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Existence and convergence results for a class of nonexpansive type mappings in hyperbolic spaces

Rajendra Pant [a] and Rameshwar Pandey [b]

[a] Department of Pure and Applied Mathematics, University of Johannesburg, Auckland Park 2006, South Africa (pant.rajendra@gmail.com; rpant@uj.ac.za)
[b] Department of Mathematics, Visvesvaraya National Institute of Technology, Nagpur 440010, India (rampandey0502@gmail.com)

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Abstract

We consider a wider class of nonexpansive type mappings and present some fixed point results for this class of mappings in hyperbolic spaces. Indeed, first we obtain some existence results for this class of mappings. Next, we present some convergence results for an iteration algorithm for the same class of mappings. Some illustrative non-trivial examples have also been discussed.

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1. Introduction

A mapping p from the set of reals $\mathbb{R}$ to a metric space (E, ρ) is said to be *metric embedding* if $\rho(p(m), p(n)) = |m - n|$ for all $m, n \in \mathbb{R}$. The image of set $\mathbb{R}$ under a metric embedding is called a metric line. The image of a real interval $[a, b] = \{t \in \mathbb{R} : a \leq t \leq b\}$ under metric embedding is called a *metric segment*. Assume that (E, ρ) has a family $\mathcal{F}$ of metric lines such that for each pair $u, v \in E$ $(u \neq v)$ there is a unique metric line in $\mathcal{F}$ which passes through u and v. This metric line determines a unique metric segment joining u and

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v. This segment is denoted by $[u,v]$ and this is an isometric image of the real interval $[0,\rho(u,v)]$. We denote by $\gamma u \oplus (1-\gamma)v$, the unique point w of $[u,v]$ which satisfies

$$\rho(u,w) = (1-\gamma)\rho(u,v) \quad \text{and} \quad \rho(w,v) = \gamma\rho(u,v),$$

where $\gamma \in [0,1]$. Such a metric space with a family of metric segments is called a *convex metric space* [37]. Further, if we have

$$\rho(\gamma u \oplus (1-\gamma)v, \gamma w \oplus (1-\gamma)z) \leq \gamma\rho(u,w) + (1-\gamma)\rho(v,z)$$

for all $u,v,w,z \in E$, then E is said to be a *hyperbolic metric space* [42].

Hyperbolic spaces are more general than normed spaces and CAT(0) spaces. These spaces are nonlinear. Indeed, all normed linear and CAT(0) spaces are hyperbolic spaces (cf. [28, 29, 33]). As nonlinear examples, one can consider the Hadamard manifolds [6] and the Hilbert open unit ball equipped with the hyperbolic metric [14].

A mapping $T : E \to E$ is said to be nonexpansive if $\rho(T(u), T(v)) \leq \rho(u,v)$, for all $u,v \in E$. $F(T)$ denotes set of fixed points of T. The study of existence of fixed point of nonexpansive type mappings has been of great interest in nonlinear analysis (cf. [15, 46, 3, 4, 7, 8, 10, 11, 27, 35, 39, 41]). Fixed point theory of nonexpansive mappings in hyperbolic spaces has been extensively studied (cf. [47, 5, 38, 13, 16, 42, 31, 32]).

In the present paper, we consider a wider class of nonexpansive type mapping which properly generalizes some well-known classes of nonexpansive type mappings in hyperbolic spaces. We present some existence and convergence results. Since hyperbolic spaces are more general spaces, our theorems extend, generalize and complement many results in the literature.

2. Preliminaries

Let us recall the following definition which is due to Kohlenbach [31]:

Definition 2.1 ([31])**.** A triplet (E,ρ,H) is said to be a hyperbolic metric space if (E,ρ) is a metric space and $H : E \times E \times [0,1] \to E$ is a function such that for all $u,v,w,z \in E$ and $\beta,\gamma \in [0,1]$, the following hold:

(K1) $\rho(z, H(u,v,\beta)) \leq (1-\beta)\rho(z,u) + \beta\rho(z,v)$;
(K2) $\rho(H(u,v,\beta), H(u,v,\gamma)) = |\beta-\gamma|\rho(u,v)$;
(K3) $H(u,v,\beta) = H(v,u,1-\beta)$;
(K4) $\rho(H(u,z,\beta), H(v,w,\beta)) \leq (1-\beta)\rho(u,v) + \beta\rho(z,w)$.

The set $seg[u,v] := \{H(u,v,\beta); \beta \in [0,1]\}$ is called the metric segment with endpoints u and v. Now onwards, we write $H(u,v,\beta) = (1-\beta)u \oplus \beta v$. A subset K of E is said to be convex if $(1-\beta)u \oplus \beta v \in K$, for all $u,v \in K$ and $\beta \in [0,1]$. When there is no ambiguity, we write (E,ρ) for (E,ρ,H).

Definition 2.2 ([12, 20])**.** Let (E,ρ) be a hyperbolic metric space. For any $a\in E$, $r>0$ and $\epsilon>0$. Set

$$\delta(r,\epsilon)=\inf\left\{1-\frac{1}{r}\rho\left(\frac{1}{2}u\oplus\frac{1}{2}v,a\right);\rho(u,a)\leq r,\rho(v,a)\leq r,\rho(u,v)\geq r\epsilon\right\}.$$

We say that E is uniformly convex if $\delta(r,\epsilon)>0$, for any $r>0$ and $\epsilon>0$.

Definition 2.3 ([47, 9])**.** A hyperbolic metric space (E,ρ) is said to be strictly convex if for any $u,v,a\in E$ and $\alpha\in(0,1)$;

$$\rho(\alpha u\oplus(1-\alpha)v,a)=\rho(u,a)=\rho(v,a),$$

then we must have $u=v$.

Every uniformly convex hyperbolic metric space is strictly convex [9].

Definition 2.4 ([19])**.** A hyperbolic metric space (E,ρ) is said to satisfy property (R) if for each decreasing sequence $\{F_n\}$ of nonempty bounded closed convex subsets of E, $\bigcap\limits_{n=1}^{\infty}F_n\neq\varnothing$.

Uniformly convex hyperbolic spaces satisfy the property (R), see [5].

Definition 2.5 ([45])**.** Let K be a subset of a metric space (E,ρ). A mapping $T:K\to K$ is said to satisfy Condition (I) if there exists a nondecreasing function $g:[0,\infty)\to[0,\infty)$ satisfying $g(0)=0$ and $g(r)>0$ for all $r\in(0,\infty)$ such that $\rho(u,T(u))\geq g(dist(u,F(T)))$ for all $u\in K$, here $dist(u,F(T))$ denotes the distance of u from $F(T)$, where $F(T)$ denotes the set of fixed points of T.

Definition 2.6. A sequence $\{u_n\}$ in K is said to be approximate fixed point sequence (a.f.p.s for short) for a mapping $T:K\to K$ if $\lim\limits_{n\to\infty}\rho(T(u_n),u_n)=0$.

Let K be a nonempty subset of a hyperbolic metric space (E,ρ) and $\{u_n\}$ a bounded sequence in E. For each $u\in E$, define:

- asymptotic radius of $\{u_n\}$ at u as $r(\{u_n\},u):=\limsup\limits_{n\to\infty}\rho(u_n,u)$;
- asymptotic radius of $\{u_n\}$ relative to K as
$$r(\{u_n\},K):=\inf\{r(\{u_n\},u);u\in K\};$$
- asymptotic centre of $\{u_n\}$ relative to K by
$$A(\{u_n\},K):=\{u\in K;r(\{u_n\},u)=r(\{u_n\},K)\}.$$

Lim in [34] introduced the concept of Δ-convergence in a metric space. Kirk and Panyanak in [30] used Lim's concept in CAT(0) spaces and showed that many Banach space results involving weak convergence have precise analogs in this setting.

Definition 2.7 ([30])**.** A bounded sequence $\{u_n\}$ in E is said to Δ-converge to a point $u\in E$, if u is the unique asymptotic centre of every subsequence $\{u_{n_k}\}$ of $\{u_n\}$.

3. Main Results

The following definition is essentially due to Suzuki [46] in a Banach space.

Definition 3.1. Let K be a nonempty subset of a hyperbolic metric space E. A mapping $T : K \to K$ is said to satisfy condition (C) if for all $u, v \in K$,

$$\frac{1}{2}\rho(u, T(u)) \leq \rho(u, v) \quad \text{implies} \quad \rho(T(u), T(v)) \leq \rho(u, v). \tag{3.1}$$

Now, we consider a wider class of nonexpansive type mappings and present some auxiliary and existence results. We also discuss an illustrative example.

Definition 3.2. Let K be a nonempty subset of a hyperbolic metric space E. A mapping $T : K \to K$ is said to be *Reich-Suzuki type nonexpansive mappings* if there exists a $k \in [0, 1)$ such that

$$\frac{1}{2}\rho(u, T(u)) \leq \rho(u, v) \quad \text{implies}$$

$$\rho(T(u), T(v)) \leq k\rho(T(u), u) + k\rho(T(v), v) + (1 - 2k)\rho(u, v) \tag{3.2}$$

for all $u, v \in K$.

Now, we present some basic properties of the above class of mappings.

Proposition 3.3. *A mapping satisfying the condition (C) is a Reich-Suzuki type nonexpansive mapping but the converse need not be true.*

Proof. When $k = 0$, then T is a mapping satisfying the condition (C). □

The following example shows that the converse need not be true:

Example 3.4. Let $K = [-2, 2]$ be a subset of $\mathbb{R}$ endowed with the usual metric, that is, $\rho(u, v) = |u - v|$. Define $T : K \to K$ by

$$T(u) = \begin{cases} -\dfrac{u}{2}, & \text{if } u \in [-2, 0)\backslash\{\frac{-1}{8}\} \\ 0, & \text{if } u = \frac{-1}{8} \\ -\dfrac{u}{3}, & \text{if } u \in [0, 2]. \end{cases}$$

Then for $u = -\frac{1}{8}$ and $v = -\frac{1}{5}$, we have

$$\frac{1}{2}\rho(u, T(u)) = \frac{1}{16} \leq \frac{3}{40} = \rho(u, v), \text{ but } \rho(T(u), T(v)) = \frac{1}{10} > \frac{3}{40} = \rho(u, v).$$

Thus T does not satisfy condition (C). Now we show that T is a Reich-Suzuki type nonexpansive mapping with $k = \frac{1}{2}$. We consider different cases as follows:

(i) Let $u, v \in [-2, 0)\backslash\{\frac{-1}{8}\}$; we have

$$\begin{aligned} \rho(T(u), T(v)) = \frac{1}{2}|v - u| &\leq \frac{1}{2}|u| + \frac{1}{2}|v| \\ &\leq \frac{3}{4}|u| + \frac{3}{4}|v| = \frac{1}{2}\left|u + \frac{u}{2}\right| + \frac{1}{2}\left|v + \frac{v}{2}\right| \\ &= k(\rho(u, T(u)) + \rho(v, T(v))) + (1 - 2k)\rho(u, v). \end{aligned}$$

(ii) Let $u, v \in [0,2]$; we have

$$\begin{aligned}\rho(T(u),T(v)) = \frac{1}{3}|u-v| &\leq \frac{1}{3}|u| + \frac{1}{3}|v| \\ &\leq \frac{2}{3}|u| + \frac{2}{3}|v| = \frac{1}{2}\left|u + \frac{u}{3}\right| + \frac{1}{2}\left|v + \frac{v}{3}\right| \\ &= k(\rho(u,T(u)) + \rho(v,T(v))) + (1-2k)\rho(u,v).\end{aligned}$$

(iii) Let $u \in [-2,0)\backslash\{\frac{-1}{8}\}$ and $v \in [0,2]$; we have

$$\begin{aligned}\rho(T(u),T(v)) = |\frac{v}{3} - \frac{u}{2}| \leq \frac{1}{2}|u| + \frac{1}{3}|v| &\leq \frac{3}{4}|u| + \frac{2}{3}|v| \\ &= \frac{1}{2}\left|u + \frac{u}{2}\right| + \frac{1}{2}\left|v + \frac{v}{3}\right| \\ &= k(\rho(u,T(u)) + \rho(v,T(v))) \\ &\quad +(1-2k)\rho(u,v).\end{aligned}$$

(iv) Let $u \in [-2,0)\backslash\{\frac{-1}{8}\}$ and $v = \frac{-1}{8}$; we have

$$\begin{aligned}\rho(T(u),T(v)) = \frac{1}{2}|u| &\leq \frac{3}{4}|u| + \frac{1}{16} \\ &= \frac{1}{2}\rho(T(u),u) + \frac{1}{2}\rho(T(v),v) \\ &= k(\rho(u,T(u)) + \rho(v,T(v))) + (1-2k)\rho(u,v).\end{aligned}$$

(v) Let $u \in [0,2]$ and $v = \frac{-1}{8}$; we have

$$\begin{aligned}\rho(T(u),T(v)) = \frac{1}{3}|u| &\leq \frac{2}{3}|u| + \frac{1}{16} \\ &= \frac{1}{2}\rho(T(u),u) + \frac{1}{2}\rho(T(v),v) \\ &= k(\rho(u,T(u)) + \rho(v,T(v))) + (1-2k)\rho(u,v).\end{aligned}$$

Thus T is a Reich-Suzuki type nonexpansive mapping with only fixed point 0.

Notice that the space considered in the above example was a linear space. Now we present an example of a hyperbolic space which is not linear. Therefore it is a non-trivial example of a hyperbolic space.

Example 3.5 (see also [17])**.** Let $E = \{(u_1,u_2) \in \mathbb{R}^2; u_1,u_2 > 0\}$. Define $\rho : E \times E \to [0,\infty)$ by

$$\rho(u,v) = |u_1 - v_1| + |u_1u_2 - v_1v_2|$$

for all $u = (u_1,u_2)$ and $v = (v_1,v_2)$ in E. Then it can be easily seen that (E,ρ) is a metric space. Now for $\beta \in [0,1]$, define a function $H : E \times E \times [0,1] \to E$ by

$$H(u,v,\beta) = \left((1-\beta)u_1 + \beta v_1, \frac{(1-\beta)u_1u_2 + \beta v_1v_2}{(1-\beta)u_1 + \beta v_1}\right).$$

Then (E,ρ,H) is a hyperbolic metric space.

Further, suppose $K := [\frac{1}{2}, 3] \times [\frac{1}{2}, 3] \subset E$ and $T : K \to K$ be a mapping defined by

$$T(u_1, u_2) = \begin{cases} (1, 1), & \text{if } (u_1, u_2) \neq (\frac{1}{2}, \frac{1}{2}) \\ (\frac{3}{2}, \frac{3}{2}), & \text{if } (u_1, u_2) = (\frac{1}{2}, \frac{1}{2}). \end{cases}$$

Then T is a Reich-Suzuki type nonexpansive mapping which does not satisfy the condition (C).

Proof. First, we show that (E, ρ, H) is a hyperbolic metric space. For $u = (u_1, u_2)$, $v = (v_1, v_2)$, $z = (z_1, z_2)$ and $w = (w_1, w_2)$ in E :

(K1) $$\begin{aligned} \rho(z, H(u, v, \beta)) &= |z_1 - (1-\beta)u_1 - \beta v_1| + |z_1 z_2 - (1-\beta)u_1 u_2 - \beta v_1 v_2| \\ &\leq (1-\beta)|z_1 - u_1| + \beta|z_1 - v_1| + (1-\beta)|z_1 z_2 - u_1 u_2| \\ &\quad + \beta|z_1 z_2 - v_1 v_2| \\ &= (1-\beta)\rho(z, u) + \beta\rho(z, v). \end{aligned}$$

(K2) $$\begin{aligned} \rho(H(u, v, \beta), H(u, v, \gamma)) &= |(1-\beta)u_1 + \beta v_1 - (1-\gamma)u_1 - \gamma v_1| \\ &\quad + |(1-\beta)u_1 u_2 + \beta v_1 v_2 - (1-\gamma)u_1 u_2 - \gamma v_1 v_2| \\ &= |\beta - \gamma|(|u_1 - v_1| + |u_1 u_2 - v_1 v_2|) \\ &= |\beta - \gamma|\rho(u, v). \end{aligned}$$

(K3) $$\begin{aligned} H(u, v, \beta) &= \left((1-\beta)u_1 + \beta v_1, \frac{(1-\beta)u_1 u_2 + \beta v_1 v_2}{(1-\beta)u_1 + \beta v_1}\right) \\ &= \left(\beta v_1 + (1-\beta)u_1, \frac{\beta v_1 v_2 + (1-\beta)u_1 u_2}{\beta v_1 + (1-\beta)u_1}\right) \\ &= H(v, u, 1-\beta). \end{aligned}$$

(K4) $$\begin{aligned} \rho(H(u, z, \beta), H(v, w, \beta)) &= |(1-\beta)u_1 + \beta z_1 - (1-\beta)v_1 - \beta w_1| \\ &\quad + |(1-\beta)u_1 u_2 + \beta z_1 z_2 - (1-\beta)v_1 v_2 - \beta w_1 w_2| \\ &\leq (1-\beta)(|u_1 - v_1| + |u_1 u_2 - v_1 v_2|) \\ &\quad + \beta(|z_1 - w_1| + |z_1 z_2 - w_1 w_2|) \\ &= (1-\beta)\rho(u, v) + \beta\rho(z, w). \end{aligned}$$

Therefore, (E, ρ, H) is a hyperbolic metric space but not a normed linear space. Next, we show that T does not satisfies condition (C) on K. Let $u = (\frac{1}{2}, \frac{1}{2})$ and $v = (\frac{11}{10}, \frac{11}{10})$. Then

$$\frac{1}{2}\rho(u, T(u)) = \frac{3}{2} \leq \frac{156}{100} = \rho(u, v), \text{ but } \rho(T(u), T(v)) = \frac{7}{4} > \frac{156}{100} = \rho(u, v).$$

Finally, we show that T is Reich-Suzuki type nonexpansive mapping for $k = \frac{1}{2}$. We consider the following cases:

Case (i) If $u = (u_1, u_2)$, $v = (v_1, v_2) \neq (\frac{1}{2}, \frac{1}{2})$, then

$$\rho(T(u), T(v)) = 0 \leq \frac{1}{2}(\rho(u, T(u)) + \rho(v, T(v))).$$

Case (ii) If $u = (u_1, u_2) \neq (\frac{1}{2}, \frac{1}{2})$ and $v = (v_1, v_2) = (\frac{1}{2}, \frac{1}{2})$, then

$$\begin{aligned}\frac{1}{2}(\rho(u, T(u)) + \rho(v, T(v))) &= \frac{1}{2}[|u_1 - 1| + |u_1 u_2 - 1|] + \frac{1}{2}\left[1 + \frac{10}{4}\right] \\ &= \frac{1}{2}[|u_1 - 1| + |u_1 u_2 - 1|] + \frac{14}{8} \\ &\geq \frac{7}{4} = \rho(T(u), T(v)).\end{aligned}$$

Therefore T is a Reich-Suzuki type nonexpansive mapping with only fixed point $(1, 1)$. □

Proof of the following Proposition and Lemma may be completed on the pattern of [46].

Proposition 3.6. *Let K be a nonempty subset of a hyperbolic metric space E and $T : K \to K$ is a Reich-Suzuki type nonexpansive mapping with a fixed point $z \in K$. Then T is quasi-nonexpansive.*

The following lemma gives the structure of fixed point set for a Reich-Suzuki type nonexpansive mapping.

Lemma 3.7. *Let K be a nonempty subset of a hyperbolic metric space E and $T : K \to K$ is a Reich-Suzuki type nonexpansive mapping. Then $F(T)$ is closed. Moreover, if E is strictly convex and K is convex, then $F(T)$ is convex.*

The following lemmas will be useful to prove main results of this section, which are modeled on the pattern of [46] and can be proved easily. Therefore proof are omitted.

Lemma 3.8. *Let K be a nonempty subset of a hyperbolic metric space E and $T : K \to K$ is a Reich-Suzuki type nonexpansive mapping. Then for each $u, v \in K$,*

(i) $\rho(T(u), T^2(u)) \leq \rho(u, T(u))$;

(ii) *Either* $\frac{1}{2}\rho(u, T(u)) \leq \rho(u, v)$ or $\frac{1}{2}\rho(T(u), T^2(u)) \leq \rho(T(u), v)$;

(iii) *Either* $\rho(T(u), T(v)) \leq k\rho(T(u), u) + k\rho(v, T(v)) + (1 - 2k)\rho(u, v)$ or $\rho(T^2(u), T(v)) \leq k\rho(T^2(u), T(u)) + k\rho(T(v), v) + (1 - 2k)\rho(T(u), v)$.

Lemma 3.9. *Let K be a nonempty subset of a hyperbolic metric space E and $T : K \to K$ is a Reich-Suzuki type nonexpansive mapping. Then for all $u, v \in K$, we have*

$$\rho(u, T(v)) \leq \frac{(3 + k)}{(1 - k)}\rho(u, T(u)) + \rho(u, v).$$

The simplest iteration process is the well-known Picard iteration process and is defined as:

$$\begin{cases} u_1 \in Y \\ u_{n+1} = T(u_n), \quad n \in \mathbb{N}. \end{cases}$$

Even though T is nonexpansive and has a fixed point, it is possible that the Picard iteration does not converge to the fixed point of T. To overcome from such problems and to get better rate of convergence, a number of iteration processes have been introduced by many authors (cf. [1, 2, 18, 21, 22, 23, 24, 25, 36, 40, 43, 44, 48] and others). We present some convergence results for Thakur *et al*[48] iteration process. One can obtain similar results for other iteration processes using the same line of proofs.

For some fixed point $u_1 \in K$, the Thakur *et al.* iterative scheme in the framework of hyperbolic metric spaces can be defined as follows [48]:

$$\begin{cases} u_1 \in K \\ u_{n+1} = T(v_n) \\ v_n = T((1-\alpha_n)u_n \oplus \alpha_n w_n) \\ w_n = (1-\beta_n)u_n \oplus \beta_n T(u_n), \quad n \in \mathbb{N}, \end{cases} \tag{3.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0,1)$.

Lemma 3.10. *Let K be a nonempty closed convex subset of a hyperbolic metric space E and $T : K \to K$ is a Reich-Suzuki type nonexpansive mapping. Let $\{u_n\}$ be a sequence with $u_1 \in K$ defined by (3.3). For any $z \in F(T)$ the following hold:*

(i) $\max\{\rho(u_{n+1}, z), \rho(v_n, z), \rho(w_n, z)\} \leq \rho(u_n, z)$ *for all* $n \in \mathbb{N}$;
(ii) $\lim_{n\to\infty} \rho(u_n, z) =$ *exists;*
(iii) $\lim_{n\to\infty} D(u_n, F(T))$ *exists.*

Proof. Let $z \in F(T)$ be arbitrary. By Proposition 3.6 and (3.3), we have

$$\begin{aligned} \rho(w_n, z) &= \rho((1-\beta_n)u_n \oplus \beta_n T(u_n), z) \\ &\leq (1-\beta_n)\rho(u_n, z) + \beta_n \rho(T(u_n), z) \\ &\leq (1-\beta_n)\rho(u_n, z) + \beta_n \rho(u_n, z) \\ &= \rho(u_n, z). \end{aligned} \tag{3.4}$$

Further, by Proposition 3.6, (3.3) and (3.4), we have

$$\begin{aligned} \rho(v_n, z) &= \rho(T((1-\alpha_n)u_n \oplus \alpha_n w_n), z) \\ &\leq \rho((1-\alpha_n)u_n \oplus \alpha_n w_n, z) \\ &\leq (1-\alpha_n)\rho(u_n, z) + \alpha_n \rho(w_n, z) \\ &\leq (1-\alpha_n)\rho(u_n, z) + \alpha_n \rho(u_n, z) \\ &= \rho(u_n, z). \end{aligned} \tag{3.5}$$

Further, by Proposition 3.6, (3.3) and (3.5), we have

$$\begin{aligned} \rho(u_{n+1}, z) &= \rho(T(v_n), z) \\ &\leq \rho(v_n, z) \\ &\leq \rho(u_n, z). \end{aligned} \tag{3.6}$$

Combining (3.4), (3.5) and (3.6) together establishes (i). Also by (3.6) the sequence $\{\rho(u_n, z)\}$ is bounded and monotone decreasing. Therefore $\lim_{n\to\infty} \rho(u_n, z)$ exists and this proves (ii).
Now, since for each $z \in F(T)$, we have $\rho(u_{n+1}, z) \leq \rho(u_n, z)$ for all $n \in \mathbb{N}$. Taking infimum over all $z \in F(T)$, we get $D(u_{n+1}, F(T)) \leq D(u_n, F(T))$ for all $n \in \mathbb{N}$. So, the sequence $\{D(u_n, F(T))\}$ is bounded and decreasing. Therefore, $\lim_{n\to\infty} D(u_n, F(T))$ exists. □

Our next result is prefaced by the following lemmas.

Lemma 3.11 ([32]). *Let E be a complete uniformly convex hyperbolic metric space with monotone modulus of uniform convexity δ. Then every bounded sequence $\{u_n\}$ in K has a unique asymptotic centre with respect to any nonempty closed convex subset K of E.*

Lemma 3.12 ([26]). *Let (E, ρ) be a uniformly convex hyperbolic metric space with monotone modulus of uniform convexity δ. Let $z \in E$ and $\{\alpha_n\}$ be a sequence such that $0 < a \leq \alpha_n \leq b < 1$. If $\{u_n\}$ and $\{v_n\}$ are sequences in E such that $\limsup_{n\to\infty} \rho(u_n, z) \leq r$, $\limsup_{n\to\infty} \rho(v_n, z) \leq r$ and $\lim_{n\to\infty} \rho(\alpha_n v_n \oplus (1 - \alpha_n)u_n, z) = r$ for some $r \geq 0$, then we have $\lim_{n\to\infty} \rho(u_n, v_n) = 0$.*

Theorem 3.13. *Let K be a nonempty closed convex subset of a complete uniformly convex hyperbolic metric space E and $T : K \to K$ is a Reich-Suzuki type nonexpansive mapping. Let $\{u_n\}$ be a sequence with $u_1 \in K$ defined by (3.3) such that $\{\alpha_n\} \subseteq [1/2, b)$ and $\{\beta_n\} \subseteq [a, b]$ or $\{\alpha_n\} \subseteq [a, b]$ and $\{\beta_n\} \subseteq [a, 1]$ for some a, b with $0 < a \leq b < 1$. Then $F(T) \neq \varnothing$ if and only if $\{u_n\}$ is bounded and $\lim_{n\to\infty} \rho(T(u_n), u_n) = 0$.*

Proof. Suppose $\{u_n\}$ is a bounded sequence and $\lim_{n\to\infty} \rho(T(u_n), u_n) = 0$. By Lemma 3.11, $A(\{u_n\}, K) \neq \varnothing$, let $z \in A(\{u_n\}, K)$. By definition of asymptotic radius, we have

$$r(\{u_n\}, T(z)) = \limsup_{n\to\infty} \rho(u_n, T(z)).$$

Using Lemma 3.9, we have

$$\begin{aligned} r(\{u_n\}, T(z)) &= \limsup_{n\to\infty} \rho(u_n, T(z)) \\ &\leq \frac{(3+k)}{(1-k)} \limsup_{n\to\infty} \rho(T(u_n), u_n) + \limsup_{n\to\infty} \rho(u_n, z) \\ &= r(\{u_n\}, z). \end{aligned}$$

By the uniqueness of the asymptotic centre of $\{u_n\}$, we have $T(z) = z$.
Conversely, let $F(T) \neq \varnothing$ and $z \in F(T)$. Then from Lemma 3.10, $\lim_{n\to\infty} \rho(u_n, z)$ exists. Suppose

$$\lim_{n\to\infty} \rho(u_n, z) = r. \tag{3.7}$$

By (3.7) and Proposition 3.6, we have

$$\limsup_{n\to\infty} \rho(T(u_n), z) \le r. \tag{3.8}$$

By (3.7) and (3.4),

$$\limsup_{n\to\infty} \rho(w_n, z) \le \lim_{n\to\infty} \rho(u_n, z) = r. \tag{3.9}$$

Using (3.9) and Proposition 3.6,

$$\limsup_{n\to\infty} \rho(T(w_n), z) \le r. \tag{3.10}$$

By (3.3), Lemma 3.10 and Proposition 3.6, we have

$$\begin{array}{rcl}
\rho(u_{n+1}, z) & = & \rho(T(v_n), z) \\
& \le & \rho(v_n, z) \\
& = & \rho(T((1-\alpha_n)u_n \oplus \alpha_n w_n), z) \\
& \le & \rho((1-\alpha_n)u_n \oplus \alpha_n w_n, z) \\
& \le & (1-\alpha_n)\rho(u_n, z) + \alpha_n \rho(w_n, z) \\
& \le & (1-\alpha_n)\rho(u_n, z) + \alpha_n \rho(u_n, z) \\
& = & \rho(u_n, z),
\end{array}$$

or

$$\rho(u_{n+1}, z) \le \rho((1-\alpha_n)u_n \oplus \alpha_n w_n, z) \le \rho(u_n, z), \tag{3.11}$$

it implies that

$$r \le \lim_{n\to\infty} \rho((1-\alpha_n)u_n \oplus \alpha_n w_n, z) \le r,$$

Then

$$\lim_{n\to\infty} \rho((1-\alpha_n)u_n \oplus \alpha_n w_n, z) = r. \tag{3.12}$$

From (3.7), (3.9), (3.12) and Lemma 3.12, we get

$$\lim_{n\to\infty} \rho(u_n, w_n) = 0. \tag{3.13}$$

By the triangle inequality, we have

$$\rho(u_n, z) \le \rho(u_n, w_n) + \rho(w_n, z),$$

making $n \to \infty$ and using (3.13)

$$r \le \liminf_{n\to\infty} \rho(w_n, z). \tag{3.14}$$

So, by (3.9) and (3.14) we have,

$$\lim_{n\to\infty} \rho(w_n, z) = r. \tag{3.15}$$

Now, by (3.3) and Proposition 3.6, we have

$$\begin{aligned}\rho(w_n, z) &= \rho((1-\beta_n)u_n \oplus \beta_n T(u_n), z)\\ &\leq (1-\beta_n)\rho(u_n, z) + \beta_n \rho(T(u_n), z)\\ &\leq (1-\beta_n)\rho(u_n, z) + \beta_n \rho(u_n, z)\\ &= \rho(u_n, z). \end{aligned} \tag{3.16}$$

So, making $n \to \infty$ and using equation (3.15) and (3.7), we get

$$\lim_{n\to\infty} \rho((1-\beta_n)u_n \oplus \beta_n T(u_n), z) = r. \tag{3.17}$$

By (3.7), (3.8), (3.17) and Lemma 3.12, we conclude that $\lim_{n\to\infty} \rho(T(u_n), u_n) = 0$. □

Now, we present a result for Δ-convergence.

Theorem 3.14. *Let K be a nonempty closed convex subset of a complete uniformly convex hyperbolic metric space E and $T : K \to K$ be a Reich-Suzuki type nonexpansive mapping with $F(T) \neq \varnothing$. Let $\{u_n\}$ be a sequence with $u_1 \in K$ defined by (3.3) such that $\{\alpha_n\} \subseteq [1/2, b)$ and $\{\beta_n\} \subseteq [a, b]$ or $\{\alpha_n\} \subseteq [a, b]$ and $\{\beta_n\} \subseteq [a, 1]$ for some a, b with $0 < a \leq b < 1$. Then the sequence $\{u_n\}$ Δ-converges to a fixed point of T.*

Proof. By Theorem 3.13, $\{u_n\}$ is a bounded sequence. Therefore $\{u_n\}$ has a Δ-convergent subsequence. We show that every Δ-convergent subsequence of $\{u_n\}$ has a unique Δ-limit in $F(T)$. Arguing by contradiction suppose $\{u_n\}$ has two subsequences $\{u_{n_j}\}$ and $\{u_{n_k}\}$ Δ-converging to l and m, respectively. By Theorem 3.13, $\{u_{n_j}\}$ is bounded and $d(T(u_{n_j}), u_{n_j}) = 0$. We claim that $l \in F(T)$. We know that

$$r(\{u_{n_j}\}, T(l)) = \limsup_{j\to\infty} \rho(u_{n_j}, T(l)).$$

By Lemma 3.9, we have

$$\begin{aligned} r(\{u_{n_j}\}, T(l)) &= \limsup_{j\to\infty} \rho(u_{n_j}, T(l))\\ &\leq \frac{(3+k)}{(1-k)} \limsup_{j\to\infty} \rho(u_{n_j}, T(u_{n_j})) + \limsup_{j\to\infty} \rho(u_{n_j}, l)\\ &\leq r(\{u_{n_j}\}, l). \end{aligned}$$

Since the asymptotic centre of $\{u_{n_j}\}$ has a unique element, so $T(l) = l$. Similarly, $T(m) = m$. By the uniqueness of asymptotic centre of a sequence, we have

$$\begin{aligned} \limsup_{n\to\infty} \rho(u_n, l) &= \limsup_{j\to\infty} \rho(u_{n_j}, l) < \limsup_{j\to\infty} \rho(u_{n_j}, m)\\ &= \limsup_{n\to\infty} \rho(u_n, m) = \limsup_{k\to\infty} \rho(u_{n_k}, m)\\ &< \limsup_{k\to\infty} \rho(u_{n_k}, l) = \limsup_{n\to\infty} \rho(u_n, l), \end{aligned}$$

which is a contradiction, unless $l = m$. □

Theorem 3.15. *Let K be a nonempty closed and convex subset of a uniformly convex hyperbolic metric space E. Let $T : K \to K$ be a Reich-Suzuki type nonexpansive mapping with $F(T) \neq \varnothing$. Let $\{u_n\}$ be a sequence with $u_1 \in K$ defined by (3.3) such that $\{\alpha_n\} \subseteq [1/2, b)$ and $\{\beta_n\} \subseteq [a, b]$ or $\{\alpha_n\} \subseteq [a, b]$ and $\{\beta_n\} \subseteq [a, 1]$ for some a, b with $0 < a \leq b < 1$. Then the sequence $\{u_n\}$ converges strongly to a fixed point of T if and only if $\liminf_{n\to\infty} D(u_n, F(T)) = 0$.*

Proof. Suppose that $\liminf_{n\to\infty} D(u_n, F(T)) = 0$. From Lemma 3.10, $\liminf_{n\to\infty} D(u_n, F(T))$ exists, so

$$\lim_{n\to\infty} \rho(u_n, F(T)) = 0. \tag{3.18}$$

In view of (3.18), there exists a subsequence $\{u_{n_k}\}$ of the sequence $\{u_n\}$ such that $\rho(u_{n_k}, z_k) \leq \frac{1}{2^k}$ for all $k \geq 1$, where $\{z_k\}$ is a sequence in $F(T)$. By Lemma 3.10, we have

$$\rho(u_{n_{k+1}}, z_k) \leq \rho(u_{n_k}, z_k) \leq \frac{1}{2^k}. \tag{3.19}$$

Now, by the triangle inequality and (3.19), we have

$$\begin{aligned} \rho(z_{k+1}, z_k) &\leq \rho(z_{k+1}, u_{n_{k+1}}) + \rho(u_{n_{k+1}}, z_k) \\ &\leq \frac{1}{2^{k+1}} + \frac{1}{2^k} < \frac{1}{2^{k-1}}. \end{aligned}$$

A standard argument shows that $\{z_k\}$ is a Cauchy sequence. By Lemma 3.7, $F(T)$ is closed, so $\{z_k\}$ converges to some point $z \in F(T)$. Now

$$\rho(u_{n_k}, z) \leq \rho(u_{n_k}, z_k) + \rho(z_k, z).$$

Letting $k \to \infty$ implies that $\{u_{n_k}\}$ converges strongly to z. By Lemma 3.10, $\lim_{n\to\infty} \rho(u_n, z)$ exists. Hence $\{u_n\}$ converges strongly to z. The converse part is obvious. ☐

Finally, we give another strong convergence theorem.

Theorem 3.16. *Let E be a uniformly convex and complete hyperbolic metric space. Let K, T and $\{u_n\}$ be same as in Theorem 3.13. Let T satisfy the condition (I) with $F(T) \neq \varnothing$, then $\{u_n\}$ converges strongly to a fixed point of T.*

Proof. From Theorem 3.13, it follows that

$$\liminf_{n\to\infty} \rho(T(u_n), u_n) = 0. \tag{3.20}$$

Since T satisfy the condition (I), we have $\rho(T(u_n), u_n) \geq g(D(u_n, F(T)))$. From (3.20), we get

$$\liminf_{n\to\infty} g(D(u_n, F(T))) = 0.$$

Since $g : [0, \infty) \to [0, \infty)$ is a nondecreasing function with $g(0) = 0$ and $g(r) > 0$ for all $r \in (0, \infty)$, we have

$$\liminf_{n\to\infty} D(u_n, F(T)) = 0.$$

Therefore all the assumptions of Theorem 3.15 are satisfied. Hence $\{u_n\}$ converges strongly to a fixed point of T. □

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