,z) - \frac{1}{4}d(y,z). \tag{2.3}$$

Inequality (2.3) is known as the CN inequality of Bruhat and Titis [7].

Definition 2.3. A geodesic space X is said to be a CAT(0) space if all geodesic triangles satisfy the CAT(0) inequality. Equivalently, X is called a CAT(0) space if and only if it satisfies the CN inequality.
CAT(0) spaces are examples of uniquely geodesic spaces and complete CAT(0) spaces are called Hadamard spaces.

Definition 2.4 ([4])**.** Let X be a CAT(0) space. Denote the pair $(a,b) \in X \times X$ by $\overrightarrow{ab}$ and call it a vector. Then, a mapping $\langle .,. \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}\left(d^2(a,d) + d^2(b,c) - d^2(a,c) - d^2(b,d)\right) \ \forall a,b,c,d \in X$$

is called a quasilinearization mapping.

It is easily to check that $\langle \overrightarrow{ab}, \overrightarrow{ab} \rangle = d^2(a,b)$, $\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$, $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ae}, \overrightarrow{cd} \rangle + \langle \overrightarrow{eb}, \overrightarrow{cd} \rangle$ and $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle$ for all $a,b,c,d,e \in X$. A geodesic space X is said to satisfy the Cauchy-Swartz inequality if $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leq d(a,b)d(c,d) \ \forall a,b,c,d \in X$. It has been established in [4] that a geodesically connected metric space is a CAT(0) space if and only if it satisfies the Cauchy-Schwartz inequality. Examples of CAT(0) spaces includes: Euclidean spaces $\mathbb{R}^n$, Hilbert spaces, simply connected Riemannian manifolds of nonpositive sectional curvature [29], $\mathbb{R}$-trees, Hilbert ball [15], among others.

Lemma 2.5 (see [23, Lemma 7])**.** *Let X be a uniformly convex hyperbolic space with modulus of uniform convexity η. For any $c > 0$, $\epsilon \in (0,2]$, $\lambda \in [0,1]$ and $v, x, y \in X$, we have that $d(x,v) \leq c, d(y,v) \leq c$ and $d(x,y) \geq \epsilon c$ imply that*

$$d((1-\lambda)x \oplus \lambda y, v) \leq (1 - 2\lambda(1-\lambda)\eta(c,\epsilon))c.$$

If X is a CAT(0) space, then X is uniformly convex with modulus of uniform convexity $\eta(c,\epsilon) := \frac{\epsilon^2}{8}$ (see [23, Proposition 8]).

We end this subsection with the following important lemmas which characterizes CAT(0) spaces.

Lemma 2.6. *Let X be a CAT(0) space, $x, y, z \in X$ and $t, s \in [0,1]$. Then*

(i) $d(tx \oplus (1-t)y, z) \leq td(x,z) + (1-t)d(y,z)$ (see[13]).
(ii) $d^2(tx \oplus (1-t)y, z) \leq td^2(x,z) + (1-t)d^2(y,z) - t(1-t)d^2(x,y)$ (see [13]).
(iii) $d^2(tx \oplus (1-t)y, z) \leq t^2d^2(x,z) + (1-t)^2d^2(y,z) + 2t(1-t)\langle\overrightarrow{xz}, \overrightarrow{yz}\rangle$ (see [11]).
(iv) $d(tw \oplus (1-t)x, ty \oplus (1-t)z) \leq td(w,y) + (1-t)d(x,z)$ (see [6]).
(v) $z = tx \oplus (1-t)y$ implies $\langle\overrightarrow{zy}, \overrightarrow{zw}\rangle \leq t\langle\overrightarrow{xy}, \overrightarrow{zx}\rangle, \quad \forall \; w \in X$ (see [11]).
(vi) $d(tx \oplus (1-t)y, sx \oplus (1-s)y) \leq |t-s|d(x,y)$ (see [8]).

Lemma 2.7 ([37])**.** *Let X be a CAT(0) space. For any $t \in [0,1]$ and $u, v \in X$, let $u_t = tu \oplus (1-t)v$. Then, for all $x, y \in X$,*

(1) $\langle\overrightarrow{u_tx}, \overrightarrow{u_ty}\rangle \leq t\langle\overrightarrow{ux}, \overrightarrow{u_ty}\rangle + (1-t)\langle\overrightarrow{vx}, \overrightarrow{u_ty}\rangle$;
(2) $\langle\overrightarrow{u_tx}, \overrightarrow{uy}\rangle \leq t\langle\overrightarrow{ux}, \overrightarrow{uy}\rangle + (1-t)\langle\overrightarrow{vx}, \overrightarrow{ux}\rangle$ *and*
(3) $\langle\overrightarrow{u_tx}, \overrightarrow{vy}\rangle \leq t\langle\overrightarrow{ux}, \overrightarrow{vy}\rangle + (1-t)\langle\overrightarrow{vx}, \overrightarrow{vy}\rangle$.

Lemma 2.8 ([35])**.** *Let X be a CAT(0) space, $\{x_i, i = 1, 2, \ldots, N\} \subset X$ and $\alpha_i \in [0,1], \; i = 1, 2, \ldots, N$ such that $\sum_{i=1}^N \alpha_i = 1$. Then,*

$$d\left(\bigoplus_{i=1}^N \alpha_i x_i, z\right) \leq \sum_{i=1}^N \alpha_i d(x_i, z), \; \forall z \in X.$$

Remark 2.9 (see also [35]). For a CAT(0) space X, if $\{x_i, i = 1, 2, \ldots, N\} \subset X$, and $\alpha_i \in [0,1], \; i = 1, 2, \ldots, N$. Then by induction, we can write

$$\bigoplus_{i=1}^N \alpha_i x_i := (1-\alpha_N) \bigoplus_{i=1}^{N-1} \frac{\alpha_i}{1-\alpha_N} x_i \oplus \alpha_N x_N. \tag{2.4}$$

2.2. The notion of Δ-convergence.

Definition 2.10. Let $\{x_n\}$ be a bounded sequence in a geodesic metric space X. Then, the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is defined by

$$A(\{x_n\}) = \{\bar{v} \in X : \limsup_{n\to\infty} d(\bar{v}, x_n) = \inf_{v \in X} \limsup_{n\to\infty} d(v, x_n).$$

It is generally known that in a Hadamard space, $A(\{x_n\})$ consists of exactly one point. A sequence $\{x_n\}$ in X is said to be Δ-convergent to a point $\bar{v} \in X$ if $A(\{x_{n_k}\}) = \{\bar{v}\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. In this case, we write Δ-$\lim\limits_{n\to\infty} x_n = \bar{v}$ (see [12]). The concept of Δ-convergence in metric spaces was first introduced and studied by Lim [24]. Kirk and Panyanak [21] later introduced and studied this concept in CAT(0) spaces, and proved that it is very similar to the weak convergence in Banach space setting.
The following lemma is very important as regards to Δ-convergence.

Lemma 2.11 ([13]). *Every bounded sequence in a Hadamard space always have a $\triangle-$convergent subsequence.*

2.3. **Existence of solution of equilibrium problems and resolvent operators.**

Theorem 2.12 ([22, Theorem 4.1]). *Let C be a nonempty closed and convex subset of a Hadamard space X and $f : C \times C \to \mathbb{R}$ be a bifunction satisfying the following:*

- (A1) *$f(x,x) \geq 0$ for each $x \in C$,*
- (A2) *for every $x \in C$, the set $\{y \in C : f(x,y) < 0\}$ is convex,*
- (A3) *for every $y \in C$, the function $x \mapsto f(x,y)$ is upper semicontinuous,*
- (A4) *there exists a compact subset $L \subset C$ containing a point $y_0 \in L$ such that $f(x,y_0) < 0$ whenever $x \in C \backslash L$.*

Then, problem (1.1) has a solution.

In [22], the authors introduce the resolvent of the bifunction f associated with the EP (1.1). They defined a perturbed bifunction $\bar{f}_{\bar{x}} : C \times C \to \mathbb{R}$ ($\bar{x} \in X$) of f by

$$\bar{f}_{\bar{x}}(x,y) := f(x,y) - \langle \overrightarrow{x\bar{x}}, \overrightarrow{xy} \rangle, \ \forall x,y \in C. \tag{2.5}$$

The perturbed bifunction $\bar{f}$ has a unique equilibrium called the resolvent operator $J^f : X \to 2^C$ of the bifunction f (see [22]), defined by

$$J^f(x) := EP(C, \bar{f}_x) = \{z \in C : f(z,y) - \langle \overrightarrow{zx}, \overrightarrow{zy} \rangle \geq 0, \ y \in C\}, \ x \in X. \tag{2.6}$$

It was established in [22] that J^f is well defined.
We now recall the following definitions which will be needed in what follows.

Definition 2.13. Let X be a CAT(0) space. A point $x \in X$ is called a fixed point of a nonlinear mapping $T : X \to X$, if $Tx = x$. We denote the set of fixed points of T by $F(T)$. The mapping T is said to be

(i) *firmly nonexpansive* (see [19]), if

$$d^2(Tx,Ty) \leq \langle \overrightarrow{TxTy}, \overrightarrow{xy} \rangle \ \forall x,y \in X,$$

(ii) *nonexpansive*, if

$$d(Tx,Ty) \leq d(x,y) \ \forall x,y \in X.$$

From Cauchy-Schwartz inequality, it is clear that firmly nonexpansive mappings are nonexpasive.

Definition 2.14. Let X be a CAT(0) space and C be a nonempty closed and convex subset of X. A function $f : C \times C \to \mathbb{R}$ is called monotone if $f(x,y) + f(y,x) \leq 0$ for all $x,y \in C$.

Definition 2.15. Let X be a CAT(0) space. A function $f : D(f) \subseteq X \to (-\infty, +\infty]$ is said to be convex if

$$f(tx \oplus (1-t)y) \leq tf(x) + (1-t)f(y) \ \forall x,y \in X, \ t \in (0,1).$$

f is proper, if $D(f) := \{x \in X : f(x) < +\infty\} \neq \varnothing$. The function $f : D(f) \to (-\infty, \infty]$ is lower semi-continuous at a point $x \in D(f)$ if

$$f(x) \leq \liminf_{n\to\infty} f(x_n), \tag{2.7}$$

for each sequence $\{x_n\}$ in $D(f)$ such that $\lim\limits_{n\to\infty} x_n = x$; f is said to be lower semicontinuous on $D(f)$ if it is lower semi-continuous at any point in $D(f)$.

Lemma 2.16 ([22, Proposition 5.4])**.** *Suppose that f is monotone and $D(J^f) \neq \varnothing$. Then, the following properties hold.*

(i) *J^f is single-valued.*
(ii) *If $D(J^f) \supset C$, then J^f is nonexpansive restricted to C.*
(iii) *If $D(J^f) \supset C$, then $F(J^f) = EP(C, f)$.*

Theorem 2.17 ([22, Theorem 5.2])**.** *Suppose that f has the following properties*

(i) *$f(x, x) = 0$ for all $x \in C$,*
(ii) *f is monotone,*
(iii) *for each $x \in C$, $y \mapsto f(x, y)$ is convex and lower semicontinuous.*
(iv) *for each $x \in C$, $f(x, y) \geq \limsup_{t\downarrow 0} f((1-t)x \oplus tz, y)$ for all $x, z \in C$.*

Then $D(J^f) = X$ and J^f single-valued.

The following two lemmas will be very useful in establishing our strong convergence theorem.

Lemma 2.18 ([34])**.** *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a metric space of hyperbolic type X and $\{\beta_n\}$ be a sequence in [0,1] with $\liminf_{n\to\infty} \beta_n < \limsup_{n\to\infty} \beta_n < 1$. Suppose that $x_{n+1} = \beta_n x_n \oplus (1-\beta_n) y_n$ for all $n \geq 0$ and $\limsup_{n\to\infty}(d(y_{n+1}, y_n) - d(x_{n+1}, x_n)) \leq 0$. Then $\lim_{n\to\infty} d(y_n, x_n) = 0$.*

Lemma 2.19 (Xu, [38])**.** *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n \sigma_n + \gamma_n, n \geq 0,$$

where, (i) $\{\alpha_n\} \subset [0,1]$, $\sum \alpha_n = \infty$; (ii) $\lim sup\, \sigma_n \leq 0$; (iii) $\gamma_n \geq 0$; $(n \geq 0)$, $\sum \gamma_n < \infty$. Then, $a_n \to 0$ as $n \to \infty$.

3. Main results

Lemma 3.1. *Let X be a CAT(0) space, $\{x_i,\ i = 1, 2, \ldots, N\} \subset X$, $\{y_i,\ i = 1, 2, \ldots, N\} \subset X$ and $\alpha_i \in [0,1]$ for each $i = 1, 2, \ldots, N$ such that $\sum_{i=1}^N \alpha_i = 1$. Then,*

$$d\left(\bigoplus_{i=1}^N \alpha_i x_i, \bigoplus_{i=1}^N \alpha_i y_i\right) \leq \sum_{i=1}^N \alpha_i d(x_i, y_i). \tag{3.1}$$

Proof. (By induction). For $N = 2$, the result follows from Lemma 2.6 (iv). Now, assume that (3.1) holds for $N = k$, for some $k \geq 2$. Then, we prove that

(3.1) also holds for $N = k+1$. Indeed, by Remark 2.9, Lemma 2.6 (iv) and our assumption, we obtain that

$$\begin{aligned} d\left(\bigoplus_{i=1}^{k+1}\alpha_i x_i, \bigoplus_{i=1}^{k+1}\alpha_i y_i\right) &\leq (1-\alpha_{k+1})d\left(\bigoplus_{i=1}^{k}\frac{\alpha_i}{1-\alpha_{k+1}}x_i, \bigoplus_{i=1}^{k}\frac{\alpha_i}{1-\alpha_{k+1}}y_i\right) \\ &\quad +\alpha_{k+1}d(x_{k+1}, y_{k+1}) \\ &\leq \sum_{i=1}^{k+1}\alpha_i d(x_i, y_i). \end{aligned}$$

Hence, (3.1) holds for all $N \in \mathbb{N}$. □

Remark 3.2. It follows from (2.6) that the resolvent J_λ^f of the bifunction f and order $\lambda > 0$ is given as

$$J_\lambda^f(x) := EP(C, \bar{f}_x) = \{z \in C : f(z,y) + \frac{1}{\lambda}\langle \overrightarrow{xz}, \overrightarrow{zy}\rangle \geq 0,\ y \in C\},\ x \in X, \tag{3.2}$$

where $\bar{f}$ is defined in this case as

$$\bar{f}_{\bar{x}}(x,y) := f(x,y) + \frac{1}{\lambda}\langle \overrightarrow{\bar{x}\,x}, \overrightarrow{xy}\rangle,\ \forall x,y \in C,\ \bar{x} \in X. \tag{3.3}$$

Lemma 3.3. *Let C be a nonempty, closed and convex subset of a Hadamard space X and $f : C \times C \to \mathbb{R}$ be a monotone bifunction such that $C \subset D(J_\lambda^f)$ for $\lambda > 0$. Then, the following hold:*

(i) *J_λ^f is firmly nonexpansive restricted to C.*
(ii) *If $F(J_\lambda) \neq \varnothing$, then*
$$d^2(J_\lambda x, x) \leq d^2(x,v) - d^2(J_\lambda^f x, v)\ \forall x \in C,\ v \in F(J_\lambda^f).$$
(iii) *If $0 < \lambda \leq \mu$, then $d(J_\mu^f x, J_\lambda^f x) \leq \sqrt{1-\frac{\lambda}{\mu}}d(x, J_\mu^f x)$, which implies that $d(x, J_\lambda^f x) \leq 2d(x, J_\mu^f x)\ \forall x \in C$.*

Proof. (i) Let $x, y \in C$, then by Lemma (2.16) (i) and the definition of J_λ^f, we have

$$f(J_\lambda^f x, J_\lambda^f y) + \frac{1}{\lambda}\langle \overrightarrow{xJ_\lambda^f x}, \overrightarrow{J_\lambda^f xJ_\lambda^f y}\rangle \geq 0 \tag{3.4}$$

and

$$f(J_\lambda^f y, J_\lambda^f x) + \frac{1}{\lambda}\langle \overrightarrow{yJ_\lambda^f y}, \overrightarrow{J_\lambda^f yJ_\lambda^f x}\rangle \geq 0. \tag{3.5}$$

Adding (3.4) and (3.5), and noting that f is monotone, we obtain

$$\frac{1}{\lambda}\left(\langle \overrightarrow{xJ_\lambda^f x}, \overrightarrow{J_\lambda^f xJ_\lambda^f y}\rangle + \langle \overrightarrow{yJ_\lambda^f y}, \overrightarrow{J_\lambda^f yJ_\lambda^f x}\rangle\right) \geq 0,$$

which implies that

$$\langle \overrightarrow{xy}, \overrightarrow{J_\lambda^f xJ_\lambda^f y}\rangle \geq \langle \overrightarrow{J_\lambda^f xJ_\lambda^f y}, \overrightarrow{J_\lambda^f xJ_\lambda^f y}\rangle.$$

That is,

$$\langle \overrightarrow{xy}, \overrightarrow{J_\lambda^f x J_\lambda^f y} \rangle \geq d^2(J_\lambda^f x, J_\lambda^f y). \tag{3.6}$$

(ii) It follows from (3.6) and the definition of quasilinearization that

$$d^2(x, J_\lambda^f x) \leq d^2(x, v) - d^2(v, J_\lambda^f x) \ \forall x \in C, \ v \in F(J_\lambda^f).$$

(iii) Let $x \in C$ and $0 < \lambda \leq \mu$, then we have that

$$f(J_\lambda^f x, J_\mu^f x) + \frac{1}{\lambda} \langle \overrightarrow{x J_\lambda^f x}, \overrightarrow{J_\lambda^f x J_\mu^f x} \rangle \geq 0 \tag{3.7}$$

and

$$f(J_\mu^f x, J_\lambda^f x) + \frac{1}{\mu} \langle \overrightarrow{x J_\mu^f x}, \overrightarrow{J_\mu^f x J_\lambda^f x} \rangle \geq 0. \tag{3.8}$$

Adding (3.7) and (3.8), and by the monotonicity of f, we obtain that

$$\langle \overrightarrow{J_\lambda^f x x}, \overrightarrow{J_\mu^f x J_\lambda^f x} \rangle \geq \frac{\lambda}{\mu} \langle \overrightarrow{J_\mu^f x x}, \overrightarrow{J_\mu^f x J_\lambda^f x} \rangle.$$

By the definition of quasilinearization, we obtain that

$$\left(\frac{\lambda}{\mu} + 1\right) d^2(J_\mu^f x, J_\lambda^f x) \leq \left(1 - \frac{\lambda}{\mu}\right) d^2(x, J_\mu x) + \left(\frac{\lambda}{\mu} - 1\right) d^2(x, J_\lambda^f x).$$

Since $\frac{\lambda}{\mu} \leq 1$, we obtain that

$$\left(\frac{\lambda}{\mu} + 1\right) d^2(J_\mu^f x, J_\lambda^f x) \leq \left(1 - \frac{\lambda}{\mu}\right) d^2(x, J_\mu^f x),$$

which implies

$$d(J_\mu^f x, J_\lambda^f x) \leq \sqrt{1 - \frac{\lambda}{\mu}} d(x, J_\mu^f x). \tag{3.9}$$

Furthermore, by triangle inequality and (3.9), we obtain

$$d(x, J_\lambda^f x) \leq 2d(x, J_\mu^f x).$$

□

Remark 3.4. We note here that, if the bifunction f satisfies assumption (i)-(iv) of Theorem 2.17, the conclusions of Lemma 3.3 hold in the whole space X.

Lemma 3.5. *Let C be a nonempty, closed and convex subset of a Hadamard space X and T be a nonexpansive mapping on C. Let $f_i : C \times C \to \mathbb{R}$, $i = 1, 2, \ldots, N$ be a finite family of monotone bifunctions such that $C \subset D(J_\lambda^{f_i})$ for $\lambda > 0$. Then, for $\beta_i \in (0,1)$ with $\sum_{i=0}^N \beta_i = 1$, the mapping $S_\lambda : C \to C$ defined by $S_\lambda x := \beta_0 x \oplus \beta_1 J_\lambda^{f_1} x \oplus \beta_2 J_\lambda^{f_2} x \oplus \cdots \oplus \beta_N J_\lambda^{f_N} x$ for all $x \in C$, is nonexpansive and $F(T \circ S_{\lambda_2}) \subseteq \cap_{i=1}^N F(J_{\lambda_1}^{f_i}) \cap F(T)$ for $0 < \lambda_1 \leq \lambda_2$, where $S_{\lambda_2} : C \to C$ is defined by $S_{\lambda_2} x := \beta_0 x \oplus \beta_1 J_{\lambda_2}^{f_1} x \oplus \beta_2 J_{\lambda_2}^{f_2} x \oplus \cdots \oplus \beta_N J_{\lambda_2}^{f_N} x$ for all $x \in C$.*

Proof. Since f is monotone, it follows from Lemma 3.3 (i) (or Lemma 2.16 (ii)) that $J_\lambda^{f_i}$ is nonexpansive for $\lambda > 0,\ i = 1, 2, \dots, N$. Thus, by Lemma 3.1, we obtain

$$\begin{aligned} d(S_\lambda x, S_\lambda y) &\leq \beta_0 d(x,y) + \beta_1 d(J_\lambda^{f_1}x, J_\lambda^{f_1}y) + \cdots + \beta_N d(J_\lambda^{f_N}x, J_\lambda^{f_N}y) \\ &\leq \sum_{i=0}^{N} \beta_i d(x,y) \\ &= d(x,y). \end{aligned}$$

Hence, S_λ is nonexpansive.

Now, let $x \in F(T \circ S_{\lambda_2})$ and $v \in \cap_{i=1}^N F(J_{\lambda_2}^{f_i}) \cap F(T)$. Then, by Lemma 3.1, we obtain

$$\begin{aligned} d(x,v) &\leq d(S_{\lambda_2}x, v) \\ &\leq \beta_0 d(x,v) + \beta_1 d(J_{\lambda_2}^{f_1}x, v) + \cdots + \beta_N d(J_{\lambda_2}^{f_N}x, v) \\ &\leq \sum_{i=0}^{N-1} \beta_i d(x,v) + \beta_N d(J_{\lambda_2}^{f_N}x, v) \qquad (3.10) \\ &\leq d(x,v). \end{aligned}$$

From (3.10), we obtain that

$$d(x,v) = \sum_{i=0}^{N-1} \beta_i d(x,v) + \beta_N d(J_{\lambda_2}^{f_N}x, v) = (1-\beta_N)d(x,v) + \beta_N d(J_{\lambda_2}^{f_N}x, v),$$

which implies that $d(x,v) = d(J_{\lambda_2}^{f_N}x, v) = d(S_{\lambda_2}x, v) = d(\beta_0 x \oplus \beta_1 J_{\lambda_2}^{f_1}x \oplus \beta_2 J_{\lambda_2}^{f_2}x \oplus \cdots \oplus \beta_N J_{\lambda_2}^{f_N}x, v)$. Similarly, we obtain

$$d(x,v) = d(J_{\lambda_2}^{f_{N-1}}x, v) = \cdots = d(J_{\lambda_2}^{f_2}x, v) = d(J_{\lambda_2}^{f_1}x, v).$$

Thus,

$$d(x,v) = d(J_{\lambda_2}^{f_N}x, v) = \cdots = d(J_{\lambda_2}^{f_1}x, v) = d(\beta_0 x \oplus \beta_1 J_{\lambda_2}^{f_1}x \oplus \beta_2 J_{\lambda_2}^{f_2}x \oplus \cdots \oplus \beta_N J_{\lambda_2}^{f_N}x, v). \qquad (3.11)$$

Now, let $d(x,v) = c$. If $c > 0$, and there exist $\epsilon > 0$ and $i, j \in \{0, 1, 2, \dots, N\}$, $i \neq j$ such that $d(J_{\lambda_2}^{f_i}x, J_{\lambda_2}^{f_j}x) \geq \epsilon c$ (where $J_{\lambda_2}^{f_0} = I$), then by Lemma 2.5, we obtain that

$$d(\beta_0 x \oplus \beta_1 J_{\lambda_2}^{f_1}x \oplus \beta_2 J_{\lambda_2}^{f_2}x \oplus \cdots \oplus \beta_N J_{\lambda_2}^{f_N}x, v) < c = d(x,v),$$

and this contradicts (3.11). Hence, $c = 0$. This implies that $x = v$, hence

$$x \in \cap_{i=1}^N F(J_{\lambda_2}^{f_i}) \cap F(T). \qquad (3.12)$$

Since $0 < \lambda_1 \leq \lambda_2$, we obtain from Lemma 3.3 (iii) and (3.12) that

$$d(x, J_{\lambda_1}^{f_i}x) \leq 2d(x, J_{\lambda_2}^{f_i}x) = 0,\ i = 1, 2, \dots, N.$$

Hence, $x \in \cap_{i=1}^N F(J_{\lambda_1}^{f_i}) \cap F(T)$. Therefore, we conclude that $F(T \circ S_{\lambda_2}) \subseteq \cap_{i=1}^N F(J_{\lambda_1}^{f_i}) \cap F(T)$. □

We now present our strong convergence theorem.

Theorem 3.6. *Let C be a nonempty closed and convex subset of a Hadamard space X and $f_i : C \times C \to \mathbb{R}$, $i = 1, 2, \ldots, N$ be a finite family of monotone and upper semicontinuous bifunctions such that $C \subset D(J_\lambda^{f_i})$ for $\lambda > 0$. Let $T : C \to C$ be a nonexpansive mapping and $g : C \to C$ be a contraction mapping with coefficient $\tau \in (0, 1)$. Suppose that $\Gamma := \cap_{i=1}^N EP(f_i, C) \cap F(T) \neq \varnothing$ and for arbitrary $x_1 \in C$, the sequence $\{x_n\}$ is generated by*

$$\begin{cases} y_n = S_{\lambda_n} x_n := \beta_0 x_n \oplus \beta_1 J_{\lambda_n}^{f_1} x_n \oplus \beta_2 J_{\lambda_n}^{f_2} x_n \oplus \cdots \oplus \beta_N J_{\lambda_n}^{f_N} x_n, \\ x_{n+1} = \alpha_n g(x_n) \oplus \beta_n x_n \oplus \gamma_n T y_n, \quad n \geq 1, \end{cases} \tag{3.13}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$, and $\{\lambda_n\}$ is a sequence of positive real numbers satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$, $\alpha_n + \beta_n + \gamma_n = 1$ $\forall n \geq 1$,
(iii) $0 < \lambda \leq \lambda_n$ $\forall n \geq 1$ *and* $\lim_{n\to\infty} \lambda_n = \lambda$,
(iv) $\beta_i \in (0, 1)$ *with* $\sum_{i=0}^{N} \beta_i = 1$.

Then, $\{x_n\}$ converges strongly to $\bar{z} \in \Gamma$.

Proof. **Step 1:** We show that $\{x_n\}$ is bounded. Let $u \in \Gamma$, then by Lemma 2.8, we obtain that

$$\begin{aligned} d(x_{n+1}, u) &\leq \alpha_n d(g(x_n), u) + \beta_n d(x_n, u) + \gamma_n d(T y_n, u) \\ &\leq \alpha_n \tau d(x_n, u) + \alpha_n d(g(u), u) + \beta_n d(x_n, u) + \gamma_n d(y_n, u) \\ &\leq \alpha_n \tau d(x_n, u) + (\alpha_n + \beta_n) d(x_n, u) + \alpha_n d(g(u), u) \\ &= (1 - \alpha_n(1 - \tau)) d(x_n, u) + \alpha_n d(g(u), u) \\ &\leq \max\left\{ d(x_n, u) + \frac{d(g(u), u)}{1 - \tau} \right\} \\ &\vdots \\ &\leq \max\left\{ d(x_1, u) + \frac{d(g(u), u)}{1 - \tau} \right\}. \end{aligned}$$

Hence, $\{x_n\}$ is bounded. Consequently, $\{y_n\}$, $\{g(x_n)\}$ and $\{T(y_n)\}$ are all bounded.
Step 2: We show that $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$. Observe from Remark 2.9, that (3.13) can be rewritten as

$$\begin{cases} y_n = S_{\lambda_n} x_n := \beta_0 x_n \oplus \beta_1 J_{\lambda_n}^{f_1} x_n \oplus \cdots \oplus \beta_N J_{\lambda_n}^{f_N} x_n, \\ w_n = \frac{\alpha_n}{1-\beta_n} g(x_n) \oplus \frac{\gamma_n}{1-\beta_n} T y_n, \\ x_{n+1} = \beta_n x_n \oplus (1 - \beta_n) w_n, \quad n \geq 1. \end{cases} \tag{3.14}$$

Now, from (3.14), Lemma 2.6 (iv),(vi) and the nonexpansivity of T, we obtain that

$$d(w_{n+1},w_n)=d\Big(\frac{\alpha_{n+1}}{1-\beta_{n+1}}g(x_{n+1})\oplus\frac{\gamma_{n+1}}{1-\beta_{n+1}}Ty_{n+1},\frac{\alpha_n}{1-\beta_n}g(x_n)\oplus\frac{\gamma_n}{1-\beta_n}Ty_n\Big)$$
$$\leq d\Big(\frac{\alpha_{n+1}}{1-\beta_{n+1}}g(x_{n+1})\oplus\big(1-\frac{\alpha_{n+1}}{1-\beta_{n+1}}\big)Ty_{n+1},\frac{\alpha_{n+1}}{1-\beta_{n+1}}g(x_n)\oplus\big(1-\frac{\alpha_{n+1}}{1-\beta_{n+1}}\big)Ty_n\Big)$$
$$+d\Big(\frac{\alpha_{n+1}}{1-\beta_{n+1}}g(x_n)\oplus\big(1-\frac{\alpha_{n+1}}{1-\beta_{n+1}}\big)Ty_n,\frac{\alpha_n}{1-\beta_n}g(x_n)\oplus\big(1-\frac{\alpha_n}{1-\beta_n}\big)Ty_n\Big)$$
(3.15)
$$\leq\frac{\alpha_{n+1}}{1-\beta_{n+1}}\tau d(x_{n+1},x_n)+\big(1-\frac{\alpha_{n+1}}{1-\beta_{n+1}}\big)d(y_{n+1},y_n)+\Big|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\Big|d(g(x_n),Ty_n)$$

Without loss of generality, we may assume that $0<\lambda_{n+1}\leq\lambda_n\ \forall n\geq 1$. Thus, from (3.14), condition (iv), Lemma 3.1 and Lemma 3.3 (iii), we obtain

$$\begin{aligned}d(y_{n+1},y_n)=&d\big(\beta_0x_{n+1}\oplus\beta_1J^{f_1}_{\lambda_{n+1}}x_{n+1}\oplus\cdots\oplus\beta_NJ^{f_N}_{\lambda_{n+1}}x_{n+1},\ \beta_0x_n\oplus\beta_1J^{f_1}_{\lambda_n}x_n\oplus\cdots\oplus\beta_NJ^{f_N}_{\lambda_n}x_n\big)\\
\leq&\beta_0d(x_{n+1},x_n)+\sum_{i=1}^N\beta_id(J^{f_i}_{\lambda_{n+1}}x_{n+1},J^{f_i}_{\lambda_n}x_n)\\
\leq&\beta_0d(x_{n+1},x_n)+\sum_{i=1}^N\beta_id(J^{f_i}_{\lambda_{n+1}}x_{n+1},J^{f_i}_{\lambda_{n+1}}x_n)+\sum_{i=1}^N\beta_id(J^{f_i}_{\lambda_{n+1}}x_n,J^{f_i}_{\lambda_n}x_n)\\
\leq&d(x_{n+1},x_n)+\Big(\sqrt{1-\frac{\lambda_{n+1}}{\lambda_n}}\Big)\sum_{i=1}^N\beta_id(J^{f_i}_{\lambda_{n+1}}x_n,x_n)\end{aligned}$$
(3.16)
$$\leq d(x_{n+1},x_n)+\Big(\sqrt{1-\frac{\lambda_{n+1}}{\lambda_n}}\Big)\bar{M},$$

where $\bar{M}:=\sup_{n\geq1}\big\{\sum_{i=1}^N\beta_id(J^{f_i}_{\lambda_{n+1}}x_n,x_n)\big\}$. Substituting (3.16) into (3.15), we obtain that

$$\begin{aligned}d(w_{n+1},w_n)\leq&\frac{\alpha_{n+1}}{1-\beta_{n+1}}\tau d(x_{n+1},x_n)+\big(1-\frac{\alpha_{n+1}}{1-\beta_{n+1}}\big)d(x_{n+1},x_n)\\
&+\Big(\sqrt{1-\frac{\lambda_{n+1}}{\lambda_n}}\Big)\Big(1-\frac{\alpha_{n+1}}{1-\beta_{n+1}}\Big)M+\Big|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\Big|d(g(x_n),Ty_n)\\
=&\big[1-\frac{\alpha_{n+1}}{1-\beta_{n+1}}(1-\tau)\big]d(x_{n+1},x_n)+\Big(\sqrt{1-\frac{\lambda_{n+1}}{\lambda_n}}\Big)\Big(1-\frac{\alpha_{n+1}}{1-\beta_{n+1}}\Big)M\\
&+\Big|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\Big|d(g(x_n),Ty_n).\end{aligned}$$

Since $\lim\limits_{n\to\infty}\alpha_n=0$, $\lim\limits_{n\to\infty}\lambda_n=\lambda$ and $\{g(x_n)\}$, $\{Ty_n\}$ are bounded, we obtain that

$$\limsup_{n\to\infty}\big(d(w_{n+1},w_n)-d(x_{n+1},x_n)\big)\leq 0.$$

Thus, by Lemma 2.18 and condition (ii), we obtain that

$$\lim_{n\to\infty} d(w_n, x_n) = 0. \tag{3.17}$$

Hence, by Lemma 2.6 we obtain that

$$d(x_{n+1}, x_n) \le (1-\beta_n)d(w_n, x_n) \to 0, \qquad \text{as } n \to \infty. \tag{3.18}$$

Step 3: We show that $\lim_{n\to\infty} d(x_n, T(S_{\lambda_n})x_n) = 0 = \lim_{n\to\infty} d(w_n, T(S_{\lambda_n})w_n)$. Notice also that (3.13) can be rewritten as

$$x_{n+1} = \alpha_n g(x_n) \oplus (1-\alpha_n)\Big(\frac{\beta_n x_n \oplus \gamma_n T y_n}{(1-\alpha_n)}\Big), \qquad y_n = S_{\lambda_n} x_n.$$

Thus, by Lemma 2.6, we obtain that

$$d\Big(x_{n+1}, \frac{\beta_n x_n \oplus \gamma_n T y_n}{(1-\alpha_n)}\Big) \le \alpha_n d\Big(g(x_n), \frac{\beta_n x_n \oplus \gamma_n T y_n}{(1-\alpha_n)}\Big) \to 0, \quad \text{as } n \to \infty. \tag{3.19}$$

Also, from (2.1), we obtain

$$d\Big(x_n, \frac{\beta_n x_n \oplus \gamma_n T y_n}{(1-\alpha_n)}\Big) = \frac{\gamma_n}{1-\alpha_n} d(x_n, T y_n),$$

which implies from (3.18) and (3.19) that

$$\frac{\gamma_n}{1-\alpha_n} d(x_n, T y_n) \le d(x_n, x_{n+1}) + d\Big(x_{n+1}, \frac{\beta_n x_n \oplus \gamma_n T y_n}{(1-\alpha_n)}\Big) \to 0, \quad \text{as } n \to \infty.$$

Hence,

$$\lim_{n\to\infty} d(x_n, T y_n) = \lim_{n\to\infty} d(x_n, T(S_{\lambda_n})x_n) = 0. \tag{3.20}$$

Now, since $\{x_n\}$ is bounded and X is a complete CAT(0) space, then from Lemma 2.11, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\Delta - \lim_{k\to\infty} x_{n_k} = \bar{z}$. Again, since $T \circ S_{\lambda_n}$ is nonexpansive (and every nonexpansive mapping is demiclosed), it follows from (3.20), condition (iii), Lemma 3.5 and Lemma 2.16 (iii) that $\bar{z} \in F(T \circ S_{\lambda_n}) \subseteq \cap_{i=1}^{N} F(J_\lambda^{f_i}) \cap F(T) = \Gamma$.

$$\begin{aligned} d(w_n, T(S_{\lambda_n})w_n) &\le d(w_n, x_n) + d(x_n, T(S_{\lambda_n})x_n) + d(T(S_{\lambda_n})x_n, T(S_{\lambda_n})w_n) \\ &\le 2d(w_n, x_n) + d(x_n, T(S_{\lambda_n} x_n) \to 0, \text{ as } n \to \infty. \end{aligned} \tag{3.21}$$

Step 4: We show that $\limsup_{n\to\infty} \langle \overrightarrow{g(\bar{z})\bar{z}}, \overrightarrow{x_n \bar{z}} \rangle \le 0$.

Now, define $T_n x = \beta_n x \oplus (1-\beta_n)w$, where $w = \frac{\alpha_n}{(1-\beta_n)} g(x) \oplus \frac{\gamma_n}{(1-\beta_n)} T(S_{\lambda_n})x$, then T_n is a contractive mapping for each $n \ge 1$. Thus, there exists a unique fixed point z_n of T_n $\forall n \ge 1$. That is,

$z_m = \beta_m z_m \oplus (1-\beta_m) w_m$, where $w_m = \frac{\alpha_m}{(1-\beta_m)} g(z_m) \oplus \frac{\gamma_m}{(1-\beta_m)} T(S_{\lambda_m}) z_m$.

Moreover, $\lim_{m\to\infty} z_m = \overline{z} \in \Gamma$ (see [31]).

Thus, we obtain that

$$\begin{aligned} d(z_m, w_n) &= d(\beta_m z_m \oplus (1-\beta_m) w_m, w_n) \\ &\le \beta_m d(z_m, w_n) + (1-\beta_m) d(w_m, w_n), \end{aligned}$$

which implies that

$$d(z_m, w_n) \leq d(w_m, w_n). \tag{3.22}$$

From (3.22) and Lemma 2.6(v), we obtain that

$$\begin{aligned}
d^2(w_m, w_n) &= \langle \overrightarrow{w_m w_n}, \overrightarrow{w_m w_n} \rangle \\
&= \langle \overrightarrow{w_m T(S_{\lambda_m}) z_m}, \overrightarrow{w_m w_n} \rangle + \langle \overrightarrow{T(S_{\lambda_m}) z_m w_n}, \overrightarrow{w_m w_n} \rangle \\
&\leq \frac{\alpha_m}{(1-\beta_m)} \langle \overrightarrow{g(z_m) T(S_{\lambda_m}) z_m}, \overrightarrow{w_m w_n} \rangle + \langle \overrightarrow{T(S_{\lambda_m} z_m) w_n}, \overrightarrow{w_m w_n} \rangle \\
&= \frac{\alpha_m}{(1-\beta_m)} \langle \overrightarrow{g(z_m) T(S_{\lambda_m} z_m)}, \overrightarrow{w_m z_m} \rangle + \frac{\alpha_m}{(1-\beta_m)} \langle \overrightarrow{g(z_m) w_n}, \overrightarrow{z_m w_n} \rangle \\
&+ \frac{\alpha_m}{(1-\beta_m)} \langle \overrightarrow{w_n T(S_{\lambda_m} z_m)}, \overrightarrow{z_m w_m} \rangle + \langle \overrightarrow{T(S_{\lambda_m} z_m) T(S_{\lambda_m} w_n)}, \overrightarrow{w_m w_m} \rangle \\
&+ \langle \overrightarrow{T(S_{\lambda_m} w_m)}, \overrightarrow{w_m w_n} \rangle \\
&\leq \frac{\alpha_m}{(1-\beta_m)} d(g(z_m), T(S_{\lambda_m} z_m)) d(w_m, z_m) + \frac{\alpha_m}{(1-\beta_m)} \langle \overrightarrow{g(z_m) z_m}, \overrightarrow{z_m w_n} \rangle \\
&+ \frac{\alpha_m}{(1-\beta_m)} \langle \overrightarrow{z_m T(S_{\lambda_m} z_m)}, \overrightarrow{z_m w_n} \rangle + d(T(S_{\lambda_m} z_m), T(S_{\lambda_m} w_n)) d(w_m, w_n) \\
&+ d(T(S_{\lambda_m} w_n), w_n) d(w_m, w_n) \\
&\leq \frac{\alpha_m}{(1-\beta_m)} d(g(z_m), T(S_{\lambda_m} z_m)) d(w_n, z_m) + \frac{\alpha_m}{(1-\beta_m)} \langle \overrightarrow{g(z_m) z_m}, \overrightarrow{z_m w_n} \rangle \\
&+ \frac{\alpha_m}{(1-\beta_m)} \langle \overrightarrow{z_m T(S_{\lambda_m} z_m)}, \overrightarrow{z_m w_n} \rangle + d(z_m, w_m) d(w_m, w_n) + d(T(S_{\lambda_m} w_n), w_n) d(w_n, w_m) \\
&\leq \frac{\alpha_m}{(1-\beta_m)} d(g(z_m), T(S_{\lambda_m} z_m)) d(w_n, z_m) + \frac{\alpha_m}{(1-\beta_m)} \langle \overrightarrow{g(z_m) z_m}, \overrightarrow{z_m w_n} \rangle \\
&+ \frac{\alpha_m}{(1-\beta_m)} d(z_m, T(S_{\lambda_m} z_m)) d(w_m, z_m) + d(w_m, w_n) + d(T(S_{\lambda_m} w_n), w_n) d(w_n, w_m),
\end{aligned}$$

which implies that

$$\begin{aligned}
\langle \overrightarrow{g(z_m) z_m}, \overrightarrow{w_n z_m} \rangle &\leq d(g(z_m), T(S_{\lambda_m}) z_m) d(w_n, z_m) + d(z_m, T(S_{\lambda_m}) z_m) d(z_m, w_m) \\
&+ \frac{(1-\beta_m)}{\alpha_m} d(T(S_{\lambda_n}) w_n, w_n) d(w_m, w_m).
\end{aligned}$$

Thus, taking lim sup as $n \to \infty$ first, then as $m \to \infty$, it follows from (3.17),(3.20) and (3.21) that

$$\limsup_{m\to\infty} \limsup_{n\to\infty} \langle \overrightarrow{g(z_m) z_m}, \overrightarrow{w_n z_m} \rangle \leq 0. \tag{3.23}$$

Furthermore,

$$\begin{aligned}
\langle \overrightarrow{g(\overline{z}) \overline{z}}, \overrightarrow{x_n \overline{z}} \rangle &= \langle \overrightarrow{g(\overline{z}) g(z_m)}, \overrightarrow{x_n \overline{z}} \rangle + \langle \overrightarrow{g(z_m) z_m}, \overrightarrow{x_n w_n} \rangle + \langle \overrightarrow{g(z_m) z_m}, \overrightarrow{w_n z_m} \rangle + \langle \overrightarrow{g(z_m) z_m}, \overrightarrow{z_m \overline{z}} \rangle + \langle \overrightarrow{z_m \overline{z}}, \overrightarrow{x_n \overline{z}} \rangle \\
&\leq d(g(\overline{z}), g(z_m)) d(x_n, \overline{z}) + d(g(z_m), z_m) d(x_n, w_n) + \langle \overrightarrow{g(z_m) z_m}, \overrightarrow{w_n z_m} \rangle \\
&+ d(g(z_m), z_m) d(z_m, \overline{z}) + d(z_m, \bar{z}) d(x_n, \bar{z}) \\
&\leq (1+\tau) d(z_m, \overline{z}) d(x_n, \bar{z}) + \langle \overrightarrow{g(z_m) z_m}, \overrightarrow{w_n z_m} \rangle + [d(x_n, w_n) + d(z_m, \overline{z})] d(g(z_m), z_m),
\end{aligned}$$

which implies from (3.17), (3.23) and the fact that $\lim_{m\to\infty} z_m = \overline{z}$, that

$$\begin{aligned}\limsup_{n\to\infty}\langle\overrightarrow{g(\overline{z})\overline{z}},\overrightarrow{x_n\overline{z}}\rangle &= \limsup_{m\to\infty}\limsup_{n\to\infty}\langle\overrightarrow{g(\overline{z})\overline{z}},\overrightarrow{x_n\overline{z}}\rangle\\ &\le \limsup_{m\to\infty}\limsup_{n\to\infty}\langle\overrightarrow{g(z_m)z_m},\overrightarrow{w_n z_m}\rangle \le 0. \end{aligned} \tag{3.24}$$

Step 5: Lastly, we show that $\{x_n\}$ converges strongly to $\overline{z}\in\Gamma$.
From Lemma 2.7, we obtain that

$$\begin{aligned}\langle\overrightarrow{w_n\overline{z}},\overrightarrow{x_n\overline{z}}\rangle &\le \frac{\alpha_n}{(1-\beta_n)}\langle\overrightarrow{g(x_n)\overline{z}},\overrightarrow{x_n\overline{z}}\rangle + \frac{\gamma_n}{(1-\beta_n)}\langle\overrightarrow{T(S_{\lambda_n})x_n\overline{z}},\overrightarrow{x_n\overline{z}}\rangle\\ &\le \frac{\alpha_n}{(1-\beta_n)}\langle\overrightarrow{g(x_n)g(\overline{z})},\overrightarrow{x_n\overline{z}}\rangle + \frac{\alpha_n}{(1-\beta_n)}\langle\overrightarrow{g(\overline{z})\overline{z}},\overrightarrow{x_n\overline{z}}\rangle + \frac{\gamma_n}{(1-\beta_n)}d(T(S_{\lambda_n})x_n,\overline{z})d(x_n,\overline{z})\\ &\le \frac{\alpha_n}{(1-\beta_n)}\tau d^2(x_n,\overline{z}) + \frac{\alpha_n}{(1-\beta_n)}\langle\overrightarrow{g(\overline{z})\overline{z}},\overrightarrow{x_n\overline{z}}\rangle + (1-\frac{\alpha_n}{1-\beta_n})d^2(x_n,\overline{z})\\ &= \left[\frac{\alpha_n}{(1-\beta_n)}\tau + (1-\frac{\alpha_n}{1-\beta_n})\right]d^2(x_n,\overline{z}) + \frac{\alpha_n}{(1-\beta_n)}\langle\overrightarrow{g(\overline{z})\overline{z}},\overrightarrow{x_n\overline{z}}\rangle.\end{aligned}$$

Thus, from Lemma 2.6, we have

$$\begin{aligned}d^2(x_{n+1},\overline{z}) &\le \beta_n d^2(x_n,\overline{z}) + (1-\beta_n)d^2(w_n,\overline{z})\\ &= \beta_n d^2(x_n,\overline{z}) + (1-\beta_n)\langle\overrightarrow{w_n\overline{z}},\overrightarrow{w_n\overline{z}}\rangle\\ &= \beta_n d^2(x_n,\overline{z}) + (1-\beta_n)[\langle\overrightarrow{w_n\overline{z}},\overrightarrow{w_n x_n}\rangle + \langle\overrightarrow{w_n\overline{z}},\overrightarrow{x_n\overline{z}}\rangle]\\ &\le [\beta_n + \alpha_n\tau + \gamma_n]d^2(x_n,\overline{z}) + (1-\beta_n)\langle\overrightarrow{w_n\overline{z}},\overrightarrow{w_n x_n}\rangle + \alpha_n\langle g(\overline{z})\overline{z}, x_n\overline{z}\rangle\\ &\le (1-\alpha_n(1-\tau))d^2(x_n,\overline{z}) + \alpha_n(1-\tau)\left[\frac{1}{1-\tau}\langle\overrightarrow{g(\overline{z})\overline{z}},\overrightarrow{x_n\overline{z}}\rangle\right] + (1-\beta_n)d(w_n,x_n)M.\end{aligned} \tag{3.25}$$

By (3.17) and applying Lemma 2.19 to (3.25), we obtain that $\{x_n\}$ converges strongly to $\bar{z}$. □

Corollary 3.7. *Let C be a nonempty closed and convex subset of a Hadamard space X and $f_i : C\times C\to\mathbb{R}$, $i = 1,2,\dots,N$ be a finite family of monotone and upper semicontinuous bifunctions such that $C\subset D(J_\lambda^{f_i})$ for $\lambda>0$. Let $g : C\to C$ be a contraction mapping with coefficient $\tau\in(0,1)$. Suppose that $\Gamma := \cap_{i=1}^N EP(f_i,C)\neq\varnothing$ and for arbitrary $x_1\in C$, the sequence $\{x_n\}$ is generated by*

$$\begin{cases} y_n = S_{\lambda_n}x_n := \beta_0 x_n\oplus\beta_1 J_{\lambda_n}^{f_1}x_n\oplus\beta_2 J_{\lambda_n}^{f_2}x_n\oplus\cdots\oplus\beta_N J_{\lambda_n}^{f_N}x_n,\\ x_{n+1} = \alpha_n g(x_n)\oplus\beta_n x_n\oplus\gamma_n y_n, \quad n\ge 1,\end{cases} \tag{3.26}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$, and $\{\lambda_n\}$ is a sequence of positive real numbers satisfying the following conditions:

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$,
(ii) $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1$, $\alpha_n+\beta_n+\gamma_n = 1$ $\forall n\ge 1$,
(iii) $0<\lambda\le\lambda_n$ $\forall n\ge 1$ *and* $\lim_{n\to\infty}\lambda_n = \lambda$,

(iv) $\beta_i \in (0,1)$ *with* $\sum_{i=0}^{N} \beta_i = 1$.

Then, $\{x_n\}$ *converges strongly to* $\bar{z} \in \Gamma$.

Corollary 3.8. *Let* C *be a nonempty closed and convex subset of a Hadamard space* X *and* $f : C \times C \to \mathbb{R}$ *be a monotone and upper semicontinuous bifunction such that* $C \subset D(J^f_\lambda)$ *for* $\lambda > 0$. *Let* $T : C \to C$ *be a nonexpansive mapping and* $g : C \to C$ *be a contraction mapping with coefficient* $\tau \in (0,1)$. *Suppose that* $\Gamma := EP(f, C) \cap F(T) \neq \varnothing$ *and for arbitrary* $x_1 \in C$, *the sequence* $\{x_n\}$ *is generated by*

$$\begin{cases} y_n = J^f_{\lambda_n} x_n, \\ x_{n+1} = \alpha_n g(x_n) \oplus \beta_n x_n \oplus \gamma_n T y_n, \quad n \geq 1, \end{cases} \tag{3.27}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *are sequences in* $(0,1)$, *and* $\{\lambda_n\}$ *is a sequence of positive real numbers satisfying the following conditions:*

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < \liminf\limits_{n\to\infty} \beta_n \leq \limsup\limits_{n\to\infty} \beta_n < 1$, $\alpha_n + \beta_n + \gamma_n = 1$ $\forall n \geq 1$,

(iii) $0 < \lambda \leq \lambda_n$ $\forall n \geq 1$ *and* $\lim\limits_{n\to\infty} \lambda_n = \lambda$.

Then, $\{x_n\}$ *converges strongly to* $\bar{z} \in \Gamma$.

4. Application to optimization problems

In this section, we give application of our results to solve some optimization problems. Throughout this section, X is a Hadamard space and C is a nonempty closed and convex subset of X.

4.1. **Minimization problem.** Let $h : X \to \mathbb{R}$ be a proper convex and lower semicontimnuous function. Consider the bifunction $f_h : C \times C \to \mathbb{R}$ defined by

$$f_h(x, y) = h(y) - h(x), \ \forall x, y \in C.$$

Then, f_h is monotone and upper semicontinuous (see [22]). Moreover, $\mathrm{EP}(C, f_h) = \arg\min_C h$, $J^{f_h} = \mathrm{prox}^h$ and $D(\mathrm{prox}^h) = X$ (see [22]). Now, consider the following finite family of minimization problem and fixed point problem:

$$\text{Find } x \in F(T) \text{ such that } h_i(x) \leq h_i(y), \ \forall y \in C, \ i = 1, 2 \ldots, N, \tag{4.1}$$

where T is a nonexpansive mapping. Thus, by setting $J^{f_i}_{\lambda_n} = \mathrm{prox}^{h_i}_{\lambda_n}$ in Algorithm (3.13), we can apply Theorem 3.6 to approximate solutions of problem (4.1).

4.2. **Variational inequality problem.** Let $S : C \to C$ be a nonexpansive mapping. Now define the bifunction $f_S : C \times C \to \mathbb{R}$ by $f_S(x, y) = \langle \overrightarrow{Sxx}, \overrightarrow{xy} \rangle$. Then, f_S is monotone and $J^{f_S} = J^S$ (see [20, 3]). Consider the following finite family of variational inequality and fixed point problems:

$$\text{Find } x \in F(T) \text{ such that } \langle \overrightarrow{S_i x x}, \overrightarrow{xy} \rangle \geq 0, \ \forall y \in C, \ i = 1, 2 \ldots, N, \tag{4.2}$$

where T is a nonexpansive mapping on C. Thus, by setting $J^{f_i}_{\lambda_n} = J^{S_i}_{\lambda_n}$ in Algorithm (3.13), we can apply Theorem 3.6 to approximate solutions of problem (4.2).

4.3. **Convex feasibility problem.** Let $C_i, i = 1, 2, \ldots, N$ be a finite family of nonempty closed and convex subsets of C such that $\cap_{i=1}^N C_i \neq \varnothing$. Now, consider the following convex feasibility problem:

$$\text{Find } x \in F(T) \text{ such that } x \in \cap_{i=1}^N C_i. \tag{4.3}$$

We know that the indicator function $\delta_C : X \to \mathbb{R}$ defined by

$$\delta_C(x) = \begin{cases} 0, & \text{if } x \in C, \\ +\infty, & \text{otherwise} \end{cases}$$

is a proper convex and lower semicontinuous function. By letting $\delta_C = h$ and following similar argument as in Subsection 4.1, we obtain that f_{δ_C} is monotone and upper semicontinuous, and $J^{f_{\delta_C}} = \text{prox}^{\delta_C} = P_C$. Therefore, by setting $J^{f_i} = P_{C_i},\ i = 1, 2, \ldots, N$ in Algorithm (3.13), we can apply Theorem 3.6 to approximate solutions of (4.3).

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