Solving linear and quadratic random matrix differential equations using: A mean square approach. The non-autonomous case

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Abstract

This paper is aimed to extend, the non-autonomous case, the results recently given in the paper [1] for solving autonomous linear and quadratic random matrix differential equations. With this goal, important deterministic results like the Abel-Liouville-Jacobi's formula, are extended to the random scenario using the so-called L_p-random matrix calculus. In a first step, random time-dependent matrix linear differential equations are studied and, in a second step, random non-autonomous Riccati matrix differential equations are solved using the hamiltonian approach based on dealing with the extended underlying linear system. Illustrative numerical examples are also included.
Keywords: mean square random calculus, L_p-random matrix calculus, random non-autonomous Riccati matrix differential equation, analytic-numerical solution

1. Introduction

In the recent paper [1] linear and quadratic random autonomous differential equations were motivated and studied in the L_p-random sense. In that paper, all the coefficients were assumed to be random matrices rather than matrix stochastic processes, hence in [1] coefficients do not depend on time. Based on the well-known linear hamiltonian approach, (see [2] and [3] for excellent references about Riccati differential equations and the hamiltonian approach), the solution of the initial value problem for a general class of Riccati random quadratic matrix equations is obtained in terms of the blocks of the solution stochastic process of the underlying random linearized problem.

In this paper, we address the solution in the L_p-random sense of the non-autonomous Riccati matrix differential initial value problem (IVP)

$$W'(t) + W(t)\,A(t) + D(t)\,W(t) + W(t)\,B(t)\,W(t) - C(t) = 0\,, \quad W(0) = W_0\,, \tag{1}$$

where the variable coefficient matrices $A(t) \in \mathrm{L}_p^{n\times n}(\Omega)$, $D(t) \in \mathrm{L}_p^{m\times m}(\Omega)$, $B(t) \in \mathrm{L}_p^{n\times m}(\Omega)$, $C(t) \in \mathrm{L}_p^{m\times n}(\Omega)$, the initial condition $W_0 \in L_p^{m\times n}(\Omega)$ and the unknown $W(t) \in \mathrm{L}_p^{m\times n}(\Omega)$ are matrix stochastic processes whose size are specified in the superindexes and defined in certain $\mathrm{L}_p^{r\times s}(\Omega)$ spaces, that will be specified later. It is important to underline that in (1), the meaning of the derivative $W'(t)$ is understood in the p-th mean sense, that is, a kind of strong random convergence that it will be introduced in Section 2. It is convenient to highligth that using the $\mathrm{L}_p^{r\times s}(\Omega)$-random approach is not equivalent to deal with the averaged deterministic problem based on taking the expectations in every entry of the matrices that define the differential equation (1). Even more, from a practical point of view, it is more realistic to consider the random approach rather than the determinisitic since when modelling input data of the Riccati equation (1) are usually fixed after measurments, hence having errors. We point out that the content of this paper may be regarded as a continuation of [1, 4, 5]. Finally, we highlight some recent and interesting contributions dealing with scalar random Riccati-type differential equations by means of $\mathrm{L}_p(\Omega)$-random calculus or alternative techniques [6, 7], for example.

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The organization of this paper is as follows. Section 2 is devoted to extend some stochastic results presented in section 2 of [1] and to introduce new ones as well. These new results are addressed to establish a random analogous of the Abel-Liouville-Jacobi's formula that will play a key role to deal with the non-autonomous random case. In Section 3 the random non-autonomous matrix linear problem is treated, including the bilateral case. In Section 4 the random non-autonomous Riccati matrix equation is solved based on the extended underlying linear problem, including a procedure for the numerical solution inspired in the results of [4] that were obtained for the non-autonomous deterministic counterpart. In Section 5 the theoretical results obtained throughout the paper are illustrated by means of several numerical examples. Finally, conclusions are drawn in Section 6.

2. New results on L_p-random matrix calculus

The aim of this section is to establish new results belonging the so called $\mathrm{L}_p(\Omega)$-random matrix calculus that will be required later for solving both non-autonomous linear systems (see Section 3) and non-autonomous nonlinear random Riccati-type matrix differential equations of the form (1) (see Section 4). This section can be viewed as continuation of the contents introduced in [1, Sec.2]. For the sake of consistency, hereinafter we will keep the same notation introduced in [1]. For ease of presentation, it is convenient to remember that given a complete probability space, $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathrm{L}_p^{m\times n}(\Omega)$ denotes the set of all real random matrices $X = (x_{i,j})_{m\times n}$ such as $x_{i,j} : \Omega \longrightarrow \mathbb{R}$, $1 \le i \le m$, $1 \le j \le n$, are real random variables (r.v.'s) satisfying that

$$\left\|x_{i,j}\right\|_p = \left(\mathrm{E}\left[|x_{i,j}|^p\right]\right)^{1/p} < +\infty, \quad p \ge 1, \tag{2}$$

where $\mathrm{E}[\cdot]$ denotes the expectation operator. It can be proved that $(\mathrm{L}_p^{m\times n}(\Omega), \|\cdot\|_p)$, where

$$\|X\|_p = \sum_{i=1}^{m}\sum_{j=1}^{n}\left\|x_{i,j}\right\|_p, \quad \mathrm{E}\left[|x_{i,j}|^p\right] < +\infty, \tag{3}$$

is a Banach space. Notice that no confusion is possible between the common notation used for the $\|\cdot\|_p$ in (2) and in (3) because they act on scalar r.v.'s (denoted by lower case letters) and random matrices (denoted by capital case letters), respectively. In the case that $m = n = 1$, both norms are the same and $(\mathrm{L}_p^{1\times 1}(\Omega) \equiv \mathrm{L}_p(\Omega), \|\cdot\|_p)$ represents the Banach space of real r.v.'s with finite absolute moments of order p about the origin, being $p \ge 1$ fixed, [8]. In [9] a number of results corresponding to $p = 4$ (fourth random calculus) and its relationship with $p = 2$ (mean square calculus) are established and applied to solve scalar random differential equations. In [10] a scalar random Riccati differential equation whose nonlinear coefficient is assumed to be an analytic stochastic process is solved using the $\mathrm{L}_p(\Omega)$-random scalar calculus.

Given $\mathcal{T} \subset \mathbb{R}$, a family of t-indexed r.v.'s, say $\{x(t) : t \in \mathcal{T}\}$, is called a p-stochastic process (p-s.p.) if for each $t \in \mathcal{T}$, the r.v. $x(t) \in \mathrm{L}_p(\Omega)$. This definition can be extended to matrix s.p.'s $X(t) = (x_{i,j}(t))_{m\times n}$ of $\mathrm{L}_p^{m\times n}(\Omega)$, which are termed p-matrix s.p.'s, if $x_{i,j}(t) \in \mathrm{L}_p(\Omega)$ for every $1 \le i \le m$ and $1 \le j \le n$.

The definitions of continuity, differentiability and integrability of p-matrix s.p.'s follow in a straightforwardly manner using the $\|\cdot\|_p$-norm introduced in (3). As a simple but illustrative example that will be invoked later when showing more sophisticated examples in Section 5, below we show how to prove the p-differentiability of a matrix s.p. of $\mathrm{L}_p^{n\times n}(\Omega)$.

Example 1. *Let a be an absolutely continuous r.v. defined on the bounded interval (a_1, a_2), i.e., $a_1 \le a(\omega) \le a_2$ for every $\omega \in \Omega$, and let us denote by $f_a(a)$ the probability density function (p.d.f.) of the r.v. a. Let us define the following matrix s.p.*

$$H(t;a) = \begin{bmatrix} h_{1,1}(t;a) & h_{1,2}(t;a) \\ h_{2,1}(t;a) & h_{2,2}(t;a) \end{bmatrix} = \begin{bmatrix} \exp(at) & \cosh(at) \\ \sinh(at) & \exp(-at) \end{bmatrix}, \qquad t \in [0,T].$$

On the one hand, by the definition of the random matrix p-norm (see (3)*) one gets*

$$\|H(t;a)\|_p = \sum_{i=1}^{2}\sum_{j=1}^{2}\left\|h_{i,j}(t;a)\right\|_p = \left\|\exp(at)\right\|_p + \|\cosh(at)\|_p + \|\sinh(at)\|_p + \left\|\exp(-at)\right\|_p.$$

On the other hand, if we denote

$$M_{t,p} := \max\{M_{t,p}^{i,j} \,:\, 1 \leq i,j \leq 2\}, \quad \textit{where} \quad M_{t,p}^{i,j} := \max_{\omega\in\Omega}\{(h_{i,j}(t;a(\omega)))^p\} < +\infty.$$

It is clear that

$$\left(\left\|h_{i,j}(t;a)\right\|_p\right)^p = \mathrm{E}\left[(h_{i,j}(t;a))^p\right] = \int_{a_1}^{a_2} (h_{i,j}(t;a))^p f_a(a)\,\mathrm{d}a \leq M_{t,p}^{i,j} < +\infty, \qquad \textit{for every } t \in [0,T],$$

where in the last step we have used that the integral of $f_a(a)$ over $[a_1,a_2]$ is just 1 because it is a p.d.f. This proves that $H(t;a) \in \mathrm{L}_p^{2\times 2}(\Omega)$ for every $p \geq 1$. Moreover the $H(t;a)$ is p-differentiable being its p-derivative

$$H'(t;a) = \begin{bmatrix} h'_{1,1}(t;a) & h'_{1,2}(t;a) \\ h'_{2,1}(t;a) & h'_{2,2}(t;a) \end{bmatrix} = \begin{bmatrix} a\exp(at) & a\sinh(at) \\ a\cosh(at) & -a\exp(-at) \end{bmatrix}, \qquad t \in [0,T].$$

Indeed, observe that

$$\left\|\frac{H(t+\Delta t;a)-H(t;a)}{\Delta t} - H'(t,a)\right\|_p = \sum_{i=1}^{2}\sum_{j=1}^{2}\left\|\frac{h_{i,j}(t+\Delta t;a)-h_{i,j}(t;a)}{\Delta t} - h'_{i,j}(t;a)\right\|_p,$$

and, for example for $i=j=1$,

$$\begin{aligned}\left(\left\|\frac{h_{1,1}(t+\Delta t;a)-h_{1,1}(t;a)}{\Delta t} - h'_{1,1}(t;a)\right\|_p\right)^p &= \mathrm{E}\left[\left(\frac{\exp(a(t+\Delta t))-\exp(at)}{\Delta t} - a\exp(at)\right)^p\right] \\ &= \mathrm{E}\left[\left(\frac{\exp(a(t+\Delta t))-\exp(at)(1+a\Delta t)}{\Delta t}\right)^p\right] = O((\Delta t)^p) \xrightarrow[\Delta t\to 0]{} 0,\end{aligned}$$

and the same can be shown for the rest of the components $h'_{1,2}(t;a)$, $h'_{2,1}(t;a)$ and $h'_{2,2}(t;a)$.

The following two key inequalities for scalar r.v.'s will be used extensively throughout this paper (see [1])

$$\|x\|_r \leq \|x\|_s, \quad 1 \leq r \leq s, \quad x \in \mathrm{L}_s(\Omega), \tag{4}$$

and

$$\|xy\|_p \leq \|x\|_{2p}\,\|y\|_{2p}, \quad x,y \in \mathrm{L}_{2p}(\Omega). \tag{5}$$

As a consequence of inequality (4), the $\mathrm{L}_p(\Omega)$ spaces are embedded according to the following relationship, that will be play a key role throughout the this paper,

$$\mathrm{L}_s(\Omega) \subset \mathrm{L}_r(\Omega), \qquad 1 \leq r \leq s. \tag{6}$$

The following result may be regarded as a matrix adaptation of the fundamental theorem of mean square calculus but generalized to the p-norm defined in (3), [11, p.104]. We omit its proof since it follows the same argument shown in [11, p.104] but working componentwise and using the p-norm defined in (2) instead of particularizing this norm for $p = 2$.

Proposition 1. *Let $Z(t) \in \mathrm{L}_p^{m\times m}(\Omega)$ be a p-differentiable matrix s.p. and assume that $Z'(t)$ is p-integrable, then*

$$Z(t) - Z(0) = \int_0^t Z'(s)\,\mathrm{d}s\,.$$

Definition 1. Let $\{\ell_{i,j}(t) \,:\, t \in \mathcal{T} \subset \mathbb{R},\ 1 \leq i,j \leq m\}$ be scalar s.p.'s. The trace, $\mathrm{tr}(L(t))$, of the square matrix s.p. $L(t) = (\ell_{i,j}(t))_{m\times m}$ is defined by the sum of its diagonal entries, that is,

$$\mathrm{tr}(L(t)) = \sum_{i=1}^{m} \ell_{i,i}(t)\,.$$

The following result generalizes inequality (5) to an arbitrary number of factors since it is obtained for the particular case $m = 2$. This inequality will be applied later.

Lemma 1. *Let us consider a set of scalar s.p.'s* $\{x_i(t) : t \in \mathcal{T} \subset \mathbb{R},\ 1 \le i \le m\}$ *in* $\mathrm{L}_{2^{m-1}p}(\Omega)$. *Then, for each* $t \in \mathcal{T}$, *it is verified that*

$$\left\| \prod_{i=1}^{m} x_i(t) \right\|_p \le \prod_{i=1}^{m} \|x_i(t)\|_{2^{m-1}p} \,, \tag{7}$$

and $\prod_{i=1}^{m} x_i(t)$ *belongs to* $\mathrm{L}_p(\Omega)$.

Proof. *It follows by induction over m. Let* $t \in \mathcal{T} \subset \mathbb{R}$ *be arbitrary but fixed. For* $m = 1$ *the proof is trivial since* (7) *becomes an identity. Let us assume that* (7) *is satisfied for the* $m-1$ *scalar s.p.'s* $\{x_i(t) : 1 \le i \le m-1\}$, *that is to say, the following inequality*

$$\left\| \prod_{i=1}^{m-1} x_i(t) \right\|_p \le \prod_{i=1}^{m-1} \|x_i(t)\|_{2^{m-2}p} \,, \tag{8}$$

holds provided that $\|x_i(t)\|_{2^{m-2}p} < +\infty$, *i.e.,* $x_i(t) \in \mathrm{L}_{2^{m-2}p}(\Omega)$, $1 \le i \le m-2$. *Now assuming* $m \ge 2$, *we shall prove* (7),

$$\begin{aligned}
\left\| \prod_{i=1}^{m} x_i(t) \right\|_p &= \left\| \left(\prod_{i=1}^{m-1} x_i(t) \right) x_m(t) \right\|_p \\
&\overset{(I)}{\le} \left\| \prod_{i=1}^{m-1} x_i(t) \right\|_{2p} \|x_m(t)\|_{2p} \\
&\overset{(II)}{\le} \left(\prod_{i=1}^{m-1} \|x_i(t)\|_{2^{m-2}(2p)} \right) \|x_m(t)\|_{2p} \\
&= \left(\prod_{i=1}^{m-1} \|x_i(t)\|_{2^{m-1}p} \right) \|x_m(t)\|_{2p} \\
&\overset{(III)}{\le} \left(\prod_{i=1}^{m-1} \|x_i(t)\|_{2^{m-1}p} \right) \|x_m(t)\|_{2^{m-1}p} \\
&\le \prod_{i=1}^{m} \|x_i(t)\|_{2^{m-1}p} < +\infty \,.
\end{aligned}$$

In step (I) we have applied (5) *for r.v.'s* $x = \prod_{i=1}^{m-1} x_i(t)$, $y = x_m(t)$. *Taking into account that by hypothesis* $x_i(t) \in \mathrm{L}_{2^{m-1}p}(\Omega)$, $1 \le i \le m$, *together with the proof itself, it is justified that* $\prod_{i=1}^{m-1} x_i(t)$ *and* $x_m(t)$ *are in* $\mathrm{L}_{2p}(\Omega)$, *which is required to legitimate the application of inequality* (5). *In step (II) we have applied the induction hypothesis* (8) *with de identification* $2p$ *instead of p, and finally, in step (III) we have used the Lyapunov's inequality* (4) *with* $r \equiv 2p$ *and* $s \equiv 2^{m-1}p$, $m \ge 2$ *since by hypothesis* $x_m(t) \in \mathrm{L}_{2^{m-1}p}(\Omega)$. □

Remark 1. Notice that if in Lemma 1 we consider $m-1$ scalar s.p.'s $\{x_i(t) : t \in \mathcal{T} \subset \mathbb{R},\ 1 \le i \le m-1\}$ in $\mathrm{L}_{2^{m-1}p}(\Omega)$, then (7) is still true for $p \equiv 2p$ since $\|x_i(t)\|_{2^{m-2}(2p)} = \|x_i(t)\|_{2^{m-1}p} < +\infty$. As a consequence, $\prod_{i=1}^{m-1} x_i(t) \in \mathrm{L}_{2p}(\Omega)$. This result will be used in the proof of the following lemma.

Lemma 2. *Let us consider a set of scalar s.p.'s* $\{x_i(t) : t \in \mathcal{T} \subset \mathbb{R},\ 1 \le i \le m\}$ *in* $\mathrm{L}_{2^{m-1}p}(\Omega)$ *for every* $t \in \mathcal{T}$ *and* $(2^{m-1}p)$*-differentiable, then* $\prod_{i=1}^{m} x_i(t)$ *is p-differentiable and, for each* $t \in \mathcal{T}$, *its value is*

$$\left(\prod_{i=1}^{m} x_i(t) \right)' = \sum_{i=1}^{m} \left\{ \left(\prod_{\substack{j=1 \\ j \ne i}}^{m} x_j(t) \right) x_i'(t) \right\} . \tag{9}$$

Proof. It follows by induction over m. Let $t \in \mathcal{T} \subset \mathbb{R}$ be arbitrary but fixed. For $m = 1$ the proof is trivial because both sides of (9) are the same. Let us assume that for $m \geq 2$

$$\left(\prod_{i=1}^{m-1} x_i(t) \right)' = \sum_{i=1}^{m-1} \left\{ \left(\prod_{\substack{j=1 \\ j \neq i}}^{m-1} x_j(t) \right) x_i'(t) \right\} , \tag{10}$$

holds. On the one hand, applying Remark 1 it is guaranteed that $\prod_{i=1}^{m-1} x_i(t) \in \mathrm{L}_{2p}(\Omega)$. On the other hand, due to $x_m(t) \in \mathrm{L}_{2^{m-1}p}(\Omega)$ and (6), it is known that $x_m(t) \in \mathrm{L}_{2p}(\Omega)$. Then, according to Proposition 2 of [1] (in its scalar version) for the p-derivative of the product of two $2p$-differentiable s.p.'s, one gets

$$\left(\prod_{i=1}^{m} x_i(t) \right)' = \left(\left(\prod_{i=1}^{m-1} x_i(t) \right) x_m(t) \right)' = \left(\prod_{i=1}^{m-1} x_i(t) \right)' x_m(t) + \left(\prod_{i=1}^{m-1} x_i(t) \right) x_m'(t) . \tag{11}$$

Using the induction hypothesis (10) in (11), one obtains the result

$$\begin{aligned} \left(\prod_{i=1}^{m} x_i(t) \right)' &= \left(\sum_{i=1}^{m-1} \left(\prod_{\substack{j=1 \\ j \neq i}}^{m-1} x_j(t) \right) x_i'(t) \right) x_m(t) + \left(\prod_{i=1}^{m-1} x_i(t) \right) x_m'(t) \\ &= \sum_{i=1}^{m-1} \left\{ \left(\prod_{\substack{j=1 \\ j \neq i}}^{m} x_j(t) \right) x_i'(t) \right\} + \left(\prod_{i=1}^{m-1} x_i(t) \right) x_m'(t) \\ &= \sum_{i=1}^{m} \left(\prod_{\substack{j=1 \\ j \neq i}}^{m} x_j(t) \right) x_i'(t) . \qquad \square \end{aligned}$$

In [1], we defined the determinant of a square matrix s.p. $A(t) = (a_{i,j})_{n \times n}$ as

$$\det A(t) = \sum_{\sigma \in P_n} \mathrm{sgn}(\sigma)\, a_{1,\sigma(1)}(t) \cdots a_{n,\sigma(n)}(t) , \quad \text{for each } t \in \mathcal{T} \subset \mathbb{R} ,$$

being P_n the set of all permutations of the n elements $(1, 2, \ldots, n)$, that is, the set of all permutations of the indexes defining the n columns of $A(t)$, and $\mathrm{sgn}(\sigma)$ the signature of the permutation $\sigma = (\sigma(1), \ldots, \sigma(n))$. Inasmuch as $A(t)$ is a matrix s.p. then $\det A(t)$ is a scalar s.p. Furthermore, under conditions given in Proposition 3 of [1], it is guaranteed that $\det A(t)$ is continuous in the p-norm defined by (2). The following result allows us to compute the p-derivative of the determinant of a family of s.p.'s. It can be regarded as an extension of the classical rule for differentiating the determinant whose entries are differentiable deterministic functions.

Lemma 3. *Let us consider a square matrix s.p.* $A(t) = (a_{i,j}(t))_{n \times n}$, $t \in \mathcal{T} \subset \mathbb{R}$. *Let us suppose that the scalar s.p.'s* $a_{i,j}(t)$, $i, j = 1, \ldots, n$, *lie in* $\mathrm{L}_{2^{n-1}p}(\Omega)$ *for every* $t \in \mathcal{T}$ *and are* $(2^{n-1}p)$*-differentiable for every* $t \in \mathcal{T}$. *Then, the determinant s.p. of* $A(t)$, $\det A(t)$, *is* p*-differentiable and its* p*-derivative is given by*

$$(\det A(t))' = \det \begin{bmatrix} (a_{1,1}(t))' & \cdots & (a_{1,n}(t))' \\ a_{2,1}(t) & \cdots & a_{2,n}(t) \\ \vdots & & \vdots \\ a_{n,1}(t) & \cdots & a_{n,n}(t) \end{bmatrix} + \det \begin{bmatrix} a_{1,1}(t) & \cdots & a_{1,n}(t) \\ (a_{2,1}(t))' & \cdots & (a_{2,n}(t))' \\ \vdots & & \vdots \\ a_{n,1}(t) & \cdots & a_{n,n}(t) \end{bmatrix} + \cdots + \det \begin{bmatrix} a_{1,1}(t) & \cdots & a_{1,n}(t) \\ a_{2,1}(t) & \cdots & a_{2,n}(t) \\ \vdots & & \vdots \\ (a_{n,1}(t))' & \cdots & (a_{n,n}(t))' \end{bmatrix} . \tag{12}$$

Proof. Since $a_{i,j}(t) \in \mathrm{L}_{2^{n-1}p}(\Omega)$, then $\mathrm{E}\left[|a_{i,j}(t)|^{2^{n-1}p} \right] < +\infty$, $\forall i, j : 1 \leq i, j, \leq n$, $n \geq 1$, $t \in \mathcal{T}$, and accordingly to expression (15) of [1] it is guaranteed that $(\det A(t)) \in \mathrm{L}_p(\Omega)$. Now, considering the definition of $\det A(t)$ one gets

$$(\det A(t))' = \left(\sum_{\sigma \in P_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}(t) \right)' = \sum_{\sigma \in P_n} \mathrm{sgn}(\sigma) \left(\prod_{i=1}^{n} a_{i,\sigma(i)}(t) \right)' . \tag{13}$$

Using that the n scalar s.p.'s $a_{i,\sigma(i)}(t)$, $i = 1, \ldots, n$, are in $\mathrm{L}_{2^{n-1}p}(\Omega)$ and they are $(2^{n-1}p)$-differentiable, we can apply Lemma 2 to (13) (with the identification $m \equiv n$) for each $t \in \mathcal{T}$ and then obtaining

$$\begin{aligned}(\det A(t))' &= \sum_{\sigma \in P_n} \operatorname{sgn}(\sigma) \left\{ \sum_{i=1}^{n} \left(\prod_{\substack{j=1 \\ j \neq i}}^{n} a_{i,\sigma(j)}(t) \right) a'_{j,\sigma(j)}(t) \right\} = \sum_{i=1}^{n} \left\{ \sum_{\sigma \in P_n} \operatorname{sgn}(\sigma) \left(\prod_{\substack{j=1 \\ j \neq i}}^{n} a_{i,\sigma(j)}(t) \right) a'_{j,\sigma(j)}(t) \right\} \\ &= \sum_{i=1}^{n} \det \begin{bmatrix} a_{1,1}(t) & \cdots & a_{1,n}(t) \\ \vdots & & \vdots \\ (a_{i,1}(t))' & \cdots & (a_{i,n}(t))' \\ \vdots & & \vdots \\ a_{n,1}(t) & \cdots & a_{n,n}(t) \end{bmatrix}. \qquad \square\end{aligned}$$

Proposition 2 (Abel-Liouville-Jacobi's random formula). *Let $\Phi(t) = \left(\phi_i^j(t)\right)_{n \times n}$, $t \in \mathcal{T} \subset \mathbb{R}$ be a matrix s.p. such that its entries, $\phi_i^j(t)$, are $(2^{n-1}p)$-differentiable scalar s.p.'s. Let us assume that $\Phi(t)$ verifies the random matrix linear equation $\Phi'(t) = L(t)\,\Phi(t)$, where the elements $\ell_{i,j}(t)$ of the matrix s.p. $L(t) = \left(\ell_{i,j}(t)\right)_{n \times n}$ lie in $\mathrm{L}_{2^{n-1}p}(\Omega)$ and they are $(2^{n-1}p)$-differentiable for each $t \in \mathcal{T}$. Then, the scalar s.p. $\det \Phi(t) \in \mathrm{L}_p(\Omega)$ satisfies the random first-order homogeneous linear equation*

$$(\det \Phi(t))' = \operatorname{tr}(L(t)) \, \det \Phi(t) \,.$$

Furthermore, under the following conditions

(C1) $L(t) \in \mathrm{L}_{2p}^{n \times n}(\Omega)$, *for each* $t \in \mathcal{T}$,

(C2) $\det \Phi(t_0) \in \mathrm{L}_{2p}(\Omega)$, *for* $t_0 \in \mathcal{T}$,

(C3) There exist $r > 2p$ *and* $\delta > 0$, *such that*

$$\sup_{s,s^* \in [-\delta,\delta]} \mathrm{E}\left[\left(\exp\left(\int_{x+s}^{t+s^*} \operatorname{tr}(L(u)) \, \mathrm{d}u \right) \right)^r \right] = \sup_{s,s^* \in [-\delta,\delta]} \mathrm{E}\left[\prod_{i=1}^{n} \exp\left(r \int_{x+s}^{t+s^*} \ell_{i,i}(u) \, \mathrm{d}u \right) \right] < +\infty \,,$$

it is verified that $\det \Phi(t)$ *satisfies the following identity for each t*

$$\det \Phi(t) = \det \Phi(t_0) \, \exp\left(\int_{t_0}^{t} \operatorname{tr}(L(s)) \, \mathrm{d}s \right), \quad t_0 \in \mathcal{T} \,. \tag{14}$$

Proof. Let us fix $t \in \mathcal{T}$, and without loss of generality let us consider that the matrix s.p. $\Phi(t)$ takes the form

$$\Phi(t) = \left[\Phi^1(t) \cdots \Phi^n(t) \right] = \begin{bmatrix} \phi_1^1(t) & \cdots & \phi_1^n(t) \\ \vdots & & \vdots \\ \phi_i^1(t) & \cdots & \phi_i^n(t) \\ \vdots & & \vdots \\ \phi_n^1(t) & \cdots & \phi_n^n(t) \end{bmatrix},$$

where $\Phi^j(t)$, $j = 1, \ldots, n$, denote the j-th column vector of the matrix s.p. $\Phi(t)$ and $\phi_i^j(t)$ the i-th component of the column vector $\Phi^j(t)$. Since the entries of the matrix s.p. $\Phi(t)$ are $(2^{n-1}p)$-differentiable, then accordingly to Lemma 3 the first p-derivative of the scalar s.p. $\det \Phi(t)$ exists, $(\det \Phi(t))'$, and by (12) it can be calculated as follows

$$(\det \Phi(t))' = \sum_{i=1}^{n} \det \begin{bmatrix} \phi_1^1(t) & \cdots & \phi_1^n(t) \\ \vdots & & \vdots \\ \left(\phi_i^1(t)\right)' & \cdots & \left(\phi_i^n(t)\right)' \\ \vdots & & \vdots \\ \phi_n^1(t) & \cdots & \phi_n^n(t) \end{bmatrix}. \tag{15}$$

Taking into account we are assuming that $\Phi(t)$ verifies the random matrix linear equation $\Phi'(t) = L(t)\,\Phi(t)$, then its column vectors $\Phi^j(t)$, $j = 1, \ldots, n$, also hold this equation, that is,

$$(\Phi^j(t))' = L(t)\,\Phi^j(t), \qquad \forall j = 1, \ldots, n\,. \tag{16}$$

By (16), the i-th component, $\phi_i^j(t)$, $i = 1, \ldots, n$, of each column vector s.p. $\Phi^j(t)$, $j = 1 \ldots, n$, takes the form

$$\left(\phi_i^j(t)\right)' = \sum_{k=1}^{n} \ell_{i,k}(t)\,\phi_k^j(t)\,, \quad \forall i, j = 1, \ldots, n\,. \tag{17}$$

Substituting (17) into (15), one gets

$$(\det \Phi(t))' = \sum_{i=1}^{n} \det \begin{bmatrix} \phi_1^1(t) & \cdots & \phi_1^n(t) \\ \vdots & & \vdots \\ \sum_{k=1}^{n} \ell_{i,k}(t)\,\phi_k^1(t) & \cdots & \sum_{k=1}^{n} \ell_{i,k}(t)\,\phi_k^n(t) \\ \vdots & & \vdots \\ \phi_n^1(t) & \cdots & \phi_n^n(t) \end{bmatrix}. \tag{18}$$

Note the i-th row, F_i, of the right-hand side of (18) is a linear combination of all remaining rows of (18). Then, making the elementary row operations

$$F_i - \sum_{\substack{k=1 \\ k \neq i}}^{n} \ell_{i,k}(t)\,F_k \longrightarrow F_i\,, \qquad \forall i = 1, \ldots, n,$$

and considering the standard determinant properties, one gets

$$(\det \Phi(t))' = \sum_{i=1}^{n} \det \begin{bmatrix} \phi_1^1(t) & \cdots & \phi_1^n(t) \\ \vdots & & \vdots \\ \ell_{i,i}(t)\,\phi_i^1(t) & \cdots & \ell_{i,i}(t)\,\phi_i^n(t) \\ \vdots & & \vdots \\ \phi_n^1(t) & \cdots & \phi_n^n(t) \end{bmatrix} = \left(\sum_{i=1}^{n} \ell_{i,i}(t)\right) \det \Phi(t) = \operatorname{tr}(L(t))\, \det \Phi(t)\,. \tag{19}$$

Now, let us consider the following scalar random IVP

$$\left.\begin{array}{rcl} y'(t) & = & \operatorname{tr}(L(t))\,y(t)\,, \quad t \in \mathcal{T}\,, \\ y(t_0) & = & \det \Phi(t_0), \end{array}\right\} \tag{20}$$

verifying the three conditions (C1)–(C3). Then, taking into account that $\det \Phi(t)$ verifies (19), that the $\left(2^{n-1}p\right)$-differentiability of each $\ell_{i,j}(t)$ implies the $\left(2^{n-1}p\right)$-continuity of the $\operatorname{tr}(L(t))$ and, applying an analogous reasoning to the one shown in Theorem 8 of [12], we obtain that $\det \Phi(t)$ is a solution to random IVP (20) in $\mathrm{L}_p(\Omega)$. Moreover, it is given by

$$\det \Phi(t) = \det \Phi(t_0)\, \exp\left(\int_{t_0}^{t} \operatorname{tr}(L(s))\,\mathrm{d}s\right), \quad t_0 \in \mathcal{T}\,. \quad \square$$

3. Random non-autonomous linear systems

We begin this section with the solution of the random vector IVP

$$Y'(t) = L(t)\,Y(t)\,, \quad Y(0) = Y_0\,, \quad t \in [0, T]\,, \tag{21}$$

where $L(t) \in \mathrm{L}_{2p}^{n\times n}(\Omega)$ is a matrix s.p. and $Y_0 \in \mathrm{L}_{2p}^{n\times 1}(\Omega)$. Under the hypothesis, $L(t)$ is absolutely integrable in the $2p$-norm defined by (3) (in short, $L(t)$ is $2p$-absolutely integrable) , that is,

$$\int_0^T \|L(t)\|_{2p} \, \mathrm{d}t < +\infty \,, \tag{22}$$

it is guaranteed that, by submultiplicativity property (5), $F : [0,T]\times \mathrm{L}_{2p}^{n\times 1}(\Omega) \longrightarrow \mathrm{L}_{p}^{n\times 1}(\Omega)$, defined by $F(t,Y) = L(t)Y$, satisfies

$$\|F(t,Y_1) - F(t,Y_2)\|_p = \|L(t)(Y_1 - Y_2)\|_p \le \|L(t)\|_{2p} \; \|Y_1 - Y_2\|_{2p} \;.$$

Thus, function $F(t,Y)$ is p-Lipschitzian and by Theorem 10.6.1 of [13, p.292] which holds for abstract Banach spaces, the random vector IVP (21) admits a unique $\mathrm{L}_{p}^{n\times 1}(\Omega)$ solution in $[0,T]$.

Let us denote by $\Phi_L(t;0)$ the matrix s.p. in $\mathrm{L}_{p}^{n\times n}(\Omega)$ whose i-th column is the unique solution of problem (21) with $Y_0 = [0,\dots,0,1,0,\dots,0]^\top$, where the i-th entry is 1 and 0 elsewhere, with probability one. Then, one satisfies

$$\Phi_L'(t;0) = L(t)\,\Phi_L(t;0)\,, \quad \Phi_L(0;0) = I_n\,, \tag{23}$$

being I_n the identity matrix of size n.

Definition 2. The matrix s.p. $\Phi_L(t;0)$ satisfying (23) is referred to as the random fundamental matrix solution of the random linear system (21).

Note that if $L(t) = (\ell_{i,j}(t))$ satisfies the hypotheses of Proposition 2, then $\Phi_L(t;0)$ is invertible in $\mathrm{L}_{p}^{n\times n}(\Omega)$ in the sense introduced in the Definition 3 of [1] (see (14) and note that $\det \Phi(t_0) = I_n$ being I_n the identity matrix of size n).

For the sake of clarity in the presentation, below we introduce the following definition:

Definition 3. The linear system (21) is said to be random p-regular, $p \ge 1$, if the following conditions are satisfied:

- the matrix s.p. $L(t) \in \mathrm{L}_{2p}^{n\times n}(\Omega)$ of (21) is $2p$-absolutely integrable in $[0,T]$;
- the random fundamental matrix solution, $\Phi_L(t;0)$, and its inverse, $\Phi_L^{-1}(t;0)$, both lie in $\mathrm{L}_{p}^{n\times n}(\Omega)$ and they are p-differentiable.

Example 2. *Let $L = (\ell_{i,j})_{n\times n}$ be a random matrix for whose entries $\ell_{i,j} : \Omega \longrightarrow \mathbb{R}$ there exist positive constants $m_{i,j}$ and $h_{i,j}$ satisfying that*

$$\mathrm{E}\left[\left|\ell_{i,j}\right|^r\right] \le m_{i,j}\left(h_{i,j}\right)^r < +\infty\,, \qquad \forall r \ge 0,\;\; \forall i,j:\; 1 \le i,j \le n\,. \tag{24}$$

Then, by Section 3 *of [1], the corresponding random autonomous linear system* (23) *with $L(t) = L$ is p-regular for any $p \ge 1$ with $\Phi_L(t;0) = \exp(Lt)$ and $\Phi_L^{-1}(t;0) = \exp(-Lt)$. Moreover, as indicated in Remark 3 into Section* 3 *of [1], any bounded or truncated r.v. satisfies condition* (24)*. Therefore, important r.v.'s like binomial, uniform, beta satisfy condition* (24)*. In addition, unbounded r.v.'s like exponential, gaussian, etc. can be truncated adequately in order for this property to be satisfied. As a consequence, the set of r.v.'s satisfying condition* (24) *is, in practice, quite broad.*

Example 3. *Let us consider the random IVP* (21) *where all entries, $\ell_{i,j}(t)$, of the matrix s.p. $L(t) = \left(\ell_{i,j}(t)\right)_{n\times n}$ have s-degrees of randomness, [11, p.36],*

$$\ell_{i,j}(t) = \ell_{i,j}(t;a_1,a_2,\dots,a_s)\,.$$

Let us assume that $\ell_{i,j}(t;a_1,a_2,\dots,a_s) \in \mathrm{L}_{2p}^{n\times n}(\Omega)$ is $2p$-absolutely integrable, for each $i,j : 1 \le i,j \le n$, hence condition (22) *is guaranteed. If $Y_0 = [y_{0,1},\dots,y_{0,n}]$ is the random vector initial condition of the IVP* (21)*, then it is easy to check, throughout the approximate successive method [13], that the random fundamental matrix solution $\Phi_L(t;0)$ of* (21) *has $(s+n)$-degrees of randomness determined by the r.v.'s $a_1,a_2,\dots,a_s,y_{0,1},\dots,y_{0,n}$. By Proposition 2, $\Phi_L(t;0)$ is invertible, and assuming that $\Phi_L^{-1}(t;0) \in \mathrm{L}_{p}^{n\times n}(\Omega)$ and it is p-differentiable, then, the linear system* (21) *is p-regular.*

Below, we show an example where the random fundamental matrix solution, $\Phi_L(t;0)$, is available for the time-dependent case.

Example 4. *Let us consider the random IVP* (21) *with* $L(t) = f(t)L$*, where* $f(t)$ *is a real continuous deterministic function,* $f : [0,T] \longrightarrow \mathbb{R}$*, and* $L = \left(\ell_{i,j}\right)_{n\times n}$ *is a random matrix whose entries satisfy the condition* (24)*, hence* $L \in \mathrm{L}_{2p}^{n\times n}(\Omega)$*. Notice that* $L(t)$ *is* $2p$*-absolutely integrable in* $[0,T]$*:*

$$\int_0^T \|L(t)\|_{2p}\,\mathrm{d}t = \int_0^T \|f(t)L\|_{2p}\,\mathrm{d}t = \|L\|_{2p} \int_0^T |f(t)|\,\mathrm{d}t < +\infty\,,$$

since $\|L\|_{2p} < +\infty$ *(by assumption* (24)*, see [1]) and* $\int_0^T |f(t)|\,\mathrm{d}t < +\infty$ *(by continuity of* $f(t)$*). Also by hypothesis* (24) *and Section 3 of [1],*

$$\psi_L(t;0) = \exp\left(L\int_0^t f(s)\,\mathrm{d}s\right), \tag{25}$$

and its inverse

$$\psi_L^{-1}(t;0) = \exp\left(-L\int_0^t f(s)\,\mathrm{d}s\right), \tag{26}$$

are well-defined in $\mathrm{L}_p^{n\times n}(\Omega)$ *and it is also guaranteed that* $\psi_L(t;0)$ *and* $\psi_L^{-1}(t;0)$*, defined by* (25) *and* (26)*, respectively, are p-differentiable. Therefore, the random IVP* (21) *with* $L(t) = f(t)L$ *is random p-regular and* $\psi_L(t;0)$ *satisfies that*

$$\psi_L'(t;0) = L\,f(t)\,\exp\left(L\int_0^t f(s)\,\mathrm{d}s\right) = f(t)\,L\,\psi_L(t;0) = L(t)\,\psi_L(t;0)\,.$$

Thus, $\Phi_L(t;0) = \psi_L(t;0)$ *is its random fundamental matrix solution.*

The following result provides a closed form solution of p-regular random linear systems.

Theorem 1. *Let us assume that the random linear system* (21) *is* $2p$*-regular and let* $L(t) = \left(\ell_{i,j}(t)\right)_{n\times n}$ *be a matrix s.p. such as its entries satisfy condition* (24) *for every t. Let us suppose that the random vector s.p.* $B(t)$ *lies in* $\mathrm{L}_{2p}^{n\times 1}(\Omega)$ *and is* $2p$*-integrable, and the initial condition* $Y_0 \in \mathrm{L}_{2p}^{n\times 1}(\Omega)$*. Then*

$$X(t) = \Phi_L(t;0)\,Y_0 + \Phi_L(t;0)\int_0^t \Phi_L^{-1}(s;0)B(s)\,\mathrm{d}s\,, \tag{27}$$

satisfies the unhomogeneous problem

$$X'(t) = L(t)\,X(t) + B(t)\,, \qquad X(0) = Y_0\,,$$

where the derivative $X'(t)$ *is understood in the* $\mathrm{L}_p^{n\times 1}(\Omega)$ *sense.*

Proof. On the one hand, observe that under hypothesis $L(t) = \left(\ell_{i,j}(t)\right)_{n\times n}$ be a random matrix s.p. such as its entries satisfy condition (24) for every t fixed, it is guaranteed that $\Phi_L(t;0) \in \mathrm{L}_{2p}^{n\times n}(\Omega)$ (see [1]). On the other hand, taking derivatives of the s.p. $X(t)$ defined by (27) and, applying Proposition 2 of [1], Proposition 1 and (23), one gets

$$\begin{aligned}
X'(t) &= \Phi_L'(t;0)\,Y_0 + \Phi_L'(t;0)\int_0^t \Phi_L^{-1}(s;0)\,B(s)\,\mathrm{d}s + \Phi_L(t;0)\,\Phi_L^{-1}(t;0)\,B(t)\\
&= L(t)\,\Phi_L(t;0)Y_0 + L(t)\,\Phi_L(t;0)\int_0^t \Phi_L^{-1}(s;0)\,B(s)\,\mathrm{d}s + B(t)\\
&= L(t)\left[\Phi_L(t;0)\,Y_0 + \Phi_L(t;0)\int_0^t \Phi_L^{-1}(s;0)\,B(s)\,\mathrm{d}s\right] + B(t)\\
&= L(t)X(t) + B(t)\,.
\end{aligned}$$

In addition, by (23) one gets

$$X(0) = \Phi_L(0;0)\,Y_0 = Y_0\,. \quad \square$$

The next result provides a random analogous of the deterministic case, solved by R. Bellman in [14], where the solution of a random matrix bilateral differential equation is constructed in terms of the solution of two auxiliary random linear systems of the form (21).

Corollary 1. Let $A(t)$ and $B(t)$ be matrix s.p.'s such that $A(t) \in \mathrm{L}_{4p}^{n\times n}(\Omega)$, $B(t) \in \mathrm{L}_{2p}^{n\times n}(\Omega)$. Let $X_0 \in \mathrm{L}_{4p}^{n\times n}(\Omega)$ and let us suppose that the random linear system

$$Y'(t) = A(t)Y(t)\,, \quad Y(0) = I_n\,, \quad 0 \le t \le T\,, \tag{28}$$

is $4p$-regular, and

$$Z'(t) = (B(t))^\top \, Z(t)\,; \qquad Z(0) = I_n\,, \quad 0 \le t \le T\,, \tag{29}$$

is $2p$-regular. Then, the unique solution, $X : [0,T] \longrightarrow \mathrm{L}_p^{n\times n}(\Omega)$, of the random bilateral IVP

$$X'(t) = A(t)X(t) + X(t)B(t)\,, \qquad X(0) = X_0\,, \qquad 0 \le t \le T\,,$$

is given by

$$X(t) = \Phi_A(t;0)\, X_0\, \left(\Phi_B(t;0)\right)^\top\,, \qquad 0 \le t \le T\,,$$

where $\Phi_A(t;0)$ and $\Phi_B(t;0)$ denote the random fundamental matrix solutions of random IVP's (28)–(29), respectively.

Proof. Let $Y(t)$ and $Z(t)$ be the solution s.p.'s of the random IVP's (28) and (29) respectively, and consider the s.p. $X(t)$ defined by

$$X(t) = Y(t)\, X_0\, \left(Z(t)\right)^\top\,, \qquad 0 \le t \le T\,. \tag{30}$$

Considering the factorization $X(t) = (Y(t)X_0)\,(Z(t))^\top$ of (30) and applying Proposition 2 of [1], one follows

$$\begin{aligned} X'(t) &= (Y(t)X_0)'\,(Z(t))^\top + (Y(t)X_0)\left((Z(t))^\top\right)' \\ &= Y'(t)X_0\,(Z(t))^\top + Y(t)X_0\,(Z'(t))^\top \\ &= A(t)Y(t)X_0\,(Z(t))^\top + Y(t)X_0\,(Z(t))^\top\, B(t) \\ &= A(t)X(t) + X(t)B(t)\,. \end{aligned}$$

Notice that in the last step, we have applied (28) and (29). Moreover for the initial condition, $X(0)$, from (30), (28) and (29), one gets

$$X(0) = Y(0)X_0\,(Z(0))^\top = I_n X_0 (I_n)^\top = X_0\,.$$

Now, from Theorem 1 with $B(t)$ the null matrix of size $n \times n$, $B(t) = O_n$, we know that the solutions of random IVP's (28) and (29), are given by,

$$Y(t) = \Phi_A(t;0)\, I_n = \Phi_A(t;0)\,, \qquad Z(t) = \Phi_{B^\top}(t;0)\, I_n = \left(\Phi_B(t;0)\right)^\top\, I_n = \Phi_{B^\top}(t;0)\,, \tag{31}$$

respectively. Therefore, by (30), (31) and taking into account the p-Lipschitz property of $F(t,X) = A(t)\,X + X\,B(t)$ that guarantees the uniqueness, one gets

$$X(t) = Y(t)\, X_0\, \left(Z(t)\right)^\top = \Phi_A(t;0)\, X_0\, \left(\Phi_B(t;0)\right)^\top\,. \quad \square$$

4. Random non-autonomous Riccati matrix equation

Once random linear vector systems have been treated in the previous section, we are in a good situation to apply a random version of the linearization method developed in [2] and [4] to construct local solutions of the random time-dependent Riccati IVP (1). This approach may be regarded as a continuation of paper [1], where the random autonomous Riccati problem has been recently treated.

Consider the matrix s.p. $L(t)$ in $\mathrm{L}_{4p}^{(n+m)\times(n+m)}(\Omega)$ defined by

$$L(t) = \left[\begin{array}{c|c} A(t) & B(t) \\ \hline C(t) & -D(t) \end{array}\right], \tag{32}$$

and assume that the random matrix

$$Y_0 = \left[\begin{array}{c} I_n \\ \hline W_0 \end{array}\right], \tag{33}$$

lies in $\mathrm{L}_{4p}^{(n+m)\times n}(\Omega)$ and that random linear matrix IVP

$$Y'(t) = L(t)\, Y(t)\,, \qquad Y_0 = \left[\begin{array}{c} I_n \\ \hline W_0 \end{array}\right], \tag{34}$$

is $2p$-regular. Let us consider the block-partition of $Y(t)$ of the form

$$Y(t) = \left[\begin{array}{c} U(t) \\ \hline V(t) \end{array}\right], \qquad U(t) \in \mathrm{L}_{2p}^{n\times n}(\Omega)\,,\ V(t) \in \mathrm{L}_{2p}^{m\times n}(\Omega)\,. \tag{35}$$

Note that $U(0) = I_n$ and that if $U(t)$ is invertible in an ordinary neighbourhood of $t = 0$, $\mathcal{N}_U(0)$, and $(U(t))^{-1} \in \mathrm{L}_{2p}^{n\times n}(\Omega)$, then the s.p.

$$W(t) = V(t)\,(U(t))^{-1}\,, \qquad t \in \mathcal{N}_U(0)\,, \tag{36}$$

is well-defined and it lies in $\mathrm{L}_{p}^{m\times n}(\Omega)$. Assuming that $V(t)$ and $(U(t))^{-1}$ are $2p$-differentiable, by (36), Proposition 2 and Corollary 1 of [1], one gets that $W(t)$ is p-differentiable in $\mathcal{N}_U(0)$ with

$$W'(t) = V'(t)\,(U(t))^{-1} + V(t)\left((U(t))^{-1}\right)' = V'(t)\,(U(t))^{-1} - V(t)\,(U(t))^{-1}\,U'(t)\,(U(t))^{-1}\,, \quad t \in \mathcal{N}_U(0)\,. \tag{37}$$

Let us consider the block-partition of the random fundamental matrix solution $\Phi_L(t;0)$ of the random linear IVP (34), of the form

$$\Phi_L(t;0) = \left[\begin{array}{c|c} \Phi_{1,1}(t;0) & \Phi_{1,2}(t;0) \\ \hline \Phi_{2,1}(t;0) & \Phi_{2,2}(t;0) \end{array}\right], \tag{38}$$

with

$$\Phi_{1,1}(t;0) \in \mathrm{L}_{4p}^{n\times n}(\Omega),\ \Phi_{1,2}(t;0) \in \mathrm{L}_{4p}^{n\times m}(\Omega),\ \Phi_{2,1}(t;0) \in \mathrm{L}_{4p}^{m\times n}(\Omega),\ \Phi_{2,2}(t;0) \in \mathrm{L}_{4p}^{m\times m}(\Omega)\,. \tag{39}$$

From the definition of $2p$-regularity, (35) and (38) one gets

$$U(t) = \Phi_{1,1}(t;0) + \Phi_{1,2}(t;0)W_0\,; \quad V(t) = \Phi_{2,1}(t;0) + \Phi_{2,2}(t;0)W_0\,, \quad t \in \mathcal{N}_U(0)\,. \tag{40}$$

Then, $W(t)$ defined by (36), can be written in the form

$$W(t) = \left(\Phi_{2,1}(t;0) + \Phi_{2,2}(t;0)W_0\right)\left(\Phi_{1,1}(t;0) + \Phi_{1,2}(t;0)W_0\right)^{-1}\,, \quad t \in \mathcal{N}_U(0)\,. \tag{41}$$

From (32), (34), (35) and (37), it follows that

$$\begin{aligned} W'(t) &= V'(t)\,(U(t))^{-1} - V(t)\,(U(t))^{-1}\,U'(t)\,(U(t))^{-1} \\ &= \{C(t)U(t) - D(t)V(t)\}\,(U(t))^{-1} - V(t)\,(U(t))^{-1}\,U'(t)\,(U(t))^{-1} \\ &= C(t) - D(t)W(t) - W(t)\,\{A(t)U(t) + B(t)V(t)\}\,(U(t))^{-1} \\ &= C(t) - D(t)W(t) - W(t)A(t) - W(t)B(t)W(t)\,, \end{aligned}$$

with $W(0) = V(0)\,(U(0))^{-1} = W_0$. As factors $V(t)$ and $(U(t))^{-1}$ of $W(t)$, both lie in $\mathrm{L}_{2p}^{m\times n}(\Omega)$ and $\mathrm{L}_{2p}^{n\times n}(\Omega)$ respectively, then by Proposition 1 of [1] $W(t)$ lies in $\mathrm{L}_{p}^{m\times n}(\Omega)$.

Summarizing, the following result has been established:

Theorem 2. *Let us assume that matrix s.p.* $L(t)$ *defined by* (32)*, lie in* $\mathrm{L}_{4p}^{(n+m)\times(n+m)}(\Omega)$ *, and that the random matrix* Y_0 *defined by* (33)*, lie in* $\mathrm{L}_{4p}^{(n+m)\times n}(\Omega)$*. Let us further assume that the random linear matrix IVP* (34) *is* $2p$*-regular, and consider the block-entries* $\Phi_{i,j}(t;0)$ *of the random fundamental matrix solution* $\Phi(t;0)$ *defined by* (38)–(39)*. Let* $U(t)$ *and* $V(t)$ *be defined by* (40) *with* $U(0) = I_n$ *and* $V_0 = W_0 \in \mathrm{L}_{4p}^{m\times n}(\Omega)$*. If* $\mathcal{N}_U(0)$ *is an ordinary neighbourhood of* $t = 0$ *where* $U(t) \in \mathrm{L}_{2p}^{n\times n}(\Omega)$ *is* $2p$*-differentiable, invertible and* $(U(t))^{-1} \in \mathrm{L}_{2p}^{n\times n}(\Omega)$ *is* $2p$*-differentiable, then* $W(t)$ *defined by* (41) *is a solution of random Riccati IVP* (1) *in* $\mathrm{L}_{p}^{m\times n}(\Omega)$*.*

Remark 2. As it also occurs in the deterministic case, in dealing with non-autonomous IVP's, the fundamental matrix solution of a linear system is not available, in general. Thus, it is convenient to have the possibility of constructing reliable numerical approximations. Random linear multistep methods, for scalar problems, have been proposed in [15] and they can be extended to the random matrix framework in a similar way to the one developed in [4] in a non-trivial way. From the practical point of view, hereinafter we will consider the particular multistep matrix method (2.28) of [4]

$$Y_{k+1} - Y_k = \frac{h}{2}\left\{L(t_{k+1})Y_{k+1} - L(t_k)Y_k\right\}\,, \quad Y_0 = \left[\frac{I_n}{W_0}\right], \tag{42}$$

for solving the random linear IVP (21), where $t_{k+1} = t_k + h$, $0 \le k \le N-1$, $t_0 = 0$, $t_k \in [0, T]$, such that $Nh = T$. Solving (42), see (2.34) of [4] for small enough value of h, one gets the random approximations

$$\left.\begin{aligned} Y_0 &= \left[\frac{I_n}{W_0}\right], \\ Y_k &= \prod_{j=0}^{k-1}\left\{\left(I_{n+m} - \frac{h}{2}L(t_{k-j})\right)^{-1}\left(I_{n+m} + \frac{h}{2}L(t_{k-j-1})\right)\right\}Y_0\,, \quad 1 \le k \le N\,. \end{aligned}\right\} \tag{43}$$

Approximations (43) for the linear IVP (34) can be used to generate a sequence of approximations of the random non-autonomous Riccati IVP (1), see (2.40) of [4]. In fact, if $[I_n, O_{n\times m}]Y_k$ is invertible, being $O_{n\times m}$ the null matrix of size $n \times m$, and both $[O_{m\times n}, I_m]Y_k$ and $[I_n, O_{n\times m}]Y_k$ lie in $\mathrm{L}_{2p}^{m\times n}(\Omega)$ and $\mathrm{L}_{2p}^{n\times n}(\Omega)$, respectively, then

$$W_k = \{[O_{m\times n}, I_m]Y_k\}\,\{[I_n, O_{n\times m}]Y_k\}^{-1}\,, \qquad k = 1, 2, \ldots, N\,, \tag{44}$$

are random matrix approximations of the solution $W(t)$ of problem (1). This numerical procedure will be used in the subsequent section to compare the approximations of the mean and standard deviation of the solution s.p. to the random Riccati matrix IVP (1) constructed using the approach studied throughout this section.

5. Numerical examples

This section is devoted to illustrate the theoretical development previously exhibited by means of several examples where randomness is considered through a wide variety of probabilistic distributions. We emphasize that both scalar and random Riccati matrix differential equations are studied in the examples. Computations have been carried out using the software `Mathematica`.

Example 5. *Let us consider the following random scalar IVP based on a non-autonomuous Riccati differential equation*

$$w'(t) + a\,\exp(-t)\,(w(t))^2 - a\,\exp(-t) = 0, \quad 0 < t \le T, \qquad w(0) = w_0. \tag{45}$$

This IVP is a particular case of (1) *taking* $m = n = 1$ *and*

$$W(t) = w(t)\,, \quad W(0) = w_0\,, \quad A(t) = a\,, \quad B(t) = a\,\exp(-t)\,, \quad C(t) = a\,\exp(-t)\,, \quad D(t) = -a\,. \tag{46}$$

We will assume that both input parameters, a *and* w_0*, in the random IVP* (45)*, are independent, positive, and bounded or truncated r.v.'s defined in a common complete probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*. For the sake of clarity in the presentation, we split the construction of the approximations to the expectation and standard deviation of the solution s.p. to the random IVP* (45) *in several steps.*

Step 1. Construction of the auxiliary random linear vector IVP to (45)*.*

According to (32)–(34) *and* (46)*, in this example, we have the following extended random linear system*

$$Y'(t) = L(t)\,Y(t)\,, \quad Y_0 = \begin{bmatrix} 1 \\ \hline w_0 \end{bmatrix}, \tag{47}$$

where the matrix s.p. $L(t)$ *is defined by*

$$L(t) = \left[\begin{array}{c|c} A(t) & B(t) \\ \hline C(t) & -D(t) \end{array}\right] = \left[\begin{array}{c|c} a & a\,\exp(-t) \\ \hline a\,\exp(-t) & a \end{array}\right]. \tag{48}$$

As by hypothesis a *and* w_0 *both are either bounded or truncated r.v.'s, then the random vector* Y_0*, defined in* (47)*, lies in* $\mathrm{L}_{4p}^{2\times 1}(\Omega)$ *and the matrix s.p.* $L(t)$*, defined in* (48)*, is in* $\mathrm{L}_{4p}^{2\times 2}(\Omega)$ *for every* $t \in T$*.*

Moreover, as the matrix s.p. $L(t)$*, given by* (48)*, commutes with its integral, that is,*

$$\begin{aligned} L(t)\left(\int_0^t L(s)\,\mathrm{d}s\right) &= \begin{bmatrix} a^2\exp(-2t)\,(-1+\exp(t)+t\exp(2t)) & a^2\exp(-t)\,(-1+\exp(t)+t) \\ a^2\exp(-t)\,(-1+\exp(t)+t) & a^2\exp(-2t)\,(-1+\exp(t)+t\exp(2t)) \end{bmatrix} \\ &= \left(\int_0^t L(s)\,\mathrm{d}s\right) L(t)\,, \end{aligned}$$

then, it is known its random fundamental matrix solution $\Phi_L(t;0)$ *is given by (see [16])*

$$\Phi_L(t;0) = \exp\left(\int_0^t L(s)\,\mathrm{d}s\right).$$

Moreover, it can be seen that

$$\begin{aligned} \Phi_L(t;0) &= \left[\begin{array}{c|c} \Phi_{1,1}(t;0) & \Phi_{1,2}(t;0) \\ \hline \Phi_{2,1}(t;0) & \Phi_{2,2}(t;0) \end{array}\right] \\ &= \begin{bmatrix} \exp(at)\cosh\left[a(-1+\exp(-t)\right] & \exp(at)\sinh\left[a(1-\cosh(t)+\sinh(t))\right] \\ \exp(at)\sinh\left[a(1-\cosh(t)+\sinh(t))\right] & \exp(at)\cosh\left(a(-1+\exp(-t))\right) \end{bmatrix}. \end{aligned} \tag{49}$$

Now, we are going to check that the random linear vector IVP (47) *is* $2p$*-regular.*

- *The matrix s.p.* $L(t) = \left(\ell_{i,j}(t)\right)_{2\times 2}$*, given by* (48)*, only depends on the r.v.* a*, which is taken either bounded or truncated. Then, by Example 2 each entry* $\ell_{i,j}(t)$ *of* $L(t)$ *verifies condition* (24) *for every* $t \in T$*, and consequently,* $L(t) \in \mathrm{L}_{2p}^{2\times 2}(\Omega)$*, i.e., it is guaranteed that* $L(t)$ *is* $2p$*-absolutely integrable in* $[0, T]$*:*

$$\int_0^T \|\ell_{i,j}(t)\|_{2p}\,\mathrm{d}t = \int_0^T \left(\mathrm{E}[|\ell_{i,j}(t)|^{2p}]\right)^{1/(2p)}\,\mathrm{d}t \le \int_0^T \left(m_{i,j}(h_{i,j})^{2p}\right)^{1/(2p)}\,\mathrm{d}t = (m_{i,j})^{1/(2p)} h_{i,j}\,T < +\infty\,.$$

- *It can be checked that the inverse,* $\Phi_L^{-1}(t;0)$*, of the random fundamental matrix solution* $\Phi_L(t;0)$ *defined by* (49)*, exists in an ordinary neighbourhood of* $t = 0$ *and it takes the following form*

$$\Phi_L^{-1}(t;0) = \begin{bmatrix} \exp(-at)\cosh\left[a(1-\cosh(t)+\sinh(t))\right] & -\exp(-at)\sinh\left[a(1-\cosh(t)+\sinh(t))\right] \\ -\exp(-at)\sinh\left[a(1-\cosh(t)+\sinh(t))\right] & \exp(-at)\cosh\left[a(1-\cosh(t)+\sinh(t))\right] \end{bmatrix}. \tag{50}$$

  Notice that $\Phi_L^{-1}(0;0) = I_2$*. Moreover, based on the same argument previously shown about boundedness of the random input parameter* a*, it is easy to check that* $\Phi_L(t;0)$ *and* $\Phi_L^{-1}(t;0)$ *both lie in* $\mathrm{L}_p^{2\times 2}(\Omega)$*.*

- *Using an analogous argument to the one exhibited in the Example 1, it is straightforward to prove the* p*-differentiability of matrices s.p.'s* $\Phi_L(t;0)$ *and* $\Phi_L^{-1}(t;0)$ *defined by* (49) *and* (50)*, respectively.*

Step 2. Construction of the solution s.p. of the random scalar Riccati IVP (45).

According to (36), (38), (40) *and* (49), *the solution s.p. of the scalar random Riccati* (45), $w(t)$, *can be expressed in a closed form by terms of the random parameters a and* w_0

$$
\begin{aligned}
w(t) &= V(t)\,(U(t))^{-1} = \frac{\Phi_{2,1}(t;0) + \Phi_{2,2}(t;0)w_0}{\Phi_{1,1}(t;0) + \Phi_{1,2}(t;0)w_0} \\
&= \frac{\exp(at)\,\{w_0\cosh[a(-1+\exp(-t)] + \sinh[a(1-\cosh(t)+\sinh(t))]\}}{\exp(at)\cosh[a(-1+\exp(-t)] + \exp(at)w_0\sinh[a(1-\cosh(t)+\sinh(t))]}\,, \quad t \in \mathcal{N}_U(0)\,.
\end{aligned}
$$

Note that the parameter w_0 *lies in* $\mathrm{L}_{4p}(\Omega)$ *as well as the four block-entries* $\Phi_{i,j}(t;0)$, $1 \le i, j \le 2$, *of the random fundamental matrix solution* $\Phi_L(t;0)$ *given by* (49).

Finally, taking into account the hypotheses of Theorem 2, it remains to check that $U(t) \in \mathrm{L}_{2p}(\Omega)$ *is* $2p$*-differentiable and invertible and that its inverse* $(U(t))^{-1} \in \mathrm{L}_{2p}(\Omega)$ *is also* $2p$*-differentiable. These conditions can be checked following an analogous reasoning like the one showed in Example 1. We here omit because its checking is only cumbersome.*

Step 3. Computation of the expectation of solution s.p. of (45).

Denote by $f_a(a)$ *and* $f_{w_0}(w_0)$ *the probability density functions of r.v.'s a and* w_0, *respectively. Compute the expectation of* $w(t)$ *as follows*

$$\mathrm{E}[w(t)] = \int_{\mathbb{R}^2} w(t) f_a(a) f_{w_0}(w_0)\,\mathrm{d}a\,\mathrm{d}w_0\,.$$

Step 4. Computation of the standard deviation of solution s.p. of (45).

Determine the standard deviation by the expression

$$\sigma[w(t)] = +\sqrt{\mathrm{E}\left[(w(t))^2\right] - (\mathrm{E}[w(t)])^2}\,,$$

computing, firstly, the following expectation

$$\mathrm{E}\left[(w(t))^2\right] = \int_{\mathbb{R}^2} (w(t))^2 f_a(a) f_{w_0}(w_0)\,\mathrm{d}a\ \mathrm{d}w_0\,.$$

In Figure 1 and Figure 2, the expectation, $\mathrm{E}[w(t)]$, *and the expectation plus/minus the standard deviation,* $\mathrm{E}[w(t)] \pm \sigma[w(t)]$, *of the solution s.p. to the random scalar Riccati IVP* (45) *for different choices of the input r.v.'s a and* w_0 *have been plotted.*

Example 6. *Let us consider the random Riccati IVP* (1) *for the following election of the data*

$$
\left.
\begin{aligned}
W(t) &= \begin{bmatrix} w_1(t) \\ w_2(t) \\ w_3(t) \end{bmatrix}, \quad W_0 = \begin{bmatrix} 1 \\ w_{2,0} \\ 0 \end{bmatrix}, \quad A(t) = \frac{t^2}{2}\,a\,, \quad B(t) = t^2 \begin{bmatrix} -\frac{1}{2} & 0 & \frac{b}{2} \end{bmatrix}, \\
C(t) &= t^2 \begin{bmatrix} 0 \\ \frac{-1}{2} \\ 0 \end{bmatrix}, \quad D(t) = \frac{t^2}{2} \begin{bmatrix} -1 & 0 & 0 \\ 0 & d & 0 \\ 1 & 0 & 1 \end{bmatrix}.
\end{aligned}
\right\}. \tag{51}
$$

We will assume that the input parameters a, b, d and $w_{2,0}$ *are r.v.'s. The parameter a has a beta distribution of parameters* $\alpha = 3$ *and* $\beta = 2$, $a \sim Be(3;2)$; *b has an exponential distribution of parameter* $\lambda = 1$ *truncated at the interval* $[1,2]$, $b \sim Exp_{[1,2]}(1)$; *d has a uniform distribution on the interval* $[2,4]$, $d \sim U(2,4)$ *and, finally,* $w_{2,0}$ *has a beta distribution of parameters* $\alpha = 1$ *and* $\beta = 2$, $w_{2,0} \sim Be(1;2)$. *We will assume that all the input parameters are independent r.v.'s.*

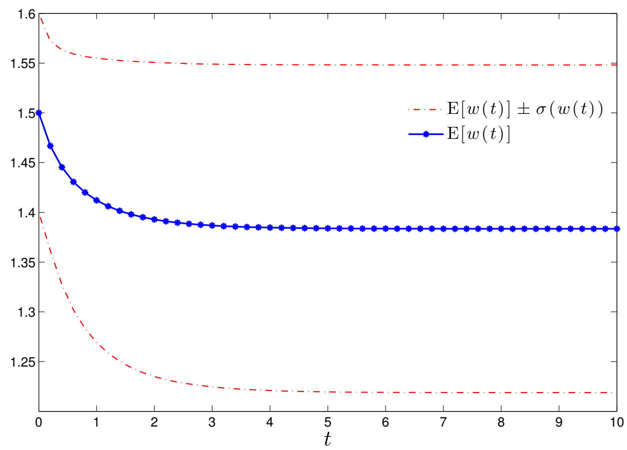

Figure 1: The expectation, $\mathrm{E}[w(t)]$, and the expectation plus/minus the standard deviation, $\mathrm{E}[w(t)] \pm \sigma[w(t)]$, of the solution s.p. to the random scalar Riccati IVP (45) for the following choice of the input r.v.'s: $a \sim \mathrm{Be}(0.2; 1)$ (a has a beta distribution of parameters $(0.2; 1)$) and $w_0 \sim \mathrm{N}_{[1,2]}(1.5; 0.1)$ (w_0 has a gaussian distribution of parameters $(1.5; 0.1)$ truncated on the interval $[1, 2]$). The expectation has been plotted on the time domain $t \in [0, 10]$ in the context of Example 5.

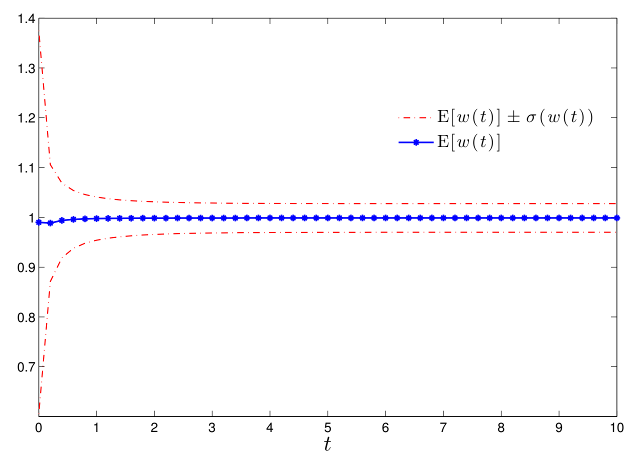

Figure 2: The expectation, $\mathrm{E}[w(t)]$, and the expectation plus/minus the standard deviation, $\mathrm{E}[w(t)] \pm \sigma[w(t)]$, of the solution s.p. to random scalar Riccati IVP (45) for the following choice of the input r.v.'s: $a \sim \mathrm{Gamma}(2; 3)$ (a has a gamma distribution of parameters $(2; 3)$) and $w_0 \sim \mathrm{Exp}_{[0.5,2]}(1.5)$ (w_0 has an exponential distribution of parameter $\lambda = 1.5$ truncated on the interval $[0.5, 2]$). The expectation has been plotted on the time domain $t \in [0, 10]$ in the context of Example 5.

Step 1. Construction of the auxiliary random linear vector IVP of (1) *with the data* (51)*.*

The extended random linear vector system (32)–(34)*, associated to* (1) *with data* (51)*, takes the form*

$$Y'(t) = L(t)\, Y(t)\,, \qquad Y_0 = \left[\begin{array}{c} 1 \\ \hline 1 \\ w_{2,0} \\ 0 \end{array}\right]. \tag{52}$$

Note that, it is verified that the random vector Y_0, defined in (52)*, lies in $\mathrm{L}_{4p}^{4\times 1}(\Omega)$ because Y_0 satisfies condition* (24) *since $w_{2,0}$ is a bounded r.v.*

In Eq. (52) *we have chosen the matrix s.p. $L(t)$ as the product of the real continuous deterministic function, $f(t) = t^2/2$, and the following random matrix L verifying condition* (24)

$$L = \begin{bmatrix} a & -1 & 0 & b \\ 0 & 1 & 0 & 0 \\ -1 & 0 & -d & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix},$$

because its entries a, b and d are bounded r.v.'s. Hence, the random coefficient matrix $L(t)$ takes the form

$$L(t) = f(t)\, L = \frac{t^2}{2}\, L, \qquad L(t) \in \mathrm{L}_{4p}^{4\times 4}(\Omega).$$

The block-partition of $L(t)$ is given by

$$L(t) = \left[\begin{array}{c|c} A(t) & B(t) \\ \hline C(t) & -D(t) \end{array}\right] = \left[\begin{array}{c|ccc} \frac{a\,t^2}{2} & -\frac{t^2}{2} & 0 & \frac{b\,t^2}{2} \\ \hline 0 & \frac{t^2}{2} & 0 & 0 \\ -\frac{t^2}{2} & 0 & -\frac{d\,t^2}{2} & 0 \\ 0 & -\frac{t^2}{2} & 0 & -\frac{t^2}{2} \end{array}\right]. \tag{53}$$

As we shown in Example 4, the random linear vector IVP (52) *with $L(t) = f(t)\, L$ is $2p$-regular, and the random fundamental matrix solution, $\Phi_L(t; 0)$, is given by*

$$\begin{aligned} \Phi_L(t;0) &= \exp\left(L \int_0^t f(s)\,\mathrm{d}s\right) = \exp\left(L \int_0^t \frac{s^2}{2}\,\mathrm{d}s\right) = \exp\left(L\,\frac{t^3}{6}\right) \\ &= \left[\begin{array}{c|c} \Phi_{1,1}(t;0)_{1\times 1} & \Phi_{1,2}(t;0)_{1\times 3} \\ \hline \Phi_{2,1}(t;0)_{3\times 1} & \Phi_{2,2}(t;0)_{3\times 3} \end{array}\right]. \end{aligned} \tag{54}$$

It can be seen that, the block-entries $\Phi_{i,j}(t;0)$, $1 \le i, j, \le 2$, of $\Phi_L(t;0)$ in (54) *are*

$$\Phi_{1,1}(t;0)_{1\times 1} = \exp(at^3/6), \tag{55}$$

$$\Phi_{1,2}(t;0)^{\top}{}_{3\times 1} = \begin{bmatrix} \dfrac{\exp(-t^3/6)\left\{b - ab + (2 + 2a + b + ab)\exp(t^3/3) - 2(1 + a + b)\exp(1/6(1+a)t^3)\right\}}{2(-1 + a^2)} \\ 0 \\ \dfrac{b\exp(-t^3/6)(-1 + \exp(1/6(1+a)t^3))}{1 + a} \end{bmatrix}, \tag{56}$$

$$\Phi_{2,1}(t;0)_{3\times 1} = \begin{bmatrix} 0 \\ -\dfrac{\exp(at^3/6) - \exp(-dt^3/6)}{a + d} \\ 0 \end{bmatrix}, \tag{57}$$

$$\Phi_{2,2}(t;0)_{3\times 3} = \begin{bmatrix} \Phi_{2,2}^1(t;0) & \Phi_{2,2}^2(t;0) & \Phi_{2,2}^3(t;0) \end{bmatrix}, \tag{58}$$

where the column vectors $\Phi_{2,2}^j(t;0)$, $1 \le j \le 3$, of block-entry $\Phi_{2,2}(t;0)$ in (58) *are the following expressions*

$$\Phi^{1}_{2,2}(t;0) = \begin{bmatrix} \exp(t^3/6) \\ \frac{b\exp(-t^3/6)}{2(1+a)(-1+d)} - \frac{(2+b)\exp(t^3/6)}{2(-1+a)(1+d)} + \frac{(1+a+b)\exp(at^3/6)}{(-1+a^2)(a+d)} + \frac{(-1-b+d)\exp(-dt^3/6)}{(a+d)(-1+d^2)} \\ -\sinh(t^3/6) \end{bmatrix}, \tag{59}$$

$$\Phi^{2}_{2,2}(t;0) = \begin{bmatrix} 0 \\ \exp(-dt^3/6) \\ 0 \end{bmatrix}, \tag{60}$$

$$\Phi^{3}_{2,2}(t;0) = \begin{bmatrix} 0 \\ -\frac{b\exp(-1/6)(1+d)t^3)\left\{(1+a)\exp(t^3/6)-(a+d)\exp(dt^3/6)+(-1+d)\exp(1/6(1+a+d)t^3)\right\}}{(1+a)(-1+d)(a+d)} \\ \exp(-t^3/6) \end{bmatrix}. \tag{61}$$

The p-differentiability of matrix s.p. $\Phi_L(t;0)$ *given by* (54)–(61) *and its inverse can be justified following an analogous reasoning to the one shown in the Example 1.*

Step 2. Construction of the solution s.p. of the random Riccati IVP (1) *with data given in* (51).

According to (36), (40) *and* (54)–(61)*, the solution s.p.,* $W(t) = [w_1(t)\ w_2(t)\ w_3(t)]^\top$*, of the random Riccati IVP* (1) *with the data* (51)*, can be expressed in a closed form as follows*

$$\begin{bmatrix} w_1(t) \\ w_2(t) \\ w_3(t) \end{bmatrix} = V(t)(U(t))^{-1} = \left(\Phi_{2,1}(t;0) + \Phi_{2,2}(t;0)\begin{bmatrix} 0 \\ w_{2,0} \\ 0 \end{bmatrix}\right)_{3\times 1} \left(\Phi_{1,1}(t;0) + \Phi_{1,2}(t;0)\begin{bmatrix} 0 \\ w_{2,0} \\ 0 \end{bmatrix}\right)^{-1}_{1\times 1}. \tag{62}$$

Note that parameter $W_0 = \begin{bmatrix} 0 \\ w_{2,0} \\ 0 \end{bmatrix} \in \mathrm{L}^{3\times 1}_{4p}(\Omega)$ *because r.v.* $w_{2,0}$ *verifies condition* (24) *since it is bounded.*

Again the $2p$*-differentiable of the s.p.* $U(t) \in \mathrm{L}_{2p}(\Omega)$

$$U(t) = \frac{\exp(-t^3/6)}{2(-1+a^2)}\left\{2(1+a)\left[\exp(t^3/3) + (-2+a)\exp(1/6(1+a)t^3)\right] \right.$$
$$\left. + b\left[1 - a + (1+a)\exp(t^3/3) - 2\exp(1/6(1+a)t^3)\right]\right\},$$

and its inverse, $(U(t))^{-1} \in \mathrm{L}_{2p}(\Omega)$*, follows in broad outline the same arguments shown in Example 1. Here, details are omitted because they are quite cumbersome.*

Step 3. Computation of the expectation of the solution s.p. of the IVP (1) *with the data given in* (51).

Compute the expectation of each one of the three components of the solution s.p. $W(t) = [w_1(t)\ w_2(t)\ w_3(t)]^\top$ *obtained from* (62)*, as follows*

$$\mathrm{E}[w_i(t)] = \int_{\mathbb{R}^4} w_i(t) f_a(a) f_b(b) f_d(d) f_{w_{2,0}}(w_{2,0})\, \mathrm{d}a\, \mathrm{d}b\, \mathrm{d}d\, \mathrm{d}w_{2,0}, \qquad i = 1, 2, 3, \tag{63}$$

where we denote by, $f_a(a)$, $f_b(b)$, $f_d(d)$ *and* $f_{w_{2,0}}(w_{2,0})$*, the probability density functions of r.v.'s* a*,* b*,* d *and* $w_{2,0}$*, respectively.*

Step 4. Computation of the standard deviation of solution s.p. of the IVP (1) *with data given in* (51).

Compute the following expectations

$$\mathrm{E}\left[(w_i(t))^2\right] = \int_{\mathbb{R}^4} (w_i(t))^2 \, f_a(a) f_b(b) f_d(d) f_{w_{2,0}}(w_{2,0}) \, \mathrm{d}a \, \mathrm{d}b \, \mathrm{d}d \, \mathrm{d}w_{2,0}, \quad i = 1,2,3\,.$$

Afterwards, computing the standard deviations according to

$$\sigma\left[w_i(t)\right] = +\sqrt{\mathrm{E}\left[(w_i(t))^2\right] - \left(\mathrm{E}\left[w_i(t)\right]\right)^2}, \quad i = 1,2,3\,, \tag{64}$$

where $\mathrm{E}\left[w_i(t)\right]$ *is given by* (63)*.*

In Figure 3, we have plotted the expectations $\mathrm{E}[w_i(t)]$*,* $i = 1,2,3$*, and plus/minus the standard deviations,* $\mathrm{E}[w_i(t)] \pm \sigma[w_i(t)]$*,* $i = 1,2,3$*, of the three components of the vector solution s.p.* $W(t) = [w_1(t)\; w_2(t)\; w_3(t)]^\top$*, given by* (62)*, of the random Riccati IVP* (1) *with the data* (51)*.*

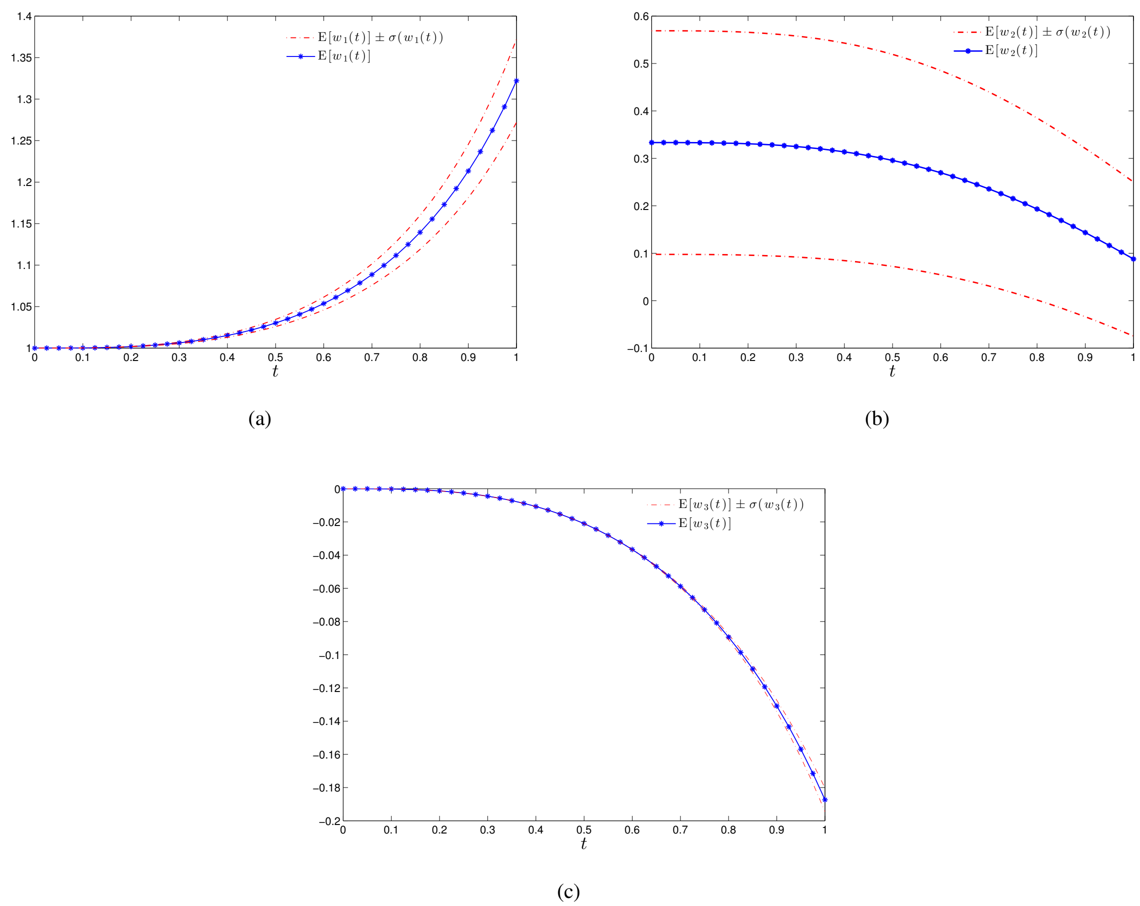

Figure 3: Evolution of the expectations $\mathrm{E}[w_1(t)]$ (plot(a)), $\mathrm{E}[w_2(t)]$ (plot(b)) and $\mathrm{E}[w_3(t)]$ (plot(c)), of the solution s.p. $W(t) = [w_1(t)\; w_2(t)\; w_3(t)]^\top$ of the Riccati (1), given by (62), on the time domain $t \in [0,1]$ in the context of Example 6.

Finally, we are going to compare the values of expectation and standard deviation of the solution s.p. $W(t)$*, defined in* (62) *as a closed form, versus the numerical approximations,* W_k*, obtained by the particular random multistep matrix method* (43)–(44)*. Note that in* (43)*, it must be guaranteed the existence of the inverse of the matrices*

$$\left(I_{n+m} - \frac{h}{2} L(t_{k-j})\right), \quad 1 \le k-j \le N\,, \;\; n = 1,\; m = 3\,, \tag{65}$$

where matrix s.p.'s $L(t_{k-j})$ are defined by (53)*. In fact, the matrices of* (65) *are invertible due to the positivity of the time-step h and the r.v.'s a, b and d.*

In Table 1, we collected the exact values of the expectations and standard deviations, in a fixed time T (so we use the so-called "approximation in the fixed station sense"), for the three components of solution s.p. $W(t) = [w_1(t)\, w_2(t)\, w_3(t)]^\top$*, denoted by* $\mathrm{E}[w_i(t)]$*,* $i = 1, 2, 3$*, and* $\sqrt{Var[w_i(T)]}$*,* $i = 1, 2, 3$*, respectively. Those values have been compared with their respective numerical expectations and numerical standard deviations, denoted by* $\mathrm{E}[w_{i,N}(T)]$ *and* $\sqrt{Var[w_{i,N}(T)]}$*, respectively, in the same fixed time* $T = Nh$*, considering* $N = 50$ *fixed. Then, for the following values of* $T \in \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1\}$*, and* $N = 50$*, the time-step h has been determined. The components* $w_{i,N}(T)$*,* $i = 1, 2, 3$*, of the numerical solution* $W_k(T)$*, have been computed in each time instant T using* (43)–(44) *for* $k = N = 50$*, that is*

$$W_N = \{[O_{3\times1}, I_3]Y_N\}\ \{[I_1, O_{1\times3}]Y_N\}^{-1}\ , \quad N = 50\,, \tag{66}$$

where the time T is reached. Note that in (66)*, the scalar r.v.* $[I_1, O_{1\times3}]Y_N$ *is invertible, and both* $[O_{3\times1}, I_3]Y_N$ *and* $[I_1, O_{1\times3}]Y_N$ *lie in* $\mathrm{L}_{2p}^{3\times1}(\Omega)$ *and* $\mathrm{L}_{2p}(\Omega)$*, respectively.*

In Table 1, the numerical values of the relative errors for the expectations, $RelErr_{\mu_i}(T)$*,* $i = 1, 2, 3$*, and the standard deviations,* $RelErr_{\sigma_i}(T)$*,* $i = 1, 2, 3$*, have been computed according to the following expressions*

$$RelErr_{\mu_i}(T) = \left| \frac{\mathrm{E}[w_i(T)] - \mathrm{E}[w_{i,N}(T)]}{\mathrm{E}[w_i(T)]} \right|, \qquad RelErr_{\sigma_i}(T) = \left| \frac{\sqrt{Var[w_i(T)]} - \sqrt{Var[w_{i,N}(T)]}}{\sqrt{Var[w_i(T)]}} \right|, \quad i = 1, 2, 3\,. \tag{67}$$

Computations have been carried out using different fixed stations T and time steps h. From the numerical values, we observe that both relative errors, for every component of the solution s.p., take very small values. This shows that the numerical values for the expectation and the standard deviations obtained from the closed form solution (62) *are quite good.*

6. Conclusions

In this paper one completes the closed form solution of the random non-autonomous Riccati matrix type IVP's, initiated in [1] for the autonomous case. The study of the random non-autonomous matrix linear case has required the random analogous of the Abel-Liouville-Jacobi's formula that is interesting itself and will be used in forthcoming works. The potential application to develop numerical methods starting from the analytic solution has been shown through appropriate results and numerical examples.

Acknowledgements

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		$\mathrm{E}[w_i(T)]$	$\mathrm{E}[w_{i,N}(T)]$	$\mathrm{RelErr}_{\mu_i}(T)$	$\sqrt{\mathrm{Var}[w_i(T)]}$	$\sqrt{\mathrm{Var}[w_{i,N}(T)]}$	$\mathrm{RelErr}_{\sigma_i}(T)$
	$i=1$	1.0002e+00	1.0002e+00	4.6728e-08	0	0	0
$T=0.1$	$i=2$	3.3302e-01	3.3302e-01	0	2.3560e-01	2.3560e-01	3.0000e-07
($h=0.002$)	$i=3$	-1.6668e-04	-1.6671e-04	2.0001e-04	0	0	0
	$i=1$	1.0019e+00	1.0019e+00	3.7725e-07	0	0	0
$T=0.2$	$i=2$	3.3085e-01	3.3085e-01	1.4000e-06	2.3489e-01	2.3489e-01	3.0000e-07
($h=0.004$)	$i=3$	-1.3340e-03	-1.3343e-03	2.0011e-04	0	0	0
	$i=1$	1.0063e+00	1.0064e+00	1.3048e-06	8.4591e-04	8.4600e-04	2.3257e-04
$T=0.3$	$i=2$	3.2499e-01	3.2499e-01	4.8000e-06	2.3297e-01	2.3297e-01	2.0000e-06
($h=0.006$)	$i=3$	-4.5083e-03	-4.5092e-03	2.0041e-04	3.7895e-06	3.7910e-06	4.3172e-04
	$i=1$	1.0152e+00	1.0152e+00	3.2403e-06	2.1527e-03	2.1532e-03	2.0995e-04
$T=0.4$	$i=2$	3.1375e-01	3.1375e-01	1.1000e-05	2.2931e-01	2.2931e-01	4.8000e-06
($h=0.008$)	$i=3$	-1.0714e-02	-1.0716e-02	2.0113e-04	2.2719e-05	2.2728e-05	4.0788e-04
	$i=1$	1.0302e+00	1.0303e+00	6.8135e-06	4.3292e-03	4.3302e-03	2.1189e-04
$T=0.5$	$i=2$	2.9569e-01	2.9569e-01	2.0100e-05	2.2348e-01	2.2348e-01	8.7000e-06
($h=0.01$)	$i=3$	-2.1022e-02	-2.1027e-02	2.0273e-04	8.8339e-05	8.8375e-05	4.0785e-04
	$i=1$	1.0537e+00	1.0537e+00	1.3066e-05	7.7370e-03	7.7387e-03	2.2050e-04
$T=0.6$	$i=2$	2.6978e-01	2.6977e-01	3.0600e-05	2.1524e-01	2.1524e-01	1.1000e-05
($h=0.012$)	$i=3$	-3.6600e-02	-3.6607e-02	2.0611e-04	2.6874e-04	2.6885e-04	4.1359e-04
	$i=1$	1.0886e+00	1.0886e+00	2.3758e-05	1.2835e-02	1.2838e-02	2.3680e-04
$T=0.7$	$i=2$	2.3556e-01	2.3555e-01	3.8000e-05	2.0459e-01	2.0458e-01	1.3600e-05
($h=0.014$)	$i=3$	-5.8807e-02	-5.8819e-02	2.1294e-04	6.9563e-04	6.9365e-04	2.8458e-03
	$i=1$	1.1395e+00	1.1396e+00	4.1890e-05	2.0500e-02	2.0505e-02	2.6277e-04
$T=0.8$	$i=2$	1.9321e-01	1.9320e-01	3.1800e-05	1.9180e-01	1.9180e-01	6.2000e-06
($h=0.016$)	$i=3$	-8.9393e-02	-8.9413e-02	2.2617e-04	1.6113e-03	1.6089e-03	1.4978e-03
	$i=1$	1.2134e+00	1.2135e+00	7.2653e-05	3.2097e-02	3.2107e-02	3.0664e-04
$T=0.9$	$i=2$	1.4355e-01	1.4355e-01	1.5000e-05	1.7747e-01	1.7747e-01	1.4800e-05
($h=0.018$)	$i=3$	-1.3088e-01	-1.3091e-01	2.5006e-04	3.4613e-03	3.4638e-03	7.2867e-04
	$i=1$	1.3220e+00	1.3222e+00	1.2535e-04	5.0139e-02	5.0158e-02	3.8335e-04
$T=1$	$i=2$	8.7824e-02	8.7842e-02	2.0260e-04	1.6245e-01	1.6246e-01	6.8930e-05
($h=0.02$)	$i=3$	-1.8738e-01	-1.8743e-01	2.9568e-04	7.1046e-03	7.1104e-03	8.1559e-04

Table 1: Values of the exact expectations, $\mathrm{E}[w_i(T)]$, $i = 1,2,3$, and exact standard deviations, $\sqrt{\mathrm{Var}[w_i(T)]}$, $i = 1,2,3$, using (63)–(64), for the three components of the solution s.p., $W(T)$, given by (62), to the random Riccati matrix IVP (1) in the context of Example 6. These values are computed in some time instants $T \in \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1\}$ using the corresponding time-step h such as $Nh = T$ for $N = 50$ fixed. Numerical expectations, $\mathrm{E}[w_{i,N}(T)]$, $i = 1,2,3$, and numerical standard deviations, $\sqrt{\mathrm{Var}[w_{i,N}(T)]}$, $i = 1,2,3$, of the vector numerical solution $W_N(T)$, computed by (43)–(44) and (66), are shown too. To compare the numerical values of both approximations to the expectation and the standard deviation, their relative errors, $\mathrm{RelErr}_{\mu_i}(T)$, $i = 1,2,3$ and $\mathrm{RelErr}_{\sigma_i}(T)$, $i = 1,2,3$, respectively, have also been computed using (67).

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