A nonlinear dynamic age-structured model of e-commerce in Spain: Stability analysis of the equilibrium by delay and stochastic perturbations

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Abstract

First, we propose a deterministic age-structured epidemiological model to study the diffusion of e-commerce in Spain. Afterwards, we determine the parameters (death, birth and growth rates) of the underlying demographic model as well as the parameters (transmission of the use of e-commerce rates) of the proposed epidemiological model that best fit real data retrieved from the Spanish National Statistical Institute. Motivated by the two following facts: first the dynamics of acquiring the use of a new technology as e-commerce is mainly driven by the feedback after interacting with our peers (family, friends, mates, mass media, etc.), hence having a certain delay, and second the inherent uncertainty of sampled real data and the social complexity of the phenomena under analysis, we introduce aftereffect and stochastic perturbations in the initial deterministic model. This leads to a delayed stochastic model for e-commerce. We then investigate sufficient conditions in order to guarantee the stability in probability of the equilibrium point of the dynamic e-commerce delayed stochastic model. Our theoretical findings are numerically illustrated using real data.

Keywords: Delayed stochastic nonlinear system of differential equations, age-structured epidemiological model, Lyapunov stochastic stability analysis, e-commerce diffusion model.

1. Introduction

Electronic commerce (in short e-commerce) is the use of advanced electronic technology for a wide range of on-line business activities for goods and services. E-commerce is gradually extending to the economic mainstream and business core aspects. E-commerce has provided a new way of doing business all over the world using the Internet. Modelling the diffusion of

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e-commerce is extremely important for business investors and policymakers for effective planning and better understanding the dynamics of this complex transactional process. A number of mathematical models have been proposed to study e-commerce using different approaches. Here, we highlight contributions based upon Making-Decision Theory mainly oriented towards the design of recommender systems [1, 2] and the measure of quality quality of services and business [3, 4]. These contributions rely on operational research (decision making support systems, multi-criteria optimization, etc.) and statistical techniques (bayesian analysis, Petri nets, etc.). Pioneering contributions dealing with the social diffusion of new technologies using mathematical models based on differential equations include [5, 6, 7]. From the point of view of dynamical systems, the study of e-commerce has been analysed in a few contributions. In [8] the authors present a competition model of e-commerce sites and they perform a planar qualitative analysis. Afterwards, Li Yanhui and Zhu Siming explored the effects of competition in e-commerce web sites via mathematical models based on ordinary differential equations, [9, 10]. These interesting studies include a qualitative equilibrium analysis and numerical simulations of the competition dynamics. In [11] some of the authors of the present paper proposed an age-structured compartmental mathematical model (similar to the ones used to model epidemics [12]) to describe the dynamics of e-commerce using real data from the Spanish National Statistical Institute (INE). This study is performed by combining two mathematical models, the first one is a demographic model providing certain demographic parameters required in the formulation of the second model, which is addressed to describe the diffusion of e-commerce. According to the available data from the Spanish INE, population was divided into six cohorts. The results obtained in [11] are quite good despite predictions were performed on the horizon 2010–2012 by fitting sampled data corresponding to only three available years at that time (2006–2008). The inclusion of the age-structured model is a difficult issue that we are going to consider in this paper. Some interesting contributions where the age structure has been considered in the context of mathematical modeling, can be seen, for example, in [13, 14, 15, 16].

We are aware that significant features of e-commerce are not contained in the formulation of the mathematical model proposed in [11]. On the one hand, according to [17], our habits are influenced by the habits of the people in our social network. This can be also applied to the habit of the use of e-commerce that can be transmitted by peer pressure or social contact among family, friends, mates, etc. However, the adoption of this technology does not take place immediately after such encounters, but it requires a certain time lag (delay). On the other hand, the success of *contagion* depends on a number of complex human and business factors whose nature is random (social contacts, purchase behavior, personality, confidence, impulsiveness, technology integration, etc., [18, 19]). Furthermore, real data required to fit the proposed model contains sampling errors and hence uncertainty. These reasons aim us to propose an epidemiological model to describe the dynamics of the use of e-commerce in Spain that considers in its formulation both delay and randomness. There are two main approaches to deal with delay: first, random fractional differential equations [20] and second, random delay differential equations [21]. In this paper, we follow the latter approach. As it has been reported in previous contributions [11], the use of technologies, and in particular the e-commerce, is strongly related to the age of users. This key feature must be taking into account in the mathematical formulation as we did in our previous contribution [11]. At this point is important to point out that we have made the decision of aggregating data from Spanish INE into two subpopulations, Group 1: persons aged $15-44$ years old (y.o.) and Group 2: persons aged $45-74$ y.o. Apart from the feasibility of the subsequent mathematical treatment of the mathematical model, this decision has been made in agreement with the significant differences of the use of e-commerce between these two age groups found in data collected from Spanish INE [22, 23]. Furthermore, it can be checked from this statistical source that the percentage of people younger than 14 y.o. and older than 74 y.o. buying by the Internet is practically negligible. Thus, in this paper we propose a mathematical model for studying the dynamics of e-commerce that combines the aforementioned bare-bones factors: an underlying age-structured demographical model, peer-pressure (contagion) to account for the diffusion of this technology, delay and randomness effects.

As we will see later, we consider uncertainty via stochastic perturbations from the equilibrium point since our model must be able to capture eventual changes that may happen about the steady point because of social and business factors affecting the dynamics of the e-commerce. This is a key issue in our subsequent analysis both from a practical and theoretical standpoints. Indeed, if the mathematical model is reliable, it is expected the numerical results in real-world (using real data of Spanish e-commerce) remain stable except perhaps in the case of large perturbations while the stability analysis has an intrinsic mathematical interest. Both questions lead to investigate the maximum size of stochastic perturbations in order to guarantee the stochastic stability of the equilibrium point.

More specifically, we will assume that the dynamics of e-commerce model with delay is exposed to additive stochastic perturbations of White Noise-type that are directly proportional to the deviation of the current state of the system from the steady state or equilibrium point. From a mathematical modelling standpoint and, on the basis of the Limit Central Theorem, it must be pointed out that the large number of independent random factors, previously mentioned, that may affect the dynamics of e-commerce diffusion supports the consideration of White Noise process which is a Gaussian stationary process with constant spectral density [24, 25]. Such type of stochastic perturbations first was proposed in [26, 27]. One of the key points of this hypothesis is that the equilibrium point is the solution of the stochastic system too. In this case, the influence of the stochastic perturbations on the considered system is small enough in the neighborhood of the equilibrium point and big enough if the system state is far enough from the equilibrium point.

The considered nonlinear system is then linearized in the neighborhood of the positive equilibrium point, and sufficient condition for asymptotic mean square stability of the zero solution of the constructed linear system is obtained via the Kolmanovskii-Shaikhet general method of Lyapunov functionals construction (GMLFC), that is used for stability investigation of stochastic functional-differential and difference equations [28, 29, 30, 21]. This way of stability investigation was successfully used in different mathematical models formulated via systems with delays: SIR epidemic model [26], predator-prey model [27, 31], social epidemic models [32, 33] and Nicholson blowflies model [34], for example.

On the basis of the aforementioned approach, the main objective of this paper is twofold. First, from an applied standpoint, to propose a mathematical model able to describe the diffusion of e-commerce in Spain using real data and, second, from a mathematical point of view, to perform a stability analysis of the equilibrium by delay and stochastic perturbations. As a consequence, the new model can be regarded, in some aspects, as an extension of the one presented in [11] since in its formulation it includes delay and randomness, but reducing the number of subpopulations of the underlying demographic model. As currently the available statistical data compiled by the Spanish INE has been updated and enlarged with respect to the ones used in [11], the fitting of the new proposed model is expected to be better and, therefore also our updated predictions.

This paper is organized as follows. In Section 2 the deterministic dynamic model of the e-commerce with delay is built including the underlying demographic model. Parameters of this deterministic model are adjusted using real data of the use of e-commerce in Spain. Section 3 is devoted to compute the equilibrium points of this deterministic model. In Section 4 we introduce randomness into the age-structured mathematical model for e-commerce with delay and key stochastic tools that are required to complete later the stability analysis are shown. Section 5 is addressed to provide sufficient conditions for stability in probability of the equilibrium point of the delayed stochastic model. In Section 6, we carry out numerical simulations of the delayed stochastic model using real data from Spanish INE showing agreement with our theoretical findings. Conclusions are drawn in Section 7.

2. Deterministic age-structured mathematical model for e-commerce with delay

This section is divided into two parts. Subsection 2.1 is addressed to introduce the underlying demographic model, while Subsection 2.2 is devoted to construct a mathematical model with delay, which integrates the demographic one for describing the dynamics of e-commerce in Spain.

2.1. Demographic model

Age of individuals is a key feature that must be taken into account in the mathematical modelling of e-commerce [35]. In order to set that the corresponding delayed stochastic model be mathematically tractable but retaining the main features of the underlying demographic model, we have made the decision of aggregating population data collected from the Spanish INE into two cohorts [22], people aged between 15 – 44 y.o. and between 45 – 74 y.o. The division into these two specific cohorts has been made because the significant differences in the use of e-commerce according to available data reported by the Spanish INE [23]. Therefore, let us define:

- Group 1 ($G_1(t)$): Percentage of Spanish population between 15 and 44 y.o. at the time instant t (in years).
- Group 2 ($G_2(t)$): Percentage of Spanish population between 45 and 74 y.o. at the time instant t (in years).

According to [12], the following system of differential equations describes the demographic evolution in each t for the two different age groups,

$$\begin{cases} \dot{G}_1(t) = \mu - c_1 G_1(t) - d_1 G_1(t), \\ \dot{G}_2(t) = \qquad c_1 G_1(t) - d_2 G_2(t), \end{cases} \tag{1}$$

where μ is the yearly birth rate (assuming that yearly death rate of people under 14 y.o. is negligible), c_1 is the yearly growth rate from $G_1(t)$ to $G_2(t)$, d_1 is the yearly death rate in the first group $G_1(t)$ and d_2 is the rate of people coming out from the second group $G_2(t)$ of people aged between 45 – 74 y.o, by death or because they become older than 74 y.o. If we assume that $G_1(t)$ and $G_2(t)$ are constant over the time, then, their derivatives $\dot{G}_1(t) = \dot{G}_2(t) = 0$ and from the first equation of (1), we have that

$$c_1 G_1 = \mu - d_1 G_1 \Longrightarrow c_1 = \frac{\mu}{G_1} - d_1. \tag{2}$$

Now, from the second equation of (1), we obtain

$$c_1 G_1 = d_2 G_2 \Longrightarrow \mu - d_1 G_1 = d_2 G_2 \Longrightarrow d_2 = \frac{\mu - d_1 G_1}{G_2}. \tag{3}$$

Example 2.1. *From Spanish INE [22], the average birth rate μ, the average death rate for people in the group G_1, d_1, the average percentage of people in the age groups G_1 and G_2, in the period* $2007 - 2015$ *are* $\mu = 0.010110$, $d_1 = 5.7333 \times 10^{-4}$ $G_1 = 0.5495$ *and* $G_2 = 0.4505$.

Then, from (2) *and* (3), $c_1 = 0.0178252$ *and* $d_2 = 0.0217424$.

With the obtained values of c_1 and d_2, the proportion of the subpopulations G_1 and G_2 remain constant over the time.

2.2. Electronic commerce model with delay

In this section, we propose a mathematical model for describing the dynamics of the use of e-commerce in Spain. As we will see later, in the formulation of this model we will consider the key feature of delay that takes place in the contagion process among peers (users and non-users) to spread the use of this technology. Furthermore, it must be noticed that this model is built on the basis of the demographic model (1).

	Group G_1 (15 − 44 y.o.)		Group G_2 (45 − 74 y.o.)	
Time	No use e-commerce	Use e-commerce	No use e-commerce	Use e-commerce
t_1 = Dec 2007	0.4154	0.1415	0.3823	0.0608
t_2 = Dec 2008	0.3955	0.1790	0.3822	0.0433
t_3 = Dec 2009	0.3652	0.2039	0.3755	0.0554
t_4 = Dec 2010	0.3425	0.2158	0.3781	0.0636
t_5 = Dec 2011	0.3242	0.2284	0.3730	0.0744
t_6 = Dec 2012	0.2891	0.2546	0.3716	0.0847
t_7 = Dec 2013	0.2668	0.2661	0.3718	0.0953
t_8 = Dec 2014	0.2258	0.2958	0.3568	0.1216
t_9 = Dec 2015	0.1891	0.3230	0.3459	0.1412

Table 1: Data of use of e-commerce in Spanish during the period 2007 − 2015. Data are aggregated in two groups depending on the age of the users: Group 1 (G_1) and Group 2 (G_2) are made up of people aged between 15 − 44 and 45 − 74 years old (y.o.), respectively. Source [23].

In Table 1, we can find data retrieved from Spanish INE [23] about the users and non-users of the e-commerce, per age group, from 2007 to 2015 in Spain.

In order to state the mathematical model, now we introduce the following notation:

- $N_i = N_i(t)$, $i = 1, 2$, denotes the percentage of people belonging to group $G_i = G_i(t)$, who have not used e-commerce at the time instant t (in years).

- $Y_i = Y_i(t)$, $i = 1, 2$, denotes the percentage of people belonging to group $G_i = G_i(t)$, who have used e-commerce at the time instant t (in years).

For the first ($i = 1$) age group of 15 − 44 y.o., we assume that a non-user of e-commerce at the time instant t, $N_1(t)$, becomes a user of this technology because the influence (*contagion*) of their peers that are users of e-commerce, $Y_1(t)$. This process is modelled via the non-linear term $\beta_1 N_1 Y_1$. Therefore, we implicitly assume *Population Mixing*, a usual hypothesis in continuous epidemiological models [12, 36]. The parameter β_1 represents the contagious or diffusion rate of e-commerce. This parameter embeds the probability that encounters among peers (users $Y_1(t)$ and non-users $N_1(t)$) be successful. A similar reasoning applies to the second ($i = 2$) age group 45 − 74. To formulate the mathematical model, we write the instantaneous variation of the percentage of non-users and users of e-commerce at the time instant t for each age group, $N_i'(t)$ and $Y_i'(t)$, $i = 1, 2$, using the so-called *Balance Mass Principle,* widely applied in Mathematical Epidemiology, to model the spread of a disease [36]. Also we are going to assume, in order to not increase the complexity of the model, that the non-users of e-commerce can only be contagied by peers of the same age group.Then taking into account the underlying demographic model (which involves the parameters μ, c_1, d_1 and d_2), the dynamics of the e-commerce can be stated via the following system of non-linear differential equations

$$\begin{cases} \dot{N}_1(t) = \mu - c_1 N_1(t) - d_1 N_1(t) - \beta_1 N_1(t) Y_1(t), \\ \dot{Y}_1(t) = \beta_1 N_1(t) Y_1(t) - c_1 Y_1(t) - d_1 Y_1(t), \\ \dot{N}_2(t) = c_1 N_1(t) - d_2 N_2(t) - \beta_2 N_2(t) Y_2(t), \\ \dot{Y}_2(t) = c_1 Y_1(t) - d_2 Y_2(t) + \beta_2 N_2(t) Y_2(t). \end{cases} \tag{4}$$

In Fig. 1 we can see the diagram of the proposed age-structured mathematical model for the diffusion of e-commerce in Spain. According with the demographic model (1), $N_1(t) + Y_1(t) = G_1(t) = \text{constant}$ and $N_2(t) + Y_2(t) = G_2(t) = \text{constant}$, and

$$N_1(t) + Y_1(t) + N_2(t) + Y_2(t) = 1. \tag{5}$$

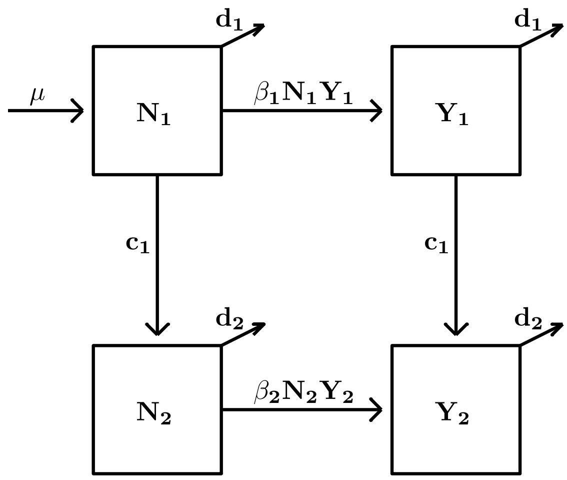

Figure 1: Compartmental diagram of the dynamic model for e-commerce in Spain given in (4). The boxes represent the subpopulations and the arrows represent the transitions among subpopulations.

Taking into account (5), model (4) can be rewritten in the following equivalent and simplified form

$$\begin{cases} \dot{N}_1(t) = \mu - c_1 N_1(t) - d_1 N_1(t) - \beta_1 N_1(t) Y_1(t), \\ \dot{Y}_1(t) = \beta_1 N_1(t) Y_1(t) - c_1 Y_1(t) - d_1 Y_1(t), \\ \dot{N}_2(t) = c_1 N_1(t) - d_2 N_2(t) - \beta_2 N_2(t) \left(1 - N_1(t) - Y_1(t) - N_2(t)\right). \end{cases} \tag{6}$$

Starting with the deterministic model (6) and using Particle Swarm Optimization (PSO) technique [37], we can estimate the diffusion parameters β_1 and β_2 that best fit, in the mean square sense, data given in Table 1, to model (6). Estimates obtained for these two parameters are $\beta_1 = 0.348385, \beta_2 = 0.061091$.

As indicated in Section 1, it is assumed that the adoption of e-commerce technology by a non-user takes place by *contagion* of his/her peers. This contagion happens after (physical or virtual) encounters between non-users and users, thus requiring a certain lag time. This fact motivates the introduction of delays to model this key feature. Of course, it must be noticed that not all encounters between peers are successful. The probability of success is implicitly embedded in the contagion parameters β_i, $i = 1, 2$. These facts lead us to introduce delay in the initial model (6) using the approach developed in [21]. Then, model (6) is transformed into the following one

$$\begin{cases} \dot{N}_1(t) = \mu - c_1 N_1(t) - d_1 N_1(t) - \beta_1 N_1(t) \displaystyle\int_0^\infty Y_1(t-s)\,\mathrm{d}k_1(s), \\ \dot{Y}_1(t) = -c_1 Y_1(t) - d_1 Y_1(t) + \beta_1 N_1(t) \displaystyle\int_0^\infty Y_1(t-s)\,\mathrm{d}k_1(s), \\ \dot{N}_2(t) = c_1 N_1(t) - d_2 N_2(t) - \beta_2 N_2(t) \displaystyle\int_0^\infty (1 - N_1 - Y_1 - N_2)(t-s)\,\mathrm{d}k_2(s), \end{cases} \tag{7}$$

where $k_i(s)$, $i = 1, 2$, are non-decreasing functions such that $\int_0^\infty \mathrm{d}k_i(s) = 1$.

3. Existence of equilibrium points

One of the main mathematical properties that should posses the deterministic non-linear dynamical model is stability. In this section, we calculate equilibrium points $(N_1^*, Y_1^*, N_2^*, Y_2^*)$ of equations (6) that must satisfy the following non-linear system of algebraic equations:

$$\begin{cases} 0 &=& \mu - c_1 N_1^* - d_1 N_1^* - \beta_1 N_1^* Y_1^*, \\ 0 &=& \beta_1 N_1^* Y_1^* - c_1 Y_1^* - d_1 Y_1^*, \\ 0 &=& c_1 N_1^* - d_2 N_2^* - \beta_2 N_2^* (1 - N_1^* - Y_1^* - N_2^*), \\ Y_2^* &=& 1 - N_1^* - N_2^* - Y_1^*. \end{cases} \tag{8}$$

It is easy to see that the two first equations of (8) give the following two equilibria: $(N_1^*, Y_1^*) = (\frac{\mu}{c_1+d_1}, 0)$, that has no practical interest, and

$$N_1^* = \frac{c_1 + d_1}{\beta_1}, \quad Y_1^* = \frac{\mu}{c_1 + d_1} - \frac{c_1 + d_1}{\beta_1}. \tag{9}$$

By (9), the third equation (8) can be represented in the form

$$(N_2^*)^2 - A N_2^* + B = 0, \tag{10}$$

where

$$A = 1 - \tfrac{\mu}{c_1+d_1} + \tfrac{d_2}{\beta_2}, \quad B = \tfrac{c_1(c_1+d_1)}{\beta_1 \beta_2}. \tag{11}$$

Taking into account that $N_1^* + Y_1^* < 1$ and $Y_1^* > 0$, one gets

$$\mu < c_1 + d_1 < \sqrt{\mu \beta_1}. \tag{12}$$

Thus, from $\mu < c_1 + d_1$ one derives $A > 0$. From positiveness of A and B it follows that the equation (10) cannot have negative roots and by the condition $A^2 > 4B$ or

$$1 + \frac{d_2}{\beta_2} > 2\sqrt{\frac{c_1(c_1 + d_1)}{\beta_1\beta_2}} + \frac{\mu}{c_1 + d_1},$$

have two positive roots.

Lemma 3.1. *If the condition* (12) *holds then $A^2 > 4B$ and therefore the equation* (10) *has two roots such that*

$$N^*_{21} = \frac{A + \sqrt{A^2 - 4B}}{2} > G_2 > N^*_{22} = \frac{A - \sqrt{A^2 - 4B}}{2}, \tag{13}$$

where $G_2 = 1 - \frac{\mu}{c_1+d_1}$.

Proof : It is evident that $A^2 = (G_2 + \frac{d_2}{\beta_2})^2 \geq 4\frac{d_2}{\beta_2}G_2$. So, it is enough to note that via (11), $c_1G_1 = d_2G_2$, $G_1 = \frac{\mu}{c_1+d_1}$ and (12) we have

$$B = \frac{d_2G_2(c_1 + d_1)^2}{\mu\beta_1\beta_2} < \frac{d_2}{\beta_2}G_2, \tag{14}$$

that proves $A^2 - 4AB > 0$. To prove (13) note as $A = G_2 + \frac{d_2}{\beta_2}$, we have that $N^*_{21} > G_2$ is equivalent to $\sqrt{A^2 - 4B} > G_2 - \frac{d_2}{\beta_2}$ and $N^*_{22} < G_2$ is equivalent to $\sqrt{A^2 - 4B} > \frac{d_2}{\beta_2} - G_2$. So, it is enough to show that $A^2 - 4B > (G_2 - \frac{d_2}{\beta_2})^2$, that is equivalent to (14). The proof is completed. □

Remark 3.1. *Via* (13)*, the positive equilibrium* $(N^*_1, Y^*_1, N^*_2, Y^*_2)$*, is defined by* (9) *and* $N^*_2 = N^*_{22}$, $Y^*_2 = 1 - N^*_1 - Y^*_1 - N^*_2$.

4. Stochastic perturbations, centralization and linearization

As it has been motivated in Section 1, we assume that the dynamics of the use of e-commerce is subject to independent and complex factors whose nature is random. Thus, the equilibrium of the proposed mathematical model (4) is also affected by randomness. According to Central Limit Theorem, Gaussian distribution is a suitable probabilistic pattern to describe such a type of uncertainty. In order to take into account this key feature, henceforth we will assume that system (7) is exposed to stochastic perturbations of White Noise type, hence Gaussian, that we will denote by $(\dot{W}_1(t), \dot{W}_2(t), \dot{W}_3(t))$, which are directly proportional to the deviation of system state at $(N_1(t), Y_1(t), N_2(t))$ from the equilibrium point (N^*_1, Y^*_1, N^*_2), that is,

$$\begin{cases} \dot{N}_1(t) = \mu - c_1N_1(t) - d_1N_1(t) - \beta_1N_1(t)\displaystyle\int_0^\infty Y_1(t-s)\,\mathrm{d}k_1(s) + \sigma_1(N_1(t) - N^*_1)\dot{W}_1(t), \\ \dot{Y}_1(t) = -c_1Y_1(t) - d_1Y_1(t) + \beta_1N_1(t)\displaystyle\int_0^\infty Y_1(t-s)\,\mathrm{d}k_1(s) + \sigma_2(Y_1(t) - Y^*_1)\dot{W}_2(t), \\ \dot{N}_2(t) = c_1N_1(t) - d_2N_2(t) - \beta_2N_2(t)\displaystyle\int_0^\infty (1 - N_1 - Y_1 - N_2)(t-s)\,\mathrm{d}k_2(s) + \sigma_3(N_2(t) - N^*_2)\dot{W}_3(t). \end{cases} \tag{15}$$

Here, $W_1(t), W_2(t), W_3(t)$ are mutually independent standard Wiener processes. The stochastic differential equations of system (15) are understood in Itô sense, [38].

To centralize system (15) in the equilibrium point, let us introduce the change of variable

$$X_1(t) = N_1(t) - N_1^*, \quad X_2(t) = Y_1(t) - Y_1^*, \quad X_3(t) = N_2(t) - N_2^*.$$

Substituting this into (15) and using (8), we obtain

$$\begin{cases} \dot{X}_1(t) = -(c_1 + d_1 + \beta_1 Y_1^*)X_1(t) - \beta_1(X_1(t) + N_1^*)\int_0^\infty X_2(t-s)\,\mathrm{d}k_1(s) + \sigma_1 X_1(t)\dot{W}_1(t), \\ \dot{X}_2(t) = \beta_1 Y_1^* X_1(t) - (c_1 + d_1)X_2(t) + \beta_1(X_1(t) + N_1^*)\int_0^\infty X_2(t-s)\,\mathrm{d}k_1(s) + \sigma_2 X_2(t)\dot{W}_2(t), \\ \dot{X}_3(t) = c_1 X_1(t) - (d_2 + \beta_2 Y_2^*)X_3(t) + \beta_2(X_3(t) + N_2^*)\int_0^\infty (X_1 + X_2 + X_3)(t-s)\,\mathrm{d}k_2(s) + \sigma_3 X_3(t)\dot{W}_3(t). \end{cases} \tag{16}$$

It is clear that stability of the equilibrium of the system (15) is equivalent to stability of the zero solution of the system (16).

Rejecting the nonlinear terms in (16), we obtain the linear part of the system (16)

$$\begin{cases} \dot{Z}_1(t) = -(c_1 + d_1 + \beta_1 Y_1^*)Z_1(t) - \beta_1 N_1^* J_1(Z_{2t}) + \sigma_1 Z_1(t)\dot{W}_1(t), \\ \dot{Z}_2(t) = \beta_1 Y_1^* Z_1(t) - (c_1 + d_1)Z_2(t) + \beta_1 N_1^* J_1(Z_{2t}) + \sigma_2 Z_2(t)\dot{W}_2(t), \\ \dot{Z}_3(t) = c_1 Z_1(t) - (d_2 + \beta_2 Y_2^*)Z_3(t) + \beta_2 N_2^*(J_2(Z_{1t}) + J_2(Z_{2t}) + J_2(Z_{3t}) + \sigma_3 Z_3(t)\dot{W}_3(t), \end{cases} \tag{17}$$

where

$$J_i(Z_{jt}) = \int_0^\infty Z_j(t-s)\mathrm{d}k_i(s), \quad i = 1, 2, \quad j = 1, 2, 3. \tag{18}$$

5. Stability of the equilibrium point

This section is addressed to establish sufficient conditions for asymptotic mean square stability of the zero solution of linear system (17) associated to the nonlinear system (16), that are also sufficient conditions for stability in probability of the zero solution of the nonlinear system (16). Therefore, such conditions are sufficient conditions for stability in probability of the equilibrium point (N_1^*, Y_1^*, N_2^*) of system (15), [21].

Putting $Z(t) = \mathrm{col}(Z_1(t), Z_2(t), Z_3(t))$, rewrite the system (17) in the matrix form

$$\dot{Z}(t) = AZ(t) + A_1 J_1(Z_t) + A_2 J_2(Z_t) + \sum_{i=1}^{3} C_i Z(t)\dot{W}_i(t), \tag{19}$$

where the matrix C_i has the element $c_{ii} = \sigma_i$ and all other elements are zeros,

$$A = \begin{pmatrix} -(c_1 + d_1 + \beta_1 Y_1^*) & 0 & 0 \\ \beta_1 Y_1^* & -(c_1 + d_1) & 0 \\ c_1 & 0 & -(d_2 + \beta_2 Y_2^*) \end{pmatrix},$$
$$A_1 = \begin{pmatrix} 0 & -\beta_1 N_1^* & 0 \\ 0 & \beta_1 N_1^* & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \beta_2 N_2^* & \beta_2 N_2^* & \beta_2 N_2^* \end{pmatrix}. \tag{20}$$

Following the GMLFC for stability investigation of (19), we consider the auxiliary equation without memory [21],

$$\dot{Z}(t) = AZ(t) + \sum_{i=1}^{3} C_i Z(t)\dot{W}_i(t). \tag{21}$$

Note that the first equation of (21) depends on $Z_1(t)$ only, the second equation of (21) depends on $Z_1(t)$ and $Z_2(t)$ only, and the third equation of (21) depends on $Z_1(t)$ and $Z_3(t)$ only. So, using Remark 2.7 of [21, p.49], we obtain the following result.

Lemma 5.1. *If*

$$c_1 + d_1 + \beta_1 Y_1^* > \frac{1}{2}\sigma_1^2, \quad c_1 + d_1 > \frac{1}{2}\sigma_2^2, \quad d_2 + \beta_2 Y_2^* > \frac{1}{2}\sigma_3^2, \tag{22}$$

then the zero solution of the equation (21) *is asymptotically mean square stable.*

Remark 5.1. *Note that the first inequality in* (22) *is the necessary and sufficient condition for asymptotic mean square stability of the first equation of the system* (21).

Lemma 5.2. *Let $R \in \mathbf{R}^{n\times n}$ be a positive definite matrix, $z = \int_D y(s)\mu(\mathrm{d}s)$, where $z, y(s) \in \mathbf{R}^n$, $\mu(\mathrm{d}s)$ is some measure on D such that $\mu(D) < \infty$ and the integral is defined in the Lebesgue sense. Then*

$$z'Rz \le \mu(D)\int_D y'(s)Ry(s)\mu(\mathrm{d}s). \tag{23}$$

Proof : The inequality (23) follows from the Cauchy-Schwarz inequality:

$$z'Rz = |R^{1/2}z|^2 = \left|\int_D R^{1/2}y(s)\mu(\mathrm{d}s)\right|^2 \le \int_D \mu(\mathrm{d}s)\int_D |R^{1/2}y(s)|^2\mu(\mathrm{d}s) = \mu(D)\int_D y'(s)Ry(s)\mu(\mathrm{d}s).$$

The proof is completed. □

Theorem 5.3. *Let A, A_1, A_2 be matrices defined in* (20) *and there exist positive definite matrices P, R_1, R_2 such that the linear matrix inequality (LMI) $\Phi < 0$ holds, where*

$$\Phi = \begin{pmatrix} \Phi_{11} & PA_1 & PA_2 \\ * & -R_1 & 0 \\ * & * & -R_2 \end{pmatrix}, \quad \Phi_{11} = PA + A'P + P_\sigma + R_1 + R_2, \quad P_\sigma = \begin{pmatrix} p_{11}\sigma_1^2 & 0 & 0 \\ 0 & p_{22}\sigma_2^2 & 0 \\ 0 & 0 & p_{33}\sigma_3^2 \end{pmatrix}, \tag{24}$$

p_{ii}, $i = 1, 2, 3$, are the diagonal elements of the matrix P. Then the equilibrium $(N_1^*, Y_1^*, N_2^*, Y_2^*)$ of the system (15) *is stable in probability.*

Proof : Let L be the generator of the equation (19). For the functional $V_1(t) = Z'(t)PZ(t)$ we have

$$\begin{aligned} LV_1(t) &= 2Z'(t)P(AZ(t) + A_1J_1(Z_t) + A_2J_2(Z_t)) + Z'(t)P_\sigma Z(t) \\ &= Z'(t)(PA + A'P + P_\sigma)Z(t) + 2\sum_{i=1}^{2} Z'(t)PA_iJ_i(Z_t). \end{aligned}$$

Consider the additional functional

$$V_2(t) = \sum_{i=1}^{2}\int_0^\infty \int_{t-s}^{t} Z'(\tau)R_iZ(\tau)\,\mathrm{d}k_i(s), \tag{25}$$

and note that by (18), (23) and $\int_0^\infty \mathrm{d}k_i(s) = 1$

$$J_i'(Z_t)R_iJ_i(Z_t) \le \int_0^\infty Z'(t-s)R_iZ(t-s)\,\mathrm{d}k_i(s). \tag{26}$$

So, by (25) and (26) we have

$$\begin{aligned} LV_2(t) &= \sum_{i=1}^{2}\left(Z'(t)R_iZ(t) - \int_0^\infty Z'(t-s)R_iZ(t-s)\,\mathrm{d}k_i(s)\right) \\ &\le Z'(t)(R_1+R_2)Z(t) - \sum_{i=1}^{2} J_i'(Z_t)R_iJ_i(Z_t). \end{aligned}$$

As a result for the functional $V = V_1 + V_2$ we obtain

$$\begin{aligned} LV(t) &\le Z'(t)(PA + A'P + P_\sigma + R_1 + R_2)Z(t) \\ &+ 2\sum_{i=1}^{2} Z'PA_iJ_i(Z_t) - \sum_{i=1}^{2} J_i'(Z_t)R_iJ_i(Z_t) = \eta'(t)\Phi\eta(t), \end{aligned}$$

where the matrix Φ is defined in (24) and $\eta(t) = \mathrm{col}\{Z(t), J_1(Z_t), J_2(Z_t)\}$. So, the constructed functional $V(t)$ is positive definite and $LV(t)$ via $\Phi < 0$ is negative definite that provides asymptotic mean square stability of the zero solution of the linear equation (19), and at the same time stability in probability of the zero solution of the nonlinear system (16), [21], that is equivalent to stability in probability of the equilibrium $(N_1^*, Y_1^*, N_2^*, Y_2^*)$ of the system (15). The proof is completed. □

Remark 5.2. (Schur complement). *The symmetric matrix* $\begin{bmatrix} A & B \\ B' & C \end{bmatrix}$ *is negative definite if and only if the matrices C and $A - BC^{-1}B'$ are both negative definite.*

Via Schur complement the LMI $\Phi < 0$ is equivalent to the Riccati matrix inequality

$$PA + A'P + P_\sigma + \sum_{i=1}^{2}(R_i + PA_iR_i^{-1}A_i'P) < 0.$$

Example 5.1. *Solving LMI $\Phi < 0$ via MATLAB by the values of the parameters* $\mu = 0.010110$, $c_1 = 0.0178252$, $d_1 = 5.7333 \times 10^{-4}$, $d_2 = 0.0217424$, $\beta_1 = 0.348385$, $\beta_2 = 0.061091$, $G_1 = 0.5495$, $G_2 = 0.4505$, *it was shown that the equilibrium*

$$(N_1^*, Y_1^*, N_2^*, Y_2^*) = (0.052811, 0.496689, 0.019584, 0.430916) \tag{27}$$

saves stability in probability for $\sigma_1 = 0.2890$, $\sigma_2 = 0.1730$, $\sigma_3 = 0.3061$. *In agreement with* (22), *we obtain* $\sigma_1 < \sqrt{2(c_1 + d_1 + \beta_1 Y_1^*)} = 0.6188$, $\sigma_2 < \sqrt{2(c_1 + d_1)} = 0.1918$, $\sigma_3 < \sqrt{2(d_2 + \beta_2 Y_2^*)} = 0.3101$.

6. Numerical simulations using real data of e-commerce in Spain

This section is devoted to carry out simulations of the stochastic model with discrete delay $h > 0$ obtained from (15) by $\mathrm{d}k_i(s) = \delta(s-h)\mathrm{d}s$, where $\delta(s)$ is Dirac's function. The model parameters μ, c_1, d_i, β_i, $i = 1, 2$, perturbations σ_i, $i = 1, 2, 3$ and equilibrium point $(N_1^*, Y_1^*, N_{22}^*, Y_2^*)$ are given in Example 5.1. Our goal in this section is to check that our simulations are in agreement with real data for Spanish INE collected in Table 1. To perform simulations, we will discretize the stochastic system with delay (15) by applying an Euler-Maruyama type numerical scheme for equations with delay [21, pp. 309–310]. This yields

$$\left\{\begin{array}{rcl} N_{1,i+1} & = & N_{1,i} + \Delta t\,(\mu - c_1 N_{1,i} - d_1 N_{1,i} - \beta_1 N_{1,i} Y_{1,i-m}) + \sigma_1 \left(N_{1,i} - N_1^*\right)(W_{1,i+1} - W_{1,i}), \\ Y_{1,i+1} & = & Y_{1,i} + \Delta t\,(-c_1 Y_{1,i} - d_1 Y_{1,i} + \beta_1 N_{1,i} Y_{1,i-m}) + \sigma_2 \left(Y_{1,i} - Y_1^*\right)(W_{2,i+1} - W_{2,i}), \\ N_{2,i+1} & = & N_{2,i} + \Delta t\,(c_1 N_{1,i} - d_2 N_{2,i} - \beta_2 N_{2,i}(1 - N_{1,i-m} - Y_{1,i-m} - N_{2,i-m})) \\ & & +\sigma_3 (N_{2,i} - N_2^*)\,(W_{3,i+1} - W_{3,i}), \end{array}\right. \tag{28}$$

where Δt is the step of discretization, m is the discretized delay, i.e. $m = h/\Delta t$, $N_{1,i} = N_1(i)$, $Y_{1,i} = Y_1(i)$ and $N_{2,i} = N_2(i)$, $i = 0, 1, 2, \ldots$ In (28), $W_{k,i} = W_k(i)$, $k = 1, 2, 3$, are simulated trajectories of the Wiener process (the algorithm of simulation is described in [21, Section 2.1.1]).

In Figure 2, we show 500 simulations or trajectories of stochastic model with delay formulated in (15) taking $\Delta t = 1$ year and delay $h = 1$ year, because one year is the time step. We can see that the prediction through the mean of the solution of the proposed model are quite well captured in both age groups and for users and non-users of this technology. Finally, it is very important to observe that with respect to stability our simulations converge towards the equilibrium point (27), then showing full agreement with our theoretical findings. We have needed to plot these simulations beyond 2100 year to illustrate the stability of all the subpopulations of the compartmental model, in particular, to subpopulation $N_2(t)$ whose stabilization is slower.

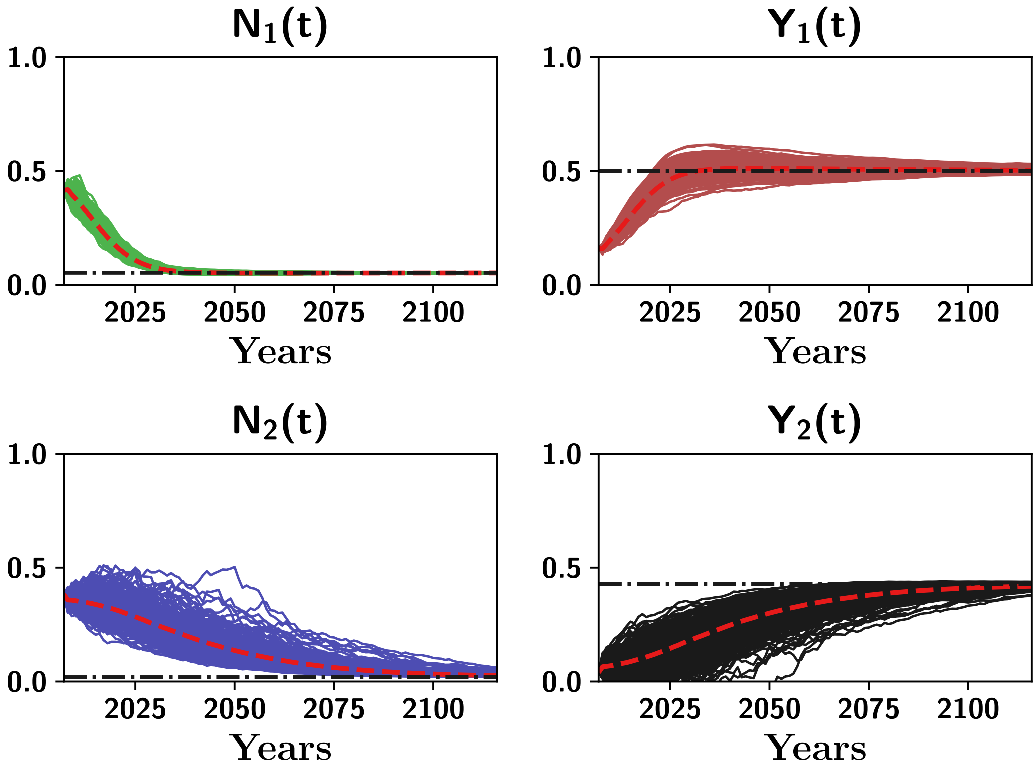

Figure 2: Simulation of 500 trajectories of the approximated solution stochastic process modelling the dynamics of e-commerce according to delayed stochastic system (15). Approximations have been constructed using the numerical scheme (28) taking $\Delta t = 1$ year and delay $h = 1$ year. Red line represents the average of the trajectories, and the black one represents the equilibrium point (27).

7. Conclusions

In this paper, we have proposed an age-structured mathematical model based on a system of non-linear differential equations with delay to describe the dynamics of e-commerce in Spain using real data. Our main goal has been to perform an analysis of the stability of the model and the dynamics of the spread, obviously subject to many random factors. Therefore, we have introduced stochastic perturbations about the equilibrium point and we have established sufficient conditions in order to guarantee the stochastic stability. A key point to conduct this kind of analysis has been to divide the underlying age-structured model into only two subpopulations by aggregating sampled data from the Spanish National Statistical Institute. The theoretical results have shown a strong agreement with real data.

Acknowledgements

This work has been partially supported by the Ministerio de Economía y Competitividad grant MTM2017-89664-P.

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