Weighted G-Drazin inverses and a new pre-order on rectangular matrices

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Abstract

This paper deals with weighted G-Drazin inverses, which is a new class of matrices introduced to extend (to the rectangular case) G-Drazin inverses recently considered by Wang and Liu for square matrices. First, we define and characterize weighted G-Drazin inverses. Next, we consider a new pre-order defined on complex rectangular matrices based on weighted G-Drazin inverses. Finally, we characterize this pre-order and relate it to the minus partial order and to the weighted Drazin pre-order.

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1 Introduction and background

Equivalence relations and partial orders are well-known types of relations defined over a set. Their usefulness is indisputable in the whole Mathematics. However, it does not

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occur the same with pre-orders which are also important tools but their usefulness in not so known. We recall that a pre-order over a nonempty set is a binary relation that is reflexive and transitive. Pre-orders are more general relations than equivalence relations and partial orders. Equivalence relations and partial orders are special cases of pre-orders which satisfy, moreover, the symmetric or the anti-symmetric property, respectively. We highlight their importance by mentioning only a few examples of pre-orders. For instance, the reachability relationship in any directed graph ($x \preceq y$ if and only if x is reachable from y) gives rise to a pre-order [23]. Every finite topological space gives rise to a pre-order on its points; for example a finite space is an Alexandroff space (that is, the set of all the open sets is closed under arbitrary intersections) and on an Alexandroff space it can be defined a pre-order in a natural way ($x \preceq y$ if and only if $U_x \subseteq U_y$, where U_x denotes the intersection of all open sets containing the point x) [16]. In Computer Science, for example, many-one and Turing reductions are pre-orders on complexity classes; the binary relation considered on the set of terms, defined by $t_1 \preceq t_2$ if and only if a subterm of t_2 is a substitution instance of t_1, is called the encompassment pre-order [9, 11].

Another important kind of binary relations are defined on matrices and known as matrix partial orders and matrix pre-orders [17]. Our interest will be concentrated on this last class of relations. For instance, mathematical morphology used in digital image processing requires the concepts of the supremum and infimum of a set of matrices, which is given by the Löwner partial order on the set of symmetric matrices [5]. The invariance properties that this matrix partial order satisfies allow the authors to define morphological operators which are crucial in the analysis of noise suppression, edge detection, shape analysis, etc. with a wide range of applications in the study of medical imaging, geological sciences, among others, as mentioned in [5].

Let $\mathbb{C}^{m\times n}$ be the set of $m\times n$ complex matrices. For a given $A \in \mathbb{C}^{m\times n}$, the symbols A^*, $\mathcal{R}(A)$, and $r(A)$ denote the conjugate transpose, the range, and the rank of A, respectively. As usual, when $m = n$, A^{-1}, $ind(A)$, and I_n denote the inverse of A, the index of A, and the identity matrix of size $n \times n$, respectively. The subscript will be omitted when no confusion is caused. The direct sum of two matrices A and B will be written as $A \oplus B$.

Let $A \in \mathbb{C}^{m\times n}$. A matrix $X \in \mathbb{C}^{n\times m}$ is called a $\{1\}-$inverse of A if $AXA = A$. The set of all $\{1\}-$inverses of A is denoted by $A\{1\}$ (see [6]).

Throughout this paper, a nonzero matrix $W \in \mathbb{C}^{n\times m}$ will be fixed and used as a weight.

Definition 1.1 *Let $A \in \mathbb{C}^{m\times n}$. A matrix $X \in \mathbb{C}^{m\times n}$ is a weighted Drazin inverse of A if $AWX = XWA$, $X = XWAWX$, and $(AW)^{k+1}XW = (AW)^k$ with $k = ind(AW)$.*

This matrix X always exists, is unique, and will be denoted by $X = A^{D,W}$ (see [8]). When $m = n$ and $W = I_n$ then $X = A^{D,I_n} = A^D$, the Drazin inverse of A. Moreover, the equalities $A^{D,W}W = (AW)^D$ and $WA^{D,W} = (WA)^D$ hold (see [8]).

Definition 1.2 *Let $A \in \mathbb{C}^{n\times n}$. A matrix $X \in \mathbb{C}^{n\times n}$ is called a G-Drazin inverse of A if $AXA = A$, $A^{k+1}X = A^k$, and $XA^{k+1} = A^k$ with $k = ind(A)$.*

This matrix X always exists, but in general it is not unique. The set of all G-Drazin inverses of A is denoted by $A\{GD\}$; an element of this set is denoted by A^{GD} (see [25]).

The following binary relations are well known [17, 15, 25].

Definition 1.3 *Let $A, B \in \mathbb{C}^{m\times n}$.*

- *The minus partial order is defined by $A \leq^- B$ if and only if $r(B-A) = r(B)-r(A)$, for $A, B \in \mathbb{C}^{m\times n}$. It is known that $A \leq^- B$ if and only if there exists $A^- \in A\{1\}$ such that $A^-A = A^-B$ and $AA^- = BA^-$ [17].*
- *The weighted Drazin pre-order, denoted by $\preceq^{D,W}$, was defined in [15] by $A \preceq^{D,W} B$ if and only if $(AW)A^{D,W} = (BW)A^{D,W}$ and $A^{D,W}(WA) = A^{D,W}(WB)$ (see also [14]). If $m = n$ and $W = I_n$ then $\preceq^{D,W}$ is called the Drazin pre-order and is denoted by $\preceq^d$ (see [17]).*

Definition 1.4 *For $A, B \in \mathbb{C}^{n\times n}$, the G-Drazin partial order was defined in [25] as follows: $A \preceq^{GD} B$ if and only if there exists a G-Drazin inverse A^{GD} of A such that $A^{GD}A = A^{GD}B$ and $AA^{GD} = BA^{GD}$.*

For a most extensive study on generalized inverses, matrix partial orders, and pre-orders the authors refer the reader to [1, 2, 3, 6, 7, 10, 12, 13, 14, 18, 19, 20, 21, 22, 24, 26].

This paper is organized as follows. Section 2 defines and characterizes the weighted G-Drazin inverse of a rectangular matrix. Moreover, a new characterization for (ordinary) G-Drazin inverses is given in Corollary 2.1. Section 3 introduces and characterizes the weigthed G-Drazin pre-order $\preceq_W^{GD}$ on rectangular matrices. This relation is based on the weighted G-Drazin inverse and extends to the rectangular case the G-Drazin partial order defined in [25]. In addition, some properties of this pre-order and connections with the minus order and the weigthed Drazin pre-order are given. Theorem 3.4 provides several equivalent conditions to characterize the weigthed G-Drazin pre-order, and condition (VI) leads up to a new characterization of the G-Drazin pre-order.

2 Weighted G-Drazin inverses

The G-Drazin inverse of a square matrix was defined in [25]. In this section, we generalize this concept to rectangular matrices. Since G-Drazin inverse can be only computed for square matrices, we are going to consider a weight matrix W of adequate size in such a way the involved powers can be performed. In this manner, the products AW and WA corresponds to square matrices, their powers are well-defined; and we can proceed with the generalization of G-Drazin inverses to rectangular matrices.

Let $A \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$. The following fact can be easily shown by induction: $(AW)^k A = A(WA)^k$ for any nonnegative integer k.

Since the case $W = O$ will give only trivial results, throughout this paper we will assume $W \neq O$ not being explicitly mentioned.

Definition 2.1 *Let $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$, $k_1 = ind(AW)$, $k_2 = ind(WA)$, and $k = max\{k_1, k_2\}$. A matrix $X \in \mathbb{C}^{m\times n}$ is a W-weighted G-Drazin inverse of A if the following conditions are satisfied:*

(1w) $WAWXWAW = WAW$,

(2rw) $(AW)^{k+1}(XW) = (AW)^k$,

(2lw) $(WX)(WA)^{k+1} = (WA)^k$.

The set of all W-weighted G-Drazin inverses of A will be denoted by $A\{W-GD\}$. Sometimes a fixed or a general element of the set $A\{W-GD\}$ will be denoted by A^{W-GD}.

Notice that if we set $m = n$ and $W = I_n$ then Definition 2.1 recovers Definition 1.1 given in [25].

Remark 2.1 Let $A \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$.

(a) If $A = O$ then any matrix of suitable size is a W-weighted G-Drazin inverse of A.

(b) $A\{W - GD\} \subseteq WAW\{1\}$.

(c) If AW and WA are nilpotent matrices then $WAW\{1\} \subseteq A\{W - GD\}$.

The following Theorem proves that for a given matrix A, a W-weighted G-Drazin inverse of A always exists and, in general, it is not unique. The first part of this theorem is well known (see [27, 28]). We include its proof here for the sake of completeness.

Theorem 2.1 *Let $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$, $k_1 = ind(AW)$, and $k_2 = ind(WA)$. Then there exist nonsingular matrices $P \in \mathbb{C}^{m\times m}$ and $Q \in \mathbb{C}^{n\times n}$ such that*

$$A = P(A_1 \oplus A_2)Q^{-1} \qquad \text{and} \qquad W = Q(W_1 \oplus W_2)P^{-1}, \tag{1}$$

where A_1 and W_1 are $t \times t$ nonsingular matrices, and A_2W_2 and W_2A_2 are nilpotent matrices of indices k_1 and k_2, respectively.

Moreover, $X \in A\{W - GD\}$ if and only if

$$X = P \begin{bmatrix} (W_1A_1W_1)^{-1} & X_{12} \\ X_{21} & X_2 \end{bmatrix} Q^{-1}, \tag{2}$$

with $X_{12}W_2 = O$, $W_2X_{21} = O$, and $X_2 \in W_2A_2W_2\{1\}$.

In particular, if $m = n$ and $AW = WA$ then $Q = P$. In this case, if $W = I_n$ then $W_1 = I_t$ and $W_2 = I_{n-t}$.

Proof. Suppose that one of the following two disjoint situations $m \neq n$ or $m = n$ with $AW \neq WA$ holds. If we consider the core-nilpotent decomposition of AW and WA (see [6]) we have that there exists nonsingular matrices $P \in \mathbb{C}^{m\times m}$ and $Q \in \mathbb{C}^{n\times n}$ such that

$$AW = P(C \oplus N)P^{-1} \qquad \text{and} \qquad WA = Q(R \oplus T)Q^{-1}$$

where C and R are nonsingular matrices of the same order (since the number of nonzero eigenvalues of AW and WA (counting multiplicities) are coincide) and N and T are nilpotent matrices of indices $k_1 = ind(AW)$ and $k_2 = ind(WA)$, respectively. Let $k = max\{k_1, k_2\}$. By the above observation, $(AW)^k A = A(WA)^k$ holds.

Partitioning

$$A = P \begin{bmatrix} A_1 & A_{12} \\ A_{21} & A_2 \end{bmatrix} Q^{-1} \qquad \text{and} \qquad W = Q \begin{bmatrix} W_1 & W_{12} \\ W_{21} & W_2 \end{bmatrix} P^{-1}$$

with appropriate sizes, we have that

$$(AW)^k A = P \begin{bmatrix} C^k A_1 & C^k A_{12} \\ O & O \end{bmatrix} Q^{-1} \qquad \text{and} \qquad A(WA)^k = P \begin{bmatrix} A_1 R^k & O \\ A_{21} R^k & O \end{bmatrix} Q^{-1}$$

and then $A_{12} = O$ and $A_{21} = O$. Hence,

$$AW = P \begin{bmatrix} C & O \\ O & N \end{bmatrix} P^{-1} = P \begin{bmatrix} A_1 W_1 & A_1 W_{12} \\ A_2 W_{21} & A_2 W_2 \end{bmatrix} P^{-1}$$

from where we have $C = A_1 W_1$, $W_{12} = O$, and $N = A_2 W_2$. Analogously, we obtain $R = W_1 A_1$, $W_{21} = O$, and $T = W_2 A_2$. Thus,

$$A = P(A_1 \oplus A_2)Q^{-1} \qquad \text{and} \qquad W = Q(W_1 \oplus W_2)P^{-1},$$

where A_1 and W_1 are nonsingular matrices, and $A_2 W_2$ and $W_2 A_2$ are nilpotent matrices of indices $k_1 = ind(AW)$ and $k_2 = ind(WA)$, respectively.

In order to find formula (2), let $X \in A\{W - GD\}$ and suppose that

$$X = P \begin{bmatrix} X_1 & X_{12} \\ X_{21} & X_2 \end{bmatrix} Q^{-1}.$$

Then, it is easy to see that

$$WAWXWAW = Q \begin{bmatrix} W_1 C X_1 R W_1 & W_1 C X_{12} T W_2 \\ W_2 N X_{21} R W_1 & W_2 N X_2 T W_2 \end{bmatrix} P^{-1}.$$

Since the equality $WAWXWAW = WAW$ holds, it then follows that $X_1 = (W_1 A_1 W_1)^{-1}$, $X_{12} W_2 A_2 W_2 = O$, $W_2 A_2 W_2 X_{21} = O$, and $X_2 \in W_2 A_2 W_2\{1\}$.

Now, $(AW)^k = P(C^k \oplus O)P^{-1}$, while

$$(AW)^{k+1} XW = P \begin{bmatrix} C^{k+1} X_1 W_1 & C^{k+1} X_{12} W_2 \\ O & O \end{bmatrix} P^{-1},$$

then by (2rw) we obtain $X_{12}W_2 = O$. Analogously, from (2lw),

$$(WA)^k = Q \begin{bmatrix} R^k & O \\ O & O \end{bmatrix} Q^{-1}, \quad \text{and} \quad WX(WA)^{k+1} = Q \begin{bmatrix} W_1X_1R^{k+1} & O \\ W_2X_{21}R^{k+1} & O \end{bmatrix} Q^{-1}.$$

We arrive at $W_2X_{21} = O$. Hence,

$$X = P \begin{bmatrix} (W_1A_1W_1)^{-1} & X_{12} \\ X_{21} & X_2 \end{bmatrix} Q^{-1},$$

with $X_{12}W_2 = O$, $W_2X_{21} = O$, and $X_2 \in W_2A_2W_2\{1\}$ as required. The converse is a simple computation.

On the other hand, if $m = n$ and $WA = AW$, this last matrix can be represented with only one core-nilpotent decomposition. In this case, the proof is analogous to the previous one with $Q = P$. If, moreover, $W = I_n$ then $W_1 = I_t$ and $W_2 = I_{n-t}$. ■

Notice that if $m = n$ and $W = I_n$, from Theorem 2.1 we recover the form (3.2) in [25].

Example 2.1 Les us consider the matrices $A = \begin{bmatrix} 1 & 1 \end{bmatrix}$ and $W = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Then $AW = [1]$ and $WA = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = Q \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} Q^{-1}$, with $Q = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$. Hence,

$$A = [1] \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad W = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} [1]$$

and

$$A^{W-GD} = [1] \begin{bmatrix} 1 & X_{12} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 + X_{12} \end{bmatrix},$$

with $X_{12} \in \mathbb{C}$.

Theorem 2.2 *Let $A, X \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$, $k_1 = ind(AW)$, $k_2 = ind(WA)$, and $k = max\{k_1, k_2\}$. Then, the following conditions are equivalent:*

(a) $X \in A\{W - GD\}$.

(b) $WAWXWAW = WAW$ and $W(AW)^kXW = WXW(AW)^k$.

Proof. Let $A, X \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$, and $k = max\{k_1, k_2\}$ and suppose that A and W are written as in (1).

(a) $\Longrightarrow$ (b) By Theorem 2.1 we have that $X \in A\{W - GD\}$ if and only if

$$X = P \begin{bmatrix} (W_1A_1W_1)^{-1} & X_{12} \\ X_{21} & X_2 \end{bmatrix} Q^{-1},$$

with $X_{12}W_2 = O$, $W_2X_{21} = O$, and $X_2 \in W_2A_2W_2\{1\}$.

By induction on k, it is easy to prove the equality $W_1(A_1W_1)^{k-1} = A_1^{-1}(A_1W_1)^k$ for $k \geq 0$. Hence, by making some computations we have

$$W(AW)^kXW = Q(W_1(A_1W_1)^{k-1} \oplus O)P^{-1}$$

and

$$WXW(AW)^k = Q(A_1^{-1}(A_1W_1)^k \oplus O)P^{-1},$$

which gives the required equality.

(b) $\Longrightarrow$ (a) Consider the partition

$$X = P \begin{bmatrix} X_1 & X_{12} \\ X_{21} & X_2 \end{bmatrix} Q^{-1}.$$

Then,

$$W(AW)^kXW = Q \begin{bmatrix} W_1(A_1W_1)^kX_1W_1 & W_1(A_1W_1)^kX_{12}W_2 \\ O & O \end{bmatrix} P^{-1} \quad \text{and}$$

$$WXW(AW)^k = Q \begin{bmatrix} W_1X_1W_1(A_1W_1)^k & O \\ W_2X_{21}W_1(A_1W_1)^k & O \end{bmatrix} P^{-1}.$$

Since $W(AW)^kXW = WXW(AW)^k$, we obtain $X_{12}W_2 = O$ and $W_2X_{21} = O$.

On the other hand, $WAW = Q(W_1A_1W_1 \oplus W_2A_2W_2)P^{-1}$ and

$$WAWXWAW = Q(W_1A_1W_1X_1W_1A_1W_1 \oplus W_2A_2W_2X_2W_2A_2W_2)P^{-1}.$$

Thus, from $WAWXWAW = WAW$ we have $X_1 = (W_1A_1W_1)^{-1}$ and $X_2 \in W_2A_2W_2\{1\}$. Therefore, by Theorem 2.1 we conclude $X \in A\{W - GD\}$. ■

From Theorem 2.2 we can give an easier condition for the Definition 1.1 given in [25]. Observe that we only need to check two conditions instead of three as in [25, Definition 1.1].

Corollary 2.1 *Let $A, X \in \mathbb{C}^{n\times n}$ and $k = ind(A)$. Then, the following conditions are equivalent:*

(a) $X \in A\{GD\}$.

(b) $AXA = A$ *and* $A^k X = XA^k$,

(c) $AXA = A$, $XA^{k+1} = A^k$, *and* $A^{k+1}X = A^k$.

3 A new weighted matrix pre-order

In order to introduce a new binary relation that extends the G-Drazin partial order to the rectangular case, we consider the following definition.

Definition 3.1 *Let $W \in \mathbb{C}^{n\times m}$ and $A, B \in \mathbb{C}^{m\times n}$. It is said that $A \preceq_W^{GD} B$ if there exist $X_1, X_2 \in A\{W - GD\}$ such that $WAWX_1 = WBWX_1$ and $X_2WAW = X_2WBW$.*

Setting $n = m$ and $W = I_n$, our Definition 3.1 becomes Definition 3.1 in [25].

Theorem 3.1 *Let $W \in \mathbb{C}^{n\times m}$ and $A, B \in \mathbb{C}^{m\times n}$. Then the following conditions are equivalent:*

(a) $A \preceq_W^{GD} B$.

(b) *there exists $X \in A\{W - GD\}$ such that $WAWX = WBWX$ and $XWAW = XWBW$.*

Proof. For (a) $\Rightarrow$ (b), let $X_1, X_2 \in A\{W - GD\}$ such that $WAWX_1 = WBWX_1$ and $X_2WAW = X_2WBW$. Then $X_1 \in WAW\{1\}$ and $X_2 \in WAW\{1\}$, therefore $X_1WAWX_2 \in WAW\{1\}$. Set $X = X_1WAWX_2$. By Definition 2.1, it is easy to see that $X \in A\{W - GD\}$. Moreover, $WAWX = WBWX$ and $XWAW = XWBW$.

(b) $\Rightarrow$ (a) is trivial. ■

Remark 3.1 Let $A, B, X \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$, $P \in \mathbb{C}^{m\times m}$, and $Q \in \mathbb{C}^{n\times n}$, with P and Q nonsingular matrices. Then:

(a) $O \preceq_W^{GD} B$ for every matrix B. In order to check it, it is enough to take $O^{W-GD} = O$.

(b) $X \in A\{W - GD\}$ if and only if $P^{-1}XQ \in P^{-1}AQ\{(Q^{-1}WP) - GD\}$. It follows directly from Definition 2.1.

(c) $A \preceq_W^{GD} B$ if and only if $P^{-1}AQ \preceq_{Q^{-1}WP}^{GD} P^{-1}BQ$.

It follows from Definition 3.1 and the previous item.

Since $A\{W - GD\} \subseteq WAW\{1\}$, we have the following lemma.

Lemma 3.1 *Let $W \in \mathbb{C}^{n\times m}$ and $A, B \in \mathbb{C}^{m\times n}$. If $A \preceq_W^{GD} B$ then $WAW \leq^- WBW$.*

Theorem 3.2 *Let $W \in \mathbb{C}^{n\times m}$ and $A, B \in \mathbb{C}^{m\times n}$. If A and W are written as in* (1) *then the following conditions are equivalent:*

(a) $A \preceq_W^{GD} B$.

(b) $B = P \begin{bmatrix} A_1 & B_3 \\ B_4 & B_2 \end{bmatrix} Q^{-1}$, for some matrices B_2, B_3, and B_4 such that $W_2B_4 = O$, $B_3W_2 = O$, and $W_2A_2W_2 \leq^- W_2B_2W_2$.

In this case, $Z \in B\{W - GD\}$ if and only if

$$Z = P \begin{bmatrix} (W_1A_1W_1)^{-1} & Z_{12} \\ Z_{21} & Z_2 \end{bmatrix} Q^{-1}$$

with $Z_{12}W_2 = O$, $W_2Z_{21} = O$, and $Z_2 \in B_2\{W_2 - GD\}$.

Proof. Let $W \in \mathbb{C}^{n\times m}$ and $A, B \in \mathbb{C}^{m\times n}$. Suppose that A and W are written as in (1).

(a) $\Longrightarrow$ (b) Since $A \preceq_W^{GD} B$, there exists $X \in A\{W - GD\}$ such that $WAWX = WBWX$ and $XWAW = XWBW$. By Theorem 2.1 we have

$$X = P \begin{bmatrix} (W_1A_1W_1)^{-1} & X_{12} \\ X_{21} & X_2 \end{bmatrix} Q^{-1},$$

where $X_{12}W_2 = O$, $W_2X_{21} = O$, and

$$\text{(i)} \qquad X_2 \in W_2A_2W_2\{1\}.$$

Thus,

$$WAWX = Q \begin{bmatrix} I_t & W_1A_1W_1X_{12} \\ O & W_2A_2W_2X_2 \end{bmatrix} Q^{-1}.$$

Let us consider the following partition of B:

$$B = P \begin{bmatrix} B_1 & B_3 \\ B_4 & B_2 \end{bmatrix} Q^{-1}.$$

Hence

$$WBWX = Q \begin{bmatrix} W_1B_1A_1^{-1}W_1^{-1} & W_1B_1W_1X_{12} + W_1B_3W_2X_2 \\ W_2B_4A_1^{-1}W_1^{-1} & W_2B_4W_1X_{12} + W_2B_2W_2X_2 \end{bmatrix} Q^{-1}.$$

Therefore, from $WAWX = WBWX$ we obtain $B_1 = A_1$, $W_2B_4 = O$, and

$$\text{(ii)} \qquad W_2A_2W_2X_2 = W_2B_2W_2X_2.$$

Analogously, by using the equality $XWAW = XWBW$ and by making some calculations we have

$$XWAW = P \begin{bmatrix} I_t & O \\ X_{21}W_1A_1W_1 & X_2W_2A_2W_2 \end{bmatrix} P^{-1}$$

and

$$XWBW = P \begin{bmatrix} I_t & W_1^{-1}A_1^{-1}B_3W_2 \\ X_{21}W_1A_1W_1 & X_{21}W_1B_3W_2 + X_2W_2B_2W_2 \end{bmatrix} P^{-1},$$

thus $B_3W_2 = O$ and

$$\text{(iii)} \qquad X_2W_2A_2W_2 = X_2W_2B_2W_2.$$

From (i), (ii), and (iii) we have $W_2A_2W_2 \leq^- W_2B_2W_2$.

(b) $\Longrightarrow$ (a) Suppose that there exist matrices B_2, B_3, and B_4 such that

$$A = P \begin{bmatrix} A_1 & O \\ O & A_2 \end{bmatrix} Q^{-1}, \quad W = Q \begin{bmatrix} W_1 & O \\ O & W_2 \end{bmatrix} P^{-1}, \quad \text{and} \quad B = P \begin{bmatrix} A_1 & B_3 \\ B_4 & B_2 \end{bmatrix} Q^{-1},$$

with $W_2B_4 = O$, $B_3W_2 = O$, and $W_2A_2W_2 \leq^- W_2B_2W_2$. Then, there exists $X_2 \in W_2A_2W_2\{1\}$ such that $W_2A_2W_2X_2 = W_2B_2W_2X_2$ and $X_2W_2A_2W_2 = X_2W_2B_2W_2$. Let us consider the matrix

$$X = P((W_1A_1W_1)^{-1} \oplus X_2)Q^{-1}.$$

By Theorem 2.1, it is clear that $X \in A\{W - GD\}$ (i). Since $W_2A_2W_2X_2 = W_2B_2W_2X_2$, we have $WAWX = WBWX$ (ii). And from $X_2W_2A_2W_2 = X_2W_2B_2W_2$, we obtain $XWAW = XWBW$. Thus, $A \preceq_W^{GD} B$ follows from (i), (ii), and (iii).

On the other hand, let $Z \in B\{W - GD\}$ and consider

$$Z = P \begin{bmatrix} Z_1 & Z_{12} \\ Z_{21} & Z_2 \end{bmatrix} Q^{-1}.$$

Since $WBWZWBW = WBW$ we have $Z_1 = (W_1A_1W_1)^{-1}$ and $Z_2 \in W_2B_2W_2\{1\}$. Moreover, there exist matrices E and F of appropriate sizes, such that

$$(BW)^h = P \begin{bmatrix} (A_1W_1)^h & O \\ E & (B_2W_2)^h \end{bmatrix} P^{-1} \quad \text{and} \quad (WB)^h = Q \begin{bmatrix} (W_1A_1)^h & F \\ O & (W_2B_2)^h \end{bmatrix} Q^{-1},$$

where $h_1 = ind(BW)$, $h_2 = ind(WB)$, and $h = max\{h_1, h_2\}$. Hence,

$$(BW)^{h+1} = P \begin{bmatrix} (A_1W_1)^{h+1} & O \\ EA_1W_1 & (B_2W_2)^{h+1} \end{bmatrix} P^{-1}$$

and

$$(WB)^{h+1} = Q \begin{bmatrix} (W_1A_1)^{h+1} & W_1A_1F \\ O & (W_2B_2)^{h+1} \end{bmatrix} Q^{-1}.$$

Therefore,

$$(BW)^{h+1}ZW = P \begin{bmatrix} (A_1W_1)^h & (A_1W_1)^{h+1}Z_{12}W_2 \\ E + (B_2W_2)^{h+1}Z_{21}W_1 & E(A_1W_1)Z_{12}W_2 + (B_2W_2)^{h+1}Z_2W_2 \end{bmatrix} P^{-1}$$

and

$$WZ(WB)^{h+1} = Q \begin{bmatrix} (W_1A_1)^h & F + W_1Z_{12}(W_2B_2)^{h+1} \\ W_2Z_{21}(W_1A_1)^{h+1} & W_2Z_{21}W_1A_1F + W_2Z_2(W_2B_2)^{h+1} \end{bmatrix} Q^{-1}.$$

By comparing block matrices in the equality $(BW)^{h+1}ZW = (BW)^h$, we have $Z_{12}W_2 = O$ and $(B_2W_2)^{h+1}Z_2W_2 = (B_2W_2)^h$. Since A_1W_1 is nonsingular, it is clear that $ind(BW) = ind(B_2W_2)$. Similarly, by comparing blocks in $WZ(WB)^{h+1} = (WB)^h$, we obtain $W_2Z_{21} = O$, $W_2Z_2(W_2B_2)^{h+1} = (W_2B_2)^h$ and $ind(WB) = ind(W_2B_2)$ as well. Hence, $Z_2 \in B_2\{W_2 - GD\}$.

The converse is an easy computation. $\blacksquare$

Notice that if we set $m = n$ and $W = I_n$, we recover Theorem 3.1 in [25] and the complete set of G-Drazin inverses of B is provided.

Corollary 3.1 *Let $A, B \in \mathbb{C}^{n\times n}$. If $A = P(C \oplus N)P^{-1}$, with P and C nonsingular matrices and N nilpotent of index $k = ind(A)$, then the following conditions are equivalent:*

(a) $A \preceq^{GD} B$.

(b) $B = P(C \oplus B_2)P^{-1}$, for some matrix B_2 such that $N \leq^{-} B_2$.

In this case, $Z \in B\{GD\}$ if and only if

$$Z = P(C^{-1} \oplus Z_2)Q^{-1},$$

with $Z_2 \in B_2\{GD\}$.

Corollary 3.2 *Let $W \in \mathbb{C}^{n\times m}$ and $A, B \in \mathbb{C}^{m\times n}$. If $A \preceq^{GD}_{W} B$ then $B\{W - GD\} \subseteq A\{W - GD\}$.*

Proof. Let A and W be written as in (1). Since $A \preceq^{GD}_{W} B$, by Theorem 3.2 we have

$$B = P \begin{bmatrix} A_1 & B_3 \\ B_4 & B_2 \end{bmatrix} Q^{-1},$$

for some matrices B_2, B_3, and B_4 such that $W_2B_4 = O$, $B_3W_2 = O$, and $W_2A_2W_2 \leq^{-} W_2B_2W_2$. The last inequality implies $W_2B_2W_2\{1\} \subseteq W_2A_2W_2\{1\}$ (see [17]).

If $Z \in B\{W - GD\}$ then, by Theorem 3.2,

$$Z = P \begin{bmatrix} (W_1A_1W_1)^{-1} & Z_{12} \\ Z_{21} & Z_2 \end{bmatrix} Q^{-1},$$

with $Z_{12}W_2 = O$, $W_2Z_{21} = O$, and $Z_2 \in B_2\{W_2 - GD\}$. By Remark 2.1 (b) we have $Z_2 \in W_2B_2W_2\{1\}$, then $Z_2 \in W_2A_2W_2\{1\}$. Therefore, by Theorem 2.1, $Z \in A\{W - GD\}$. $\blacksquare$

In general, the relation $\preceq^{GD}_{W}$ is not antisymmetric as the following example shows.

Example 3.1 *Let us consider the matrices A, B, and W given by*

$$A = \left[\begin{array}{cc|cc} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right], \quad W = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right], \quad B = \left[\begin{array}{cc|cc} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 5 \end{array}\right].$$

By Theorem 3.2, it is clear that $A \preceq_W^{GD} B$ and $B \preceq_W^{GD} A$, but $A \neq B$.

Theorem 3.3 *Let $W \in \mathbb{C}^{n\times m}$. The binary relation $\preceq_W^{GD}$ defined on $\mathbb{C}^{m\times n}$ is a pre-order, and it will be called the W-weighted G-Drazin pre-order.*

Proof. By Definition 3.1, it is immediate that $\preceq_W^{GD}$ satisfies the reflexive property.

Let $A, B, C \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$ such that $A \leq_W^{GD} B$ and $B \leq_W^{GD} C$.

Since $B \leq_W^{GD} C$, by Theorem 3.1, there exists $Z \in B\{W - GD\}$ such that

$$WBWZ = WCWZ, \tag{3}$$

$$ZWBW = ZWCW. \tag{4}$$

Since $A \leq_W^{GD} B$, by Theorem 3.2 there exist nonsingular matrices P, Q such that $A = P(A_1 \oplus A_2)Q^{-1}$, $W = Q(W_1 \oplus W_2)P^{-1}$, and $B = P \begin{bmatrix} A_1 & B_3 \\ B_4 & B_2 \end{bmatrix} Q^{-1}$, with A_1, W_1 nonsingular matrices, A_2W_2 and W_2A_2 nilpotent matrices, $W_2B_4 = O$, $B_3W_2 = O$, and $W_2A_2W_2 \leq^- W_2B_2W_2$.

Moreover,

$$Z = P \begin{bmatrix} (W_1A_1W_1)^{-1} & Z_{12} \\ Z_{21} & Z_2 \end{bmatrix} Q^{-1},$$

with $Z_{12}W_2 = O$, $W_2Z_{21} = O$, and $Z_2 \in B_2\{W_2 - GD\} \subseteq W_2B_2W_2\{1\}$.

Suppose that $C = P \begin{bmatrix} C_1 & C_3 \\ C_4 & C_2 \end{bmatrix} Q^{-1}$. Some calculations yield to

$$WBWZ = Q \begin{bmatrix} I & W_1A_1W_1Z_{12} \\ O & W_2B_2W_2Z_2 \end{bmatrix} Q^{-1}$$

and

$$WCWZ = Q\begin{bmatrix} W_1C_1W_1(W_1A_1W_1)^{-1} & W_1C_1W_1Z_{12} + W_1C_3W_2Z_2 \\ W_2C_4(W_1A_1)^{-1} & W_2C_4W_1Z_{12} + W_2C_2W_2Z_2 \end{bmatrix} Q^{-1}.$$

From (3) we obtain $C_1 = A_1$, $W_2C_4 = 0$, and $W_2B_2W_2Z_2 = W_2C_2W_2Z_2$.

On the other hand,

$$ZWBW = P\begin{bmatrix} I & O \\ Z_{21}W_1A_1W_1 & Z_2W_2B_2W_2 \end{bmatrix} P^{-1}$$

and

$$ZWCW = P\begin{bmatrix} I & (A_1W_1)^{-1}C_3W_2 \\ Z_{21}W_1A_1W_1 & Z_{21}W_1C_3W_2 + Z_2W_2C_2W_2 \end{bmatrix} P^{-1}.$$

Now, from (4), we obtain $C_3W_2 = O$ and $Z_2W_2B_2W_2 = Z_2W_2C_2W_2$.

Therefore, $C = P\begin{bmatrix} A_1 & C_3 \\ C_4 & C_2 \end{bmatrix} Q^{-1}$, with $W_2C_4 = 0$, $C_3W_2 = O$, and $W_2B_2W_2 \leq^- W_2C_2W_2$. Since $\leq^-$ is transitive, $W_2A_2W_2 \leq^- W_2C_2W_2$. Hence, by Theorem 3.2, $A \leq_W^{GD} C$ holds. ■

Theorem 3.4 *Let $W \in \mathbb{C}^{n\times m}$, $A, B \in \mathbb{C}^{m\times n}$, $k_1 = ind(AW)$, $k_2 = ind(WA)$, and $k = max\{k_1, k_2\}$. If A and W are written as in* (1) *then the following conditions are equivalent:*

(I) $A \preceq_W^{GD} B$.

(II) $B = P\begin{bmatrix} A_1 & B_3 \\ B_4 & B_2 \end{bmatrix} Q^{-1}$, *for some matrices B_2, B_3, and B_4 such that $W_2B_4 = O$, $B_3W_2 = O$, and $W_2A_2W_2 \leq^- W_2B_2W_2$.*

(III) $WAW \leq^- WBW$, $\mathcal{R}(((AW)^kBW)^*) \subseteq \mathcal{R}(((AW)^k)^*)$, *and* $\mathcal{R}(WB(WA)^k) \subseteq \mathcal{R}((WA)^k)$.

(IV) $WAW \leq^- WBW$, $(AW)^kBW = (AW)^{k+1}$, *and* $WB(WA)^k = (WA)^{k+1}$.

(V) The following conditions are assumed to hold simultaneouly

(V.a) There exists $A^{W-GD} \in A\{W-GD\}$ such that
$WAWA^{W-GD}WBW = WBWA^{W-GD}WAW = WAW$, *and*

(V.b) For every $X \in A\{W-GD\}$, $WXW(AW)^kBW = WB(WA)^kWXW$ hold.

(VI) $WAW \leq^- WBW$ and $W(AW)^kBW = WB(WA)^kW$.

(VII) $WAW \leq^- WBW$ and $WA^{D,W}WBW = WBWA^{D,W}W$.

(VIII) $WAW \leq^- WBW$ and $(WA)^DWBW = WBW(AW)^D$.

Proof. (I) $\Leftrightarrow$ (II) follows from Theorem 3.2.

(II) $\Longrightarrow$ (IV) Suppose that $A = P(A_1 \oplus A_2)Q^{-1}$, $W = Q(W_1 \oplus W_2)P^{-1}$, and $B = P\begin{bmatrix} A_1 & B_3 \\ B_4 & B_2 \end{bmatrix}Q^{-1}$, with $W_2B_4 = O$, $B_3W_2 = O$, and $W_2A_2W_2 \leq^- W_2B_2W_2$. Then, there exists $X_2 \in W_2A_2W_2\{1\}$ such that $W_2A_2W_2X_2 = W_2B_2W_2X_2$ and $X_2W_2A_2W_2 = X_2W_2B_2W_2$. Let us consider the matrix $X = P((W_1A_1W_1)^{-1} \oplus X_2)Q^{-1}$. It is easy to see that $X \in WAW\{1\}$, $WAWX = WBWX$, and $XWAW = XWBW$, this is, $WAW \leq^- WBW$. Moreover, $(AW)^kBW = (AW)^{k+1}$ and $WB(WA)^k = (WA)^{k+1}$ follow by a simple computation.

(IV) $\Longrightarrow$ (III) From $(AW)^kBW = (AW)^{k+1}$ we have $((AW)^kBW)^* = ((AW)^{k+1})^*$, and then

$\mathcal{R}(((AW)^kBW)^*) = \mathcal{R}(((AW)^{k+1})^*) = \mathcal{R}(((AW)^k)^*(AW)^*) \subseteq \mathcal{R}(((AW)^k)^*)$.

Similarly, from $WB(WA)^k = (WA)^{k+1}$ we have $\mathcal{R}(WB(WA)^k) = \mathcal{R}((WA)^{k+1}) \subseteq \mathcal{R}((WA)^k)$.

(III) $\Longrightarrow$ (II) Suppose that A and W are written as in (1) and let $B = P\begin{bmatrix} B_1 & B_3 \\ B_4 & B_2 \end{bmatrix}Q^{-1}$.

It is easy to see that

$$(AW)^kBW = P\begin{bmatrix} (A_1W_1)^kB_1W_1 & (A_1W_1)^kB_3W_2 \\ O & O \end{bmatrix}P^{-1},$$

and, consequently,

$$((AW)^kBW)^* = (P^{-1})^*\begin{bmatrix} ((A_1W_1)^kB_1W_1)^* & O \\ ((A_1W_1)^kB_3W_2)^* & O \end{bmatrix}P^*$$

and

$$((AW)^k)^* = (P^{-1})^*(((A_1W_1)^k)^* \oplus O)P^*.$$

Consider the block matrix

$$E = \begin{bmatrix} ((AW)^k BW)^* & ((AW)^k)^* \end{bmatrix}.$$

It is clear that

$$r(E) = r\left(\begin{bmatrix} ((A_1W_1)^k B_1W_1)^* & ((A_1W_1)^k)^* \\ ((A_1W_1)^k B_3W_2)^* & O \end{bmatrix}\right).$$

Since the block $((A_1W_1)^k)^*$ is nonsingular, it is easy to see that

$$\begin{aligned} r(E) &= r\left(\begin{bmatrix} O & ((A_1W_1)^k)^* \\ ((A_1W_1)^k B_3W_2)^* & O \end{bmatrix}\right) \\ &= r\left(((A_1W_1)^k)^*\right) + r\left(((A_1W_1)^k B_3W_2)^*\right). \end{aligned}$$

Since $\mathcal{R}(((AW)^k BW)^*) \subseteq \mathcal{R}(((AW)^k)^*)$, each column of $((AW)^k BW)^*$ is a linear combination of the columns of $((AW)^k)^*$. Hence, we have $r(E) \leq r(((AW)^k)^*)$, that is, $r\left(((A_1W_1)^k)^*\right) + r\left(((A_1W_1)^k B_3W_2)^*\right) \leq r(((AW)^k)^*)$. Thus, $r((A_1W_1)^k B_3W_2) = 0$, which implies $B_3W_2 = O$.

Additionally,

$$WB(WA)^k = Q\begin{bmatrix} W_1B_1(W_1A_1)^k & O \\ W_2B_4(W_1A_1)^k & O \end{bmatrix}Q^{-1} \quad \text{and} \quad (WA)^k = Q((W_1A_1)^k \oplus O)Q^{-1}.$$

Now, we consider the block matrix

$$F = \begin{bmatrix} WB(WA)^k & (WA)^k \end{bmatrix}.$$

Following an analogous reasoning to the previous one, the hypothesis $\mathcal{R}(WB(WA)^k) \subseteq \mathcal{R}((WA)^k)$ implies $W_2B_4 = O$.

On the other hand, from $WAW \leq^- WBW$ we have $r(WBW - WAW) = r(WBW) - r(WAW)$. Considering the block forms of WBW and WAW we arrive at

$$r(WBW - WAW) = r\left((W_1(B_1 - A_1)W_1 \oplus W_2(B_2 - A_2)W_2)\right) = a + c,$$

where $a = r(B_1 - A_1)$ and $c = r(W_2(B_2 - A_2)W_2)$.

Moreover, $r(WBW) = r(B_1) + r(W_2B_2W_2)$ and $r(WAW) = r(A_1) + r(W_2A_2W_2)$, hence $r(WBW) - r(WAW) = b + d$ where $b = r(B_1) - r(A_1)$ and $d = r(W_2B_2W_2) - r(W_2A_2W_2)$. Summarizing, we have $a + c = b + d$ with

$$b = r(B_1) - r(A_1) \leq r(B_1 - A_1) = a$$

and

$$d = r(W_2B_2W_2) - r(W_2A_2W_2) \leq r(W_2(B_2 - A_2)W_2) = c.$$

Therefore, $a = b$ and $d = c$, that is, $A_1 \leq^- B_1$ and $W_2A_2W_2 \leq^- W_2B_2W_2$. In addition, $A_1, B_1 \in \mathbb{C}^{t\times t}$, A_1 is nonsingular, and $t = r(A_1) \leq r(B_1)$ imply $A_1 = B_1$. Hence, (II) holds.

(II) $\Longrightarrow$ (V) Suppose that $A = P(A_1 \oplus A_2)Q^{-1}$, $W = Q(W_1 \oplus W_2)P^{-1}$, and $B = P \begin{bmatrix} A_1 & B_3 \\ B_4 & B_2 \end{bmatrix} Q^{-1}$, with $W_2B_4 = O$, $B_3W_2 = O$, and $W_2A_2W_2 \leq^- W_2B_2W_2$. Then, there exists $X_2 \in W_2A_2W_2\{1\}$ such that $W_2A_2W_2X_2 = W_2B_2W_2X_2$ and $X_2W_2A_2W_2 = X_2W_2B_2W_2$. Moreover, by Theorem 2.1 the matrix $A^{W-GD} = P((W_1A_1W_1)^{-1} \oplus X_2)Q^{-1} \in A\{W - GD\}$.

By making some computations, it is easy to prove that $WAWA^{W-GD}WBW = WAW$ and $WBWA^{W-GD}WAW = WAW$ hold. Hence, (V.a) is shown.

On the other hand, let $X \in A\{W - GD\}$. By Theorem 2.1,

$$X = P \begin{bmatrix} (W_1A_1W_1)^{-1} & X_{12} \\ X_{21} & X_2 \end{bmatrix} Q^{-1},$$

with $X_{12}W_2 = O$, $W_2X_{21} = O$, and $X_2 \in W_2A_2W_2\{1\}$.

Now, it is easy to see that

$$WXW(AW)^kBW = Q(W_1(A_1W_1)^k \oplus O)P^{-1}$$

and

$$WB(WA)^kWXW = Q(W_1(A_1W_1)^k \oplus O)P^{-1}.$$

Since $W_1(A_1W_1)^k = (W_1A_1)^kW_1$ for every integer $k \geq 0$. This shows (V.b).

(V) $\Longrightarrow$ (II) Set $X \in A\{W-GD\}$ such that the equalities $WAWXWBW = WAW = WBWXWAW$ hold. If A and W are written as in (1) then by Theorem 2.1

$$X = P \begin{bmatrix} (W_1A_1W_1)^{-1} & X_{12} \\ X_{21} & X_2 \end{bmatrix} Q^{-1},$$

for some X_{12}, X_{21}, and X_2 such that $X_{12}W_2 = O$, $W_2X_{21} = O$, and $X_2 \in W_2A_2W_2\{1\}$. Let $B = P \begin{bmatrix} B_1 & B_3 \\ B_4 & B_2 \end{bmatrix} Q^{-1}$. Some calculations give

$$WXW(AW)^kBW = Q \begin{bmatrix} A_1^{-1}(A_1W_1)^kB_1W_1 & A_1^{-1}(A_1W_1)^kB_3W_2 \\ O & O \end{bmatrix} P^{-1}$$

and

$$WB(WA)^kWXW = Q \begin{bmatrix} W_1B_1(W_1A_1)^kA_1^{-1} & O \\ W_2B_4(W_1A_1)^kA_1^{-1} & O \end{bmatrix} P^{-1}.$$

Since $WXW(AW)^kBW = WB(WA)^kWXW$, we obtain $B_3W_2 = O$ and $W_2B_4 = O$.

On the other hand, $WAWXWBW = Q(W_1B_1W_1 \oplus W_2A_2W_2X_2W_2B_2W_2)P^{-1}$ and $WAW = Q(W_1A_1W_1 \oplus W_2A_2W_2)P^{-1}$. Since $WAWXWBW = WAW$ we conclude $B_1 = A_1$ and (i1) $W_2A_2W_2X_2W_2B_2W_2 = W_2A_2W_2$.

Analogously, by using the equality $WBWXWAW = WAW$ and by making some calculations, we get (i2) $W_2B_2W_2X_2W_2A_2W_2 = W_2A_2W_2$. Therefore, from (i1), (i2), and the fact $X_2 \in W_2A_2W_2\{1\}$ we conclude $W_2A_2W_2 \leq^- W_2B_2W_2$ (see [4, Theorem 2.1]). Hence, (II) is satisfied.

(IV) $\Longrightarrow$ (VI) It is immediate from $W(AW)^{k+1} = (WA)^{k+1}W$.

(VI) $\Longrightarrow$ (II) Suppose that A and W are written as in (1). Let $B = P \begin{bmatrix} B_1 & B_3 \\ B_4 & B_2 \end{bmatrix} Q^{-1}$.

It is easy to see that

$$W(AW)^kBW = Q \begin{bmatrix} W_1(A_1W_1)^kB_1W_1 & W_1(A_1W_1)^kB_3W_2 \\ O & O \end{bmatrix} P^{-1}$$

and

$$WB(WA)^kW = Q \begin{bmatrix} W_1B_1(W_1A_1)^{k+1}W_1 & O \\ W_2B_4(W_1A_1)^kW_1 & O \end{bmatrix} P^{-1}.$$

Since $W(AW)^kBW = WB(WA)^kW$, by equating we obtain $B_3W_2 = O$ and $W_2B_4 = O$.

Since $WAW \leq^{-} WBW$, following a similar reasoning as in last part of (III) $\Longrightarrow$ (II), we obtain $A_1 = B_1$ and $W_2A_2W_2 \leq^{-} W_2B_2W_2$.

(II) $\Longrightarrow$ (VII) Suppose that $A = P(A_1 \oplus A_2)Q^{-1}$, $W = Q(W_1 \oplus W_2)P^{-1}$, and $B = P\begin{bmatrix} A_1 & B_3 \\ B_4 & B_2 \end{bmatrix}Q^{-1}$, with $W_2B_4 = O$, $B_3W_2 = O$, and $W_2A_2W_2 \leq^{-} W_2B_2W_2$. Then, in a similar way to the first part of the proof of (II) $\Longrightarrow$ (IV), we obtain $WAW \leq^{-} WBW$. Moreover, it is easy to check that

$$WBWA^{D,W}W = WAWA^{D,W}W = WA^{D,W}WBW = Q(W_1 \oplus O)P^{-1}.$$

(VII) $\Longrightarrow$ (II) Suppose that A and W are written as in (1). Let $B = P\begin{bmatrix} B_1 & B_3 \\ B_4 & B_2 \end{bmatrix}Q^{-1}$.

It is easy to see that

$$WA^{D,W}WBW = Q\begin{bmatrix} A_1^{-1}B_1W_1 & A_1^{-1}B_3W_2 \\ O & O \end{bmatrix}P^{-1}$$

and

$$WBWA^{D,W}W = Q\begin{bmatrix} W_1B_1A_1^{-1} & O \\ W_2B_4A_1^{-1} & O \end{bmatrix}P^{-1}.$$

By equating we have $B_3W_2 = O$ and $W_2B_4 = O$. The rest of the proof continues as in the case (III) $\Longrightarrow$ (II) by using the hypothesis $WAW \leq^{-} WBW$.

(VII) $\Leftrightarrow$ (VIII) It is immediate from the equalities $A^{D,W}W = (AW)^D$ and $WA^{D,W} = (WA)^D$. ■

Corollary 3.3 *Let $A, B \in \mathbb{C}^{n\times n}$ and $k = ind(A)$. Then, the following conditions are equivalent:*

(I) $A \preceq^{GD} B$.

(II) (a) There exists $A^{GD} \in A\{GD\}$ such that $AA^{GD}B = BA^{GD}A = A$.

(b) For every $X \in A\{GD\}$, $XA^kB = BA^kX$ holds.

Note that the relations $\preceq_W^{GD}$ and $\preceq^{D,W}$ are not related to each other, in the sense that none of them implies the other one, as the following example shows.

Example 3.2 *Let us consider the matrices A, B, and W given by*

$$A = \left[\begin{array}{cc|cc} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right], \quad W = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right], \quad B = \left[\begin{array}{cc|cc} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 0 & 5 \end{array}\right].$$

It is easy to see that $ind(AW) = 1$, $ind(WA) = 2$, and $A \preceq_W^{GD} B$ but $A \not\preceq^{D,W} B$.

Consider now the matrices $\widetilde{A}$, $\widetilde{B}$, and $\widetilde{W}$ given by

$$\widetilde{A} = \left[\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \quad \widetilde{W} = \left[\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right], \quad \widetilde{B} = \left[\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

It is easy to see that $ind(\widetilde{A}\widetilde{W}) = 2$, $ind(\widetilde{W}\widetilde{A}) = 2$, and $\widetilde{A} \preceq^{D,\widetilde{W}} \widetilde{B}$ but $\widetilde{A} \not\preceq_{\widetilde{W}}^{GD} \widetilde{B}$.

The following results relate the pre-order $\preceq_W^{GD}$ to the minus partial order and the weighted Drazin inverse [8].

Lemma 3.2 *Let $W \in \mathbb{C}^{n\times m}$ and $A, B \in \mathbb{C}^{m\times n}$. If $A \preceq^{D,W} B$ and $WAW \leq^- WBW$ then $A \preceq_W^{GD} B$.*

Proof. From $A \preceq^{D,W} B$, by [15, Theorem 2.3 (c)] we have $(AW)(AW)^D = (BW)(AW)^D$ and $(WA)^D(WA) = (WA)^D(WB)$. By multiplying adequately by W we obtain

$$WAW(AW)^D = WBW(AW)^D \qquad \text{and} \qquad (WA)^D WAW = (WA)^D WBW.$$

Since $(AW)^D = A^{D,W}W$ and $(WA)^D = WA^{D,W}$, we get the equality of item (VII) in Theorem 3.4. By Theorem 3.4 we arrive at $A \preceq_W^{GD} B$. ■

Example 3.2 shows that the converse in the previous lemma does not hold and that the hypothesis $WAW \leq^- WBW$ can not be dropped.

Theorem 3.5 *Let $W \in \mathbb{C}^{n\times m}$ and $A, B \in \mathbb{C}^{m\times n}$. If $A \preceq_W^{GD} B$ then $WA^{D,W}W \leq^- WB^{D,W}W$.*

Proof. Suppose that A and W are written as in (1). By Theorem 3.2,

$$B = P \begin{bmatrix} A_1 & B_3 \\ B_4 & B_2 \end{bmatrix} Q^{-1},$$

with $W_2B_4 = O$, $B_3W_2 = O$, and $W_2A_2W_2 \leq^- W_2B_2W_2$. Then, there exists $X_2 \in W_2A_2W_2\{1\}$ such that $W_2A_2W_2X_2 = W_2B_2W_2X_2$ and $X_2W_2A_2W_2 = X_2W_2B_2W_2$. It then follows that

$$(AW)^D = P \begin{bmatrix} (A_1W_1)^{-1} & O \\ O & O \end{bmatrix} P^{-1} \quad \text{and} \quad BW = P \begin{bmatrix} A_1W_1 & O \\ B_4W_1 & B_2W_2 \end{bmatrix} P^{-1}.$$

By [6, Corollary 7.7.1], we have $(BW)^D = P \begin{bmatrix} (A_1W_1)^{-1} & O \\ L & (B_2W_2)^D \end{bmatrix} P^{-1}$, where $L = \left(I - (B_2W_2)(B_2W_2)^D\right)\left(\sum\limits_{i=0}^{m}(B_2W_2)^iB_4W_1((A_1W_1)^{-1})^i\right)((A_1W_1)^{-1})^2-$ $(B_2W_2)^DB_4W_1(A_1W_1)^{-1}$.

From $W_2B_4 = O$ we obtain $\sum\limits_{i=0}^{m}(B_2W_2)^iB_4W_1((A_1W_1)^{-1})^i = B_4W_1$ and then $L = \left(I - (B_2W_2)(B_2W_2)^D\right)B_4W_1(A_1W_1)^{-2} - (B_2W_2)^DB_4W_1(A_1W_1)^{-1}$.

By applying the distributive property, by using that a matrix commutes with its Drazin inverse, and the fact that $W_2B_4 = O$ we obtain

$$L = B_4W_1(A_1W_1)^{-2} - (B_2W_2)^DB_4W_1(A_1W_1)^{-1} = B_4W_1(A_1W_1)^{-2}.$$

The last equality follows from the fact that $(B_2W_2)^D$ is a polynomial in B_2W_2 without constant term [6, Theorem 7.5.1].

Thus, $W_2L = O$. Moreover,

$$W(AW)^D = Q \begin{bmatrix} A_1^{-1} & O \\ O & O \end{bmatrix} P^{-1} \quad \text{and} \quad W(BW)^D = Q \begin{bmatrix} A_1^{-1} & O \\ O & W_2(B_2W_2)^D \end{bmatrix} P^{-1}.$$

Hence,

$$W(BW)^D - W(AW)^D = Q \begin{bmatrix} O & O \\ O & W_2(B_2W_2)^D \end{bmatrix} P^{-1}.$$

It is immediate that $r(W(BW)^D - W(AW)^D) = r(W_2(B_2W_2)^D) = r(W(BW)^D) - r(W(AW)^D)$ since A_1^{-1} is nonsingular. Hence, $W(AW)^D \leq^- W(BW)^D$ and this is equivalent to $WA^{D,W}W \leq^- WB^{D,W}W$ due to the equality $(AW)^D = A^{D,W}W$. ■

It is well known that, for instance, for the minus partial order we have that $A \leq^- B$ if and only if $B - A \leq^- B$. Our last results analyze this property with respect to the pre-order $\preceq_W^{GD}$.

Proposition 3.1 *Let $W \in \mathbb{C}^{n\times m}$ and $A, B \in \mathbb{C}^{m\times n}$. If A and W are written as in* (1)*, $A \preceq_W^{GD} B$, and B is in the form of Theorem 3.2, then the following conditions are equivalent:*

(a) $B - A \preceq_W^{GD} B$,

(b) $B_2 - A_2 \preceq_{W_2}^{GD} B_2$.

Proof. By hypothesis and Theorem 3.4 we have $WAW \leq^- WBW$ and $W_2A_2W_2 \leq^- W_2B_2W_2$. On the other hand, it is known that these inequalities are equivalent to $W(B-A)W \leq^- WBW$ and $W_2(B_2-A_2)W_2 \leq^- W_2B_2W_2$ (see [4]). By taking into account item (VI) in Theorem 3.4, it is enough to prove $W((B-A)W)^kBW = WB(W(B-A))^kW$ if and only if $W_2((B_2 - A_2)W_2)^kB_2W_2 = W_2B_2(W_2(B_2 - A_2))^kW_2$, which is easy to see by making some calculations. ■

A similar result can be deduced for the G-Drazin partial order on square matrices.

Let $A, B \in \mathbb{C}^{n\times n}$ such that $A \preceq^{GD} B$. Recall that there exists a nonsingular matrix $P \in \mathbb{C}^{n\times n}$ such that

$$A = P(C \oplus N)P^{-1} \qquad \text{and} \qquad B = P(C \oplus T)P^{-1}, \tag{5}$$

where C is nonsingular of size $a \times a$, N is nilpotent of index $k = ind(A)$, and $N \leq^- T$ (see [25, Theorem 3.1]).

Corollary 3.4 *Let $A, B \in \mathbb{C}^{n\times n}$ such that $A \preceq^{GD} B$. If A and B can be written as in* (5) *then the following conditions are equivalent:*

(a) $B - A \preceq^{GD} B$,

(b) $T - N \preceq^{GD} T$.

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