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Extremal balleans

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ABSTRACT

A ballean (or coarse space) is a set endowed with a coarse structure. A ballean X is called normal if any two asymptotically disjoint subsets of X are asymptotically separated. We say that a ballean X is ultranormal (extremely normal) if any two unbounded subsets of X are not asymptotically disjoint (every unbounded subset of X is large). Every maximal ballean is extremely normal and every extremely normal ballean is ultranormal, but the converse statements do not hold. A normal ballean is ultranormal if and only if the Higson′s corona of X is a singleton. A discrete ballean X is ultranormal if and only if X is maximal. We construct a series of concrete balleans with extremal properties.

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1. Introduction

Let X be a set. A family $\mathcal{E}$ of subsets of $X \times X$ is called a *coarse structure* if

- each $E \in \mathcal{E}$ contains the diagonal $\triangle_X$, $\triangle_X = \{(x,x) : x \in X\}$;
- if E, $E' \in \mathcal{E}$ then $E \circ E' \in \mathcal{E}$ and $E^{-1} \in \mathcal{E}$, where $E \circ E' = \{(x,y) : \exists z((x,z) \in E, \ (z,y) \in E')\}$, $E^{-1} = \{(y,x) : (x,y) \in E\}$;
- if $E \in \mathcal{E}$ and $\triangle_X \subseteq E' \subseteq E$ then $E' \in \mathcal{E}$;
- for any $x, y \in X$, there exists $E \in \mathcal{E}$ such that $(x,y) \in E$.

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A subset $\mathcal{E}' \subseteq \mathcal{E}$ is called a *base* for $\mathcal{E}$ if, for every $E \in \mathcal{E}$, there exists $E' \in \mathcal{E}'$ such that $E \subseteq E'$. For $x \in X$, $A \subseteq X$ and $E \in \mathcal{E}$, we denote $E[x] = \{y \in X : (x,y) \in E\}$, $E[A] = \cup_{a \in A} \ E[a]$ and say that $E[x]$ and $E[A]$ are *balls of radius E around x* and A.

The pair $(X, \mathcal{E})$ is called a *coarse space* [14] or a ballean [10], [12].

Each subset $Y \subseteq X$ defines the *subballean* $(Y, \mathcal{E}_Y)$, where $\mathcal{E}_Y$ is the restriction of $\mathcal{E}$ to $Y \times Y$. A subset Y is called *bounded* if $Y \subseteq E[x]$ for some $x \in X$ and $E \in \mathcal{E}$.

Given a ballean $(X, \mathcal{E})$, a subset Y of X is called

- *large* if there exists $E \in \mathcal{E}$ such that $X = E[A]$;
- *small* if $(X \setminus A) \cap L$ is large for each large subset L;
- *thick* if, for every $E \in \mathcal{E}$, there exists $a \in A$ such that $E[a] \subseteq A$.

Every metric d on a set X defines the *metric ballean* $(X, \mathcal{E}_d)$, where $\mathcal{E}_d$ has the base $\{\{(x,y) : d(x,y) \leq r\} : r \in \mathbb{R}^+\}$. A ballean $(X, \mathcal{E})$ is called *metrizable* if there exists a metric d on X such that $\mathcal{E} = \mathcal{E}_d$. A ballean $(X, \mathcal{E})$ is metrizable if and only if $\mathcal{E}$ has a countable base [12, Theorem 2.1.1]. Let $(X, \mathcal{E})$ be a ballean. A subset U of X is called an *asymptotic neighbourhood* of a subset $Y \subseteq X$ if, for every $E \in \mathcal{E}$, $E[Y] \setminus U$ is bounded.

Two subsets Y, Z of X are called

- *asymptotically disjoint* if, for every $E \in \mathcal{E}$, $E[Y] \cap E[Z]$ is bounded;
- *asymptotically separated* if Y, Z have disjoint asymptotic neighbourhoods.

A ballean $(X, \mathcal{E})$ is called *normal* [7] if any two asymptotically disjoint subsets are asymptotically separated. Every ballean $(X, \mathcal{E})$ with linearly ordered base of $\mathcal{E}$, in particular, every metrizable ballean is normal [7, Proposition 1.1].

A function $f : X \longrightarrow \mathbb{R}$ is called *slowly oscillating* if, for any $E \in \mathcal{E}$ and $\varepsilon > 0$, there exists a bounded subset B of X such that $diam \ \ f(E[x]) < \varepsilon$ for each $x \in X \ B$. By [7, Theorem 2.2], a ballean $(X, \mathcal{E})$ is normal if and only if, for any two disjoint and asymptotically disjoint subsets Y, Z of X, there exists a slowly oscillating function $f : X \longrightarrow [0,1]$ such that $f|_Y = 0$ and $f|_Z = 1$.

We say that an unbounded ballean $(X, \mathcal{E})$ is

- *ultranormal* [1] if any two unbounded subsets of X are not asymptotically disjoint;
- *extremely normal* if any unbounded subset of X is large.

An unbounded ballean $(X, \mathcal{E})$, is called *maximal* if X is bounded in any stronger coarse structure. By [13], every unbounded subset of a maximal ballean is large and every small subset is bounded. Hence, every maximal ballean is extremely normal.

A family $\mathcal{B}$ of subsets of a set X, closed under finite unions and subsets, is called a *bornology* if $\cup\mathcal{B} = X$. For every ballean X, the family $\mathcal{B}_X$ of all bounded subsets of X is a bornology.

A ballean $(X, \mathcal{E})$ is called *discrete* (or *pseudodiscrete* [12], or *thin* [5]) if, for every $E \in \mathcal{E}$, there exists $B \in \mathcal{B}_X$ such that $E[x] = \{x\}$ for each $x \in X \setminus B$. Every bornology $\mathcal{B}$ on a set X defines the discrete ballean $(X, \{E_B : B \in \mathcal{B}\})$, where $E_B[x] = B$ if $x \in B$ and $E_B[x] = \{x\}$ if $x \in X\setminus B$. It follows that every discrete ballean X is uniquely determined by its bornology $\mathcal{B}_X$, see [12, Theorem 3.2.1]. Every discrete ballean is normal [12, Example 4.2.2].

For a ballean X, the following conditions are equivalent: X is discrete, every function $f : X \longrightarrow \{0, 1\}$ is slowly oscillating, every unbounded subset of X is thick, see [12, Theorem 3.3.1] and [3, Theorem 2.2].

An unbounded discrete ballean is called *ultradiscrete* if the family $\{X\setminus B : B \in \mathcal{B}_X\}$ is an ultrafilter on X. Every ultradiscrete ballean is maximal [12, Example 10.1.2]. Thus, we have got

ultradiscrete $\Longrightarrow$ maximal $\Longrightarrow$ extremely normal $\Longrightarrow$ ultranormal

and no one arrow can be reversed, see Section 3.

2. Characterizations

Let $(X, \mathcal{E})$ be an unbounded ballean. We endow X with the discrete topology, identify the Stone-Čech compactification βX of X with the set of all ultrafilters on X and denote $X^\sharp = \{p \in \beta X : P \text{ is unbounded for each } P \in p\}$. Given any $p, q \in X^\sharp$, we write $p \parallel q$ if there exists $E \in \mathcal{E}$ such that $E[Q] \in p$ for each $Q \in q$. By [7, Lemma 4.1], $\parallel$ is an equivalence relation on $X^\sharp$. We denote by $\sim$ the minimal (by inclusion) closed (in $X^\sharp \times X^\sharp$) equivalence on $X^\sharp$ such that $\parallel \subseteq \sim$. The compact Hausdorff space $X^\sharp/\sim$ is called the *corona* of $(X, \mathcal{E})$, it is denoted by $\nu(X, \mathcal{E})$. If every ball $\{y \in X : d(x, y) \le r\}$ in the metric space (X, d) is compact then $\nu(X, \mathcal{E}_d)$ is the Higson's corona of (X, d), see [8] and [14].

We say that a function $f : X \longrightarrow \mathbb{R}$ is *constant at infinity* if there exists $c \in \mathbb{R}$ such that, for each $\varepsilon > 0$, the set $\{x \in X : |f(x) - c| > \varepsilon\}$ is bounded. We say that f is *almost constant* if $f|_{X\setminus B}$ =const for some $B \in \mathcal{B}_X$. If the bornology $\mathcal{B}_X$ is closed under countable unions then every function, constant at infinity, is almost constant.

If f is constant at infinity then f is slowly oscillating. We denote so$(X, \mathcal{E})$ the set of all bounded slowly oscillating functions on X. For a bounded function $f : X \longrightarrow \mathbb{R}$, f^β denotes the extension of f to βX.

Theorem 2.1. *For an unbounded normal ballean $(X, \mathcal{E})$, the following conditions are equivalent:*

(1) *every function $f \in so(X, \mathcal{E})$ is constant at infinity;*
(2) *$\nu(X, \mathcal{E})$ is a singleton;*
(3) *$(X, \mathcal{E})$ is ultranormal.*

Proof. (3) $\Longrightarrow$ (1). We assume that a function $f \in$ *so* $(X, \mathcal{E})$ is not constant at infinity. Then there exists distinct $a, b \in \mathbb{R}$ and $p, q \in X^\sharp$ such that $f^\beta(p) = a$, $f^\beta(q) = b$. We put $\varepsilon = \frac{|a-b|}{4}$ and choose $P \in p$, $Q \in q$ such that $f(P) \subset (a - \varepsilon, a + \varepsilon)$, $f(Q) \subset (b - \varepsilon, b + \varepsilon)$. Given an arbitrary $E \in \mathcal{E}$, we take $B \in \mathcal{B}_X$ such that *diam* $f(E[x]) < \varepsilon$ for each $x \in X \setminus B$. It follows that $E[P \setminus B] \cap E[Q \setminus B] = \varnothing$ so P and Q are asymptotically disjoint and we get a contradiction to (3).

(1) $\Longrightarrow$ (2). Let $p, q \in X^\sharp$. By Proposition 8.1.4 from [12], $p \sim q$ if and only if $f^\beta(p) = f^\beta(q)$ for every $f \in$ *so* $(X, \mathcal{E})$.

(2) $\Longrightarrow$ (3). We assume that $(X, \mathcal{E})$ is not ultranormal and choose two disjoint and asymptotically disjoint unbounded subsets A, B of X. We chose $p, q \in X^\sharp$ such that $A \in p$, $B \in q$. Since X is normal, there exists $f \in$ *so* $(X, \mathcal{E})$ such that $f \mid_A = 0$, $f \mid_B = 1$. Then $f^\beta(p) \neq f^\beta(q)$ and $\mid \nu(X, \mathcal{E}) \mid > 1$ by Proposition 8.1.4 from [12]. □

An unbounded ballean X is called *irresolvable* [11] if X can not be partitioned into two large subsets.

Theorem 2.2. *For an unbounded discrete ballean $(X, \mathcal{E})$, the following conditions are equivalent:*

(1) $(X, \mathcal{E})$ *is ultradiscrete;*
(2) $(X, \mathcal{E})$ *is extremely normal;*
(3) $(X, \mathcal{E})$ *is ultranormal;*
(4) $(X, \mathcal{E})$ *is maximal and irresolvable.*

Proof. (1) $\Longrightarrow$ (2). We denote by p the ultrafilter $\{X \backslash B : B \in \mathcal{B}_X\}$. If A is an unbounded subset of X then $A \in p$ so $X \backslash A$ is bounded and A is large.

(2) $\Longrightarrow$ (3). Let A, C be unbounded subsets of X. Since A is large, there exists $E \in \mathcal{E}$ such that $E[A] = X$ so $C \subseteq E[A]$ and A, C are not asymptotically disjoint.

(3) $\Longrightarrow$ (1). If $(X, \mathcal{E})$ is not ultradiscrete then there exist two disjoint unbounded subsets A, C of X. Since $(X, \mathcal{E})$ is discrete, A and C are asymptotically disjoint.

(1) $\Longrightarrow$ (4). This is Theorem 10.4.5 from [12]. □

Let $\mathcal{B}$ be a bornology on a set X. Following [1], we say that a coarse structure $\mathcal{E}$ is *compatible* with $\mathcal{B}$ if $\mathcal{B}$ is the bornology of all bounded subsets of $(X, \mathcal{E})$.

Each bornology $\mathcal{B}$ on X defines two coarse structures $\Downarrow \mathcal{B}$ and $\Uparrow \mathcal{B}$, the smallest and the largest coarse structures on X compatible with $\mathcal{B}$. Clearly, $\Downarrow \mathcal{B}$ is the discrete coarse structure defined by $\mathcal{B}$, in particular, $(X, \Downarrow \mathcal{B})$ is normal.

The coarse structure $\Uparrow \mathcal{B}$ consists of all entourages $E \subseteq X \times X$ such that $E = E^{-1}$ and $E[B] \in \mathcal{B}$ for each $B \in \mathcal{B}$. But in contrast to $\Downarrow \mathcal{B}$, the coarse structure $\Uparrow \mathcal{B}$ needs not to be normal [2, Theorem 12].

If a ballean $(X, \mathcal{E})$ is maximal then $\Uparrow \mathcal{B}_X = \mathcal{E}$. It follows that every maximal ballean $(X, \mathcal{E})$ is uniquely determined by the bornology of bounded subsets: if $(X, \mathcal{E}')$ is maximal and $\mathcal{B}_{(X,\mathcal{E})} = \mathcal{B}_{(X,\mathcal{E}')}$ then $\mathcal{E} = \mathcal{E}'$.

Proposition 2.3. *Let $(X, \mathcal{E})$ be an extremely normal ballean and let $\mathcal{E}'$ be coarse structure on X such that $\mathcal{E} \subseteq \mathcal{E}'$ and $(X, \mathcal{E}')$ is maximal. Then $\mathcal{B}_{(X,\mathcal{E})} = \mathcal{B}_{(X,\mathcal{E}')}$ and $\mathcal{E}' = \Uparrow \mathcal{B}_{(X,\mathcal{E})}$.*

Proof. We assume the contrary and pick $A \in \mathcal{B}_{(X,\mathcal{E}')} \setminus \mathcal{B}_{(X,\mathcal{E})}$. Since $(X, \mathcal{E})$ is extremely normal, A is large in $(X, \mathcal{E})$. Hence, A is large in $(X, \mathcal{E}')$ so $(X, \mathcal{E}')$ is bounded and we get a contradiction to maximality of $(X, \mathcal{E}')$. ☐

Following [2], we say that a ballean $(X, \mathcal{E})$ is *relatively maximal* if $\mathcal{E} = \Uparrow \mathcal{B}_{(X,\mathcal{E})}$.

We recall that two ultrafilters $p, q \in \beta X$ are *incomparable* if, for every $f : X \longrightarrow X$, we have $f^\beta(p) \neq q$, $f^\beta(q) \neq p$.

Proposition 2.4. *Let $(X, \mathcal{E})$ be an unbounded discrete ballean such that any two distinct utrafilters in $X^\sharp$ are incomparable. Then $(X, \mathcal{E})$ is relatively maximal.*

Proof. We suppose the contrary and choose a coarse structure $\mathcal{E}'$ on X such that $\mathcal{E} \subset \mathcal{E}'$ and $\mathcal{B}_{(X,\mathcal{E})} = \mathcal{B}_{(X,\mathcal{E}')}$. Then there exists $E' \in \mathcal{E}'$ such that the set $Y = \{y \in X : |E'(y)| > 1\}$ is unbounded in $(X, \mathcal{E}')$. For each $y \in Y$, we pick $f(y) \in E'(y)$, $f(y) \neq y$ and extend f to X by $f(x) = x$ for each $x \in X \setminus Y$. We take an arbitrary ultrafilter $p \in (X, \mathcal{E})^\sharp$ such that $Y \in p$. By the assumption, $f(P)$ is bounded in $(X, \mathcal{E})$ for some $P \in p$, $P \subseteq Y$. Then $(E')^{-1}[f(P)]$ must be bounded in $(X, \mathcal{E})$ and we get a contradiction with the choice of p. ☐

3. Constructions

Given a family $\mathfrak{F}$ of subsets of $X \times X$, we denote by $\langle \mathfrak{F} \rangle$ the intersections of all coarse structures, containing each $F \cup \triangle_X$, $F \in \mathfrak{F}$ and say that $\langle \mathfrak{F} \rangle$ is *generated* by $\mathfrak{F}$. It is easy to see that $\langle \mathfrak{F} \rangle$ has a base of subsets of the form $E_0 \circ E_1 \circ \ldots \circ E_n$, where

$$F_i \in \{(F \cup \triangle_X) \cup (F \cup \triangle_X)^{-1} : F \in \mathfrak{F}\} \cup \{(x, y) \cup \triangle_X : x, y \in X\},\ n < \omega.$$

If $\mathcal{E}_1$ and $\mathcal{E}_2$ are coarse structures, we write $\mathcal{E}_1 \bigvee \mathcal{E}_2$ in place of $\langle \mathcal{E}_1 \cup \mathcal{E}_2 \rangle$. For lattices of coarse structures, see [11].

Proposition 3.1. *Let $(X, \mathcal{E})$ be an unbounded discrete ballean and let p, q be distinct ultrafilters from $X^\sharp$. If $(X, \mathcal{E})$ is relatively maximal then $f^\beta(p) \neq q$ for each $f : X \longrightarrow X$ such that, for every $B \in \mathcal{B}_{(X,\mathcal{E})}$, $f(B) \in \mathcal{B}_{(X,\mathcal{E})}$ and $f^{-1}(B) \in \mathcal{B}_{(X,\mathcal{E})}$.*

Proof. We assume the contrary and choose f such that $f^{\beta}(p) = q$. We put $F = \{(x, f(x)) : x \in X\}$ and denote by $\mathcal{E}'$ the coarse structure generated by $\mathcal{E}$ and F. Since E is discrete and E' is not discrete, we have $\mathcal{E} \subset \mathcal{E}'$.

We take $x \in X$ and $E_1, \dots, E_n \in \mathcal{E} \cup \{F \cup \triangle_X, \ \ F^{-1} \cup \triangle_X\}$. Applying an induction by n and the assumptions $f(B) \in \mathcal{B}_{(X,\mathcal{E})}$, $\ f^{-1}(B) \in \mathcal{B}_{(X,\mathcal{E})}$ for each $B \in \mathcal{B}_{(X,\mathcal{E})}$, we conclude that $E_1 \circ \dots \circ E_n[x] \in \mathcal{B}_{(X,\mathcal{E})}$. Hence, $\mathcal{B}_{(X,\mathcal{E})} = \mathcal{B}_{(X,\mathcal{E})'}$ and we get a contradiction to relative maximality of $(X, \mathcal{E})$. □

Proposition 3.2. *Every unbounded subballean of maximal (extremely normal, ultranormal) ballean is maximal (extremely normal, ultranormal).*

Proof. We prove only the first statement, the second and third are evident.

Let $(X, \mathcal{E})$ be a maximal ballean, Y be an unbounded subset of X. We assume that $(Y, \mathcal{E}_Y)$ is not maximal and choose a coarse structure $\mathcal{E}'$ on Y such that $\mathcal{E} \mid_Y \subset \mathcal{E}'$ and $(Y, \mathcal{E}')$ is not bounded. We put $\mathfrak{F} = \langle \mathcal{E} \cup \mathcal{E}' \rangle$. Since $(X, \mathcal{E})$ is maximal and $\mathcal{E} \subset \mathfrak{F}$, $(X, \mathfrak{F})$ must be bounded. On the other hand, each bounded subset B of $(Y, \mathcal{E}')$ is bounded in $(X, \mathcal{E})$ because otherwise B is large in $(X, \mathcal{E})$ so B is large in $(Y, \mathcal{E}_Y)$. Now let $x \in X$, and $E_1, \dots, E_n \in \mathcal{E} \cup \{E' \cup \triangle_X : E' \in \mathcal{E}'\}$. On induction by n, we see that $E_1 \circ \dots \circ E_n[x]$ is bounded in $(X, \mathcal{E})$. Hence $(X, \mathfrak{F})$ is not bounded and we get a contradiction. □

Proposition 3.3. *Let $\mathcal{E}, \mathcal{E}'$ be coarse structures on a set X such that $\mathcal{E} \subseteq \mathcal{E}'$. Then the following statements hold:*

(1) *if $(X, \mathcal{E})$ is extremely normal then $(X, \mathcal{E}'$ is extremely normal;*
(2) *if $(X, \mathcal{E})$ is ultranormal and $\mathcal{B}_{(X,\mathcal{E})} = \mathcal{B}_{(X,\mathcal{E}')}$ then $(X, \mathcal{E}')$ is ultranormal.*

Proof. (1) Let A be a subset of X. If A is unbounded in $(X, \mathcal{E}')$ then A is unbounded in $(X, \mathcal{E})$. If A is large in $(X, \mathcal{E})$ then A is large in $(X, \mathcal{E}')$.

(2) We assume that some unbounded subsets A, B of $(X, \mathcal{E}')$ are asymptotically disjoint in $(X, \mathcal{E})$. Then $E'(A) \backslash B \in \mathcal{B}_{(X,\mathcal{E}')}$ for each $E' \in \mathcal{E}'$. Since $\mathcal{E} \subseteq \mathcal{E}'$ and $\mathcal{B}_{(X,\mathcal{E})} = \mathcal{B}_{(X,\mathcal{E}')}$, we have $E(A) \backslash B \in \mathcal{B}_{(X,\mathcal{E}')}$ for each $E \in \mathcal{E}$ so A, B are asymptotically disjoint in $(X, \mathcal{E})$. □

Example 3.4. For every infinite regular cardinal κ, we construct a coarse structure $\mathfrak{M}_\kappa$ on κ such that $(\kappa, \mathfrak{M}_\kappa)$ is maximal and $\mathcal{B}_{(\kappa,\mathfrak{M}_\kappa)} = [\kappa]^{<\kappa}$. We denote by $\mathfrak{F}$ the family of all coverings of κ defined by the rule: $\mathcal{P} \in \mathfrak{F}$ if and only if, for each $P \in \mathcal{P}$ and $x \in \kappa$, $|P| < \kappa$ and $|\cup\{P' : x \in P', \, P' \in \mathcal{P}\}| < \kappa$. Then $\mathfrak{M}_\kappa$ is defined by the base $\{M_{\mathcal{P}} : \mathcal{P} \in \mathfrak{F}\}$, where $M_{\mathcal{P}} = \{(x, y) : x \in P, y \in P$ for some $P \in \mathcal{P}\}$. For general construction of coarse structures by means of coverings, see [9] or [12, Section 7.5]. Clearly, $\mathcal{B}_{(\kappa,\mathfrak{M}_\kappa)} = [\kappa]^{<\kappa}$ and $(\kappa, \mathfrak{M}_\kappa)$ is maximal [12, Example 10.2.1].

Let G be a group and let X be a G-space with the action $G \times X \longrightarrow X$, $(g, x) \longmapsto gx$. A bornology $\mathcal{I}$ on G is called a *group bornology* if, for any

$A, B \in \mathcal{I}$, we have $AB \in \mathcal{I}$, $A^{-1} \in \mathcal{I}$. Every group bornology $\mathcal{I}$ on G defines a coarse structure $\mathcal{E}_{\mathcal{I}}$ on X with the base $\{E_A : A \in \mathcal{I}, \ e \in A\}$, where e is the identity of G, $E_A = \{(x, y) : y \in Ax\}$. Moreover, every coarse structure on X can be defined in this way [6].

Example 3.5. We define a coarse structure $\mathcal{E}$ on ω such that the ballean $(\omega, \mathcal{E})$ is extremely normal but $(\omega, \mathcal{E})$ is not maximal. Let S_ω denotes the group of all permutations of ω, $\mathcal{I} = [S_\omega]^{<\omega}$, $\mathcal{E} = \mathcal{E}_{\mathcal{I}}$. We show that every infinite subset A of ω is large. We partition A and ω into two infinite subset $A = A_1 \cup A_2$, $W = W_1 \cup W_2$ and choose two permutations f_1, f_2 of ω so that

$$f_1(A_1) = W_1, \quad f_1(W_1) = A_1, \quad f_1(x) = x \quad \text{for each} \quad x \in \omega \setminus (A_1 \cup W_1),$$

$$f_2(A_2) = W_2, \quad f_2(W_2) = A_2, \quad f_2(x) = x \quad \text{for each} \quad x \in \omega \setminus (A_2 \cup W_2)$$

Then we put $F = \{f_1, f_2, id\}$, where id is the identity permutation. Clearly, $E_F[A] = \omega$ so $\mathcal{E}$ is extremely normal. To see that $(\omega, \mathcal{E})$ is not maximal, we note that $\mathcal{E} \subseteq \mathfrak{M}_\omega$ and choose a partition $\mathcal{P} = \{P_n : n \in \omega\}$ of ω such that $|P_n| = n$. Then, for each $n \in \mathbb{N}$, there exist $x \in \omega$ such that $M_{\mathcal{P}}[x] = n$. For each $H \in [S_\omega]^{<\omega}$ and $x \in \omega$, we have $|E_H[x]| \leq |H|$. Hence, $\mathcal{E} \subset \mathfrak{M}_{<\omega}$.

Example 3.6. Let κ be a cardinal, $\kappa > \omega$. We construct two coarse structures $\mathcal{E}, \mathcal{E}'$ on κ such that $\mathcal{E} \subset \mathcal{E}'$, $(\kappa, \mathcal{E})$ is ultranormal but not extremely normal, $(\kappa, \mathcal{E}')$ is extremely normal but not maximal. Let S_κ denotes the group of all permutations κ, $\mathcal{I} = [S_\kappa]^{<\omega}$, $\mathcal{E} = \mathcal{E}_{\mathcal{I}}$. Clearly, $\mathcal{B}_{(\kappa, \mathcal{E})} = [\kappa]^{<\omega}$. Let A, B be infinite subsets of κ. We take countable subset $A' \subseteq A$, $B' \subseteq B$ and use argument from Example 3.5 to choose $F = \{f_1, f_2, id\}$ so that $B' \subseteq E_F[A']$. It follows that A, B are not asymptotically disjoint in $(\kappa, \mathcal{E})$ so $(\kappa, \mathcal{E})$ is ultranormal. If A is a countable subset of κ then $|E_H[A]| = \omega$ for each $H \in [S_\kappa]^{<\omega}$ so $(\kappa, \mathcal{E})$ is not extremely normal.

We denote by $\mathcal{F}$ the discrete coarse structure on κ defined by the bornology $[\kappa]^{<\kappa}$ and put $\mathcal{E}' = \mathcal{F} \vee \mathcal{E}$. If A is an unbounded subset of $\mathcal{E}'$ then $|A| = \kappa$. Applying the arguments from Example 3.5, we see that $(\kappa, \mathcal{E}')$ is extremely normal. The ballean $(\kappa, \mathcal{E}')$ is not maximal because $\mathcal{E}' \subset \mathcal{F} \vee \mathcal{E}_J$, where $J = [S_\kappa]^{\leq\omega}$.

4. Comments

1. Let $(X, \mathcal{E})$ be a ballean, A and B be subsets of X. Following [9], we write $A\delta B$ if and only if there exists $E \in \mathcal{E}$ such that $A \subseteq E[B]$, $B \subseteq E[A]$.

Let $\mathcal{E}, \mathcal{E}'$ be coarse structures on a set X such that $\mathcal{B}_{(X,\mathcal{E})} = \mathcal{B}_{(X,\mathcal{E}')}$. If $\mathcal{E}$ and $\mathcal{E}'$ have linearly ordered bases and either $so(X, \mathcal{E}) = so(X, \mathcal{E}')$ or $\delta_{(X,\mathcal{E})} = \delta_{(X,\mathcal{E}')}$ then $\mathcal{E} = \mathcal{E}'$, see [4, Theorem 2.1] and [3, Theorem 4.2].

We take the coarse structures $\mathcal{E}$ and $\mathfrak{M}_\omega$ on ω from Example 3.5. By Theorem 2.1, $so(\omega, \mathcal{E}) = so(\omega, \mathfrak{M}_\omega)$. Since $\mathcal{B}_{(\omega,\mathcal{E})} = \mathcal{B}_{(\omega,\mathfrak{M}_\omega)} = [\omega]^{<\omega}$ and $(\omega, \mathcal{E}), (\omega, \mathfrak{M}_\omega)$ are extremely normal, we have $\delta_{(\omega,\mathcal{E})} = \delta_{(\omega,\mathfrak{M}_\omega)}$. By the construction, $\mathcal{E} \neq \mathfrak{M}_\omega$.

In this connection we remind Question 7.5.1 from [12]: does $\delta_{(X,\mathcal{E})} = \delta_{(X,\mathcal{E}')}$ imply $so(X,\mathcal{E}) = so(X,\mathcal{E}')$ and give the affirmative answer to this question.

Clearly, $\delta_{(X,\mathcal{E})} = \delta_{(X,\mathcal{E}')}$ implies $\mathcal{B}_{(X,\mathcal{E})} = \mathcal{B}_{(X,\mathcal{E}')}$. Assume that there exists $f \in so(X,\mathcal{E}) \setminus so(X,\mathcal{E}')$. Then there exist $\varepsilon > 0$ and $E' \in \mathcal{E}'$ such that, for each $B \in \mathcal{B}_{(X,\mathcal{E}')}$, one can find $y_B,\ z_B \in X \setminus B$ such that $(y_B, z_B) \in E'$ but $|f(y_B) - f(z_B)| > \varepsilon$ We put $Y = \{y_B : B \in \mathcal{B}_{(X,\mathcal{E}')}\}$ and choose a function $h : X \longrightarrow X$ such that $(y, h(y)) \in E'$, $|f(y) - f(h(y))| > \varepsilon$ for each $y \in Y$. Let $p \in X^\sharp$ and $Y \in p$, q be an ultrafilter with the base $\{h(P) : P \in p\}$. Then $|f^\beta(p) - f^\beta(q)| \geq \varepsilon$ and there exists $P' \in p$ such that $|f(x) - f(y)| > \frac{\varepsilon}{2}$ for all $x \in P'$, $y \in h(P')$. Since $f \in so(X,\mathcal{E})$, P' and $h(P')$ are not close in $(X,\mathcal{E})$ but P', $h(P')$ are close in $(X,\mathcal{E}')$.

2. Every group G has the natural *finitary coarse structure* $\mathcal{E}_{fin}$ with the base $\{E_F : F \in [G]^{<\omega},\ e \in F\}$, where $E_F = \{(x,y) : y \in Fx\}$. Let G be an uncountable abelian group. By [4, Corollary 3.2], $(G,\mathcal{E}_{fin})$ is not normal but every function $f \in so(G,\mathcal{E}_{fin})$ is constant at infinity. This example shows that the assumption of normality in Theorem 2.1 can not be omitted.

3. Example 3.6 shows that Proposition 2.3 does not hold for ultranormal ballean in place of extremely normal. Indeed, $(\kappa,\mathcal{E})$ is ultranormal, $\mathcal{B}_{(\kappa,\mathcal{E})} = [\kappa]^{<\omega}$, $\mathcal{E} \subset \mathcal{E}'$, $\mathcal{B}_{(\kappa,\mathcal{E}')} = [\kappa]^{<\kappa}$. We take a maximal coarse structure $\mathcal{E}''$ such that $\mathcal{E}' \subset \mathcal{E}''$. Then $\mathcal{E} \subset \mathcal{E}''$. and $\mathcal{B}_{(\kappa,\mathcal{E})} \neq \mathcal{B}_{(\kappa,\mathcal{E}'')}$.

4. Following [1], we say that a ballean $(X,\mathcal{E})$ *has bounded growth* if there is a mapping $f : X \longrightarrow \mathcal{B}_X$ such that

- $\cup_{x\in B} f(x) \in \mathcal{B}_X$ for each $B \in \mathcal{B}_X$;
- for each $E \in \mathcal{E}$, there exists $C \in \mathcal{B}_X$ such that $E[x] \subseteq f(x)$ for each $x \in X \setminus C$.

Clearly, every discrete ballean has bounded growth $(f(x) = \{x\})$. Let us take the maximal ballean $(\omega,\mathfrak{M}_\omega)$ from Example 3.4. Since each ball in $(\omega,\mathfrak{M}_\omega)$ is finite, one can use the diagonal process to show that $(\omega,\mathfrak{M}_\omega)$ is not of bounded growth.

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