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ORIGINAL PAPER

The uniform bounded deciding property and the separable quotient problem

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Abstract Saxon–Wilansky's paper *The equivalence of some Banach space problems* contains six properties equivalent to the existence of an infinite dimensional separable quotient in a Banach space with nice simplified proofs. In the frame of uniform bounded deciding property, we prove that for an infinite dimensional Banach space $(E, \|\cdot\|)$ the following properties are equivalents: 1) The unit sphere S_E contains a dense and non uniform bounded deciding subset. 2) The unit sphere S_E contains a dense and non strong norming subset. 3) $(E, \|\cdot\|)$ admits an infinite dimensional separable quotient.

Keywords Banach space · Separable quotient problem · Strong norming subset · Uniform bounded deciding subset

Mathematics Subject Classification 54B15 · 46B20

1 Introduction

In 1932 Mazur asked whether an infinite dimensional Banach space admits an infinite dimensional separable quotient. Saxon and Wilansky showed in their wonderful paper [10] six properties equivalent to the existence of an infinite dimensional separable quotient. They claim that most of these properties were known and they present simplified proofs to get all equivalences. The new equivalence obtained in [10] states that an infinite dimensional Banach space has a separable quotient if and only if it has a dense non-barrelled subspace.

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In this paper we present a direct proof of this equivalence in the context of uniform bounded deciding sets introduced in [2] (see Proposition 1 and Theorem 1).

For this paper to be selfcontained we give some properties of uniform bounded deciding subsets of a normed space in Sect. 2. In Sect. 3 we introduce the strong norming property and we prove its equivalence with the uniform bounded deciding property (Proposition 4).

Finally, in the last section we present a direct proof of the characterization of the existence of an infinite dimensional separable quotient in terms of uniform bounded deciding subsets.

2 Uniform bounded deciding sets

Unless otherwise stated we will suppose that each subspace F of a normed space $(E, \|\cdot\|)$ is endowed with the induced norm, denoted by $\|\cdot\|$, that the norm of the dual space $(E^*, \|\cdot\|)$ is the polar norm and that the scalar field is $\mathbb{R}$ or $\mathbb{C}$. $B_E(0, 1) := \{x \in E : \|x\| \leqslant 1\}$ is the closed unit ball of center 0 and radius 1. The unit sphere of center 0 is $S_E := \{x \in E : \|x\| = 1\}$.

Definition 1 A subset C of a normed space $(E, \|\cdot\|)$ is a uniform bounded deciding set (in brief *ubd* set) if each C-pointwise bounded subset M of the dual space $(E^*, \|\cdot\|)$ is norm bounded, i.e.,

$$\sup_{f\in M} |f(x)| < \infty,\ \forall x \in C \Longrightarrow \sup_{f\in M} \|f\| = \sup_{\substack{f\in M\\ x\in B_E(0,1)}} |f(x)| < \infty.$$

The norm bounded condition of M may be replaced by M is uniformly bounded in a bounded neighborhood of zero.

Example 1 By the Banach–Steinhaus theorem the unit sphere S_E of a Banach space $(E, \|\cdot\|)$ is an *ubd* set. If $(E, \|\cdot\|)$ is a reflexive Banach space then the set of exposed points is an *ubd* set [3], hence the set of extreme points of a reflexive Banach space E is an *ubd* set.

Example 2 Let $\mathscr{A}$ be an algebra of subsets of a set Ω and let $(L(\mathscr{A}), \|\cdot\|)$ be the linear span of the characteristiques functions e_A, $A \in \mathscr{A}$. The dual of $(L(\mathscr{A}), \|\cdot\|)$ with the polar norm is isomorphically isometric to the normed space $(ba(\mathscr{A}), \|\cdot\|)$ of bounded finitely additive measures with the variation norm. By the isomorphism we identify each measure $\mu \in ba(\mathscr{A})$ with the linear form defined on $L(\mathscr{A})$, named also by μ, such that $\mu(e_A) := \mu(A)$.

A subset $\mathscr{B}$ of the algebra $\mathscr{A}$ has Nikodym property ([8] and [12]) if for each subset M of $(ba(\mathscr{A}), \|\cdot\|)$

$$\sup_{\mu\in M} |\mu(A)| < \infty,\ \forall A \in \mathscr{B} \Longrightarrow \sup_{\substack{\mu\in M\\ A\in\mathscr{A}}} |\mu(A)| < \infty.$$

By [11, Propositions 1 and 2] we have

$$\sup_{A\in\mathscr{A}} |\mu(A)| \leqslant \|\mu\| = \sup_{g\in B_{L(\mathscr{A})}(0,1)} |\mu(g)| \leqslant 4 \sup_{A\in\mathscr{A}} |\mu(A)|,$$

hence a subset $\mathscr{B}$ of the algebra $\mathscr{A}$ has Nikodym property if and only if

$$\sup_{\mu\in M} |\mu(e_A)| < \infty,\ \forall A \in \mathscr{B} \Longrightarrow \sup_{\substack{\mu\in M\\ g\in B_{L(\mathscr{A})}(0,1)}} |\mu(g)| < \infty.$$

This prove that a subset $\mathscr{B}$ of the algebra $\mathscr{A}$ has Nikodym property if and only if the set of characteristiques functions $\{e_A : A \in \mathscr{B}\}$ is an *ubd* subset of $(L(\mathscr{A}), \|\cdot\|)$. In [6,8,12] there are examples of algebras $\mathscr{A}$ of subsets of a set Ω that have Nikodym property and then the set $\{e_A : A \in \mathscr{A}\}$ is an *ubd* subset of $(L(\mathscr{A}), \|\cdot\|)$.

Example 3 The classical Nikodym–Grothendieck theorem states that each σ-á lgebra $\mathscr{A}$ of subsets of a set Ω has Nikodym property. Valdivia proves in [11] that if $\{\mathscr{A}_{n_1} : n_1 \in \mathbb{N}\}$ is an increasing covering of a σ-algebra $\mathscr{A}$ then there exists a natural number m_1 such that $\mathscr{A}_{m_1}$ has Nikodym property; this property is named strong Nykodym property in [4] and [5], where additionally it is proved that if $\{\mathscr{A}_{n_1,n_2,\cdots,n_p} : n_i \in \mathbb{N}, i \in \mathbb{N}\}$ is an increasing web of a σ-álgebra $\mathscr{A}$, i.e., $\mathscr{A} = \cup\{\mathscr{A}_{n_1} : n_1 \in \mathbb{N}\}$ and $\mathscr{A}_{n_1,n_2,\cdots,n_p} = \{\mathscr{A}_{n_1,n_2,\cdots,n_p,n_{p+1}} : n_{p+1} \in \mathbb{N}\}$ for each $(n_1, n_2, \cdots, n_p) \in \cup\{\mathbb{N}^s : s \in \mathbb{N}\}$, there exists a sequence $\{m_q : q \in \mathbb{N}\}$ such that each $\mathscr{A}_{m_1,m_2,\cdots,m_q}$, $q \in \mathbb{N}$, has Nikodym property. Then each set

$$\{e_A : A \in \mathscr{A}_{m_1,m_2,\cdots,m_q}\},$$

is an *ubd* subset of $(L(\mathscr{A}), \|\cdot\|)$.

From *ubd* definition it follows that if two subsets C and D of a normed space $(E, \|\cdot\|)$ have the same linear span then C is *ubd* if and only if D is *ubd*. In particular, for a subset C of a normed space $(E, \|\cdot\|)$ the following three conditions are equivalent:

1. C is *ubd*.
2. $C\backslash\{0\}$ is *ubd*.
3. $\{\|x\|^{-1} x : x \in C\backslash\{0\}\}$ is *ubd*.

Hence we may restrict our attention to the *ubd* sets that are subsets of the unit sphere $S_E := \{x \in E : \|x\| = 1\}$ of a normed space $(E, \|\cdot\|)$.

Note that if C is an *ubd* set of a normed space $(E, \|\cdot\|)$ then span C is a dense subset of $(E, \|\cdot\|)$, because if there would exists $x \in B_E(0, 1)\backslash\overline{\text{span}\, C}$ then, by Hahn–Banach theorem, there exists $f_n \in E^*$ such that $f_n(x) = n$ and $f_n(\text{span}\, C) = \{0\}$, for each $n \in \mathbb{N}$. Then

$$\sup_{n\in\mathbb{N}} |f_n(x)| = 0,\ \forall x \in C \text{ and } \sup_{n\in\mathbb{N}} \|f_n\| = \infty,$$

is a contradiction with the fact that C is an *ubd* set.

The following proposition gives a natural characterization of ubd sets by barrelledness. Recall that a normed space $(F, \|\cdot\|)$ is barrelled if $(F, \|\cdot\|)$ verifies the Banach Steinhaus theorem, i.e., each F pointwise bounded subset H of the dual space $(F^*, \|\cdot\|)$ is norm bounded. By polarity this property is equivalent to the fact that each absorbent, absolutely convex and closed subset of $(F, \|\cdot\|)$ is a neighborhood of zero in $(F, \|\cdot\|)$. Each absorbent, absolutely convex and closed subset of a topological vector space is called a *barrel*. Therefore a normed space $(F, \|\cdot\|)$ is barrelled if each barrel is neigborhood of zero.

In the proof of the next proposition we will use the well known fact that if $F := \text{span}\, C$ is a dense subspace of $(E, \|\cdot\|)$ then the map $\varphi : (E^*, \|\cdot\|) \to (F^*, \|\cdot\dot{\|})$ defined by the restriction to F is an isomorphism isometric. Therefore we may identify $(E^*, \|\cdot\|)$ and $(F^*, \|\cdot\dot{\|})$.

Proposition 1 *A subset C of a normed space $(E, \|\cdot\|)$ is ubd if and only if* $(\text{span}\, C, \|\cdot\|)$ *is a dense and barrelled subspace of* $(E, \|\cdot\|)$.

Proof Let's suppose that C is an *ubd* subset of $(E, \|\cdot\|)$. Then $F := \text{span}\, C$ is a dense subspace of $(E, \|\cdot\|)$. Let $\varphi : (E^*, \|\cdot\|) \to (F^*, \|\cdot\dot{\|})$ be the restriction to F and H a subset of F^* pointwise bounded in C. As C is an *ubd* subset then $\varphi^{-1}(H)$ is norm bounded in $(E, \|\cdot\|)$, hence H is a norm bounded subset of $(F, \|\cdot\|) = (\text{span}\, C, \|\cdot\|)$ and we get that $(\text{span}\, C, \|\cdot\|)$ is barrelled.

To prove the converse, let us suppose that $(F, \|\cdot\|) = (\operatorname{span} C, \|\cdot\|)$ is a dense and barrelled subspace of $(E, \|\cdot\|)$. If H is a subset of $(E^*, \|\cdot\|)$ pointwise bounded on C then, by barrelledness, $\varphi(H)$ is norm bounded in $(F^*, \|\cdot\|)$. As φ is an isometry H is a norm bounded subset of $(E^*, \|\cdot\|)$. Therefore C is an *ubd* subset. □

3 Strong norming sets

Let D be an absorbing absolutely convex subset of a normed space $(E, \|\cdot\|)$. It is well known that the Minkowski functional p_D of D

$$p_D(x) := \inf\{\lambda : \lambda \in \mathbb{R}^+,\ x \in \lambda D\},$$

verfies that

$$\{x \in E : p_D(x) < 1\} \subset D \subset \{x \in E : p_D(x) \leqslant 1\}.$$

Hence p_D defines in E a norm equivalent to $\|\cdot\|$ if and only if D is a bounded neighborhood of $(E, \|\cdot\|)$.

Recall that the polar of a subset C of $(E, \|\cdot\|)$ is the set

$$C^{\mathrm{o}} := \{x^* \in E^* : \left|x^*(x)\right| \leqslant 1, \forall x \in C\},$$

and the bipolar

$$C^{\mathrm{oo}} := \{x \in E : \left|x^*(x)\right| \leqslant 1, \forall x^* \in C^{\mathrm{o}}\},$$

verifies that $C^{\mathrm{oo}} = \overline{\operatorname{abcx} C}$, i.e., the closure of the absolutely convex hull of C, where the absolutely convex hull of C is the set

$$\operatorname{abcx} C := \{\Sigma_{i=1}^n \lambda_i x_i : x_i \in C,\ \Sigma_{i=1}^n |\lambda_i| \leqslant 1, n \in \mathbb{N}\}.$$

From the trivial fact that in a dual pair $\langle E, E^*\rangle$ a set is bounded if and only if its polar is a neighborhood of 0 and from the equality $C^{\mathrm{ooo}} = C^{\mathrm{o}}$ it follows that for a subset C of a normed space $(E, \|\cdot\|)$ the following equivalent conditions are equivalent:

1. $C^{\mathrm{oo}} = \overline{\operatorname{abcx} C}$ is a bounded neighborhood of $(E, \|\cdot\|)$, i.e., there exists $0 < r < R$ such that
$$\{x : x \in E,\ \ \|x\| < r\} \subset C^{\mathrm{oo}} \subset \{x : x \in E,\ \ \|x\| < R\}. \tag{1}$$
2. The polar set C^{o} is a bounded neighborhood of $(E^*, \|\cdot\|)$.

By polarity, this equivalence also follows by the equivalence between (1) and the next relation (2)

$$\left\{x^* : x^* \in E,\ \ \left\|x^*\right\| < R^{-1}\right\} \subset C^{\mathrm{o}} \subset \left\{x^* : x^* \in E^*,\ \ \left\|x^*\right\| < r^{-1}\right\}. \tag{2}$$

These properties motivated the following definition.

Definition 2 A subset C of a normed space $(E, \|\cdot\|)$ is a norming set if its bipolar, $C^{\mathrm{oo}} = \overline{\operatorname{abcx} C}$, is a bounded neighborhood of zero in $(E, \|\cdot\|)$.

Hence C is norming if and only if it verifies (1) and if and only if the Minkowsky functional $p_{C^{\mathrm{oo}}}$ is a norm equivalent to the norm of $(E, \|\cdot\|)$. From the equivalence between (1) and (2) it directly follows the next proposition.

Proposition 2 *For a subset C of a normed space $(E, \|\cdot\|)$ the following conditions are equivalent:*

1. *C is norming.*
2. *C^{o} is norming.*
3. *C and C^{o} are bounded subsets of $(E, \|\cdot\|)$ and $(E^*, \|\cdot\|)$, respectively.*
4. *C^{oo} and C^{o} are neighborhoods of the origin in $(E, \|\cdot\|)$ and in $(E^*, \|\cdot\|)$, respectively.*

The next proposition proves that the projection of a norming set on the unit sphere is a norming set.

Proposition 3 *If C is a norming subset of a normed space $(E, \|\cdot\|)$ then $\{\|x\|^{-1} x : x \in C\backslash\{0\}\}$ is a norming subset of the unit sphere S_E.*

Proof We may suppose that C verifies (1). Then

$$\{x : x \in E, \ \ \|x\| < rR^{-1}\} \subset R^{-1}C^{\mathrm{oo}} \subset \{x : x \in E, \ \ \|x\| < 1\},$$

hence $R^{-1}\|x\| \leqslant 1$, for each $x \in C^{\mathrm{oo}}$, and, in particular $R^{-1} \leqslant \|x\|^{-1}$, for each $x \in C\backslash\{0\}$. Then

$$R^{-1}C^{\mathrm{oo}} = R^{-1}(C\backslash\{0\})^{\mathrm{oo}} \subset \{\|x\|^{-1} x : x \in C\backslash\{0\}\}^{\mathrm{oo}} \subset \{x : x \in E, \ \ \|x\| \leqslant 1\},$$

and we get that $\{\|x\|^{-1} x : x \in C\backslash\{0\}\}$ is a norming subset of the unit sphere S_E because:

$$\{x : x \in E, \ \ \|\cdot\| < rR^{-1}\} \subset \{\|x\|^{-1} x : x \in C\backslash\{0\}\}^{\mathrm{oo}} \subset \{x : x \in E, \ \ \|\cdot\| \leqslant 1\}.$$

□

Definition 3 A subset C of a normed space $(E, \|\cdot\|)$ is strong norming (s-norming, in brief) if each increasing covering $(C_m)_m$ of C contains a norming set C_n.

Each s-norming is norming, hence it is bounded and the next proposition characterizes strong norming subsets as bounded subsets with *ubd* property.

Proposition 4 *A subset C of a normed space $(E, \|\cdot\|)$ is strong norming if and only if C is a bounded ubd set.*

Proof Let C be an s-norming subset of $(E, \|\cdot\|)$. We know that C is bounded. If M is a C-pointwise bounded subset of $(E^*, \|\cdot\|)$ and

$$C_m := \{x \in C : |f(x)| \leqslant m, \ \ \forall f \in M\},$$

then

$$m^{-1}M \subset C_m^{\mathrm{o}}, \tag{3}$$

and $\{C_m : m \in \mathbb{N}\}$ is an increasing covering of the s-norming set C, hence there exists C_n that is a norming set and then, by Proposition 2, the polar set C_n^{o} is a bounded subset of $(E^*, \|\cdot\|)$. By (3) with $m = n$ we get that M is a bounded subset of $(E^*, \|\cdot\|)$, hence C is an *ubd* subset of $(E, \|\cdot\|)$.

Let's suppose that C is not a s-norming subset of $(E, \|\cdot\|)$. If C is unbounded the proof is done. Therefore we may suppose that C is a bounded set with an increasing covering $(C_m)_m$ of non norming subsets. By Proposition 2 each C_m^{o} is an unbounded subset of $(E^*, \|\cdot\|)$, hence there exists $f_m \in C_m^{\mathrm{o}}$ such that $\|f_m\| > m$. As the norm unbounded sequence $(f_m)_m$ is pointwise bounded on C then C is not an *ubd* subset of $(E, \|\cdot\|)$. □

This Proposition implies that if C is a non *ubd* subset of the unit sphere of a normed space then C contains a non norming subset.

4 Application: a note on the separable quotient open problem

As mentioned in the Introduction, in the next theorem we present a direct proof of the characterization of the existence of an infinite dimensional separable quotient in a Banach space in terms of uniform bounded deciding subsets.

Theorem 1 *For an infinite dimensional Banach space $(E, \|\cdot\|)$ the following properties are equivalents:*

1. $(E, \|\cdot\|)$ *contains a dense subspace F that is non-ubd.*
2. $(E, \|\cdot\|)$ *admits an infinite dimensional separable quotient.*

Proof Let us suppose that F is a dense and non-*ubd* subspace of $(E, \|\cdot\|)$. Then there exists an unbounded subset H in $(E^*, \|\cdot\|)$ that is pointwise bounded in F. Hence the closed absolutely convex set $B := H^{\circ}$ is not neighborhood of 0 in $(E, \|\cdot\|)$ and span B is dense in $(E, \|\cdot\|)$, because $F \subset \operatorname{span} B$.

It is well known that the codimension of span B in $(E, \|\cdot\|)$ cannot be countable, because if $\{x_n : n \in \mathbb{N}\}$ were a cobase of span$B$ in E then the equality

$$E = \cup\{n \operatorname{abcx}(B \cup \{x_1, x_2, \dots, x_n\}) : n \in \mathbb{N}\},$$

and Baire theorem imply that there exists $p \in \mathbb{N}$ such that $B \cup \{x_1, x_2, \dots, x_p\}$ is a neigborhood of 0 in $(E, \|\cdot\|)$. Then B is a neighborhood of 0 in span$B$. Therefore, as B is a closed subset of $(E, \|\cdot\|)$ and span B is dense in $(E, \|\cdot\|)$, we get the contradiction that B would be a neighborhood of 0 in $(E, \|\cdot\|)$, which is a contradiction.

The infinite codimension of span B in E and the Hahn–Banach separation theorem enables us to obtain two sequences $\{x_n : n \in \mathbb{N}\}$ in S_E and $\{f_n : n \in \mathbb{N}\}$ in E^* such that:

1. $f_n(x_n) = 1$, for $n \geqslant 1$.
2. $x_1 \in S_E \backslash \operatorname{span} B$ and $x_n \in (S_E \cap f_1^{\perp} \cap f_2^{\perp} \cap \cdots \cap f_{n-1}^{\perp}) \backslash \operatorname{span}\, (B \cup \{x_1, x_2, \cdots, x_{n-1}\})$, for $n > 1$.
3. $|f_1(x)| \leqslant 2^{-1}$, for each $x \in B$, and $|f_n(x + a_1x_1 + a_2x_2 + \cdots + a_{n-1}x_{n-1})| \leqslant 2^{-n}$, for each $x \in B$ and each $|a_i| \leqslant 1$, $1 \leqslant i < n$.

In fact, we select $x_1 \in S_E \backslash \operatorname{span} B$ and apply Hahn–Banach separation theorem to $\{x_1\}$ and $2B$ to determine $f_1 \in E^*$ such that $f_1(x_1) = 1$ and $|f_1(v)| \leqslant 1$, for each $v \in 2B$. The first step of the inductive process is done, because

$$|f_1(x)| \leqslant \frac{1}{2}, \text{ for each } x \in B. \tag{4}$$

Let us suppose that we have determined $\{x_m : m < n\}$ and $\{f_m : m < n\}$ for some $n > 1$. As the codimension of span B in $(E, \|\cdot\|)$ is infinite we may select

$$x_n \in \left(S_E \cap f_1^{\perp} \cap f_2^{\perp} \cap \cdots \cap f_{n-1}^{\perp}\right) \backslash \operatorname{span}\, (B \cup \{x_1, x_2, \dots, x_{n-1}\})\,,$$

and apply Hahn–Banach separation theorem to $\{x_n\}$ and

$$2^n \{B + a_1x_1 + a_2x_2 + \cdots + a_{n-1}x_{n-1} : |a_i| \leqslant 1, 1 \leqslant i < n\}\,,$$

to determine $f_n \in E^*$ such that $f_n(x_n) = 1$ and $|f_n(v)| \leqslant 1$, for each

$$v \in 2^n \{B + a_1x_1 + a_2x_2 + \cdots + a_{n-1}x_{n-1} : |a_i| \leqslant 1, 1 \leqslant i < n\}\,.$$

Then the inductive process is done because

$$|f_n(x + a_1x_1 + a_2x_2 + \cdots + a_{n-1}x_{n-1})| \leqslant \frac{1}{2^n}, \tag{5}$$

for each $x \in B$ and each $|a_i| \leqslant 1$, $1 \leqslant i < n$.

Again, with a new induction, we determine for each $x \in B$ a convergente series $\Sigma_{i=1}^{\infty} a_i x_i$, with $|a_i| \leqslant 2^{-i}$, $i \in \mathbb{N}$, such that

$$x + \Sigma_{i=1}^{\infty} a_i x_i \in \cap \left\{ f_n^{\perp} : n \in \mathbb{N} \right\}.$$

In fact, from $f_1(x_1) = 1$ and (4) we deduce that

$$f_1\,(x + a_1x_1) = f_1(x) + a_1 = 0,$$

is verified if and only

$$a_1 = -f_1(x), \text{ with } |a_1| \leqslant 2^{-1}.$$

Suppose that a_i, $1 \leqslant i < n$, has been determined such that $|a_i| \leqslant 2^{-i}$ and

$$f_i\,(x + a_1x_1 + \cdots + a_ix_i) = 0, \text{ for each } i < n.$$

Then, from $f_n(x_n) = 1$ and (5) we deduce that the equality

$$f_n(x + a_1x_1 + \cdots + a_{n-1}x_{n-1} + a_nx_n) = f_n(x + a_1x_1 + \cdots + a_{n-1}x_{n-1}) + a_n = 0, \tag{6}$$

is verified if and only if

$$a_n = -f_n(x + a_1x_1 + \cdots + a_{n-1}x_{n-1}), \text{ with } |a_n| \leqslant 2^{-n}.$$

To finish the induction we only need to notice that by construction $\Sigma_{i=n+1}^{\infty} a_i x_i \in f_n^{\perp}$, for each $n \in \mathbb{N}$, hence, by (6) we deduce that

$$f_n(x + \Sigma_{i=1}^{\infty} a_i x_i) = f_n(x + \Sigma_{i=1}^{n} a_i x_i) + f(\Sigma_{i=n+1}^{\infty} a_i x_i) = 0.$$

Finally, let

$$z_p := x + \Sigma_{i=p+1}^{\infty} a_i x_i.$$

Then

$$\left\| z_p - x \right\| \leqslant \Sigma_{i=p+1}^{\infty} |a_i| \leqslant \frac{1}{2^p} \Rightarrow x = \lim_p z_p.$$

Let $F := \bigcap\{f_n^{\perp} : n \in \mathbb{N}\}$. From

$$z_p = -\Sigma_{i=1}^{p} a_i x_i + x + \Sigma_{i=1}^{\infty} a_i x_i \in \operatorname{span}\{x_i, i \in \mathbb{N}\} + F,$$

we deduce that

$$B \subset \overline{\operatorname{span}\{x_i, i \in \mathbb{N}\} + F}.$$

This relation and the density of span B in $(E, \|\cdot\|)$ imply that $\operatorname{span}\{x_i, i \in \mathbb{N}\} + F$ is a dense subspace of $(E, \|\cdot\|)$. Consequently, if φ is the quotient map of $(E, \|\cdot\|)$ onto the quotient space $(E/F, \|\cdot\|_{E/F})$ then $(\varphi(\operatorname{span}\{x_i, i \in \mathbb{N}\}), \|\cdot\|_{E/F})$ is a separable infinite dimensional subspace of $(E/F, \|\cdot\|_{E/F})$, where $\|\cdot\|_{E/F}$ is the quotient norm.

Conversely, suppose that F is a closed subspace of $(E, \|\cdot\|)$ such that $(E/F, \|\cdot\|_{E/F})$ is a separable infinite dimensional Banach space. Then there exists in S_E a sequence $\{x_n : n \in \mathbb{N}\}$ of linear independent vectors such that the sum of span $\{x_n : n \in \mathbb{N}\}$ and F is direct and span $\{x_n : n \in \mathbb{N}\} \oplus F$ is dense in $(E, \|\cdot\|)$. By the Hahn–Banach theorem for each $n \in \mathbb{N}$ there exists $f_n \in E^*$ such that $f_n(x_{n+1}) = n\,\|x_n\|$ and $f_n(\{x_1, \cdots, x_n\} + F) = \{0\}$.

Then span $\{x_n : n \in \mathbb{N}\} \oplus F$ is a non-*ubd* subset, because the norm-unbounded sequence $\{f_n : n \in \mathbb{N}\}$ is pointwise bounded in span $\{x_n : n \in \mathbb{N}\} \oplus F$. $\square$

From this theorem and Proposition 4 follows the next corollary.

Corollary 1 *For an infinite dimensional Banach space $(E, \|\cdot\|)$ the following properties are equivalent:*

1. *The unit sphere S_E contains a dense and non* ubd *subset.*
2. *The unit sphere S_E contains a dense and non strong norming subset.*
3. *$(E, \|\cdot\|)$ admits an infinite dimensional separable quotient.*

A sequence $(y_n)_n$ in the dual E^* of a Banach space $(E, \|\cdot\|)$ is *pseudobounded* if it is point-wise bounded on a dense subspace F of E and $\sup_n \|y_n\| = \infty$ [9]. It is obvious that E^* contains a pseudobounded sequence if and only if S_E contains a dense and non ubd subset. Therefore from last Corollary we get that an infinite dimensional Banach space $(E, \|\cdot\|)$ has an infinite dimensional separable quotient if and only if its dual E^* contains a pseudobounded squence. This equivalence is contained in [9, Theorem 3].

In [1, Theorem 15] is proved that every infinite dimensional dual Banach space $(E^*, \|\cdot\|)$ has an infinite dimensional separable quotient. From Theorem 1 follows that the unit sphere S_{E^*} contains a non ubd subset. This result motivates the following problem.

Problem 1 Obtain a direct method to determine a dense and non *ubd* subset in the unit sphere S_{E^*} of the dual of a Banach space $(E, \|\cdot\|)$.

References

1. Argyros, S.A., Dodos, P., Kanellopoulos, V.: Unconditional families in Banach spaces. Math. Ann. **341**, 15–38 (2008)
2. Fernández, J., Hui, S., Shapiro, H.: Unimodular functions and uniform boundedness. Publ. Mat. **33**, 139–146 (1989)
3. Font, V.P.: On exposed and smooth points of convex bodies in Banach spaces. Bull. London Math. Soc. **28**, 51–58 (1996)
4. Kąkol, J., López-Pellicer, M.: On Valdivia strong version of Nikodym boundedness property. J. Math. Anal. Appl. **446**, 1–17 (2017)
5. López-Alfonso, S., Mas, J., Moll, S.: Nikodym boundedness property and webs in σ-algebras. RACSAM Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. **110**, 711–722 (2016)
6. López-Alfonso, S.: On Schachermayer and Valdivia results in algebras of Jordan measurable sets. RACSAM Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. **110**, 799–808 (2016)
7. Nygaard, O.: A strong uniform boundedness principle in Banach spaces. Proc. Am. Math. Soc. **129**, 861–863 (2001)
8. Schachermayer, W.: On some classical measure-theoretic theorems for non-sigma-complete Boolean algebras. Dissertationes Math. (Rozprawy Mat.) **214**, 33 (1982)
9. Śliwa, W.: The separable quotient problem and the strongly normal sequences. J. Math. Soc. Jpn. **64**, 387–397 (2012)
10. Saxon, S.A., Wilansky, A.: The equivalence of some Banach space problems. Colloq. Math. **37**, 217–226 (1977)
11. Valdivia, M.: On certain barrelled normed spaces. Ann. Inst. Fourier **29**, 39–56 (1979)
12. Valdivia, M.: On Nikodym boundedness property. RACSAM Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. **107**, 355–372 (2013)