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Additional Information

Semilocal convergence of double step Secant method under weaker Lipschitz conditions in Banach spaces

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Abstract The semilocal convergence of double step Secant method to approximate a locally unique solution of a nonlinear equation is described in Banach space setting. Majorizing sequences are used under the assumption that the first-order divided differences of the involved operator satisfies the weaker Lipschitz and the center-Lipschitz continuity conditions. A theorem is established for the existence-uniqueness region along with the estimation of error bounds for the solution. Our work improves the results derived in [1] in more stringent Lipschitz and center Lipschitz conditions and gives finer majorizing sequences. Also, an example is worked out where the conditions of [1] fail but our works. Numerical examples including nonlinear elliptic differential equations and integral equations are worked out. It is found that our conditions enlarge the convergence domain of the solution. Finally, taking a nonlinear system of m equations, the Efficiency Index (EI) and the Computational Efficiency Index (CEI) of double step Secant method are computed and its comparison with respect to other similar existing iterative methods are summarized in the tabular forms.

Keywords Semilocal convergence; Double step Secant method; Divided differences; Majorizing sequences; Error bounds; Efficiency index; Computational efficiency

Mathematics Subject Classification (2000) 47H17 · 65J15

1 Introduction

Consider approximating a locally unique solution ρ^* of

$$H(x) = 0. \tag{1}$$

where, $H : \mathcal{D} \subseteq \mathcal{X} \rightarrow \mathcal{Y}$ is a Fréchet differentiable nonlinear operator. \mathcal{X}, \mathcal{Y} are Banach spaces and \mathcal{D} be an open nonempty convex subset of \mathcal{X} . This is one of the most challenging problems in applied mathematics and engineering sciences. Many real life applications in diverse areas such as equilibrium theory, optimization, elasticity, etc. often reduce to solving these equations involving several parameters. Practical problems when formulated mathematically often use integral equations, boundary value problems and differential equations, etc. and require solving scalar equations or system of equations for their solutions. The solutions of discrete dynamical systems also require solving them in order to represent the equilibrium states of these systems. They have

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gained an added advantage due to the existence of high-speed computational devices which solve them faster and with more accuracy. Many researchers [13–15] have proposed many direct and iterative methods and their convergence analysis for (1). Good convergence properties, efficiency and numerical stability are the generally used criteria to select a suitable method. When their convergence domains are small, the additional hypotheses are used to enlarge them. Generally, local, semilocal and global convergence analysis using either majorizing sequences or recurrence relations are established for (1). The local and the semilocal convergence use information given at the solution and at the initial point, respectively. Likewise, the error bounds to ρ^* are also estimated. The quadratically convergent Newton's method [7] used to solve (1) is given by

$$x_{n+1} = x_n - \Gamma_n H(x_n), \quad n = 0, 1, 2, \dots \quad (2)$$

where, $x_0 \in \mathcal{D}$ and $\Gamma_n = H'(x_n)^{-1} \in L(\mathcal{Y}, \mathcal{X})$. Here, $L(\mathcal{Y}, \mathcal{X})$ denotes the set of bounded linear operators from \mathcal{Y} into \mathcal{X} . Sufficient conditions for the semilocal convergence with existence ball and estimation of error bounds for ρ^* are given in [15]. In [8], a family representing third order iterative methods for (1) is given by

$$x_{\alpha, n+1} = x_{\alpha, n} - \left[I + \frac{1}{2} G_H(x_{\alpha, n}) [I - \alpha G_H(x_{\alpha, n})]^{-1} \right] H'(x_{\alpha, n})^{-1} H(x_{\alpha, n}) \quad n = 0, 1, 2, \dots \quad (3)$$

where, $x_{\alpha, 0}$ is the starting iterate and $G_H(x) = H'(x)^{-1} H''(x) H'(x)^{-1} H(x)$. This family contains the Chebyshev method ($\alpha = 0$), the Halley method ($\alpha = 1/2$) and the Super-Halley method ($\alpha = 1$), respectively.

Recently, importance of iterative methods of higher orders are also realized as there exists many applications which require quick convergence, for example applications involving stiff system of equations. Moreover, they are also of theoretical interest in establishing the existence and uniqueness for solutions by lower order iterative methods. As a result, a number of papers are written for higher order iterative methods for solving (1) and establishing their convergence analysis. In addition to these single step iterations, a number of multi step iterations and their convergence analysis are also developed for solving (1). A double step iteration known as the King-Werner iteration of order $1 + \sqrt{2}$ for (1) is studied in [9, 10, 12]. Its semilocal convergence analysis is discussed using majorizing sequences under the Lipschitz continuous Fréchet derivative of H . It is given by

$$\left. \begin{aligned} x_n &= x_{n-1} - H' \left(\frac{x_{n-1} + y_{n-1}}{2} \right)^{-1} H(x_{n-1}), \\ y_n &= x_n - H' \left(\frac{x_{n-1} + y_{n-1}}{2} \right)^{-1} H(x_n), \quad n = 0, 1, 2, \dots \end{aligned} \right\} \quad (4)$$

where, $x_0, y_0 \in \mathcal{D}$ are the starting iterates. Another double step Secant iteration for solving (1) in Banach space setting is described in [1]. It is given by

$$\left. \begin{aligned} x_{n+1} &= x_n - A_n^{-1} H(x_n), \\ y_{n+1} &= x_{n+1} - A_n^{-1} H(x_{n+1}), \quad n = 0, 1, 2, \dots \end{aligned} \right\} \quad (5)$$

where, $x_0, y_0 \in \mathcal{D}$ and $A_n = [x_n, y_n; H]$. The first order divided difference $[x, y; H] : \mathcal{D} \times \mathcal{D} \rightarrow L(\mathcal{X}, \mathcal{Y})$ of H satisfies $[x, y; H](x - y) = H(x) - H(y)$ for each $x, y \in \mathcal{D}, x \neq y$. If H is Fréchet differentiable then $H'(x) = [x, x; H]$ for each $x \in \mathcal{D}$.

The aim of this paper is to establish the semilocal convergence of (5) to approximate a locally unique solution of a nonlinear equation in Banach space setting. Majorizing sequences are used under the assumption that the first-order divided differences of the involved operator satisfies the weaker Lipschitz and the center-Lipschitz continuity conditions. A theorem is established for the existence-uniqueness region along with the estimation of error bounds for the solution. Our work

improves the results derived in [1] in more stringent Lipschitz and center Lipschitz conditions and gives finer majorizing sequences. Also, an example is worked out where the conditions of [1] fail but our works. Numerical examples including nonlinear elliptic differential equations and integral equations are worked out. It is found that our conditions enlarge the convergence domain of the solution. Finally, taking a nonlinear system of m equations, the Efficiency Index (EI) and the Computational Efficiency Index (CEI) of (5) are computed and its comparison with respect to other similar existing iterative methods are summarized in the tabular forms.

The paper is arranged as follows. Introduction forms the Section 1. In Section 2, the semilocal convergence analysis of (5) under weaker Lipschitz and center Lipschitz continuity condition on divided differences of order one of the involved operator is established. The existence region, uniqueness region and error bounds for the solution are found. In Section 3, taking a nonlinear system of m equations, the Efficiency Index (EI) and the Computational Efficiency Index (CEI) of (5) are computed and their comparison with respect to other similar existing iterative methods are summarized in the tabular forms. In Section 4, numerical examples including nonlinear elliptic differential equations and integral equations are given to show the suitability of our approach. Finally, conclusions are included in Section 5.

2 Semilocal convergence of double step Secant method

In this section, we shall give the semilocal convergence analysis of double step Secant method (5). Let $k > 0, k_0 > 0, k_1 > 0, k_2 \geq 0, \eta \geq 0$ and $s \geq 0$ be non negative parameters. The triplet (H, x_0, y_0) belongs to the class $\mathcal{C}(k, k_0, k_1, k_2, \eta, s)$ if

1. $\|x_0 - y_0\| \leq s$ for $x_0, y_0 \in \mathcal{D}$.
2. $A_0^{-1} \in L(\mathcal{Y}, \mathcal{X})$.
3. $\|A_0^{-1}H(x_0)\| \leq \eta$.
4. $\|A_0^{-1}([x, y; H] - [x_0, y_0; H])\| \leq k_0\|x - x_0\| + k\|y - y_0\|$
5. $\|A_0^{-1}([x, y; H] - [u, v; H])\| \leq k_1\|x - u\| + k_2\|y - v\|$,

the last two conditions represent the weaker center Lipschitz and Lipschitz continuity conditions $\forall x, y, u, v \in \mathcal{D}$. Let $\mathcal{B}(x_0, R)$ and $\bar{\mathcal{B}}(x_0, R)$ represent the open and closed balls with center x_0 and radius R , respectively. Define the sequences $\{l_n\}$ and $\{r_n\}$ by

$$\left. \begin{aligned} l_0 &= 0, r_0 = s, \\ l_1 &= \eta, r_1 = l_1(1 + k_0l_1 + kr_0), \\ l_2 &= l_1 \left(1 + \frac{k_0l_1 + kr_0}{1 - (k_0l_1 + k(r_1 + r_0))} \right), \end{aligned} \right\} \quad (6)$$

and for $n = 1, 2, 3, \dots$

$$r_{n+1} = l_{n+1} + \frac{k_1(l_{n+1} - l_n) + k_2(r_n - l_n)}{1 - (k_0l_n + k(r_n + r_0))}(l_{n+1} - l_n), \quad (7)$$

$$l_{n+2} = l_{n+1} + \frac{k_1(l_{n+1} - l_n) + k_2(r_n - l_n)}{1 - (k_0l_{n+1} + k(r_{n+1} + r_0))}(l_{n+1} - l_n). \quad (8)$$

Lemma 1 Let ξ be the unique root of the polynomial $g(t)$ defined by

$$g(t) = kt^3 + k_0t^2 + (k_1 + k_2)(t - 1) \quad (9)$$

If

$$\frac{k_0(l_1 - l_0) + kr_0}{1 - (k_0(l_1 - l_0) + k(r_1 + r_0))} \leq \xi \quad (10)$$

and

$$0 < \frac{k_0 l_1}{1 - \xi} + \frac{k l_1}{1 - \xi} + k r_0 < 1, \quad (11)$$

then the sequences $\{r_n\}$, $\{l_n\}$ are increasing, bounded above by $l^{**} = \frac{l_1}{1 - \xi}$ and converges to the least upper bound l^* such that $l_1 \leq l^* \leq l^{**}$. Also, for $n \geq 1$, we get

$$\left. \begin{aligned} 0 &\leq r_{n+1} - l_{n+1} \leq \xi(l_{n+1} - l_n) \\ 0 &\leq l_{n+2} - l_{n+1} \leq \xi(l_{n+1} - l_n) \\ l_n &\leq r_n \end{aligned} \right\} \quad (12)$$

Proof: From (9), we get $g(0) = -(k_1 + k_2)$ and $g(1) = (k + k_0)$. This implies that g has a root ξ in $(0, 1)$. We shall prove (11) by mathematical induction. For $l_1 = 0$, $l_n = r_n = 0$ follows from (6),(7) and (8) and (12) holds for each $n = 1, 2, \dots$. For other values of $l_1 = \eta > 0$, (12) holds for each n , if

$$0 < \frac{k_1(l_{n+1} - l_n) + k_2(r_n - l_n)}{1 - (k_0 l_n + k(r_n + r_0))} \leq \xi \quad (13)$$

$$0 < \frac{k_1(l_{n+1} - l_n) + k_2(r_n - l_n)}{1 - (k_0 l_{n+1} + k(r_{n+1} + r_0))} \leq \xi \quad (14)$$

$$l_n \leq r_n \quad (15)$$

which follows from (6), (7) and (8). This gives $0 \leq r_n - l_n \leq \xi^n(l_1 - l_0)$ and $0 \leq l_{n+1} - l_n \leq \xi^n(l_1 - l_0)$. Now, using the definition of sequence $\{l_n\}$ and $\{r_n\}$, we can write

$$\begin{aligned} r_n &\leq l_n + \xi^n(l_1 - l_0) \leq l_{n-1} + \xi^{n-1}(l_1 - l_0) + \xi^n(l_1 - l_0) \\ &\leq l_1 + \xi(l_1 - l_0) + \dots + \xi^n(l_1 - l_0) = \frac{1 - \xi^{n+1}}{1 - \xi}(l_1 - l_0) < l^{**} \end{aligned} \quad (16)$$

and similarly

$$l_{n+1} \leq \frac{1 - \xi^{n+1}}{1 - \xi}(l_1 - l_0) < l^{**}. \quad (17)$$

Thus, we get

$$\frac{1}{1 - (k_0 l_n + k(r_n + r_0))} < \frac{1}{1 - (k_0 l_{n+1} + k(r_{n+1} + r_0))}$$

Hence, to show (13), (14) and (15), it can only be shown that

$$\frac{k_1(l_{n+1} - l_n) + k_2(r_n - l_n)}{1 - (k_0 l_{n+1} + k(r_{n+1} + r_0))} \leq \xi. \quad (18)$$

To show (18), using (16) and (17), it is sufficient to show that

$$l_1 \left((k_1 + k_2)\xi^{n-1} + k_0 \frac{1 - \xi^{n+1}}{1 - \xi} + k \frac{1 - \xi^{n+2}}{1 - \xi} \right) + k r_0 - 1 \leq 0$$

This motivates us to define the recurrent function $f_n(t)$ on $(0, 1)$ given by

$$f_n(t) = l_1 \left((k_1 + k_2)t^{n-1} + k_0 \frac{1 - t^{n+1}}{1 - t} + k \frac{1 - t^{n+2}}{1 - t} \right) + k r_0 - 1 \quad (19)$$

We need a relationship between two consecutive functions f_n and f_{n+1} . Replacing n by $n + 1$ in (19), we get

$$f_{n+1}(t) = f_n(t) + g(t)t^{n-1}l_1 \quad (20)$$

where, $g(t)$ is defined in (9). This gives $f_n(\xi) \leq 0, \forall, n \geq 0$. Define f_∞ on $(0, 1)$ by

$$f_\infty(t) = \lim_{n \rightarrow \infty} f_n(t) = \frac{k_0 l_1}{1-t} + \frac{k l_1}{1-t} + k r_0 - 1 \quad (21)$$

We also have by the definition of ξ , (20) and (21) that

$$f_\infty(\xi) = f_{n+1}(\xi) = f_n(\xi) \quad \text{for each } n$$

Using (11), it gives $f_\infty(\xi) \leq 0$. Thus, the sequence $\{l_n\}$ is increasing, bounded above by l^{**} and as such it converges to the least upper bound l^* . Thus, the Lemma 1 is proved.

Theorem 1 *Let $H : \mathcal{D} \subseteq \mathcal{X} \rightarrow \mathcal{Y}$ belongs to the class \mathcal{C} and assume that the Lemma 1 holds. Starting with suitably chosen $x_0, y_0 \in \mathcal{D}$, the sequences $\{x_n\}$ and $\{y_n\}$ generated by (5) are well defined, belongs and converges to a unique solution ρ^* of (1) in $\overline{\mathcal{B}}(x_0, l^*)$. Also, for $n = 0, 1, \dots$, we have*

$$\left. \begin{aligned} \|y_n - x_n\| &\leq r_n - l_n \\ \|x_{n+1} - x_n\| &\leq l_{n+1} - l_n \end{aligned} \right\} \quad (22)$$

and

$$\|\rho^* - x_n\| \leq l^* - l_n. \quad (23)$$

Further, if there exists $R > l^*$ such that $\mathcal{B}(x_0, R) \subseteq \mathcal{D}$ and $k_0 l^* + k(R + r_0) < 1$, then ρ^* is the unique solution of (1) in $\mathcal{B}(x_0, R) \cap \mathcal{D}$.

Proof: Theorem 1 can be proved by mathematical induction. For $n = 0$, (22) directly follows from the definition of \mathcal{C} and $y_0, x_1 \in \overline{\mathcal{B}}(x_0, l^*)$. For $n = 1$, using weaker center Lipschitz continuity condition and (5), we get

$$\begin{aligned} \|y_1 - x_1\| &= \|A_0^{-1} H(x_1)\| \\ &= \|A_0^{-1}([x_1, x_0; H](x_1 - x_0) - [x_0, y_0; H](x_1 - x_0))\| \\ &\leq (k_0 \|x_1 - x_0\| + k \|x_0 - y_0\|) \|x_1 - x_0\| \\ &\leq (k_0 l_1 + k s) l_1 = r_1 - l_1 \end{aligned}$$

Also,

$$\begin{aligned} \|I - A_0^{-1} A_1\| &= \|A_0^{-1} (A_0 - A_1)\| \\ &\leq (k_0 \|x_1 - x_0\| + k \|y_1 - y_0\|) \\ &\leq (k_0 \|x_1 - x_0\| + k (\|y_1 - x_1\| + \|x_1 - x_0\| + \|x_0 - y_0\|)) \\ &\leq (k_0 l_1 + k(r_1 - l_1 + l_1 - l_0 + r_0 - l_0)) = (k_0 l_1 + k(r_1 + r_0)) \end{aligned}$$

Using Banach Lemma, this gives

$$\|A_1^{-1} A_0\| \leq \frac{1}{1 - (k_0 l_1 + k(r_1 + r_0))}$$

and

$$\begin{aligned} \|x_2 - x_1\| &\leq \|A_1^{-1} A_0\| \|A_0^{-1} (H(x_1) - H(x_0) + H(x_0))\| \\ &= \|A_1^{-1} A_0\| \|A_0^{-1} ([x_1, x_0; H](x_1 - x_0) - [x_0, y_0; H](x_1 - x_0))\| \\ &\leq \frac{1}{1 - (k_0 l_1 + k(r_1 + r_0))} (k_0 l_1 + k r_0) l_1 = l_2 - l_1 \end{aligned}$$

Following in the similar manner, we get for $n \geq 2$,

$$\|A_n^{-1}A_0\| \leq \frac{1}{1 - (k_0l_n + k(r_n + r_0))}$$

This gives

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|A_n^{-1}H(x_n)\| \\ &\leq \|A_n^{-1}A_0\| \|A_0^{-1}(H(x_n) - H(x_{n-1}) + H(x_{n-1}))\| \\ &\leq \|A_n^{-1}A_0\| \|A_0^{-1}([x_n, x_{n-1}; H](x_n - x_{n-1}) - [x_{n-1}, y_{n-1}; H](x_n - x_{n-1}))\| \\ &\leq \frac{k_1(l_n - l_{n-1}) + k_2(r_{n-1} - l_{n-1})}{1 - (k_0l_n + k(r_n + r_0))} (l_n - l_{n-1}) = l_{n+1} - l_n \end{aligned}$$

and

$$\begin{aligned} \|y_{n+1} - x_{n+1}\| &= \|A_n^{-1}H(x_{n+1})\| \\ &\leq \|A_n^{-1}A_0\| \|A_0^{-1}(H(x_{n+1}) - H(x_n) + H(x_n))\| \\ &= \|A_n^{-1}A_0\| \|A_0^{-1}([x_{n+1}, x_n; H](x_{n+1} - x_n) - [x_n, y_n; H](x_{n+1} - x_n))\| \\ &\leq \frac{k_1(l_{n+1} - l_n) + k_2(r_n - l_n)}{1 - (k_0l_n + k(r_n + r_0))} (l_{n+1} - l_n) = r_{n+1} - l_{n+1}. \end{aligned}$$

Now,

$$\|x_{n+1} - x_0\| \leq \|x_{n+1} - x_n\| + \dots + \|x_1 - x_0\| \leq l_{n+1} - l_n + \dots + l_1 - l_0 = l_{n+1} < l^*$$

and

$$\|y_{n+1} - x_0\| \leq \|y_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| + \dots + \|x_1 - x_0\| \leq r_{n+1} - l_{n+1} + \dots + l_1 - l_0 = r_{n+1} < l^*$$

This shows that $x_{n+1}, y_{n+1} \in \overline{\mathcal{B}}(x_0, l^*)$. It remains to show that ρ^* is a solution of (1). Since,

$$\|A_0^{-1}H(x_{n+1})\| \leq (k_1(l_{n+1} - l_n) + k_2(r_n - l_n)) (l_{n+1} - l_n) \quad (24)$$

This implies $H(\rho^*) = 0$ on taking limit as $n \rightarrow \infty$ in (24). Now to show uniqueness of the solution, suppose y^* be another solution of (1) such that $H(y^*) = 0$. For $T = [\rho^*, y^*; H]$, we get

$$\|I - A_0^{-1}T\| \leq k_0\|\rho^* - x_0\| + k(\|y^* - x_0\| + \|x_0 - y_0\|) \leq k_0l^* + kR + kr_0 < 1$$

From Banach Lemma, this implies that T is invertible and we get $\rho^* = y^*$.

3 Analysis of computational efficiency of (5)

In this section, the analysis of computational efficiency of (5) is carried out. This requires computations of efficiency index (EI) and computational efficiency index (CEI) of (5). The term Efficiency index (EI) introduced by Ostrowski [11] is defined as $EI = q^{\frac{1}{r}}$, where q and r denote the order of convergence of an iterative method and the total number of functions and their derivatives evaluations, respectively. The Computational efficiency index (CEI) of an iterative method introduced by Traub [16] is defined as $CEI = q^{\frac{1}{op}}$, where, q is the order of convergence of an iterative method and op denotes the number of products and divisions per iteration. In the

case of \mathcal{R}^m , the operator H of (5) requires m functions evaluations, it's derivative H' requires m^2 functions evaluations. The divided difference operator is given for $1 \leq i, j \leq m$ by

$$[u, v; H]_{ij} = \frac{1}{u_j - v_j} (H_i(u_1, \dots, u_{j-1}, u_j, v_{j+1}, \dots, v_m) - H_i(u_1, \dots, u_{j-1}, v_j, v_{j+1}, \dots, v_m)),$$

It requires $m^2 - m$ functions evaluations and m^2 divisions. To solve a system of m linear equations in m unknowns by LU decomposition method, we require $m(m-1)(2m-1)/6$ products and $m(m-1)/2$ divisions and $m(m-1)$ products and m divisions for resolution of the two triangular systems. Many iterative methods using divided differences are available in literature. The Secant method [16] is given by

$$x_{n+1} = x_n - [x_n, x_{n-1}; H]^{-1} H(x_n), \quad n = 0, 1, 2, \dots$$

for x_{-1}, x_0 being the starting points. We denote it by ϕ_0 . The iterative method described in [4] is given by

$$\begin{aligned} y_n &= x_n - [x_{n-1}, x_n; H]^{-1} H(x_n), \\ x_{n+1} &= y_n - [x_{n-1}, x_n; H]^{-1} H(y_n), \quad n = 0, 1, 2, \dots \end{aligned} \quad (25)$$

for x_{-1}, x_0 being the starting points. We denote this method by ϕ_1 . Two other iterative methods described in [6] are given by

$$\begin{aligned} y_n &= x_n - [x_{n-1}, x_n; H]^{-1} H(x_n), \\ x_{n+1} &= y_n - [y_n, x_n; H]^{-1} H(y_n), \quad n = 0, 1, 2, \dots \end{aligned} \quad (26)$$

and

$$\begin{aligned} y_n &= x_n - [x_{n-1}, x_n; H]^{-1} H(x_n), \beta \neq 0. \\ z_n &= (1 - \beta)x_n + \beta y_n \\ x_{n+1} &= y_n - [z_n, x_n; H]^{-1} H(y_n), \quad n = 0, 1, 2, \dots \end{aligned} \quad (27)$$

for x_{-1}, x_0 being the starting points. We denote them by ϕ_2 and ϕ_3 , respectively. The following Table 1 summarize the no. of functions evaluations (r_i), operational costs (op_i) and order of convergence (q_i) of the iterative methods denoted by ϕ_0, ϕ_1, ϕ_2 and ϕ_3 . Denoting the (5) by

Table 1 No. of functions evaluations, operational costs and order of convergence

i	ϕ	r_i	op_i	q_i
0	ϕ_0	m^2	$\frac{1}{3}(m^3 + 6m^2 - m)$	$\frac{1+\sqrt{5}}{2}$
1	ϕ_1	$m^2 + m$	$\frac{1}{3}(m^3 + 9m^2 - m)$	2
2	ϕ_2	$2m^2$	$\frac{1}{3}(2m^3 + 12m^2 - 2m)$	$1 + \sqrt{2}$
3	ϕ_3	$2m^2 + m$	$\frac{1}{3}(2m^3 + 12m^2 + m)$	$1 + \sqrt{3}$

ϕ_4 , it is seen that it requires $m^2 + m$ functions evaluations, $\frac{1}{6}m(m-1)(2m-1)$ products and $\frac{1}{2}m(m-1)$ divisions for LU decomposition and $2m^2$ products and division for four triangular systems of resolutions. So the total computational costs comes out be $\frac{1}{3}(m^3 + 9m^2 - m)$ which is same as obtained for ϕ_1 . The comparison of EI and CEI of $\phi_i, i = 0, 1, 2, 3, 4$ are plotted in Fig. 1 and Fig. 2.

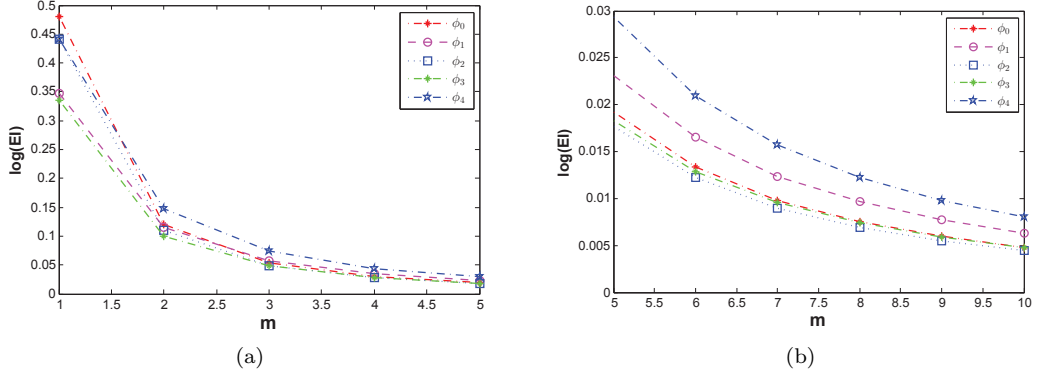


Fig. 1 (a) EI if $m = 1, 2, 3, 4, 5$ (b) EI if $m \geq 5$

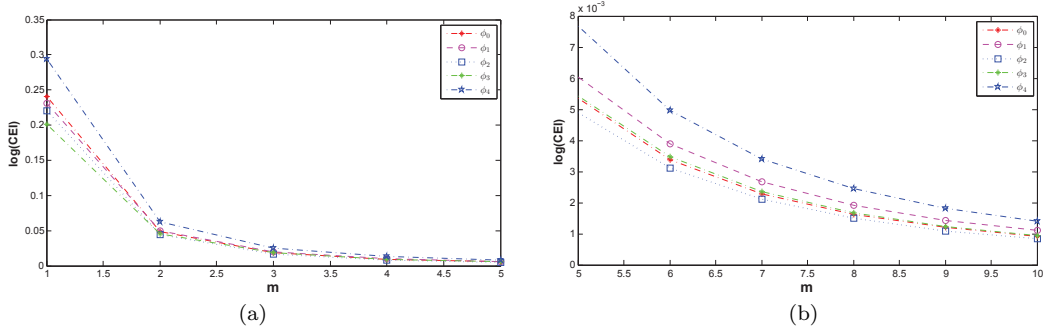


Fig. 2 (a) CEI if $m = 1, 2, 3, 4, 5$ (b) CEI if $m \geq 5$

4 Numerical Examples

Example 1 Let $\mathcal{X} = \mathcal{Y} = \mathcal{R}$ and consider H on $\mathcal{D} = \mathcal{B}(x_0, 1)$ given by

$$H(x) = x^3 - \theta, \quad \theta \in \mathcal{R}$$

Take $x_0 = 1$ and we let y_0 free such that all the assumptions of the Theorem 1 for convergence are satisfied. This can be obtained by finding a relation between θ and y_0 . From (5), we get $s = |1 - y_0|$, $\eta = \frac{|1 - \theta|}{1 + y_0 + y_0^2}$, $k_0 = 6$, $k = 3$, $k_1 = 8$, $k_2 = 4$. Taking horizontal axis for y_0 and vertical axis for θ , the following Fig.3 shows the convergence domains of our approach termed as Approach-1 and that given in [1] termed as Approach-2. It is clear that Approach-1 gives larger domain of convergence compared to that obtained by Approach-2. For comparison of the error bounds for both Approach-1 and Approach-2, we take $\mathcal{D} = \mathcal{B}(x_0, 0.3)$, $\theta = 0.75$ and $y_0 = 0.95$. This gives $s = 0.05$, $\eta = 0.0876$, $k = 0.8063$, $k_0 = 1.6126$, $k_1 = 1.8230$, $k_2 = 0.9115$. The following Table 2 summarizes the comparison of error bounds. It is clear that error bounds obtained by Approach-1 are better than those obtained by Approach-2.

Example 2 ([3]) Let $\mathcal{X} = \mathcal{Y} = \mathcal{C}[0, 1]$ be the space of continuous functions on $[0, 1]$ equipped with max-norm. Let $\mathcal{D} = \{x \in \mathcal{C}[0, 1]; \|x\| \leq R\}$, such that $R = 2$. Consider H on \mathcal{D} , given by

$$H(x)(t) = x(t) - f(t) - \frac{1}{8} \int_0^1 G_1(s, t) x(s)^3 ds, \quad x \in \mathcal{C}[0, 1] \text{ and } t \in [0, 1]$$

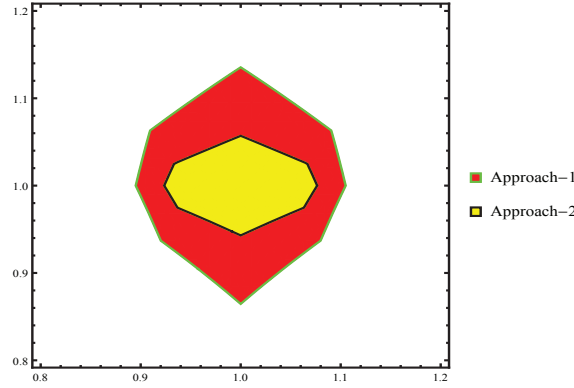


Fig. 3 Convergence domains by Approach-1 and Approach-2

Table 2 Comparison of error bounds for Approach-1 and Approach-2

n	$l_{n+1} - l_n$	$r_n - l_n$	$p_{n+1} - p_n$	$q_n - p_n$
0	0.0876	0.05	0.0876	0.05
1	$2.1486e - 02$	$1.5844e - 02$	$3.2355e - 02$	$2.1991e - 02$
2	$1.6513e - 03$	$1.5610e - 03$	$5.33154e - 03$	$5.3315e - 03$
3	$1.0540e - 05$	$1.0499e - 05$	$1.9788e - 04$	$1.9788e - 04$
4	$4.3684e - 10$	$4.3683e - 10$	$2.7729e - 07$	$2.7729e - 07$
5	0	0	$5.4484e - 13$	$5.4484e - 13$

where, $f \in C[0, 1]$ and the kernel G_1 given by

$$G_1(s, t) = \begin{cases} t(1-s) & \text{if } t \leq s, \\ s(1-t) & \text{if } s \leq t, \end{cases}$$

is the Green's function.

Now, $H'(x)$ is given by

$$(H'(x)) w(t) = w(t) - \frac{3}{8} \int_0^1 G_1(s, t) x(s)^2 w(s) ds, w \in C[0, 1] \text{ and } t \in [0, 1].$$

If $x_0(t) = f(t) = t$ and $y_0(t) = 2t$ then $\|H(x_0)\| \leq \frac{1}{64}$. It can be easily seen that for $y, z \in \mathcal{D}$, we get

$$\|H'(y) - H'(z)\| \leq \frac{3}{64} \|y^2 - z^2\|$$

and

$$\begin{aligned} \|[x, y; H] - [u, v; H]\| &\leq \frac{3}{64} \int_0^1 \|H'(y + \gamma(x - y)) - H'(v + \gamma(u - v))\| d\gamma \\ &\leq \frac{1}{64} (\|x^2 - u^2\| + \|y^2 - v^2\| + \|xy - uv\|) \end{aligned}$$

From this, we get $\eta = 0.0178571, k = 0.0714285, k_0 = 0.125, k_1 = 0.142857, k_2 = 0.0714285$ and $r_0 = 1$, which satisfies all the assumptions of Theorem 1 for convergence. The error bounds obtained from Approach-1 and Approach-2 are compared in Table 3,. It is clear that Approach-1 gives better error bounds than those obtained by Approach-2.

Table 3 Comparison of error bounds of Approach-1 and Approach-2

n	$l_{n+1} - l_n$	$r_n - l_n$	$p_{n+1} - p_n$ [1]	$q_n - p_n$ [1]
0	0.0178571	1	0.0178571	1
1	$1.4220e - 03$	$1.3154e - 03$	$2.6109e - 03$	$2.5966e - 03$
2	$4.5687e - 07$	$4.5678e - 07$	$2.2319e - 06$	$2.2319e - 06$
3	$4.8360e - 14$	$4.8360e - 14$	$1.6362e - 12$	$1.6362e - 12$

Example 3 ([2]) Consider the partial differential equation arising in the theory of gas dynamics [14], given by

$$\Delta u = u^3, \quad u = u(\xi_1, \xi_2) \quad (28)$$

where,

$$\Delta = \frac{\partial^2}{\partial \xi_1^2} + \frac{\partial^2}{\partial \xi_2^2}$$

is the two dimensional Laplace operator defined in the rectangular domain $\{(\xi_1, \xi_2) \in \mathcal{R}^2; 0 \leq \xi_1 \leq 1, 0 \leq \xi_2 \leq 1\}$ with the Dirichlet boundary condition given by

$$\begin{aligned} u(\xi_1, 0) &= 2\xi_1^2 - \xi_1 + 1, & u(\xi_1, 1) &= 2, \\ u(0, \xi_2) &= 2\xi_2^2 - \xi_2 + 1, & u(1, \xi_2) &= 2. \end{aligned} \quad (29)$$

Using central divided differences scheme, (28) can be transformed into a system of nonlinear equations given by

$$u_{i+1,j} - 4u_{i,j} + u_{i,j+1} + u_{i,j-1} + u_{i-1,j} - h^2 u_{i,j}^3 = 0, \quad i = 1, \dots, n, \quad j = 1, \dots, m$$

where, $u(i, j)$ denotes $u(\xi_{1,i}, \xi_{2,j})$, $\xi_{1,i} = ih$ with $i = 0, 1, \dots, n$, $\xi_{2,j} = jk$ with $j = 0, 1, \dots, m$ are the nodes in both variables, being $h = \frac{1}{n}$ and $k = \frac{1}{m}$. Taking $n = m = 5$, we generate a 6×6 mesh. Boundary values can be obtained from (29) and to find interior points, we transform interior values as $x_1 = u_{1,1}, x_2 = u_{2,1}, x_3 = u_{3,1}, x_4 = u_{4,1}, x_5 = u_{1,2}, x_6 = u_{2,2}, x_7 = u_{3,2}, x_8 = u_{4,2}, x_9 = u_{1,3}, x_{10} = u_{2,3}, x_{11} = u_{3,3}, x_{12} = u_{4,3}, x_{13} = u_{1,4}, x_{14} = u_{2,4}, x_{15} = u_{3,4}, x_{16} = u_{4,4}$. So, for $\mathcal{X} = \mathcal{Y} = \mathcal{R}^{16}$ and $\mathcal{Y} = \mathcal{R}^{16}$, the system can be expressed as

$$H(x) = Ax + h^2 \phi(x) - b = 0, \quad (30)$$

$$\text{where, } h = \frac{1}{5}, A = \begin{pmatrix} B & -I_4 & 0 & 0 \\ -I_4 & B & -I_4 & 0 \\ 0 & -I_4 & B & -I_4 \\ 0 & 0 & -I_4 & B \end{pmatrix}, B = \begin{pmatrix} 4 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 4 \end{pmatrix},$$

$\phi(x) = (x_1^3, x_2^3, \dots, x_{16}^3)^T$, I_4 is the 4×4 identity matrix and $b = (\frac{44}{25}, \frac{23}{25}, \frac{28}{25}, \frac{87}{25}, \frac{23}{25}, 0, 0, 2, \frac{28}{25}, 0, 0, 2, \frac{87}{25}, 2, 2, 4)^T$. Now, for $y = (y_1, y_2, \dots, y_{16})$, we get

$$\begin{aligned} \|H'(x) - H'(y)\| &= \|3h^2 \text{diag}\{x_1^2 - y_1^2, x_2^2 - y_2^2, \dots, x_{16}^2 - y_{16}^2\}\| \\ &\leq 3h^2 \max_{1 \leq i \leq 16} \|x_i^2 - y_i^2\| \leq 3h^2 \|x^2 - y^2\|. \end{aligned}$$

This gives

$$\begin{aligned} \|[x, y; H] - [u, v; H]\| &\leq \int_0^1 \|H'(y + \gamma(x - y)) - H'(v + \gamma(u - v))\| d\gamma \\ &\leq h^2 (\|x^2 - u^2\| + \|y^2 - v^2\| + \|xy - uv\|) \end{aligned}$$

If we choose $x_0 = (\frac{4}{5}, \frac{4}{5}, \dots, \frac{4}{5})^T$ and $y_0(i) = x_0(i) + 0.1$, then we get $s = 0.1, \eta = 1.0425, k = 0.1084, k_0 = 0.2169, k_1 = 0.2410, k_2 = 0.1205$. It is easy to see that the sufficient condition given in [1] is violated as for $L_0 = 0.2169, t_1 = 1.0425$ and $s_0 = 0.1$ the value of $1 - \frac{2L_0 t_1}{1 - L_0 s_0} = 0.5377$ which is not greater than or equal to $\alpha = 0.6666$. So, we can not ensure the convergence of (5) from Approach-2. However, the conditions of Approach-1 are satisfied and error bounds obtained for it are given in Table 4. Next, we use (5) to solve (30) and the approximate solution is given in Table 5 using the stopping criteria $\|x_n - x_{n-1}\| \leq 10^{-15}$.

Table 4 Error bounds for Approach-1

n	$l_{n+1} - l_n$	$r_n - l_n$
1	0.4029	0.2504
2	$9.9774e - 02$	$8.1506e - 02$
3	$6.9079e - 03$	$6.5728e - 03$
4	$3.4807e - 05$	$3.4695e - 05$
5	$8.9732e - 10$	$8.9730e - 10$

Table 5 Approximate solution by Approach-1

i	ρ_i^*	i	ρ_i^*	i	ρ_i^*	i	ρ_i^*
1	0.967514648571165	5	1.073142808305482	9	1.255308661675940	13	1.547504427760980
2	1.073142808305482	6	1.199182696602124	10	1.359712017969179	14	1.602945733655613
3	1.255308661675940	7	1.359712017969179	11	1.481965315289151	15	1.669313085344323
4	1.547504427760980	8	1.602945733655613	12	1.669313085344323	16	1.778410018624668

Interpolating the value of Table 5, we get the numerical approximate solution which can be seen by Fig. 4.

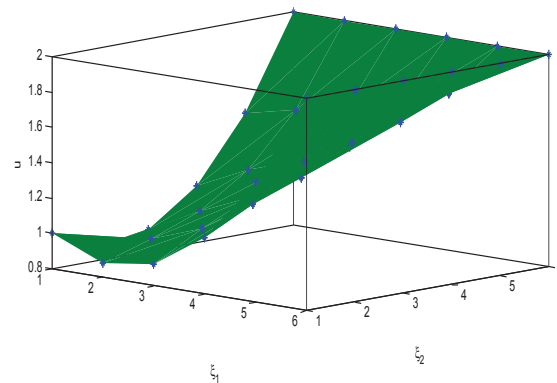


Fig. 4 Approximated solution by Approach-1

5 Conclusions

The semilocal convergence of double step Secant method to approximate a locally unique solution of a nonlinear equation in Banach space is described. Majorizing sequences are used assuming first order divided differences of the involved operator satisfying weaker Lipschitz and the center-Lipschitz continuity conditions. It is found that these conditions enlarge existence-uniqueness region and give finer majorizing sequences. A theorem is established for the existence-uniqueness region along with the estimation of error bounds for the solution. Our work improves the results derived in [1] in more stringent Lipschitz and center Lipschitz conditions. Also, an example is worked out where the conditions of [1] fail but those of our works. Numerical examples including nonlinear elliptic differential equations and integral equations are worked out. It is found that our conditions enlarge the convergence domain of the solution. The Computational efficiency index (CEI) and Efficiency index(EI) of the method is analyzed and found to be better compared to those given for some of the existing methods. Numerical examples including nonlinear elliptic differential equations and integral equations are solved. On comparing with the similar work described in [1], a substantial improvement on the location of the solution and more precise error bounds are found. It is also found that some of numerical examples work with our approach that is not suitable for one given in [1]. This validates the efficiency and suitability of our approach.

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