Closed-form eigensolutions of nonviscously, nonproportionally damped systems based on continuous damping sensitivity

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Abstract

In this paper, nonviscous, nonproportional, vibrating structures are considered. Nonviscously damped systems are characterized by dissipative mechanisms which depend on the history of the response velocities via hereditary kernel functions. Solutions of the free motion equation lead to a nonlinear eigenvalue problem involving mass, stiffness and damping matrices. Viscoelasticity leads to a frequency dependence of this latter. In this work, a novel closed-form expression to estimate complex eigenvalues is derived. The key point is to consider the damping model as perturbed by a continuous fictitious parameter. Assuming then the eigensolutions as function of this parameter, the computation of the eigenvalues sensitivity leads to an ordinary differential equation, from whose solution arises the proposed analytical formula. The resulting expression explicitly depends on the viscoelasticity (frequency derivatives of the damping function), the non-proportionality (influence of the modal damping matrix off-diagonal terms). Eigenvectors are obtained using existing methods requiring only the corresponding eigenvalue. The method is validated using a numerical example which compares proposed with exact ones and with those determined from the linear first order approximation in terms of the damping matrix. Frequency response functions are also plotted showing that the proposed approach is valid even for moderately or highly damped systems.

Keywords: nonviscous damping, nonproportionality, eigenvalues and eigenvectors, closed-form expression, nonclassical damping, symmetric systems, viscoelasticity

1. Introduction

Nonviscous damping materials are widely used for vibration control within many applications of mechanical, civil and aeronautical engineering. These type of energy dissipation devices can also be known as viscoelastic damping. The physical modeling of vibrating structures under viscoelastic damping results in a complex problem since energy dissipation is characterized by hereditary mechanisms: damping forces are function of the time-history of the velocity response. In mathematical terms, this behavior is represented by convolution integrals involving the degrees-of-freedom (dof) velocities over certain kernel functions. Hence, time-domain response is governed by the following system of linear integro-differential equations

$$\mathbf{M}\ddot{\mathbf{u}} + \int_{-\infty}^{t} \boldsymbol{\mathcal{G}}(t-\tau)\dot{\mathbf{u}} \, \mathrm{d}\tau + \mathbf{K}\mathbf{u} = \mathfrak{f}(t) \quad , \quad \mathbf{u}(0) = \mathbf{u}_0 \ , \quad \dot{\mathbf{u}}(0) = \dot{\mathbf{u}}_0 \tag{1}$$

where $\mathbf{u}(t) \in \mathbb{R}^n$ represents the array containing the n dof's, $\mathbf{M} \in \mathbb{R}^{n\times n}$ and $\mathbf{K} \in \mathbb{R}^{n\times n}$ are the mass and stiffness matrices. We assume $\mathbf{M}$ to be positive definite and $\mathbf{K}$ positive semidefinite; $\boldsymbol{\mathcal{G}}(t) \in \mathbb{R}^{n\times n}$ is the viscoelastic damping matrix in the time domain containing the hereditary kernel functions, which must satisfy the necessary conditions given by Golla and Hughes [1] to induce a dissipative behavior. The viscous

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damping can be considered as a particular case with $\boldsymbol{\mathcal{G}}(t) \equiv \mathbf{C}\,\delta(t)$, where $\mathbf{C}$ is the viscous damping matrix and $\delta(t)$ the Dirac's delta function. The time-domain response governed by Eqs. (1) is closely related to the eigensolutions of the associated nonlinear eigenvalue problem [2]. Due to this nonlinearity (induced by a frequency-dependent damping matrix), the search of eigensolutions is in general much more expensive from a computational point o view than that of classical viscous damping. In this paper, our challenge is to deduce closed-form approximations which, on one hand, takes into account the main features of a nonviscously damped system (viscoelasticity and nonproportionality) and, on the other hand, it only requires the computational complexity needed for solving the undamped eigenproblem (natural frequencies and normal modes).

The Laplace transform of the free-motion equation (1) leads to a nonlinear eigenvalue problem in the frequency domain. This nonlinearity arises from the frequency dependency of the viscoelastic function in the Laplace domain, $\mathbf{G}(s) = \mathcal{L}\{\boldsymbol{\mathcal{G}}(t)\}$. In general, the s-dependent functions within the damping matrix $\mathbf{G}(s)$ can be of different nature as long as they satisfy the necessary conditions given by Golla and Hughes [1] to describe a real dissipative motion. However, two viscoelastic models have been traditionally used for practical applications: nonviscous models based on exponential kernels proposed by Biot [3] and those based on the fractional derivatives studied by Bagley and Torvik [4, 5].

Several methods to solve the general nonlinear eigenvalue problem exist in the bibliography. Ruhe [6], Yang [7] and Singh [8] proposed methodologies based on the Taylor series expansion of the transcendental matrices combined with Newton's eigenvalue iteration method. Williams and Kennedy [9] obtained numerical solutions using on the parabolic interpolation of the determinant of the eigenvalue problem. Daya and Potier-Ferry [10], Duigou et al. [11] and Boudaoud et al. [12] developed techniques based on the asymptotic perturbation theory to determine complex frequencies and eigenvectors. Voss [13, 14] developed two algorithms based on the shift-and-invert Arnoldi's technique and on the Jacobi-Davidson method, respectively. References [15, 16, 17] describe how to transform multiple dof systems based on the Biot's model into a into a extended linear system, which can be solved using state-space techniques. For lightly nonproportional systems, Adhikari and Pascual [18, 19] published an iterative method based on the first and second order Taylor series expansion of the modal damping function. Lázaro et al. [20] proposed a recursive approach using the fixed-point iteration. References [21, 22] exploits the damping parameters as mathematical variables in certain domain obtaining solutions for both proportional and nonproportional systems. In the same direction, Lázaro et al. [23] derived a closed from expression for the complex eigenvalues of frame structures with viscoelasic layers based on fractional derivatives and assuming light nonproportionality. In these works the derivatives of the eigensolutions respect of certain damping parameter play a special role. The generalization of derivatives of eigenvalues and eigenvectors for viscoelastic structures was analyzed by Adhikari [24, 25]. Cortés and Elejebarrieta [26, 27] used Adhikari's solutions in an recursive numerical approach, valid even for highly damped systems. Li et al. [28, 29] proposed a new method for eigensensitivity analysis based on a new form of normalization. Singh [30] has proposed recently a new numerical approach to estimate simultaneously eigenvalues and eigenvectors using a iterative scheme. Lewandowski [31] developed a recursive numerical method using a perturbation parameter, valid for a special type of viscoelastic damper based on fractional derivatives.

In the present paper, a closed-form expression of the complex eigenvalues for nonviscously nonproportionally damped symmetric systems is derived. In the bibliography, numerous methods based on iterative procedures are provided. Those most relevant are described in the previous paragraph. In this work, we appeal the added value of having analytical forms valid for any nonviscous damping model independently on its nature. In fact, our derivations lead to formulas which explicitly depend on the entrees of the modal damping matrix and on its s-derivatives. We find two advantages in our proposal respect to those methods based on iterative schemes: On one hand, we dispose of a mathematical expression which is explicitly expressing how the eigenvalues depend on the damping parameters. And, on the other hand, the only computational requirements are those needed for solving the natural frequencies and the normal modes of the undamped problem. Recently, Lázaro [32] has deduced a closed-form expression valid for nonproportionally viscously symmetric damped systems. The current work generalizes that paper introducing the viscoelasticity. The method is validated through a multiple degrees-of-freedom system with various damping models with different nature, considering two levels of damping. Additionally, we compare the eigenvalues and frequency response functions with those determined using the linear first order approximation proposed by Woodhouse [33].

2. Eigensolutions of nonviscous and nonproportional systems

In general, the set of eigenvalues and eigenvectors of a linear dynamic system contains itself the complete information needed to construct frequency- and time-domain solutions. The free motion equations are obtained from $\mathfrak{f}(t) \equiv \mathbf{0}$ and $\mathfrak{u}_0 = \dot{\mathfrak{u}}_0 = \mathbf{0}$ in Eq. (1). Checking solutions of the form $\mathfrak{u}(t) = \bar{\mathbf{u}}e^{st}$ we obtain

$$\left[s^2\mathbf{M} + s\mathbf{G}(s) + \mathbf{K}\right]\bar{\mathbf{u}} \equiv \mathbf{D}(s)\bar{\mathbf{u}} = \mathbf{0} \tag{2}$$

where $\mathbf{D}(s) \in \mathbb{C}^{n\times n}$ is the dynamic stiffness matrix. The main difference between viscous and nonviscous systems is found in the nature of the solution of Eq. (2). Assuming that there are not repeated eigenvalues, nonviscous systems are characterized by having $m = 2n + r$ eigenvalues arranged as

$$\{s_1, \ldots, s_n, s_1^*, \ldots, s_n^*, s_{2n+1}, \ldots, s_{2n+r}\} \tag{3}$$

The subset $\{s_j, s_j^*\}$, $1 \leq j \leq n$ are n pairs of complex-conjugate eigenvalues, under the hypothesis that no overdamped modes exist. The rest $\{s_j\}$, $2n+1 \leq j \leq m$ are negative real eigenvalues, characteristic of nonviscous damping governed by a Biot's dissipative model [34, 35].

The jth eigenvector associated to eigenvalue s_j verifies

$$\mathbf{D}(s_j)\,\mathbf{u}_j = \mathbf{0}\ , \quad 1 \leq j \leq m \tag{4}$$

where $(\bullet)^T$ denotes the matrix transpose. In the context of nonviscously damped systems, the $2n$ complex eigensolutions $\{s_j, \mathbf{u}_j, s_j^*, \mathbf{u}_j^*\}_{j=1}^n$ are known as *elastic modes*, while $\{s_j, \mathbf{u}_j\}_{j=2n+1}^m$ are *nonviscous modes* without oscillatory nature. In the references [36, 37], two methods to determine these eigenvalues can be found for proportional and nonproportional systems, respectively.

The undamped eigenmodes play an important role in the construction of the damped solution. For $\boldsymbol{\mathcal{G}}(t) \equiv \mathbf{0}$ we define the undamped dynamic stiffness matrix as $\mathbf{D}_u(s) = s^2\mathbf{M} + \mathbf{K}$. The natural frequencies ω_j, $1 \leq j \leq n$ are the roots of $\det\left[\mathbf{D}_u(\mathrm{i}\omega)\right] = 0$. Associated to each natural frequency there exist undamped eigenvectors defined as the subspace solution of the ill-conditioned linear systems

$$\mathbf{D}_u(\mathrm{i}\omega_j)\,\mathbf{x}_j = \mathbf{0}\ , \quad 1 \leq j \leq n \tag{5}$$

where $\mathrm{i} = \sqrt{-1}$ denotes the imaginary unit. From the previous equations the orthogonality relations for symmetric systems can be derived [38] obtaining

$$\mathbf{x}_k^T\mathbf{M}\,\mathbf{x}_j = \delta_{kj}\ , \quad \mathbf{x}_k^T\mathbf{K}\,\mathbf{x}_j = \delta_{kj}\omega_j^2\ , \quad 1 \leq j, k \leq n \tag{6}$$

where δ_{kj} is the Kronecker delta. Once the complete set of eigenmodes (eigenvalues and eigenvectors) are determined, analytical closed-form expressions of both frequency- and time-domain solutions are available from the work of Adhikari [2]. Denoting by $\mathbf{u}(s) = \mathcal{L}\{\mathfrak{u}(t)\}$ and $\mathbf{f}(s) = \mathcal{L}\{\mathfrak{f}(t)\}$ to the Laplace transforms of response and external force, then we have

$$\mathbf{u}(s) = \mathbf{H}(s)\left[\mathbf{f}(s) + \mathbf{M}\dot{\mathfrak{u}}_0 + s\mathbf{M}\mathfrak{u} + \mathbf{G}(s)\mathfrak{u}_0\right] \tag{7}$$

$$\mathfrak{u}(t) = \sum_{j=1}^{n}\left[\gamma_j\psi_j(t)\mathbf{u}_j + \gamma_j^*\psi_j^*(t)\mathbf{u}_j^*\right] + \sum_{j=2n+1}^{m}\left[\gamma_j\psi_j(t)\mathbf{u}_j\right] \tag{8}$$

where $\mathbf{H}(s) = \mathbf{D}^{-1}(s)$ represents the transfer function in the Laplace domain, which can efficiently be expressed in terms of the eigenmodes as

$$\mathbf{H}(s) = \sum_{j=1}^{n} \left[\frac{\gamma_j \mathbf{u}_j \mathbf{u}_j^T}{s - s_j} + \frac{\gamma_j^* \mathbf{u}_j^* \mathbf{u}_j^{*T}}{s - s_j^*} \right] + \sum_{j=2n+1}^{m} \frac{\gamma_j \mathbf{u}_j \mathbf{u}_j^T}{s - s_j} \tag{9}$$

and the coefficient γ_j

$$\gamma_j = \left[\mathbf{u}_j^T \frac{\partial \mathbf{D}(s_j)}{\partial s} \mathbf{u}_j \right]^{-1} \quad , \quad 1 \leq j \leq m \tag{10}$$

Additionally, the time functions in Eq. (8) are

$$\psi_j(t) = \int_{\tau=0}^{t} e^{s_j(t-\tau)} \mathbf{u}_j^T \left[\mathfrak{f}(\tau) + \boldsymbol{\mathcal{G}}(\tau) \mathbf{u}_0 \right] d\tau + e^{s_j t} \mathbf{u}_j^T \left[\mathbf{M} \dot{\mathbf{u}}_0 + s_j \mathbf{M} \mathbf{u}_0 \right] \tag{11}$$

Note that according to the previous expressions, both the time- and the frequency-domain solutions result from the superposition of elastic and nonviscous modes. It is known [33, 2, 39] that, in general, the contribution of nonviscous modes is very small compared with that of elastic modes. Thus, for the majority of the physical systems its effect, represented by terms from $j = 2n+1$ to $j = m$ in Eqs. (8) and (9), can be neglected for the response calculation. For instance, Adhikari [2] and Lázaro [37] shown that the effect in the frequency response function associated to nonviscous modes is between one and three orders of magnitude smaller than those from elastic modes. The weight of the nonviscous modes effect in the response becomes higher as the damping level increases.

3. Continuous damping sensitivity and the closed-form solution of eigenvalues

The highest computational effort of the response calculation is focused on solving the nonlinear eigenvalue problem of Eq. (2). We attempt in this point the derivation of a closed-form approach of the elastic modes valid for nonproportional and nonviscous systems. For that, we will assume that the damped system presents light or moderate damping, which is a common hypothesis for the majority of the oscillating systems.

Let us define the following eigenvalue problem depending on a fictitious continuous parameter $p \in [0, 1]$

$$\left[s^2 \mathbf{M} + p\, s \mathbf{G}(s) + \mathbf{K} \right] \bar{\mathbf{u}} \equiv \boldsymbol{\mathcal{D}}(s, p)\, \bar{\mathbf{u}} = \mathbf{0} \tag{12}$$

Evaluating at $p = 0$ we have $\boldsymbol{\mathcal{D}}(s, 0) = \mathbf{D}_u(s)$ and Eq. (12) leads to the undamped problem. On the other side, at $p = 1$ the dynamic stiffness matrix is that of the damped problem $\boldsymbol{\mathcal{D}}(s, 1) = \mathbf{D}(s)$. Somewhat, the parameter p collects the physical meaning of damping as a perturbation of the undamped problem. If the dissipative capacity is not too high (lightly or moderately damped systems), it will be expected that the evaluation at $p = 1$ can be a good approach of the exact one.

The parameter p will be manipulated as a continuous variable so that we can define the complex eigenvalues (associated to the elastic modes) of Eq. (12) as $2n$ functions $\{\lambda_j(p),\, \lambda_j^*(p)\}$, $1 \leq j \leq n$. Associated to the jth eigenvalue we have the corresponding jth eigenvector, denoted by $\{\mathbf{U}_j(p)\}$, $1 \leq j \leq n$. In mathematical terms, we can define these functions as

$$\lambda_j(p) : [0, 1] \subset \mathbb{R} \to \mathbb{C} \ , \qquad \mathbf{U}_j(p) : [0, 1] \subset \mathbb{R} \to \mathbb{C}^n \ , \quad 1 \leq j \leq n \tag{13}$$

In Table 1 the eigenvalues and eigenvectors for values $p = 0, 1$ are shown. Since $s = \lambda_j(p)$ and, $\bar{\mathbf{u}} = \mathbf{U}_j(p)$ are solutions of Eq. (12) we can write, for $0 \leq p \leq 1$

$$\boldsymbol{\mathcal{D}}(\lambda_j(p), p)\, \mathbf{U}_j(p) \quad = \quad \mathbf{0} \tag{14}$$

Undamped	$p = 0$	$\lambda_j(0) = \mathrm{i}\omega_j$	$\mathbf{U}_j(0) = \mathbf{x}_j$
Damped	$p = 1$	$\lambda_j(1) = s_j$	$\mathbf{U}_j(1) = \mathbf{u}_j$

Table 1: Definition of eigenvalues and eigenvectos for the boundary values of parameter p. $\mathrm{i} = \sqrt{-1}$ denotes the imaginary unit.

Taking derivatives respect to p on Eq. (14) and using the chain rule

$$\left[\frac{\partial \boldsymbol{\mathcal{D}}(\lambda_j(p), p)}{\partial s}\frac{\mathrm{d}\lambda_j}{\mathrm{d}p} + \frac{\partial \boldsymbol{\mathcal{D}}(\lambda_j(p), p)}{\partial p}\right]\mathbf{U}_j(p) + \boldsymbol{\mathcal{D}}(\lambda_j(p), p)\frac{\mathrm{d}\mathbf{U}_j}{\mathrm{d}p} = \mathbf{0} \tag{15}$$

Premultiplying now by $\mathbf{U}_j^T(p)$

$$\mathbf{U}_j^T(p)\left[\frac{\partial \boldsymbol{\mathcal{D}}(\lambda_j(p), p)}{\partial s}\frac{\mathrm{d}\lambda_j}{\mathrm{d}p} + \frac{\partial \boldsymbol{\mathcal{D}}(\lambda_j(p), p)}{\partial p}\right]\mathbf{U}_j(p) + \mathbf{U}_j^T(p)\boldsymbol{\mathcal{D}}(\lambda_j(p), p)\frac{\mathrm{d}\mathbf{U}_j}{\mathrm{d}p} = 0 \tag{16}$$

From the symmetry of the system, we have that $\mathbf{U}_j^T(p)\boldsymbol{\mathcal{D}}(\lambda_j(p), p) = \mathbf{0}^T$ and, hence, the second term of the above equation vanishes allowing to find the derivative $\mathrm{d}\lambda_j/\mathrm{d}p$

$$\frac{\mathrm{d}\lambda_j}{\mathrm{d}p} = -\frac{\mathbf{U}_j^T(p)\,\dfrac{\partial \boldsymbol{\mathcal{D}}(\lambda_j(p), p)}{\partial p}\,\mathbf{U}_j(p)}{\mathbf{U}_j^T(p)\,\dfrac{\partial \boldsymbol{\mathcal{D}}(\lambda_j(p), p)}{\partial s}\,\mathbf{U}_j(p)} \tag{17}$$

From the definition of $\boldsymbol{\mathcal{D}}(s, p)$, the derivative $\frac{\partial \boldsymbol{\mathcal{D}}(s,p)}{\partial p} = s\mathbf{G}(s)$ and, therefore

$$\frac{\mathrm{d}\lambda_j}{\mathrm{d}p} = -\lambda_j(p)\,\frac{\mathbf{U}_j^T(p)\,\mathbf{G}(\lambda_j(p))\,\mathbf{U}_j(p)}{\mathbf{U}_j^T(p)\,\frac{\partial \boldsymbol{\mathcal{D}}}{\partial s}(\lambda_j(p), p)\,\mathbf{U}_j(p)} \tag{18}$$

Introducing the notation

$$\mathcal{W}_j(p) = \mathbf{U}_j^T(p)\,\mathbf{G}(\lambda_j(p))\,\mathbf{U}_j(p) \quad , \quad \mathcal{T}_j(p) = \mathbf{U}_j^T(p)\,\frac{\partial \boldsymbol{\mathcal{D}}(\lambda_j(p), p)}{\partial s}\,\mathbf{U}_j(p) \tag{19}$$

Eq. (17) can be expressed in a compact form as

$$\frac{\mathrm{d}\lambda_j}{\mathrm{d}p} = -\lambda_j\,\frac{\mathcal{W}_j(p)}{\mathcal{T}_j(p)} \tag{20}$$

We see that λ_j can be read as the solution of a differential equation. Obviously, λ_j is implicitly inside $\mathcal{W}_j(p)$ and $\mathcal{T}_j(p)$. Additionally, both of them depend on the damped eigenvectors. Consequently, exact integration of Eq. (20) is not available. However, appealing now to the hypothesis of light damping, $\mathcal{W}_j(p)$ and $\mathcal{T}_j(p)$ can be expanded in p retaining linear terms and avoiding higher order ones. Thus, we have

$$\mathcal{W}_j(p) \approx \mathcal{W}_j(0) + p\,\frac{\mathrm{d}\mathcal{W}_j(0)}{\mathrm{d}p} \quad , \quad \mathcal{T}_j(p) \approx \mathcal{T}_j(0) + p\,\frac{\mathrm{d}\mathcal{T}_j(0)}{\mathrm{d}p} \tag{21}$$

The values of $\mathcal{W}_j(0)$ and $\mathcal{T}_j(0)$ can be calculated evaluating their expressions at $p = 0$, that is at the undamped system. Therefore, according to Table 1 and to the orthogonality relations of Eq. (6), we have

$$\mathcal{W}_j(0) = \mathbf{U}_j^T(0)\,\mathbf{G}(\lambda_j(0))\,\mathbf{U}_j(0) = \mathbf{x}_j^T\,\mathbf{G}(\mathrm{i}\omega_j)\,\mathbf{x}_j = \Gamma_{jj}(\mathrm{i}\omega_j) \tag{22}$$

$$\mathcal{T}_j(0) = \mathbf{U}_j^T(0)\,\frac{\partial \boldsymbol{\mathcal{D}}(\lambda_j(0), 0)}{\partial s}\,\mathbf{U}_j(0) = \mathbf{x}_j^T\,(2\mathrm{i}\omega_j\mathbf{M})\,\mathbf{x}_j = 2\mathrm{i}\omega_j \tag{23}$$

where $\Gamma_{kj}(s) = \mathbf{x}_k^T \, \mathbf{G}(s) \, \mathbf{x}_j$ denotes the entrees of the damping matrix in the modal space of the undamped system. In the Eq. (23) the value of the derivative $\partial \boldsymbol{\mathcal{D}}/\partial s = 2s\mathbf{M} + p\,[\mathbf{G} + s\partial\mathbf{G}/\partial s]$ has been used. The following paragraphs are devoted to calculate expressions for $\mathrm{d}\mathcal{W}_j(0)/\mathrm{d}p$ and $\mathrm{d}\mathcal{T}_j(0)/\mathrm{d}p$.

Taking derivatives in the definition of $\mathcal{W}_j(p)$ and using the chain rule

$$\frac{\mathrm{d}\mathcal{W}_j}{\mathrm{d}p} = \frac{\mathrm{d}\mathbf{U}_j^T}{\mathrm{d}p}\,\mathbf{G}(\lambda_j(p))\,\mathbf{U}_j(p) + \mathbf{U}_j^T(p)\,\frac{\partial \mathbf{G}(\lambda_j(p))}{\partial s}\,\mathbf{U}_j(p)\,\frac{\mathrm{d}\lambda_j}{\mathrm{d}p} + \mathbf{U}_j^T(p)\,\mathbf{G}(\lambda_j(p))\,\frac{\mathrm{d}\mathbf{U}_j}{\mathrm{d}p} \tag{24}$$

Evaluating at $p = 0$

$$\frac{\mathrm{d}\mathcal{W}_j(0)}{\mathrm{d}p} = \frac{\mathrm{d}\mathbf{U}_j^T(0)}{\mathrm{d}p}\,\mathbf{G}(\mathrm{i}\omega_j)\,\mathbf{x}_j + \mathbf{x}_j^T\,\frac{\partial \mathbf{G}(\mathrm{i}\omega_j)}{\partial s}\,\mathbf{x}_j\,\frac{\mathrm{d}\lambda_j(0)}{\mathrm{d}p} + \mathbf{x}_j^T\,\mathbf{G}(\mathrm{i}\omega_j)\,\frac{\mathrm{d}\mathbf{U}_j(0)}{\mathrm{d}p} \tag{25}$$

The value of $\mathrm{d}\lambda_j(0)/\mathrm{d}p$ can be obtained directly from its definition in Eq. (18) and from Eqs. (22) and (23). Thus

$$\frac{\mathrm{d}\lambda_j(0)}{\mathrm{d}p} = -(\mathrm{i}\omega_j)\frac{\mathcal{W}_j(0)}{\mathcal{T}_j(0)} = -\frac{\Gamma_{jj}(\mathrm{i}\omega_j)}{2} \tag{26}$$

To calculate the eigenvector derivative $\mathrm{d}\mathbf{U}_j(0)/\mathrm{d}p$ at $p = 0$ we use the expressions deduced by Adhikari [25] for symmetric systems. The expressions evaluated at $p = 0$ are

$$\frac{\mathrm{d}\mathbf{U}_j(0)}{\mathrm{d}p} = a_{jj}\mathbf{x}_j + \frac{\Gamma_{jj}(\mathrm{i}\omega_j)}{4\mathrm{i}\omega_j}\mathbf{x}_j - \mathrm{i}\omega_j \sum_{\substack{k=1\\k\neq j}}^{n} \frac{\Gamma_{kj}(\mathrm{i}\omega_j)}{\omega_k^2 - \omega_j^2}\,\mathbf{x}_k \tag{27}$$

where

$$a_{jj} = -\frac{1}{4}\left[\frac{\partial \Gamma_{jj}(\mathrm{i}\omega_j)}{\partial s} + \frac{\Gamma_{jj}(\mathrm{i}\omega_j)}{\mathrm{i}\omega_j}\right] \tag{28}$$

The derivation of these expressions from those of Adhikari [25] for our particular case can be found in Appendix A. We can introduce the above results into Eq. (25) resulting

$$\begin{aligned}
\frac{\mathrm{d}\mathcal{W}_j(0)}{\mathrm{d}p} &= \frac{\mathrm{d}\mathbf{U}_j^T(0)}{\mathrm{d}p}\,\mathbf{G}(\mathrm{i}\omega_j)\,\mathbf{x}_j + \mathbf{x}_j^T\,\mathbf{G}(\mathrm{i}\omega_j)\,\frac{\mathrm{d}\mathbf{U}_j(0)}{\mathrm{d}p} + \mathbf{x}_j^T\,\frac{\partial \mathbf{G}(\mathrm{i}\omega_j)}{\partial s}\,\mathbf{x}_j\,\frac{\mathrm{d}\lambda_j(0)}{\mathrm{d}p} \\
&= \left(a_{jj}\mathbf{x}_j^T + \frac{\Gamma_{jj}(\mathrm{i}\omega_j)}{4\mathrm{i}\omega_j}\mathbf{x}_j^T - \mathrm{i}\omega_j \sum_{\substack{k=1\\k\neq j}}^{n} \frac{\Gamma_{kj}(\mathrm{i}\omega_j)}{\omega_k^2 - \omega_j^2}\,\mathbf{x}_k^T\right)\mathbf{G}(\mathrm{i}\omega_j)\mathbf{x}_j \\
&+ \mathbf{x}_j^T\,\mathbf{G}(\mathrm{i}\omega_j)\left(a_{jj}\mathbf{x}_j + \frac{\Gamma_{jj}(\mathrm{i}\omega_j)}{4\mathrm{i}\omega_j}\mathbf{x}_j - \mathrm{i}\omega_j \sum_{\substack{k=1\\k\neq j}}^{n} \frac{\Gamma_{kj}(\mathrm{i}\omega_j)}{\omega_k^2 - \omega_j^2}\,\mathbf{x}_k\right) \\
&+ \mathbf{x}_j^T\,\frac{\partial \mathbf{G}(\mathrm{i}\omega_j)}{\partial s}\,\mathbf{x}_j\left(-\frac{\Gamma_{jj}(\mathrm{i}\omega_j)}{2}\right)
\end{aligned} \tag{29}$$

Since $\Gamma_{kj}(s) = \mathbf{x}_k^T\mathbf{G}(s)\mathbf{x}_j$, we can express the above expression as function of $\Gamma_{kj}(s)$ and its derivative evaluated at $s = \mathrm{i}\omega_j$.

$$\frac{\mathrm{d}\mathcal{W}_j(0)}{\mathrm{d}p} = 2a_{jj}\,\Gamma_{jj}(\mathrm{i}\omega_j) + \frac{\Gamma_{jj}^2(\mathrm{i}\omega_j)}{2\mathrm{i}\omega_j} - \frac{1}{2}\frac{\partial \Gamma_{jj}(\mathrm{i}\omega_j)}{\partial s}\,\Gamma_{jj}(\mathrm{i}\omega_j) - 2\mathrm{i}\omega_j \sum_{\substack{k=1\\k\neq j}}^{n} \frac{\Gamma_{kj}^2(\mathrm{i}\omega_j)}{\omega_k^2 - \omega_j^2}$$

Substituting now the value of a_{jj} from Eq. (28)

$$\begin{aligned}\frac{\mathrm{d}\mathcal{W}_j(0)}{\mathrm{d}p} &= -\frac{\Gamma_{jj}(\mathrm{i}\omega_j)}{2}\left[\frac{\partial\Gamma_{jj}(\mathrm{i}\omega_j)}{\partial s}+\frac{\Gamma_{jj}(\mathrm{i}\omega_j)}{\mathrm{i}\omega_j}\right]+\frac{\Gamma_{jj}^2(\mathrm{i}\omega_j)}{2\mathrm{i}\omega_j}-\frac{1}{2}\frac{\partial\Gamma_{jj}(\mathrm{i}\omega_j)}{\partial s}\,\Gamma_{jj}(\mathrm{i}\omega_j)-2\mathrm{i}\omega_j\sum_{\substack{k=1\\k\neq j}}^{n}\frac{\Gamma_{kj}^2(\mathrm{i}\omega_j)}{\omega_k^2-\omega_j^2}\\ &= -\frac{\partial\Gamma_{jj}(\mathrm{i}\omega_j)}{\partial s}\,\Gamma_{jj}(\mathrm{i}\omega_j)-2\mathrm{i}\omega_j\sum_{\substack{k=1\\k\neq j}}^{n}\frac{\Gamma_{kj}^2(\mathrm{i}\omega_j)}{\omega_k^2-\omega_j^2}\equiv 2\mathrm{i}\omega_j\,(\mathrm{i}\zeta_j\nu_j-\alpha_j)\end{aligned}\tag{30}$$

where the coefficients

$$\zeta_j=\frac{\Gamma_{jj}(\mathrm{i}\omega_j)}{2\omega_j}\ ,\quad \nu_j=\frac{\partial\Gamma_{jj}(\mathrm{i}\omega_j)}{\partial s}\ ,\quad \alpha_j=\sum_{\substack{k=1\\k\neq j}}^{n}\frac{\Gamma_{kj}^2(\mathrm{i}\omega_j)}{\omega_k^2-\omega_j^2}\tag{31}$$

This form of ordering the terms will be justified later, since it is related to the physical meaning of the solution.

We focus now on the calculation of the remaining term of Eq. (21), that is $\mathrm{d}\mathcal{T}_j(0)/\mathrm{d}p$. As before, we take again derivative from its definition

$$\begin{aligned}\frac{\mathrm{d}\mathcal{T}_j}{\mathrm{d}p} &= \frac{\mathrm{d}\mathbf{U}_j^T}{\mathrm{d}p}\,\frac{\partial\boldsymbol{\mathcal{D}}(\lambda_j(p),p)}{\partial s}\,\mathbf{U}_j(p)+\mathbf{U}_j^T(p)\,\frac{\partial\boldsymbol{\mathcal{D}}(\lambda_j(p),p)}{\partial s}\,\frac{\mathrm{d}\mathbf{U}_j}{\mathrm{d}p}\\ &\quad+\mathbf{U}_j^T(p)\left[\frac{\partial^2\boldsymbol{\mathcal{D}}(\lambda_j(p),p)}{\partial s^2}\,\frac{\mathrm{d}\lambda_j}{\mathrm{d}p}+\frac{\partial^2\boldsymbol{\mathcal{D}}(\lambda_j(p),p)}{\partial s\partial p}\right]\mathbf{U}_j(p)\end{aligned}\tag{32}$$

From the definition of $\boldsymbol{\mathcal{D}}(s,p)$ the derivatives respect to s and p are

$$\begin{aligned}\frac{\partial\boldsymbol{\mathcal{D}}(s,p)}{\partial s} &= 2s\mathbf{M}+p\left[\mathbf{G}(s)+s\,\frac{\partial\mathbf{G}(s)}{\partial s}\right]\\ \frac{\partial^2\boldsymbol{\mathcal{D}}(s,p)}{\partial s^2} &= 2\mathbf{M}+p\left[2\frac{\partial\mathbf{G}(s)}{\partial s}+s\,\frac{\partial^2\mathbf{G}(s)}{\partial s^2}\right]\\ \frac{\partial^2\boldsymbol{\mathcal{D}}(s,p)}{\partial s\partial p} &= \mathbf{G}(s)+s\,\frac{\partial\mathbf{G}(s)}{\partial s}\end{aligned}\tag{33}$$

Let us evaluate at $p=0$. For that, we use the results in Table 1 and the expression of $\mathrm{d}\mathbf{U}_j(0)/\mathrm{d}p$ from Eq. (27)

$$\begin{aligned}\frac{\mathrm{d}\mathcal{T}_j(0)}{\mathrm{d}p} &= \left(a_{jj}\mathbf{x}_j^T+\frac{\Gamma_{jj}(\mathrm{i}\omega_j)}{4\mathrm{i}\omega_j}\mathbf{x}_j^T-\mathrm{i}\omega_j\sum_{\substack{k=1\\k\neq j}}^{n}\frac{\Gamma_{kj}(\mathrm{i}\omega_j)}{\omega_k^2-\omega_j^2}\,\mathbf{x}_k^T\right)(2\mathrm{i}\omega_j\mathbf{M})\;\mathbf{x}_j\\ &+\ \mathbf{x}_j^T\;(2\mathrm{i}\omega_j\mathbf{M})\left(a_{jj}\mathbf{x}_j+\frac{\Gamma_{jj}(\mathrm{i}\omega_j)}{4\mathrm{i}\omega_j}\mathbf{x}_j-\mathrm{i}\omega_j\sum_{\substack{k=1\\k\neq j}}^{n}\frac{\Gamma_{kj}(\mathrm{i}\omega_j)}{\omega_k^2-\omega_j^2}\,\mathbf{x}_k\right)\\ &+\ \mathbf{x}_j^T\left[2\mathbf{M}\;(-\Gamma_{jj}(\mathrm{i}\omega_j)/2)+\mathbf{G}(\mathrm{i}\omega_j)+\mathrm{i}\omega_j\frac{\partial\mathbf{G}(\mathrm{i}\omega_j)}{\partial s}\right]\mathbf{x}_j\end{aligned}\tag{34}$$

Using the orthogonality relations, the terms within the sums vanish due to $\mathbf{x}_k^T\mathbf{M}\mathbf{x}_j=\delta_{jk}$, where δ_{jk} denotes the Kronecker delta. After some operations and the corresponding simplifications

$$\frac{\mathrm{d}\mathcal{T}_j(0)}{\mathrm{d}p}=4\mathrm{i}\omega_j\,a_{jj}+\Gamma_{jj}(\mathrm{i}\omega_j)+\mathrm{i}\omega_j\,\frac{\partial\Gamma_{jj}(\mathrm{i}\omega_j)}{\partial s}\tag{35}$$

Finally, using the value of a_{jj} from Eq. (28) yields

$$\frac{\mathrm{d}\mathcal{T}_j(0)}{\mathrm{d}p} = 0 \tag{36}$$

Summarizing the results of Eqs. (22), (23), (30) and (36)

$$\begin{aligned} \mathcal{W}_j(p) &\approx \mathcal{W}_j(0) + p\,\frac{\mathrm{d}\mathcal{W}_j(0)}{\mathrm{d}p} = -2\mathrm{i}\omega_j\left[\mathrm{i}\zeta_j + (\alpha_j - \mathrm{i}\zeta_j\nu_j)\,p\right] \\ \mathcal{T}_j(p) &\approx \mathcal{T}_j(0) + p\,\frac{\mathrm{d}\mathcal{T}_j(0)}{\mathrm{d}p} = 2\mathrm{i}\omega_j \end{aligned} \tag{37}$$

where the coefficients $\alpha_j, \zeta_j, \nu_j \in \mathbb{C}$ have been defined in Eq. (31). Returning to Eq. (20), the relation $\mathcal{W}_j(p)/\mathcal{T}_j(p)$ can be approximated by

$$\frac{\mathcal{W}_j(p)}{\mathcal{T}_j(p)} \approx \frac{\mathcal{W}_j(0) + p\,\dfrac{\mathrm{d}\mathcal{W}_j(0)}{\mathrm{d}p}}{\mathcal{T}_j(0) + p\,\dfrac{\mathrm{d}\mathcal{T}_j(0)}{\mathrm{d}p}} = -\mathrm{i}\zeta_j - (\alpha_j - \mathrm{i}\zeta_j\nu_j)\,p \tag{38}$$

Under this approach, Eq. (20) adopts the form of an ordinary differential equation of separated variables

$$\begin{cases} \dfrac{\mathrm{d}\lambda_j}{\mathrm{d}p} \approx \lambda_j \cdot \left[\mathrm{i}\zeta_j + (\alpha_j - \mathrm{i}\zeta_j\nu_j)\,p\right] \\ \lambda_j(0) = \mathrm{i}\omega_j \end{cases} \tag{39}$$

Integrating and evaluating the solution at $p = 1$ a closed-form compact approach of the jth complex elastic eigenvalue can be derived as

$$s_j \approx \mathrm{i}\omega_j \exp\left\{\mathrm{i}\zeta_j + (\alpha_j - \mathrm{i}\zeta_j\nu_j)/2\right\} \tag{40}$$

where, rewriting the expression of the coefficients

$$\zeta_j = \frac{\Gamma_{jj}(\mathrm{i}\omega_j)}{2\omega_j}\ , \quad \nu_j = \frac{\partial\Gamma_{jj}(\mathrm{i}\omega_j)}{\partial s}\ , \quad \alpha_j = \sum_{\substack{k=1\\k\neq j}}^{n} \frac{\Gamma_{kj}^2(\mathrm{i}\omega_j)}{\omega_k^2 - \omega_j^2} \tag{41}$$

These three coefficients only depend on the natural frequencies and on the entrees of the damping matrix and its s-derivative in the modal space. Therefore, undamped eigenmodes are the only requirement for the calculation of s_j. In addition, they allow us to read the physical insight of the proposed solution.

First, ζ_j represents the damping ratio of the jth mode and consequently it is a measurement of the level of modal damping. In fact, this coefficient emerges in the first order perturbation solution of lightly nonproportional systems [33, 18, 40] as

$$s_j \approx \mathrm{i}\omega_j - \zeta_j\omega_j = \mathrm{i}\omega_j - \frac{\Gamma_{jj}(\mathrm{i}\omega_j)}{2\omega_j} \tag{42}$$

This expression was obtained by Woodhouse [33] and can also be considered as a closed–form which does not depend on the nature of damping model since is explicitly expressed as function on the entrees $\Gamma_{jk}(s)$.

Second, the coefficient ν_j is the derivative of the damping matrix evaluated at the natural frequency and is characteristic of nonviscously or viscoelastically damped systems. Its weight in the final expression depends of the named *viscoelasticity* of the damping model. Mathematically, a system present small viscoelasticity if $\mathbf{G}(s)$ does not present high variations with respect to the frequency, which is a synonym of small values of the s-derivative of the damping coefficients. A deep analysis on this property can be found in the reference [41]. For viscous damping, the elements $\Gamma_{jk}(s) = C'_{jk}$ are now the entrees of the viscous damping matrix in the modal space and consequently does not depend on the frequency. Therefore $\nu_j = 0$ for viscous damping, leading to the approximation obtained by Lázaro [32].

Finally, the coefficient α_j is formed by a sum of products of the off-diagonal elements of the damping matrix affected by the *distance* between the natural frequencies. Consequently, it contains the information related with the nonproportionality of the damping matrix. It vanishes for proportional systems, for which in general $\Gamma_{jk}(s) = 0$, for $j \neq k$. The necessary and sufficient conditions for proportional damping in nonviscous systems have been studied by Adhikari [42]. For those system which present light nonproportionality this term is expected to be small since they are characterized by [43, 2]

$$\sum_{\substack{k=1\\k\neq j}}^{n} |\Gamma_{jk}(\mathrm{i}\omega_j)| < |\Gamma_{jj}(\mathrm{i}\omega_j)| \ , \qquad \forall\ 1 \leq j \leq n \tag{43}$$

We consider that Eq. (40) is the most remarkable contribution of this paper. Such as described, the so found expression represents itself a closed-form to obtain the jth complex eigenvalue which explicitly depends on the main properties of the damping model: (i) the damping level, (ii) the nonviscousity or viscoelasticity and (iii) the nonproportionality. According to this mathematical result, it is interesting that each one of these three properties are presented as three terms or weights affecting or perturbing the undamped state. So, the complex eigenvalue can be expressed as the product.

$$s_j \approx \mathrm{i}\omega_j \cdot DL_j \cdot NP_j \cdot NV_j \tag{44}$$

where the different terms or weights are (i) $DL_j = e^{\mathrm{i}\zeta_j}$, the Damping-Level term. (ii) $NV_j = e^{-\mathrm{i}\zeta_j\nu_j/2}$, the nonviscous term (or viscoelasticity term). (iii) $NP_j = e^{\alpha_j/2}$, the nonproportional term. Now, in order to complete the proposal, let us see how to obtain the estimation of the associated eigenvectors.

4. Computation of eigenvectors

Assuming as known certain eigenvalue, the ill-conditioned linear systems (4) needs to be solved to obtain the associated eigenvectors. Adhikari [2] found closed-form expressions for eigenmodes using a numerical method based on the Neumann series expansion. The eigenvectors so calculated are

$$\mathbf{u}_j \quad \approx \quad \mathbf{x}_j - s_j \sum_{\substack{k=1\\k\neq j}}^{n} \frac{\Gamma_{kj}(s_j)}{D_k(s_j)} \mathbf{x}_k + s_j^2 \sum_{\substack{k=1\\k\neq j}}^{n} \sum_{\substack{l=1\\l\neq j\neq k}}^{n} \frac{\Gamma_{kl}(s_j)\Gamma_{lj}(s_j)}{D_k(s_j)\, D_l(s_j)} \mathbf{x}_k \tag{45}$$

where $D_j(s) = s^2 + s\Gamma_{jj}(s) + \omega_j^2$. The above expressions show the Adhikari results up to the second order approximation, in terms of the entrees of the damping matrix in the modal space $\Gamma_{jk}(s)$. According to Adhikari, higher order terms could be considered and the convergence of the series depends on the weight of the off-diagonal terms $\Gamma_{jk}(s)$, $j \neq k$ respect to those of the main diagonal $\Gamma_{jj}(s)$. This presents the great advantage that barely involves computational effort since it does not require any matrix inversion process and the results are just expressed as linear combination of the undamped eignemodes.

5. Numerical Example

The theoretical results will be validated through a numerical example consisting in a discrete six degrees-of-freedom mass-lumped dynamical system with two viscoelastic links, shown in Fig. 1. The six lumped

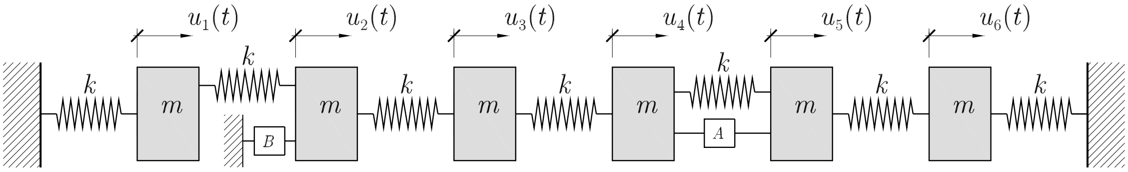

Figure 1: Example: Lumped-mass dynamical system with two viscoelastic dampers. $k = 100$ kN/m, $m = 1$ t. Viscoelastic damper A based on a five exponential kernels. Viscoelastic damper B based on a four-parameter fractional derivative-based model.

masses have $m = 10^3$ kg and are linked using springs with linear constant $k = 10^5$ N/m. The damping is introduced by two linear viscoelastic constrains.

Link A is modeled by a nonviscous damper based on exponential kernels. Mathematically, the total force reaction between degrees of freedom u_4 and u_5 is related to the relative displacement $\Delta u_{45} = u_5 - u_4$ with the following linear nonviscous model

$$\mathcal{R}_{45}(t) = k\,\Delta u_{45}(t) + \int_{-\infty}^{t} \mathcal{G}_A(t-\tau)\,\Delta\dot{u}_{45}(\tau)\,d\tau$$

The damping function $\mathcal{G}_A(t)$ is based on five exponential kernels. The time and frequency domain expressions of this function results in

$$\mathcal{G}_A(t) = \frac{1}{5}\sum_{k=1}^{5} c_k\,\mu_k\,e^{-\mu_k t}\ , \quad G_A(s) = \mathcal{L}\left\{\mathcal{G}_A(t)\right\} = \frac{1}{5}\sum_{k=1}^{5}\frac{c_k\,\mu_k}{s+\mu_k} \tag{46}$$

where μ_k represent the relaxation parameters and c_k the damping coefficient of the limit viscous model obtained doing $\mu_k \to \infty$ for all $1 \le k \le 5$. Instead of parameters c_k, the nondimensional damping ratios defined as $\xi_k = c_k/2m\omega_0$ will be used, where $\omega_0 = \sqrt{k/m} = 10$ rad/s is a reference frequency. According to Eq. (46), the relationship between reaction and displacement in the Laplace domain is

$$R_{45}(s) = [k + s\ G_A(s)]\,\Delta\bar{u}_{45}(s) \tag{47}$$

The viscoelastic constrain B linking the structure to the ground obeys to model based on the fractional derivatives, in particular a four-parameter viscoelastic model is used [44]. According to this model, the time-domain equation relating force $\mathcal{R}_{02}$ to displacement u_2 can be written as

$$\mathcal{R}_{02} + T_r^{\alpha}\frac{\mathrm{d}^{\alpha}\mathcal{R}_{02}}{\mathrm{d}t^{\alpha}} = k_B\left(u_2 + c\ T_r^{\alpha}\frac{\mathrm{d}^{\alpha}u_2}{\mathrm{d}t^{\alpha}}\right) \tag{48}$$

where c, α, T_r and k_B are the mentioned four parameters, also called storage coefficient, fractional exponent, relaxation time and linear-static rigidity, respectively. For real materials $c > 1,\ 0 < \alpha < 1,\ T_r > 0,\ k_B > 0$. The time-domain kernel function $\mathcal{G}_B(t)$ is difficult to obtain explicitly and it becomes necessary to appeal to infinite series based functions [45]. However, the damping function in the Laplace domain $G_B(s)$ can easily be determined just applying the Laplace transform to the fractional derivatives of Eq. (48) and using its properties. Denoting by $R_{02}(s) = \mathcal{L}\{\mathcal{R}_{02}(t)\}$ and $\bar{u}_2(s) = \mathcal{L}\{u_2(t)\}$ to the Laplace transform of the reaction and displacement, respectively, then

$$R_{02}(s) = k_B\frac{1 + c\,(T_r s)^{\alpha}}{1 + (T_r s)^{\alpha}}\bar{u}_2(s) \equiv [k_B + s\ G_B(s)]\,\bar{u}_2(s) \tag{49}$$

	Parameters	(Units)	DAMPING LEVEL LOW	HIGH
Constraint A	$\{\mu_j\}_{j=1}^5$	(rad/s)	{5, 8, 13, 20, 40}	
	ξ_1	(–)	0.05	0.24
	ξ_2	(–)	0.07	0.22
	ξ_3	(–)	0.03	0.28
	ξ_4	(–)	0.01	0.22
	ξ_5	(–)	0.04	0.20
Constraint B	α	(–)	0.70	0.70
	T_r	(s)	1.00E-05	1.00E-04
	c	(–)	5.20	50.50
	k_B	(kN/m)	5.00E+04	5.00E+04

Table 2: Assumed values of damping parameters to cover lowly and highly damped structures

where

$$G_B(s) = \frac{k_B}{s}\,\frac{(c-1)(T_r s)^\alpha}{1+(T_r s)^\alpha} \tag{50}$$

The free-motion equations in the Laplace domain can be obtained assembling the mass and the stiffness matrices associated with the structural configuration shown in Fig. 1, resulting

$$\left[s^2\mathbf{M} + s\mathbf{G}(s) + \mathbf{K}\right]\bar{\mathbf{u}}(s) = \mathbf{0} \tag{51}$$

where $\mathbf{M} = m\mathbf{I}_6$, and

$$\mathbf{K} = \begin{bmatrix} 2k & -k & 0 & 0 & 0 & 0 \\ -k & 2k+k_B & -k & 0 & 0 & 0 \\ 0 & -k & 2k & -k & 0 & 0 \\ 0 & 0 & -k & 2k & -k & 0 \\ 0 & 0 & 0 & -k & 2k & -k \\ 0 & 0 & 0 & 0 & -k & 2k \end{bmatrix} \quad , \quad \mathbf{G}(s) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & G_B(s) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & G_A(s) & -G_A(s) & 0 \\ 0 & 0 & 0 & -G_A(s) & G_A(s) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{52}$$

Both viscoelastic dampers induce certain level of energy dissipation depending on the values of their parameters. In particular, the damping level of the constraint A is mainly controlled by the damping rations $\xi_k = c_k/2m\omega_0$, whereas the nonviscosity directly depend on the value of the relaxation parameters, μ_k. On the other hand, although the damping behavior of constraint B is a result of a combination of the different parameters c, α and T_r, the main responsible of the level of energy-disspation is the storage parameter, c [46, 47]. Not all modes behave equally with the different damping parameters since the system present nonproportional damping. Thus, to cover cases of low and high damping, two combinations of the damping parameters will be considered ('LD' and 'HD' for Low and High Damping. See Table 2)

To measure the modal damping level of the jth mode we will use the Q_j-factor and the modal damping ratio g_j, defined as

$$Q_j = -\frac{\Im\{s_j\}}{2\Re\{s_j\}} \quad , \quad g_j = -\frac{\Re\{s_j\}}{|s_j|} \tag{53}$$

Since the damping model induces a strictly dissipative motion, $\Re\{s_j\} < 0$, for $1 \leq j \leq 6$, hence the negative sign in both definitions. Additionally, the modal damping ratio is usually presented in percentage. The exponential decay of amplitudes is directly related to the real part of eigenvalues, therefore the lower Q_j, the higher the modal damping level. Woodhouse [33] considers $Q_j \leq 10$ "as very high damping for most structural vibration applications". Otherwise, the upper bound $Q_j = \infty$ characterizes an undamped mode. Modal damping ratio behaves inversely to the Q-factor, thus $g_j = 0$ represents the undamped state and $g_j = 100\%$ is characteristic of a critically damped mode. In Table 3, Q_j-factors and modal damping ratios g_j

LOW DAMPING

Mode	Quality factor	Daming ratio	Undamped frequeny	Damping–level coef.	Non–Viscous coef.	Non–Proportional coef.
j	Q_j (–)	g_j (%)	ω_j (rad/s)	DL_j (–)	NV_j (–)	NP_j (–)
1	339	0.15%	4.45042	$1.00071 + 0.00150\mathrm{i}$	$1.00002 - 0.00002\mathrm{i}$	$1.00000 + 0.00000\mathrm{i}$
2	198	0.25%	8.67767	$1.00183 + 0.00265\mathrm{i}$	$1.00004 - 0.00012\mathrm{i}$	$1.00000 - 0.00000\mathrm{i}$
3	90	0.56%	12.46980	$1.00544 + 0.00596\mathrm{i}$	$1.00003 - 0.00031\mathrm{i}$	$1.00002 - 0.00003\mathrm{i}$
4	$1,477$	0.03%	15.63663	$1.00036 + 0.00036\mathrm{i}$	$1.00000 - 0.00003\mathrm{i}$	$1.00000 - 0.00000\mathrm{i}$
5	135	0.37%	18.01938	$1.00469 + 0.00424\mathrm{i}$	$0.99992 - 0.00050\mathrm{i}$	$1.00001 - 0.00002\mathrm{i}$
6	80	0.62%	19.49856	$1.00650 + 0.00556\mathrm{i}$	$1.00014 + 0.00073\mathrm{i}$	$1.00001 - 0.00004\mathrm{i}$

HIGH DAMPING

Mode	Quality factor	Daming ratio	Undamped frequeny	Damping–level coef.	Non–Viscous coef.	Non–Proportional coef.
j	Q_j	g_j (%)	ω_j (rad/s)	DL_j	NV_j	NP_j
1	11	4.34%	4.45042	$1.02332 + 0.04940\mathrm{i}$	$1.00316 - 0.00403\mathrm{i}$	$0.99961 + 0.00065\mathrm{i}$
2	12	4.06%	8.67767	$1.02302 + 0.04350\mathrm{i}$	$1.00034 - 0.00156\mathrm{i}$	$1.00009 + 0.00028\mathrm{i}$
3	18	2.81%	12.46980	$1.03385 + 0.04454\mathrm{i}$	$1.00336 - 0.01477\mathrm{i}$	$1.00115 - 0.00091\mathrm{i}$
4	149	0.36%	15.63663	$1.00344 + 0.00480\mathrm{i}$	$1.00021 - 0.00144\mathrm{i}$	$1.00001 - 0.00000\mathrm{i}$
5	19	2.67%	18.01938	$1.03456 + 0.03979\mathrm{i}$	$0.99538 - 0.01091\mathrm{i}$	$1.00066 - 0.00087\mathrm{i}$
6	8	6.09%	19.49856	$1.04432 + 0.04510\mathrm{i}$	$1.00520 + 0.02328\mathrm{i}$	$1.00100 - 0.00178\mathrm{i}$

Table 3: Natural undamped frequencies and damping coefficients DL_j, NV_j and NP_j —Eq. (44)—, calculated with the proposed formula. Second and third columns show the Q factors and the modal damping rations, respectively

(%) of each mode are shown for the two damping considered cases, allowing to distinguish which modes are the most affected by the nonviscous dampers. The proposed model involves three modal coefficients, namely *damping-level,* DL_j, *nonviscousity* or *viscoelasticity*, NV_j and *nonproportionality,* NP_j coefficients. As deduced, the resulting complex eigenvalue is approximated as the product $s_j \approx \mathrm{i}\omega_j \, DL_j \, NV_j \, NP_j$ (Eq. (44) rewritten, see results in Table 3). Somehow, DL_j, NV_j, NP_j represent the effect of the associated damping property on the corresponding eigenvalue. Therefore, the further from the unity the more perturbed is the complex frequency. The results are shown for both LD and HD cases. Note that the damping level coefficient DL_j is closely related to the quality factor, showing an inversely proportional tendency. Additionally, both NV_j and NP_j are in general at least one order of magnitude lower than that of DL_J since they come from second order terms (that is, product of modal damping matrix entrees, see Eqs. (41)).

In Table 4 the proposed approximations of the eigenvalues are listed and compared with the exact results and with linear first order approximation. Let us describe how these two latter eigensolutions have been obtained.

The exact eigenvalues have been determined using an iterative scheme based on a generalization of the Newton's method for nonlinear eigenproblems [6, 8]. This method is based on the linearization of the problem (2) around an initial guess, say s_{0j}. The value $s_{0j} + \delta$ is assumed to be very close to the jth eigenvalue and hence, $\mathbf{D}(s_{0j} + \delta)\,\mathbf{u}_j \approx \mathbf{0}$. Expanding this matrix up to the first order in terms of the unknown δ results in

$$\left[\mathbf{D}\left(s_{0j}\right) + \delta \, \frac{\partial \mathbf{D}\left(s_{0j}\right)}{\partial s}\right] \mathbf{u}_j \approx \mathbf{0} \tag{54}$$

The above equation represents a linear eigenvalue problem in the variable δ. Only the smallest eigenvalue of this problem (in absolute value), $\delta_j^{(0)}$, needs to be found, allowing to find the approximation of the next step as $s_j^{(1)} = s_{0j} + \delta_j^{(0)}$. The iterative process consists on building the sequence $\{s_j^{(n)}\}$, which is locally convergent with quadratic speed. The initial guess is taken as the undamped eigenvalue $s_{0j} = \mathrm{i}\omega_j$ and the process will conclude when the relative error between two consecutive iterations is less than 10^{-10}, considering the achieved solution as the exact one.

LOW DAMPING

j	Q_j	g_j	Eigenvalues, s_j: Exact	Proposed	Light Prop Damp.	Error, %: Proposed	Light Prop Damp.
1	339	0.15%	−0.0066 + 4.4537i	−0.0066 + 4.4537i	−0.00665 + 4.45360i	0.0590 + 0.0000i	1.3132 + 0.0016i
2	198	0.25%	−0.0219 + 8.6939i	−0.0219 + 8.6939i	−0.02295 + 8.69361i	0.0337 + 0.0005i	4.7036 + 0.0031i
3	90	0.56%	−0.0700 + 12.5382i	−0.0700 + 12.5383i	−0.07392 + 12.53792i	0.0369 + 0.0007i	5.5506 + 0.0022i
4	1477	0.03%	−0.0053 + 15.6422i	−0.0053 + 15.6422i	−0.00570 + 15.64225i	0.2738 + 0.0001i	7.3691 + 0.0002i
5	135	0.37%	−0.0669 + 18.1027i	−0.0670 + 18.1027i	−0.07606 + 18.10411i	0.0907 + 0.0005i	13.6956 + 0.0081i
6	80	0.62%	−0.1218 + 19.6285i	−0.1219 + 19.6282i	−0.10751 + 19.62559i	0.1253 + 0.0014i	11.7065 + 0.0149i

HIGH DAMPING

j	Q_j	g_j	Eigenvalues, s_j: Exact	Proposed	Light Prop Damp.	Error, %: Proposed	Light Prop Damp.
1	11	4.34%	−0.19866 + 4.57059i	−0.20509 + 4.56754i	−0.21710 + 4.55693i	3.236 + 0.067i	9.278 + 0.299i
2	12	4.06%	−0.36058 + 8.88435i	−0.36624 + 8.88180i	−0.37319 + 8.88326i	1.569 + 0.029i	3.496 + 0.012i
3	18	2.81%	−0.36277 + 12.92025i	−0.35557 + 12.95868i	−0.54191 + 12.90805i	1.986 + 0.297i	49.379 + 0.094i
4	149	0.36%	−0.05614 + 15.69038i	−0.05253 + 15.69392i	−0.07492 + 15.69057i	6.421 + 0.023i	33.469 + 0.001i
5	19	2.67%	−0.49489 + 18.54463i	−0.49442 + 18.57641i	−0.69372 + 18.65734i	0.096 + 0.171i	40.176 + 0.608i
6	8	6.09%	−1.25537 + 20.56851i	−1.32321 + 20.47097i	−0.83748 + 20.38467i	5.404 + 0.474i	33.288 + 0.894i

Table 4: Complex Eigenvalues calculated with the proposed formula, Eq. (40). Comparison with (a) Exact results, obtained using an iterative method described in Eq. (54) and (b) First-order linear approximation, Eq. (55). Second and third columns show the Q factors and the modal damping rations, respectively

Additionally, we are interested in comparing the proposed solution with other of the same nature, that is, explicit closed forms valid for any nonviscously, nonproportionally damped systems. We can find in the bibliography analytical approximations but built for specific damping models and supported by the light nonproportionality assumption [18, 22, 23]. On the other hand, solutions proposed for nonproportional systems are based on iterative approaches [26, 11, 21] something that is not of interest for our comparison since we look for explicit expressions. What is new of our approach respect to the above references is to provide an analytical form involving all the information of the system in a simple expression (all the entrees of the modal damping matrix are presented), valid for any nonviscous damping model and without requiring eigenvector calculation or iterative processes.

To the best author's knowledge, only the approach of Woodhouse [33] (and revisited by Adhikari [2, 48]), which coincides with the linear first order approximation (LIN), provides a closed-form depending on the values of the modal damping matrix, no matter the damping model behind. Therefore, we use it to be compared with that of our method. In the most general case, the so-called LIN approach is expressed as

$$s_j \approx \mathrm{i}\omega_j - \zeta_j\omega_j = \mathrm{i}\omega_j - \frac{1}{2}\,\mathbf{x}_j^T\,\mathbf{G}(\mathrm{i}\omega_j)\,\mathbf{x}_j \tag{55}$$

In the last two columns of Table 4 the relative error (in %) of real and imaginary part is shown. The LD and HD cases are listed within the two sub-tables. As expected, focusing on one particular mode, the higher the damping level, the higher the relative error. In addition, real part of eigenvalues (usually linearly depending on the damping coefficients) presents higher relative error than that of the imaginary part. Relative errors of the proposed method are in general much lower than those of the LIN method. Even if we compare the proposed eigenvalues obtained for HD case with those determined using LIN method for LD case, something that extends the derived formula also for moderately or highly damped systems. This behavior emerges from the fact that our closed-form expression has been derived after integrating Eq. (39), which in turn comes from the a linear expansion of functions $\mathcal{W}_j(p)$ y $\mathcal{T}_j(p)$, both defined in Eqs. (37), in terms of the fictitious damping parameter p. Therefore, its order of approximation can be considered as one unit higher than that of the linear first-order expression, Eq. (55).

As known, the eigensolutions contain all the information related to the dynamic response, both in time and in frequency domain. Together with the comparison of eigenvalues (shown in Table 4), we consider necessary to verify how the derived expression affects to the response in the frequency domain. To this end, the transfer function $\mathbf{H}(s)$ will be plotted for the three described methods using the Eq. (9) as function of frequency $s = \mathrm{i}\omega$. Therefore, and according to Eq. (9), also the complex eigenvectors need to be computed since they take part of the expression of $\mathbf{H}(s)$. On one hand, the 'Exact' eigenvectors are obtained from the recursive method described above, based on the generalized Newton's method for nonlinear eigenvalue problems, see Eq (54). On the other hand, the eigenvectors of the 'Proposed' and the 'LIN' approximations are determined from Eq. (45), introducing as s_j the corresponding eigenvalues associated to each approach.

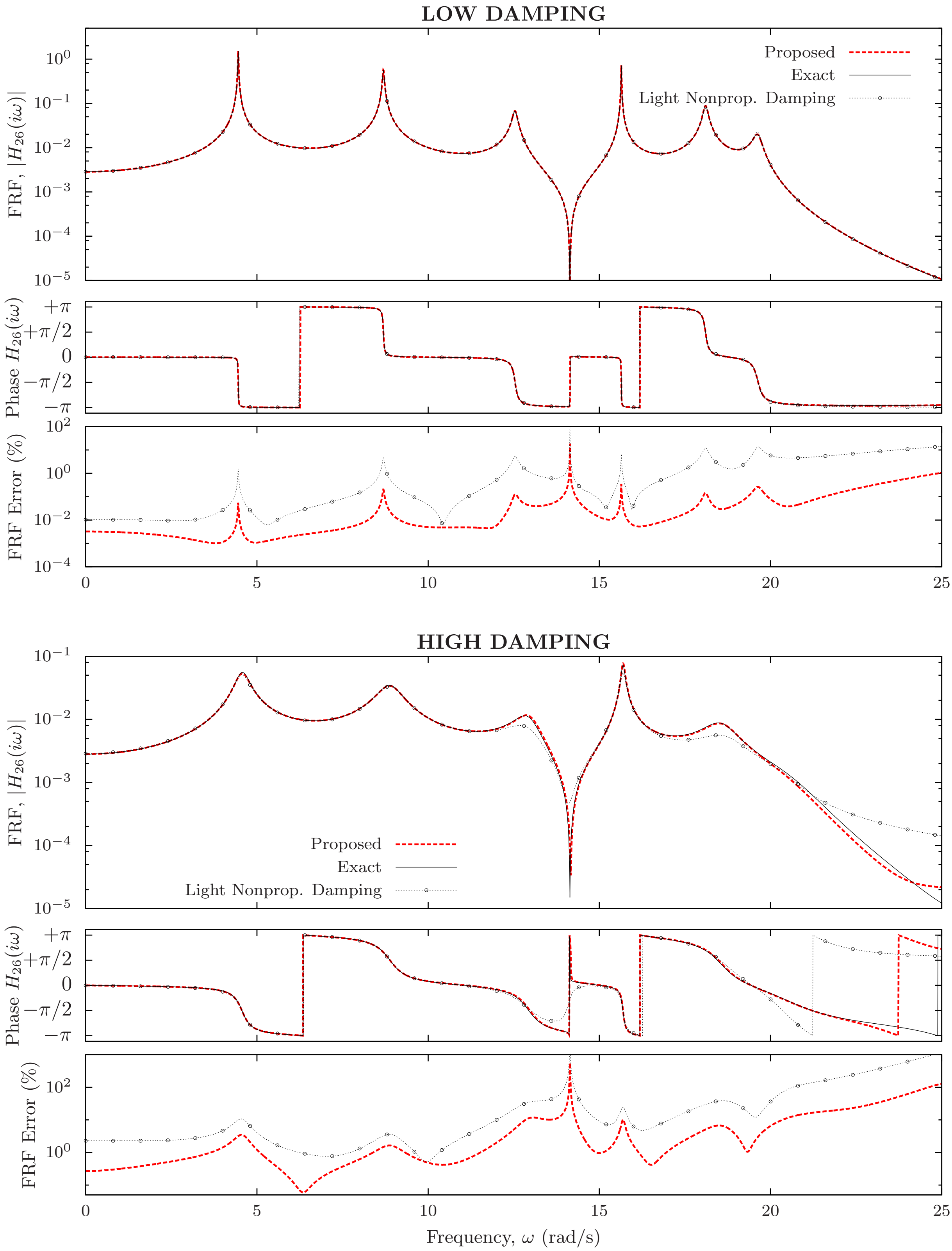

Figure 2: Frequency Response Function $H_{26}(\mathrm{i}\omega)$ calculated from Eq. (9). Eigenmodes computation: 'Exact' from Eqs. (54); 'Proposed' from Eqs. (40),(45); 'Linear First Order' approximation (LIN) from Eqs. (42),(45). Top-graphics: Low damping. Bottom-graphics: High damping

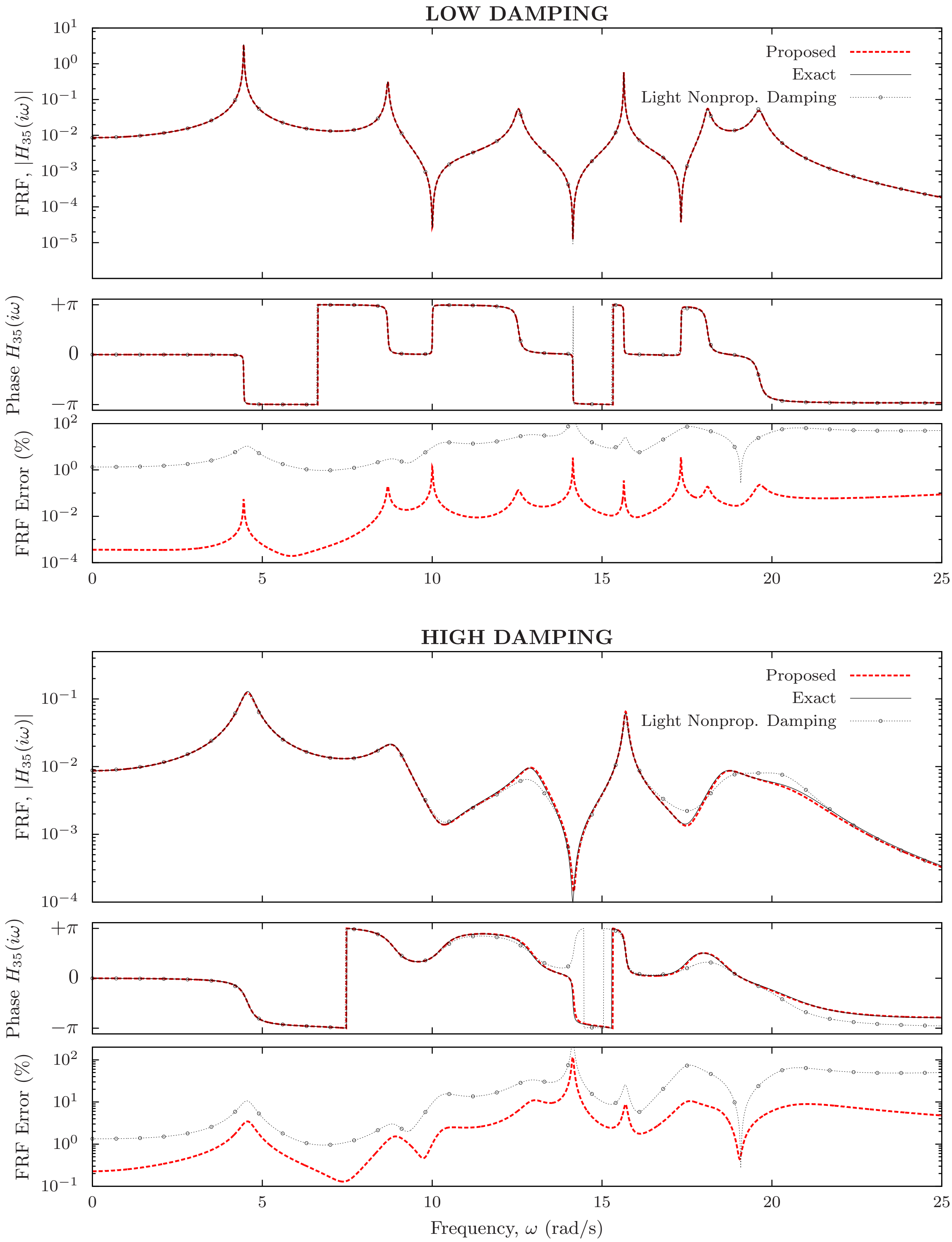

Figure 3: Frequency Response Function $H_{35}(\mathrm{i}\omega)$ calculated from Eq. (9). Eigenmodes computation: 'Exact' from Eqs. (54); 'Proposed' from Eqs. (40),(45); 'Linear First Order' approximation (LIN) from Eqs. (42),(45). Top-graphics: Low damping. Bottom-graphics: High damping

The transfer function $\mathbf{H}(\mathrm{i}\omega)$ is a 6×6 frequency-dependent symmetric matrix. Each entree $H_{jk}(\mathrm{i}\omega)$ represents the complex valued frequency response function (FRF) associated to the jth and kth degrees of freedom. Fig. 2 shows graphically the FRF $H_{26}(\mathrm{i}\omega)$ for LD and HD cases (top and bottom plots, respectively). FRF's from exact, proposed and LIN methods are plotted both in magnitude and phase together with the relative error. The latter is obtained comparing the proposed and LIN approximations with the exact one, allowing us to compare the accuracy along the frequency domain covering the range up to the sixth mode. Fig. 3 shows the same information but for the FRF $H_{35}(\mathrm{i}\omega)$. Since the derived equations are based on the hypothesis of light damping, it seems clear that the accuracy strongly depends on the level of damping after a simple comparison between the curves for LD nad HD. In fact, FRF-error plots, mathematically defined as

$$\epsilon_{jk}(\omega) = \frac{|H_{jk,\mathrm{exact}}(\mathrm{i}\omega) - H_{jk,\mathrm{aprox}}(\mathrm{i}\omega)|}{|H_{jk,\mathrm{exact}}(\mathrm{i}\omega)|} \tag{56}$$

show that the error increases from LD cases to HD cases, both in figs. 2 and 3. However, it can also be noticed within the error curves that, for the whole studied frequency range, the proposed solution is between one and three orders of magnitude more accurate than that of the LIN approximation. This fact has already been observed in the comparison of the relative errors of eigenvalues (Table 4) and it is justified in how the proposed formula has been obtained.

Our model predicts the jth complex eigenfrequency as the natural frequency somehow perturbed by the effects of the level of damping, the viscoelasticity and the nonproportionality, via DL_j, NV_j and NP_j coefficients, shown in Table 3. Since these coefficients vary for each mode, it is expected that accuracy of the FRF curves along the frequency band is neither uniform. Let us compare now the values of Table 3 and the FRF-curves of Figs. 2 and 3. We observe in Table 3 that the most affected modes by the above commented effects are the 3th, 5th and 6th ones (approximate natural frequencies 12.5, 18.0 and 19.5 rad/s, respectively). For them, the coefficients DL_j, NV_j and NP_j are the furthest from the unity. Now, in Figs. 2 and 3 (HD case) we see around these frequencies a stronger lack of accuracy for the FRF obtained with the LIN approximation. This behavior can be explained since the viscoelasticity and the nonproportionality are not considered in LIN method, together with the fact that the damping model is taken into account up to the first order of magnitude. Otherwise, the FRF curves from the proposed method fit very well with the exact solution even for highly damped systems, both in magnitude and phase, validating its derivation and the assumed hypothesis. Additionally, we remark also the theoretical value of the proposed approach. In fact, the method allows to explicitly relate complex eigenvalues with modal damping matrix entrees using for it just the computational effort needed to solve the linear undamped problem.

Future research is now focused on exploiting this method to determine as challenge higher orders of magnitude and more sophisticated closed-form derivations. Additionally, we are investigating if these new closed forms could be used to a new line of research in damping identification.

6. Conclusions

Nonviscous vibrating systems are characterized by dissipative forces depending on the time history of the degrees of freedom via hereditary kernel functions. The dynamic equilibrium leads to a system of integro-differential equations in the time domain. In the Laplace domain, eigenvalues and eigenvectors must be determined from a non-linear eigenvalue problem which in general requires high computational effort. Our research is focused on developing numerical methods for this type of problems.

In this paper, a closed-form approximated expression of the eigenvalues for nonviscous, nonproportional vibrating systems is derived. It is demonstrated that the resulting expression explicitly depends (i) on the damping level induced by the dissipative model; (ii) on the viscoelasticity of the system, i.e. the variability of the damping model in the frequency domain; (iii) on the nonproportionality of the modal damping matrix. This latter dependence involves the off-diagonal terms of the modal damping matrix together with the distance between natural frequencies. The resulting formula has the great advantage of involving only the computational complexity associated to solution of the undamped eigenvalue problem. The developments are carried out using a fictitious parameter affecting the damping model and on the basis of small damping assumption.

To validate the proposed approach, a six degrees-of-freedom lumped-mass nonproportional nonviscously damped system is analyzed. Exact eigensolutions are obtained using a iterative scheme based on the generalized Newton's method for non-linear eigenvalue problems. Additionally, we contrast the results with those from the linear first order approximation, which are also closed-form expressions constructed without discriminating the type of damping model. To this end, two cases of the level of damping are considered differentiating between 'low damping' (LD) and 'high damping' (HD). Additionally, the frequency response functions are plotted both in magnitude and phase in order to detect how our approach affects to the eigenvectors, also for both LD and HD. The proposed approach clearly depends on the level of damping, as evidenced by the increasing of the relative error of eigenvalues for LD case in contrast to those of HD. However, the proposed method presents a notably lower level than that of the linear first order approximation, keeping the accuracy within acceptable values even for moderately or highly damped systems. The sensitivity of the obtained results to the damping level also can be noticed in the frequency response functions (FRF). The different effects considered in the model (viscoelasticity and nonproportionality) enhance the estimation of the response against that one determined from the linear first order approximation. This behavior can be observed within the neighborhood of those modes most affected by the so-called coefficients of damping level, viscoelasticity and nonproportionality.

Appendix A. Eigenvectors Derivatives

Computation of eigenderivatives of eigenvalues and eigenvectors in nonviscously damped systems has been studied by Adhikari [25]. In this work, Adhikari derived general expressions of eigenvector derivatives respect to certain design parameter. We manipulate the expressions obtained by Adhikari in this Appendix in order to a better clarity in the developments made in the paper and avoiding some hard operations and simplifications.

We are interested in the p-derivative of eigenvectors of Eq. (12), say $\mathbf{U}_j(p)$. Let us rewrite the eigenrelations depending on the introduced p parameter.

$$\boldsymbol{\mathcal{D}}(\lambda_j(p), p)\,\mathbf{U}_j(p) \quad = \quad \mathbf{0} \tag{A.1}$$

The expressions calculated by Adhikari in the reference [25] are

$$\frac{\mathrm{d}\mathbf{U}_j}{\mathrm{d}p} \quad = \quad a_{jj}\,\mathbf{U}_j(p) + \sum_{\substack{k=1\\k\neq j}}^{m} a_{jk}\,\mathbf{U}_k(p) \tag{A.2}$$

where

$$\begin{aligned} a_{jk} \quad &= \quad -\frac{\mathbf{U}_k^T(p)\dfrac{\partial\boldsymbol{\mathcal{D}}(\lambda_j(p), p)}{\partial p}\mathbf{U}_j(p)}{\theta_k\,\left[\lambda_j(p) - \lambda_k(p)\right]}\;, \quad k \neq j \\ a_{jj} \quad &= \quad -\mathbf{U}_j^T(p)\,\frac{\partial^2\boldsymbol{\mathcal{D}}(\lambda_j(p), p)}{\partial s\partial p}\,\mathbf{U}_j(p)/2\theta_j \end{aligned} \tag{A.3}$$

and

$$\theta_k = \mathbf{U}_k^T(p)\,\frac{\partial\boldsymbol{\mathcal{D}}(\lambda_k(p), p)}{\partial s}\,\mathbf{U}_k(p) \tag{A.4}$$

The number m of terms in the above sum is equal to the number of poles of the matrix $\boldsymbol{\mathcal{D}}^{-1}(s,p)$. For our purposes, the expression (A.2) must be evaluated at $p=0$ (undamped problem), consequently the sum is extended to the $m=2n$ undamped modes. Additionally, we have the following results corresponding to the undamped problem ($p=0$ in Eq. (33))

$$\frac{\partial \boldsymbol{\mathcal{D}}(\mathrm{i}\omega_j,0)}{\partial s} = 2\mathrm{i}\omega_j \mathbf{M} \ , \quad \frac{\partial^2 \boldsymbol{\mathcal{D}}(\mathrm{i}\omega_j,0)}{\partial s \partial p} = \mathrm{i}\omega_j \frac{\partial \mathbf{G}(\mathrm{i}\omega_j)}{\partial s} + \mathbf{G}(\mathrm{i}\omega_j) \ , \quad \frac{\partial \boldsymbol{\mathcal{D}}(\mathrm{i}\omega_j,0)}{\partial p} = \mathrm{i}\omega_j \mathbf{G}(\mathrm{i}\omega_j) \tag{A.5}$$

Introducing these results in Eq. (A.2) and using the equivalences of Table 1 we have, after some manipulations

$$\begin{aligned}
\left.\frac{\mathrm{d}\mathbf{U}_j}{\mathrm{d}p}\right|_{p=0} &= a_{jj}\,\mathbf{U}_j(0) + \sum_{\substack{k=1\\k\neq j}}^{2n} a_{jk}\,\mathbf{U}_k(0) \\
&= a_{jj}\,\mathbf{x}_j - \sum_{\substack{k=1\\k\neq j}}^{n} \left(\frac{\mathbf{x}_k^T \frac{\partial \boldsymbol{\mathcal{D}}(\mathrm{i}\omega_j,0)}{\partial p} \mathbf{x}_j}{\mathbf{x}_k^T \frac{\partial \boldsymbol{\mathcal{D}}(i\omega_k,0)}{\partial s} \mathbf{x}_k} \right) \frac{\mathbf{x}_k}{\mathrm{i}\omega_j - i\omega_k} - \sum_{k=1}^{n} \left(\frac{\mathbf{x}_k^T \frac{\partial \boldsymbol{\mathcal{D}}(\mathrm{i}\omega_j,0)}{\partial p} \mathbf{x}_j}{\mathbf{x}_k^T \frac{\partial \boldsymbol{\mathcal{D}}(-i\omega_k,0)}{\partial s} \mathbf{x}_k} \right) \frac{\mathbf{x}_k}{\mathrm{i}\omega_j + i\omega_k} \\
&= a_{jj}\,\mathbf{x}_j - \sum_{\substack{k=1\\k\neq j}}^{n} \frac{\mathrm{i}\omega_j\,\Gamma_{kj}(\mathrm{i}\omega_j)}{2i\omega_k\,(\mathrm{i}\omega_j - i\omega_k)}\,\mathbf{x}_k + \sum_{\substack{k=1\\k\neq j}}^{n} \frac{\mathrm{i}\omega_j\,\Gamma_{kj}(\mathrm{i}\omega_j)}{2i\omega_k\,(\mathrm{i}\omega_j + i\omega_k)}\,\mathbf{x}_k + \frac{\Gamma_{jj}(\mathrm{i}\omega_j)}{4\mathrm{i}\omega_j}\,\mathbf{x}_j \\
&= a_{jj}\,\mathbf{x}_j + \frac{\Gamma_{jj}(\mathrm{i}\omega_j)}{4\mathrm{i}\omega_j}\,\mathbf{x}_j - \sum_{\substack{k=1\\k\neq j}}^{n} \frac{\mathrm{i}\omega_j\,\Gamma_{kj}(\mathrm{i}\omega_j)\,\mathbf{x}_k}{2i\omega_k} \left(\frac{1}{\mathrm{i}\omega_j - i\omega_k} - \frac{1}{\mathrm{i}\omega_j + i\omega_k} \right) \\
&= a_{jj}\,\mathbf{x}_j + \frac{\Gamma_{jj}(\mathrm{i}\omega_j)}{4\mathrm{i}\omega_j}\,\mathbf{x}_j - \mathrm{i}\omega_j \sum_{\substack{k=1\\k\neq j}}^{n} \frac{\Gamma_{kj}(\mathrm{i}\omega_j)\,\mathbf{x}_k}{\omega_k^2 - \omega_j^2}
\end{aligned} \tag{A.6}$$

Finally the coefficient a_{jj} at $p=0$ is

$$a_{jj} = -\mathbf{U}_j^T(0) \frac{\partial^2 \boldsymbol{\mathcal{D}}(\mathrm{i}\omega_j,0)}{\partial s \partial p} \mathbf{U}_j(p)/2\theta_j = -\frac{\mathbf{x}_j^T \left[\mathrm{i}\omega_j \frac{\partial \mathbf{G}(\mathrm{i}\omega_j)}{\partial s} + \mathbf{G}(\mathrm{i}\omega_j) \right] \mathbf{x}_j}{4\mathrm{i}\omega_j} = -\frac{1}{4} \left[\frac{\partial \Gamma_{jj}(\mathrm{i}\omega_j)}{\partial s} + \frac{\Gamma_{jj}(\mathrm{i}\omega_j)}{\mathrm{i}\omega_j} \right] \tag{A.7}$$

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