Projections for generalized inverses

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Abstract: Let R be a unital ring with involution. In Section 2, for given two core invertible elements $a, b \in R$, we investigate mainly the absorption law for the core inverse in virtue of the equality of the projections $aa^{\oplus}$ and $bb^{\oplus}$. In Section 3, we study several relations concerning the projections $a'a$ and bb', where $a' \in a\{1,2,4\}$ and $b' \in b\{1,2,3\}$. Some well-known results are extended to the $*$-reducing ring case. As an application, EP elements in a $*$-reducing ring are considered.

Key words: Projection, absorption law, core inverse, Moore-Penrose inverse.

AMS subject classifications: 15A09, 16W10, 16U60.

1 Introduction

Throughout this paper, R will denote a unital ring with involution, i.e., a ring with unity 1, and a mapping $a \mapsto a^*$ satisfying $(a^*)^* = a$, $(ab)^* = b^*a^*$ and $(a+b)^* = a^* + b^*$, for all $a, b \in R$. Let $a, x \in R$. If

$$(1)\ axa = a, \qquad (2)\ xax = x, \qquad (3)\ (ax)^* = ax, \qquad (4)\ (xa)^* = xa, \qquad (1.1)$$

then x is called a *Moore-Penrose inverse* of a. If such an element x exists, then it is unique and denoted by $a^{\dagger}$. The set of all Moore-Penrose invertible elements will be denoted by $R^{\dagger}$. Let $I \subset \{1,2,3,4\}$. An element $b \in R$ is called an I inverse of $a \in R$ if equalities $i \in I$ of (1.1) hold. The set of all I inverses of a will be denoted by a^I, the element a is I invertible when $a^I \neq \varnothing$ and the set of all I invertible elements will be denoted by R^I. Let $a \in R$. It can be easily proved that the set of elements $x \in R$ such that

$$axa = a, \quad xax = x \quad \text{and} \quad ax = xa$$

is empty or a singleton. If this set is a singleton, its unique element is called the *group inverse* of a and denoted by $a^{\#}$. The set of all group invertible elements will be denoted by $R^{\#}$. The subset of R of all invertible elements will be denoted by R^{-1}. We will also use the following notations: $aR = \{ax : x \in R\}$, $Ra = \{xa : x \in R\}$, ${}^{\circ}a = \{x \in R : xa = 0\}$ and $a^{\circ} = \{x \in R : ax = 0\}$.

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The notion of the core inverse for a complex matrix was introduced by Baksalary and Trenkler [1]. In [12], Rakić et al. generalized the core inverse of a complex matrix to the case of an element in R. More precisely, let $a, x \in R$, if

$$axa = a, \quad xR = aR \quad \text{and} \quad Rx = Ra^*, \tag{1.2}$$

then x is called a *core inverse* of a. If such an element x exists, then it is unique and denoted by $a^{\oplus}$. The set of all core invertible elements in R will be denoted by $R^{\oplus}$. Also, in [12] the authors defined a related inner inverse in a ring with an involution. If $a \in R$, then $x \in R$ is called a *dual core inverse* of a if

$$axa = a, \quad xR = a^*R \quad \text{and} \quad Rx = Ra.$$

If such an element x exists, then it is unique and denoted by $a_{\oplus}$. The set of all dual core invertible elements in R will be denoted by $R_{\oplus}$. It is evident that $a \in R^{\oplus}$ if and only if $a^* \in R_{\oplus}$, and in this case, one has $(a^{\oplus})^* = (a^*)_{\oplus}$. More characterizations of elements to be core invertible by equations can be found in [12, 15].

An element $a \in R$ is said to be an *EP* element if $a \in R^{\dagger} \cap R^{\#}$ and $a^{\dagger} = a^{\#}$ [2, 5]. The set of all EP elements will be denoted by R^{EP}. An element $p \in R$ is said to be a *projection* if $p^2 = p = p^*$. We will use the notation $[a, b] = ab - ba$.

In [7, Proposition 6], for two complex matrices A and B, Hartwig and Spindelböck explored equivalent conditions such that $A^{\dagger}A = BB^{\dagger}$. In [11, Theorem 2.3], for $a, b \in R^{\dagger}$, Patrício and Mendes investigated necessary and sufficient conditions such that $aa^{\dagger} = bb^{\dagger}$. If we take $c = b^{\dagger}$ in $aa^{\dagger} = bb^{\dagger}$, then by $(b^{\dagger})^{\dagger} = b$, we have $aa^{\dagger} = c^{\dagger}c$, and therefore, this form is the same as $A^{\dagger}A = BB^{\dagger}$. In [7, 11], authors investigated the equality $aa^{\dagger} = bb^{\dagger}$ under the hypothesis $a, b \in R^{\dagger}$. Motivated by [7, 11], in Section 2, we discuss when the projections $aa^{\oplus}$ and $bb^{\oplus}$ are equal, which is under the hypothesis $a, b \in R^{\oplus}$. We will extend [7, Proposition 6] of complex matrices to the case of $*$-reducing rings in Section 3. In Theorem 3.6, when $a^{(1,2,4)} \in a^{\{1,2,4\}}$ and $b^{(1,2,3)} \in b^{\{1,2,3\}}$, we give some relationships between the projections $a^{(1,2,4)}a$ and $bb^{(1,2,3)}$. Theorem 3.6 will be useful in the sequel. In Theorem 3.9, we can see that the proof becomes simple with the aid of Theorem 3.6. Note that Theorem 3.9 is a generalization of known equivalent conditions for an EP element in a $*$-reducing ring.

2 Core invertibility: the case $aa^{\oplus} = bb^{\oplus}$

Before we investigate necessary and sufficient conditions such that two core invertible elements satisfy $aa^{\oplus} = bb^{\oplus}$, some auxiliary work should be done.

Proposition 2.1. *Let $a, b \in R$. We have:*

(1) *If $a, b \in R^{\{1,3\}}$, then the following statements are equivalent:*

(i) *$aR = bR$;*

(ii) *$aa^{(1,3)} = bb^{(1,3)}$ for all $a^{(1,3)} \in a\{1,3\}$ and $b^{(1,3)} \in b\{1,3\}$;*

(iii) *$aa^{(1,3)} = bb^{(1,3)}$ for some $a^{(1,3)} \in a\{1,3\}$ and $b^{(1,3)} \in b\{1,3\}$.*

(2) *If $a, b \in R^{\{1,4\}}$, then the following statements are equivalent:*

(i) *$Ra = Rb$;*

(ii) *$a^{(1,4)}a = b^{(1,4)}b$ for all $a^{(1,4)} \in a\{1,4\}$ and $b^{(1,4)} \in b\{1,4\}$;*

(iii) *$a^{(1,4)}a = b^{(1,4)}b$ for some $a^{(1,4)} \in a\{1,4\}$ and $b^{(1,4)} \in b\{1,4\}$.*

Proof. Let us prove (1). (i) $\Rightarrow$ (ii). Suppose $aR = bR$, then $a = bx$ and $b = ay$ for some $x, y \in R$. Thus

$$a = bx = bb^{(1,3)}bx = bb^{(1,3)}a; \tag{2.1}$$
$$b = ay = aa^{(1,3)}ay = aa^{(1,3)}b. \tag{2.2}$$

Then we have

$$bb^{(1,3)} \overset{(2.2)}{=} aa^{(1,3)}bb^{(1,3)} = (aa^{(1,3)}bb^{(1,3)})^* = bb^{(1,3)}aa^{(1,3)}; \tag{2.3}$$
$$aa^{(1,3)} \overset{(2.1)}{=} bb^{(1,3)}aa^{(1,3)}. \tag{2.4}$$

A combination of (2.3) and (2.4) implies $aa^{(1,3)} = bb^{(1,3)}$.

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i). Suppose exists $a^{(1,3)} \in a\{1,3\}$ and $b^{(1,3)} \in b\{1,3\}$ such that $aa^{(1,3)} = bb^{(1,3)}$. Then we have $a = aa^{(1,3)}a = bb^{(1,3)}a$ and $b = bb^{(1,3)}b = aa^{(1,3)}b$, thus $aR = bR$.

The proof of (2) is similar to the proof of (1). $\square$

In [12, Theorem 2.14], Rakić et al. proved that an element $a \in R^{\oplus}$ satisfies

$$aa^{\oplus}a = a, \qquad a^{\oplus}aa^{\oplus} = a^{\oplus}, \qquad (aa^{\oplus})^* = aa^{\oplus}, \qquad a(a^{\oplus})^2 = a^{\oplus}, \qquad a^{\oplus}a^2 = a. \tag{2.5}$$

Also, in [12, Theorem 2.15], they proved that an element $a \in R_{\oplus}$ satisfies

$$aa_{\oplus}a = a, \qquad a_{\oplus}aa_{\oplus} = a_{\oplus}, \qquad (a_{\oplus}a)^* = a^{\oplus}a, \qquad (a^{\oplus})^2a = a_{\oplus}, \qquad a^2a_{\oplus} = a.$$

Thus a core invertible element is $\{1,2,3\}$-invertible and a dual core invertible element is $\{1,2,4\}$-invertible.

Two proofs of the next corollary can be found in [3, Proposition 1, Chapter 1] and in [12, Lemma 2.10]. We present another proof based on Proposition 2.1.

Corollary 2.2. *Let $p, q \in R$ be two projections. Then $pR = qR$ if and only if $p = q$.*

Proof. Since $p, q \in R$ are projections, then $p^2 = p = p^\dagger$ and $q^2 = q = q^\dagger$. By Proposition 2.1, $pR = qR$ implies $pp^\dagger = qq^\dagger$, that is $p = q$. The converse is clear. $\square$

In [11, Theorem 2.3], for $a, b \in R^\dagger$, the authors investigated equivalent conditions such that $aa^\dagger = bb^\dagger$. It is evident that the equality $xx^\dagger R = xR$ holds when $x \in R^\dagger$, thus the first item of next Corollary 2.3 follows from this observation and previous Corollary 2.2. The second item of next Corollary 2.3 follows from $x^*R = x^\dagger xR$, $(Rx)^* = x^*R$, and previous Corollary 2.2. If $x \in R^\# \cap R^\dagger = R^{\oplus} \cap R_{\oplus}$, by [12, Theorems 2.11 and 2.12] it follows that $xx^\dagger = xx^{\oplus}$ and $x^\dagger x = x_{\oplus}x$. The proof of next Corollary 2.3 only uses Proposition 2.1. Also, it is noteworthy that the characterization of $aa^{\oplus} = bb^{\oplus}$ ($a_{\oplus}a = b_{\oplus}b$) is generalized because only $a, b \in R^{\oplus}$ ($a, b \in R_{\oplus}$, resp.) is used.

Corollary 2.3. *Let $a, b \in R$. Then we have:*

(1) *If $a, b \in R^{\dagger}$, then $aR = bR$ is equivalent to $aa^{\dagger} = bb^{\dagger}$;*

(2) *If $a, b \in R^{\dagger}$, then $Ra = Rb$ is equivalent to $a^{\dagger}a = b^{\dagger}b$;*

(3) *If $a, b \in R^{\oplus}$, then $aR = bR$ is equivalent to $aa^{\oplus} = bb^{\oplus}$;*

(4) *If $a, b \in R_{\oplus}$, then $Ra = Rb$ is equivalent to $a_{\oplus}a = b_{\oplus}b$.*

For an idempotent p in a ring R, every $a \in R$ can be written as

$$a = pap + pa(1-p) + (1-p)ap + (1-p)a(1-p) \tag{2.6}$$

or in matrix form

$$\begin{bmatrix} pap & pa(1-p) \\ (1-p)ap & (1-p)a(1-p) \end{bmatrix}.$$

The decomposition (2.6) is known as the *Pierce decomposition.* Notice that if the idempotent p is a projection, then the above matrix representation preserves the involution.

Lemma 2.4. *Let $a, p, q \in R$. Then the following are equivalent:*

(1) *$a \in R^{\oplus}$;*

(2) [9, Theorem 3.5] *there exists a projection p such that $pa = 0$ and $a + p \in R^{-1}$;*

(3) [14, Theorem 3.3] *there exists a projection q such that $qa = 0$ and $a(1-q)+q \in R^{-1}$.*

Under these equivalence, one has that these projections p and q are unique and $p = q = 1 - aa^{\oplus}$.

We give a sketch of the proof of this lemma for the sake of completeness.

If $a \in R^{\oplus}$, define the projection $p = 1 - aa^{\oplus}$. From (2.5), it is evident that $pa = 0$, and thus, the Pierce decomposition of a with respect to p is

$$a = \begin{bmatrix} 0 & 0 \\ ap & a(1-p) \end{bmatrix}, \tag{2.7}$$

hence

$$a + p = \begin{bmatrix} p & 0 \\ ap & a(1-p) \end{bmatrix} \quad \text{and} \quad a(1-p) + p = \begin{bmatrix} p & 0 \\ 0 & a(1-p) \end{bmatrix}. \tag{2.8}$$

Observe that $a^{\oplus} \in aa^{\oplus}Raa^{\oplus} = (1-p)R(1-p)$ and $a(1-p)a^{\oplus} = 1-p = a^{\oplus}a(1-p)$. Hence $a^{\oplus}$ is the inverse of $a(1-p)$ in the ring $(1-p)R(1-p)$, and thus, by (2.8), $a+p$ and $a(1-p)+p$ are invertible in R because p is a unit in the ring pRp.

If exists a projection p such that $pa = 0$, then the Pierce decomposition of a, $a+p$, and $a(1-p)+p$ with respect to p are written in (2.7) and (2.8). Since p is a unit in the ring pRp, the above decompositions prove that $a + p \in R^{-1} \Leftrightarrow a(1-p) + p \in R^{-1} \Leftrightarrow a(1-p) \in [(1-p)R(1-p)]^{-1}$. Now, it is not difficult to prove that if $a + p \in R^{-1}$ or

$a(1-p)+p \in R^{-1}$, then a is core invertible and the inverse of $a(1-p)$ in $(1-p)R(1-p)$ is the core inverse of a.

Assume that p_1 and p_2 are two projections such that $p_i a = 0$ and $a + p_i \in R^{-1}$ for $i = 1, 2$. By the proof of the previous paragraph, the inverse of $a(1-p_i)$ in $(1-p_i)R(1-p_i)$ is $a^{\#}$ for $i = 1, 2$. Therefore, $(1-p_i)a^{\#} = a^{\#}$ and $a(1-p_i)a^{\#} = 1 - p_i$ for $i = 1, 2$. It is deduced that $aa^{\#} = 1 - p_i$ for $i = 1, 2$.

From the sketch of the proof of Lemma 2.4, if $a \in R^{\#}$, then the matrix representations of a, $a + p$, and $a(1-p) + p$ with respect to $p = 1 - aa^{\#}$ are written in (2.7) and (2.8).

Lemma 2.5. [6, Lemma 2] *Let R be any unitary ring and $e^2 = e \in R$. Then $exe + 1 - e$ is invertible in R if and only if exe is invertible in eRe with unit e, for all $x \in R$. If $(exe)^{-e}$ denotes the inverse of exe in eRe, then we have*

$$(exe)^{-e} = e(exe + 1 - e)^{-1}e$$

and

$$(exe + 1 - e)^{-1} = (exe)^{-e} + 1 - e.$$

Observe that if we represent the element $exe + 1 - e$ of the above lemma respect the idempotent e, we have $exe+1-e = \begin{bmatrix} exe & 0 \\ 0 & 1-e \end{bmatrix}$, which makes this lemma clear. Recall that the unity of the corner ring $(1-e)R(1-e)$ is $1 - e$.

The absorption law in a ring R means that for two invertible elements $a, b \in R$, we have $a^{-1}(a+b)b^{-1} = a^{-1} + b^{-1}$. In [8], Jin and Benítez investigated the absorption law for the core inverse.

Lemma 2.6. [8, Theorem 3.5] *Let $a, b \in R^{\#}$. Then the following are equivalent:*

(1) $a^{\#}(a+b)b^{\#} = a^{\#} + b^{\#}$;

(2) $aR = bR$;

(3) $^\circ a = {}^\circ b$.

In the following theorem, we will give more necessary and sufficient conditions such that the absorption law for the core inverse is valid. Let $a^\pi = 1 - aa^{\#}$.

Theorem 2.7. *Let $a, b \in R^{\#}$. Then the following are equivalent:*

(1) $aa^{\#} = bb^{\#}$;

(2) $a^\pi = b^\pi$;

(3) $a^{\#}(a+b)b^{\#} = a^{\#} + b^{\#}$;

(4) $a^\pi b = 0$ *and* $a^\pi + b \in R^{-1}$;

(5) $a^\pi b = 0$ *and* $a^\pi + b(1 - a^\pi) \in R^{-1}$;

(6) $a^\pi b = 0$ *and* $b(1-a^\pi)$ *is invertible in* $(1-a^\pi)R(1-a^\pi)$.

In this case, the expression of the subset of elements b such that $a^\pi = b^\pi$ is

$$\{b \in R : a^\pi = b^\pi\} = \{z + t : z \in (1-a^\pi)Ra^\pi, t \in [(1-a^\pi)R(1-a^\pi)]^{-1}\}.$$

Moreover, the relationship of $a^\pi + b$ and $a^\pi + b(1-a^\pi)$ is

$$(a^\pi + b)^{-1} = (b^2 + a^\pi)^{-1}[a^\pi + b(1-a^\pi)].$$

Proof. (1) $\Leftrightarrow$ (2) is trivial.

(1) $\Leftrightarrow$ (3) follows from Corollary 2.3 and Lemma 2.6.

(1) $\Rightarrow$ (4). Suppose $aa^{\oplus} = bb^{\oplus}$, or equivalently $a^\pi = b^\pi$ since (1) and (2) are equivalent. Taking into account the equality given in (2), we have $a^\pi b = (1-aa^{\oplus})b = (1-bb^{\oplus})b = 0$. Since

$$(a^\pi + b)(b^{\oplus} + 1 - b^{\oplus}b) = (1 - aa^{\oplus} + b)(b^{\oplus} + 1 - b^{\oplus}b) = (1 - bb^{\oplus} + b)(b^{\oplus} + 1 - b^{\oplus}b) = 1$$

and

$$(b^{\oplus} + 1 - b^{\oplus}b)(a^\pi + b) = (b^{\oplus} + 1 - b^{\oplus}b)(1 - bb^{\oplus} + b) = 1,$$

we have that $a^\pi + b$ is invertible.

(4) $\Rightarrow$ (2) (or (6) $\Rightarrow$ (2)). It is easy to check that a^π is a projection. By Lemma 2.4, we have $a^\pi = 1 - bb^{\oplus} = b^\pi$.

(1) $\Rightarrow$ (5). Suppose $aa^{\oplus} = bb^{\oplus}$. Then

$$\begin{aligned} a^\pi b &= (1-aa^{\oplus})b = (1-bb^{\oplus})b = 0; \\ [a^\pi + b(1-a^\pi)](b^{\oplus} + 1 - bb^{\oplus}) &= (1 - bb^{\oplus} + b^2b^{\oplus})(b^{\oplus} + 1 - bb^{\oplus}) = 1; \\ (b^{\oplus} + 1 - bb^{\oplus})[a^\pi + b(1-a^\pi)] &= (b^{\oplus} + 1 - bb^{\oplus})(1 - bb^{\oplus} + b^2b^{\oplus}) = 1. \end{aligned}$$

Thus $a^\pi + b(1-a^\pi)$ is invertible.

(5) $\Leftrightarrow$ (6) By $a^\pi b = 0$, we have $a^\pi + b(1-a^\pi) = a^\pi + (1-a^\pi)b(1-a^\pi)$. Thus by Lemma 2.5, we get the equivalence between (5) and (6).

Now, we will find the general expression of the elements $b \in R^{\oplus}$ such that $aa^{\oplus} = bb^{\oplus}$. We use the matrix representations of a^π and b with respect to the projection a^π. Then $a^\pi = \begin{bmatrix} a^\pi & 0 \\ 0 & 0 \end{bmatrix}$. Let

$$b = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$$

satisfy $aa^{\oplus} = bb^{\oplus}$ (and therefore, also (2), (3) and (4) hold). Then $a^\pi b = 0$ gives

$$0 = a^\pi b = \begin{bmatrix} a^\pi & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x & y \\ z & t \end{bmatrix} \quad \Rightarrow \quad x = y = 0.$$

Thus

$$a^\pi + b = \begin{bmatrix} a^\pi & 0 \\ z & t \end{bmatrix} \quad \Rightarrow \quad t \in [(1-a^\pi)R(1-a^\pi)]^{-1},$$

because a^π, the (1,1) entry of the last representation, is a unit in the corner ring $a^\pi Ra^\pi$. Also, it is simple to see that

$$c = \begin{bmatrix} 0 & 0 \\ z & t \end{bmatrix}, \ t \in [(1-a^\pi)R(1-a^\pi)]^{-1} \quad \Rightarrow \quad a^\pi c = 0 \text{ and } a^\pi + c \in R^{-1}.$$

Hence we have proved the characterisation of the elements b such that $a^\pi = b^\pi$ stated in the theorem.

Last, we give the relationship of $a^\pi + b$ and $a^\pi + b(1 - a^\pi)$. Since $a^\pi = b^\pi$, we have $a^\pi + b = b^\pi + b$ and $a^\pi + b(1 - a^\pi) = b^\pi + b(1 - b^\pi)$. Thus

$$[a^\pi + b(1 - a^\pi)](a^\pi + b) = [b^\pi + b(1 - b^\pi)](b^\pi + b) = b^2 + b^\pi = b^2 + a^\pi,$$

which is invertible because $a^\pi + b$ and $a^\pi + b(1 - a^\pi)$ are both invertible. Thus

$$(a^\pi + b)^{-1} = (b^2 + a^\pi)^{-1}[a^\pi + b(1 - a^\pi)].$$

□

Observe that the condition (1) of Theorem 2.7 is symmetric in a and in b. Hence we can get duplicate results.

Theorem 2.8. *Let $a, b \in R^{\oplus}$. Then the following are equivalent:*

(1) $aa^{\oplus} = bb^{\oplus}$*;*

(2) $b^\pi = a^\pi b^\pi$ *and* $b^\pi + 1 - a^\pi$ *is left invertible;*

(3) $1 - a^\pi = (1 - a^\pi)(1 - b^\pi)$ *and* $b^\pi + 1 - a^\pi$ *is left invertible;*

(4) $[a^\pi, b^\pi] = 0$, $b^\pi + 1 - a^\pi$ *is right invertible and* $a^\pi + 1 - b^\pi$ *is left invertible;*

(5) $a^\pi b^\pi$ *is Hermitian,* $b^\pi + 1 - a^\pi$ *is right invertible and* $a^\pi + 1 - b^\pi$ *is left invertible.*

Proof. The implications (1) ⇒ (2) and (1) ⇒ (4) are trivial.

(2) ⇒ (1). Suppose that $b^\pi = a^\pi b^\pi$ and $u = b^\pi + 1 - a^\pi$ is left invertible. Then we have

$$ru = 1 \quad \text{for some } r \in R. \tag{2.9}$$

Since a^π and b^π are Hermitian, then $a^\pi b^\pi = b^\pi$ implies $b^\pi a^\pi = b^\pi$. Thus

$$u(1 - a^\pi) = (b^\pi + 1 - a^\pi)(1 - a^\pi) = b^\pi(1 - a^\pi) + (1 - a^\pi)^2 = 1 - a^\pi \tag{2.10}$$

$$u(1 - b^\pi) = (b^\pi + 1 - a^\pi)(1 - b^\pi) = b^\pi(1 - b^\pi) + (1 - a^\pi)(1 - b^\pi) = 1 - a^\pi. \tag{2.11}$$

By equations (2.9), (2.10) and (2.11), we have

$$1 - a^\pi = ru(1 - a^\pi) = r(1 - a^\pi) = ru(1 - b^\pi) = 1 - b^\pi.$$

That is $aa^{\oplus} = bb^{\oplus}$.

(2) ⇔ (3). Since $(1 - a^\pi)(1 - b^\pi) = 1 - a^\pi - b^\pi + a^\pi b^\pi$, then it is obvious that $a^\pi b^\pi = b^\pi$ if and only if $(1 - a^\pi)(1 - b^\pi) = 1 - a^\pi$.

(4) ⇔ (5). If $[a^\pi, b^\pi] = 0$, then $(a^\pi b^\pi)^* = (b^\pi a^\pi)^* = a^\pi b^\pi$ and reciprocally.

(4) ⇒ (1). Assume $[a^\pi, b^\pi] = 0$ and $u = b^\pi + 1 - a^\pi$ is right invertible and $v = a^\pi + 1 - b^\pi$ is left invertible. We have

$$us = 1 \text{ and } tv = 1 \quad \text{for some } s, t \in R. \tag{2.12}$$

From $a^\pi b^\pi = b^\pi a^\pi$, we get

$$(1-b^\pi)u = (1-b^\pi)(b^\pi+1-a^\pi) = (1-b^\pi)b^\pi+(1-b^\pi)(1-a^\pi) = (1-b^\pi)(1-a^\pi). \quad (2.13)$$

As a^π and b^π commute, so does $1-a^\pi$ and $1-b^\pi$. Thus from (2.13) we get

$$(1-b^\pi)(1-a^\pi)u = (1-b^\pi)(1-a^\pi). \quad (2.14)$$

From equations (2.12), (2.13) and (2.14), we have

$$1-b^\pi = (1-b^\pi)us = (1-b^\pi)(1-a^\pi)s = (1-b^\pi)(1-a^\pi)us = (1-b^\pi)(1-a^\pi). \quad (2.15)$$

Similarly,

$$v(1-a^\pi) = (a^\pi+1-b^\pi)(1-a^\pi) = a^\pi(1-a^\pi)+(1-b^\pi)(1-a^\pi) = (1-b^\pi)(1-a^\pi), \quad (2.16)$$

and

$$v(1-b^\pi)(1-a^\pi) = (1-b^\pi)(1-a^\pi). \quad (2.17)$$

From equations (2.12), (2.16) and (2.17), we have

$$1-a^\pi = t(1-b^\pi)(1-a^\pi) = (1-b^\pi)(1-a^\pi). \quad (2.18)$$

By (2.15) and (2.18), we have $aa^{\oplus} = bb^{\oplus}$. □

Let $a \in R^{\oplus}$. If $aa^*+1-aa^{\oplus} \in R^{-1}$, then $a = (aa^*+1-aa^{\oplus})^{-1}aa^*a \in Raa^*a$, which implies that a is Moore-Penrose invertible in view of [16, Theorem 2.16]. Therefore, if $a \notin R^\dagger$, then $aa^*+1-aa^{\oplus}$ is not invertible.

Lemma 2.9. *Let $a \in R^{\oplus}$. If $aa^\pi a^* = 0$, then $aa^*+a^\pi \in R^{-1}$.*

Proof. We use the matrix representation of a with respect to the projection a^π.

$$a = \begin{bmatrix} 0 & 0 \\ aa^\pi & a(1-a^\pi) \end{bmatrix}, \ a^\pi = \begin{bmatrix} a^\pi & 0 \\ 0 & 0 \end{bmatrix}, \ a^* = \begin{bmatrix} 0 & a^\pi a^* \\ 0 & (1-a^\pi)a^* \end{bmatrix}, \ aa^* = \begin{bmatrix} 0 & 0 \\ 0 & aa^* \end{bmatrix}.$$

It is easy to check that $a(1-a^\pi)$ is invertible in $(1-a^\pi)R(1-a^\pi)$ with the inverse $a^{\oplus}$ and

$$aa^\pi a^* = 0 \Leftrightarrow a(1-a^\pi)a^* = aa^* \Leftrightarrow a(1-a^\pi)(1-a^\pi)a^* = aa^* \Leftrightarrow a(1-a^\pi)(a(1-a^\pi))^* = aa^*.$$

Thus aa^* is invertible in $(1-a^\pi)R(1-a^\pi)$ because aa^* is the product of two invertible elements in $(1-a^\pi)R(1-a^\pi)$. Since aa^* is invertible in $(1-a^\pi)R(1-a^\pi)$ if and only if $aa^*+a^\pi \in R^{-1}$, we finish the proof. □

Let $a \in R^{\oplus}$. If $aa^\pi = 0$, then $a = a^2a^{\oplus}$, and therefore, $a^{\oplus}a = a^{\oplus}(a^2a^{\oplus}) = (a^{\oplus}a^2)a^{\oplus} = aa^{\oplus}$, thus a is Moore-Penrose invertible, $a^\dagger = a^{\oplus}$, and a is EP by [12, Theorem 3.1]. Reciprocally, it is evident that if a is EP, then $a^{\oplus} = a^\dagger$ and $aa^{\oplus} = a^{\oplus}a$, which implies $aa^\pi = 0$. Notice that the condition $aa^\pi a^* = 0$ is weaker than $aa^\pi = 0$. But, it is strictly weaker, as the following example shows.

Example 2.10. Let R be the ring of 2×2 matrices whose entries are in $\mathbb{Z}_4$. In R we take the matrix transposition as the involution. Consider $a = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$. By checking the (1.1) and (1.2), it is simple to prove

$$a^\dagger = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, \qquad a^{\textcircled{\#}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Now, it is simple to compute

$$aa^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad a^\dagger a = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix},$$

which, by the definition of EP element, shows that a is not EP. In addition,

$$a^\pi = 1 - aa^{\textcircled{\#}} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad aa^\pi a^* = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} = 0.$$

But if the ring R is $*$-reducing, then $aa^\pi a^* = 0$ implies that $aa^\pi = 0$ (recall that a ring R is $*$-reducing if $x^*x = 0$ implies $x = 0$ for any $x \in R$). In fact: $0 = aa^\pi a^* = aa^\pi(aa^\pi)^*$ implies $aa^\pi = 0$. Of course, the ring considered in the above example is not reducing: it is enough to take $x = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ to see that $xx^* = 0$ and $x \neq 0$.

Theorem 2.11. *Let $a, b \in R^{\textcircled{\#}}$ with $aa^\pi a^* = 0$. Then $aa^{\textcircled{\#}} = bb^{\textcircled{\#}}$ if and only if $b^\pi = a^\pi b^\pi$ and $u = aa^* + b^\pi$ is invertible.*

Proof. We will prove the necessity and the sufficiency simultaneously. Since $aa^\pi a^* = 0$, by Lemma 2.9, we have that $aa^* + a^\pi$ is invertible. Notice that $aa^{\textcircled{\#}} = bb^{\textcircled{\#}}$, i.e., $a^\pi = b^\pi$, implies $a^\pi b^\pi = b^\pi$. Furthermore, we can use $a^\pi b^\pi = b^\pi$ to prove both the necessity and the sufficiency. Taking involution in $a^\pi b^\pi = b^\pi$ we get $b^\pi a^\pi = b^\pi$. Observe that

$$\begin{aligned}(b^\pi + 1 - a^\pi)(aa^* + a^\pi) &= b^\pi aa^* + b^\pi a^\pi + (1 - a^\pi)aa^* + (1 - a^\pi)a^\pi \\ &= b^\pi aa^{\textcircled{\#}} aa^* + b^\pi + aa^* \\ &= b^\pi(1 - a^\pi)aa^* + b^\pi + aa^* \\ &= b^\pi + aa^*.\end{aligned}$$

And, therefore, $b^\pi + 1 - a^\pi = (b^\pi + aa^*)(aa^* + a^\pi)^{-1}$. Now, by Theorem 2.8 and using that a Hermitian element is left invertible if and only if such element is invertible, we have:

$$aa^{\textcircled{\#}} = bb^{\textcircled{\#}} \;\Leftrightarrow\; b^\pi = a^\pi b^\pi \text{ and } b^\pi + 1 - a^\pi \in R^{-1} \;\Leftrightarrow\; b^\pi = a^\pi b^\pi \text{ and } b^\pi + aa^* \in R^{-1}.$$

The proof is finished. □

3 Projections $a^{(1,2,4)}a$ and $bb^{(1,2,3)}$

In this section, we will investigate projections $a^{(1,2,4)}a$ and $bb^{(1,2,3)}$. If we replace $a^{(1,2,4)}$ and $b^{(1,2,3)}$ by $a^\dagger$ and $b^\dagger$, respectively, then we can get some special corollaries, which will be useful when we discuss the projection equation $a^\dagger a = bb^\dagger$.

Lemma 3.1. *Let $a \in R$. Then we have the following results:*

(1) *If $a \in R^{\{1,2,3\}}$ and $a^{(1,2,3)} \in a\{1,2,3\}$, then $a^{(1,2,3)}b = a^{(1,2,3)}c$ if and only if $a^*b = a^*c$ for all $b, c \in R$;*

(2) *If $a \in R^{\{1,2,4\}}$ and $a^{(1,2,4)} \in a\{1,2,4\}$, then $ba^{(1,2,4)} = ca^{(1,2,4)}$ if and only if $ba^* = ca^*$ for all $b, c \in R$.*

Proof. Since the proof of (2) is similar to the proof of (1), we only prove (1). Since $a \in R^{\{1,2,3\}}$, then

$$\begin{aligned} a^{(1,2,3)}b =& a^{(1,2,3)}aa^{(1,2,3)}b = a^{(1,2,3)}(a^{(1,2,3)})^*a^*b = a^{(1,2,3)}(a^{(1,2,3)})^*a^*c = a^{(1,2,3)}c; \\ a^*b =& (aa^{(1,2,3)}a)^*b = a^*aa^{(1,2,3)}b = a^*aa^{(1,2,3)}c = a^*c. \end{aligned}$$

□

Proposition 3.2. *Let $a \in R^{\{1,2,4\}}$, $b \in R^{\{1,2,3\}}$ and $a^{(1,2,4)} \in a\{1,2,4\}$, $b^{(1,2,3)} \in b\{1,2,3\}$. Then $abb^{(1,2,3)}a^{(1,2,4)}ab = ab$ if and only if $b^{(1,2,3)}a^{(1,2,4)}abb^{(1,2,3)}a^{(1,2,4)} = b^{(1,2,3)}a^{(1,2,4)}$.*

Proof. By Lemma 3.1, we have

$$\begin{aligned} abb^{(1,2,3)}a^{(1,2,4)}ab = ab \Leftrightarrow \quad & b^*(a^{(1,2,4)}a)^*(bb^{(1,2,3)})^*a^* = b^*a^* \\ \Leftrightarrow \quad & b^*a^{(1,2,4)}abb^{(1,2,3)}a^* = b^*a^* \\ \Leftrightarrow \quad & b^{(1,2,3)}a^{(1,2,4)}abb^{(1,2,3)}a^{(1,2,4)} = b^{(1,2,3)}a^{(1,2,4)}. \end{aligned}$$

□

Corollary 3.3. *Let $a, b \in R^\dagger$. Then $abb^\dagger a^\dagger ab = ab$ if and only if $b^\dagger a^\dagger abb^\dagger a^\dagger = b^\dagger a^\dagger$.*

Lemma 3.4. *Let R be a $*$-reducing ring with $p^* = p = p^2 \in R$ and $q^* = q = q^2 \in R$. Then the following are equivalent:*

(1) $(pq)^2 = pq$;

(2) $(qp)^2 = qp$;

(3) $pq = qp$.

Proof. (1) $\Rightarrow$ (3) We use the matrix representations of p and q with respect to the projection p. Then $p = \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix}$ and $q = \begin{bmatrix} q_1 & q_2 \\ q_2^* & q_3 \end{bmatrix}$ by $q^* = q$. Then $pq = \begin{bmatrix} q_1 & q_2 \\ 0 & 0 \end{bmatrix}$ and $qp = \begin{bmatrix} q_1 & 0 \\ q_2^* & 0 \end{bmatrix}$. Now $(pq)^2 = (pq)$ and $q^2 = q$ implies, respectively, $q_1^2 = q_1$ and $q_1^2 + q_2q_2^* = q_1$. Hence, we have $q_2q_2^* = 0$, which implies $q_2 = 0$ because R is a $*$-reducing ring. Thus $pq = \begin{bmatrix} q_1 & 0 \\ 0 & 0 \end{bmatrix}$ and $qp = \begin{bmatrix} q_1 & 0 \\ 0 & 0 \end{bmatrix}$. That is $pq = qp$.

(3) $\Rightarrow$ (1) and (3) $\Rightarrow$ (2) are trivial. By the symmetry of p and q, we have (2) $\Rightarrow$ (3) with the help of (1) $\Rightarrow$ (3). □

Example 3.5. In a general ring R, the implication of $(pq)^2 = pq \Rightarrow pq = qp$ may not hold, where $p^* = p = p^2 \in R$ and $q^* = q = q^2 \in R$. Let us consider the following counterexample. Let R be the ring of 2×2 matrices over $\mathbb{Z}_4$ with the conjugate transposition as involution. Considering the matrices $p = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $q = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$, it is easy to check that $p^* = p = p^2$ and $q^* = q = q^2$ and $(pq)^2 = pq$, and yet $pq = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ and $qp = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$.

Theorem 3.6. *Let R be a $*$-reducing ring. If $a \in R^{\{1,2,4\}}$, $b \in R^{\{1,2,3\}}$ and $a^{(1,2,4)} \in a\{1,2,4\}$, $b^{(1,2,3)} \in b\{1,2,3\}$. Then the following are equivalent:*

(1) $abb^{(1,2,3)}a^{(1,2,4)}ab = ab$;

(2) $[a^{(1,2,4)}a, bb^{(1,2,3)}] = 0$;

(3) $b^{(1,2,3)}a^{(1,2,4)}abb^{(1,2,3)}a^{(1,2,4)} = b^{(1,2,3)}a^{(1,2,4)}$.

Proof. (3) $\Rightarrow$ (2). By Proposition 3.2, we have $abb^{(1,2,3)}a^{(1,2,4)}ab = ab$ if and only if $b^{(1,2,3)}a^{(1,2,4)}abb^{(1,2,3)}a^{(1,2,4)} = b^{(1,2,3)}a^{(1,2,4)}$. Define the projections $e = a^{(1,2,4)}a$ and $f = bb^{(1,2,3)}$. Multiplying by $a^{(1,2,4)}$ on the left side of $abb^{(1,2,3)}a^{(1,2,4)}ab = ab$ and multiplying by $b^{(1,2,3)}$ on the right side of $abb^{(1,2,3)}a^{(1,2,4)}ab = ab$, then we have $(ef)^2 = ef$, thus $[a^{(1,2,4)}a, bb^{(1,2,3)}] = 0$ by Lemma 3.4.

(2) $\Rightarrow$ (1) is trivial.

(1) $\Rightarrow$ (3). It is obvious by Proposition 3.2. $\square$

Corollary 3.7. [10, Theorem 2.1] *Let R be a $*$-reducing ring with $a, b \in R^\dagger$. Then the following are equivalent:*

(1) $abb^\dagger a^\dagger ab = ab$;

(2) $[a^\dagger a, bb^\dagger] = 0$;

(3) $b^\dagger a^\dagger abb^\dagger a^\dagger = b^\dagger a^\dagger$.

Lemma 3.8. [13, Lemma 8] *Let $a, b \in R$. Then:*

(1) *$aR \subseteq bR$ implies $^\circ b \subseteq {}^\circ a$ and the converse is valid whenever b is regular;*

(2) *$Ra \subseteq Rb$ implies $b^\circ \subseteq a^\circ$ and the converse is valid whenever b is regular.*

Theorem 3.9. *Let R be a $*$-reducing ring with $a, b \in R^\dagger$. Then the following are equivalent:*

(1) $a^\dagger a = bb^\dagger$;

(2) $[a^\dagger a, bb^\dagger] = 0$, *$aR \subseteq abR$ and $Rb \subseteq Rab$;*

(3) $(ab)^\dagger = b^\dagger a^\dagger$, *$aR \subseteq abR$ and $Rb \subseteq Rab$;*

(4) $abb^\dagger a^\dagger ab = ab$, *$aR \subseteq abR$ and $Rb \subseteq Rab$;*

(5) $b^\dagger a^\dagger abb^\dagger a^\dagger = b^\dagger a^\dagger$, *$aR \subseteq abR$ and $Rb \subseteq Rab$;*

(6) $[a^\dagger a, bb^\dagger] = 0$, $^\circ(ab) \subseteq {}^\circ a$ *and* $(ab)^\circ \subseteq b^\circ$*;*

(7) $(ab)^\dagger = b^\dagger a^\dagger$, $^\circ(ab) \subseteq {}^\circ a$ *and* $(ab)^\circ \subseteq b^\circ$*;*

(8) $abb^\dagger a^\dagger ab = ab$, $^\circ(ab) \subseteq {}^\circ a$ *and* $(ab)^\circ \subseteq b^\circ$*;*

(9) $b^\dagger a^\dagger abb^\dagger a^\dagger = b^\dagger a^\dagger$, $^\circ(ab) \subseteq {}^\circ a$ *and* $(ab)^\circ \subseteq b^\circ$*.*

Proof. The equivalence between (2), (4) and (5) can be obtained by Corollary 3.3 and Corollary 3.7.

(1) $\Rightarrow$ (2). Suppose $a^\dagger a = bb^\dagger$. Then

$$\begin{aligned} [a^\dagger a, bb^\dagger] &= [a^\dagger a, a^\dagger a] = 0; \\ a &= aa^\dagger a = abb^\dagger; \\ b &= bb^\dagger b = a^\dagger ab, \end{aligned}$$

that is $[a^\dagger a, bb^\dagger] = 0$, $aR \subseteq abR$ and $Rb \subseteq Rab$.

(2) $\Rightarrow$ (3). By Corollary 3.7, we have $[a^\dagger a, bb^\dagger] = 0 \Leftrightarrow abb^\dagger a^\dagger ab = ab$ and $b^\dagger a^\dagger abb^\dagger a^\dagger = b^\dagger a^\dagger$. Since $aR \subseteq abR$ and $Rb \subseteq Rab$, then

$$a = abx \text{ for some } x \in R; \tag{3.1}$$
$$b = yab \text{ for some } y \in R. \tag{3.2}$$

Thus by equations (3.1) and (3.2), we have

$$a = abb^\dagger a^\dagger abx = abb^\dagger a^\dagger a = aa^\dagger abb^\dagger = abb^\dagger; \tag{3.3}$$
$$b = yabb^\dagger a^\dagger ab = bb^\dagger a^\dagger ab = a^\dagger ab. \tag{3.4}$$

The equations (3.3) and (3.4) give that

$$\begin{aligned} abb^\dagger a^\dagger &= aa^\dagger \text{ is Hermitian;} \\ b^\dagger a^\dagger ab &= b^\dagger b \text{ is Hermitian.} \end{aligned}$$

(3) $\Rightarrow$ (1). Suppose $(ab)^\dagger = b^\dagger a^\dagger$, $aR \subseteq abR$, and $Rb \subset Rab$. There exist $x, y \in R$ such that $a = abx$ and $b = yab$. Now, $abb^\dagger a^\dagger a = abb^\dagger a^\dagger abx = ab(ab)^\dagger abx = abx = a$, and similarly, $bb^\dagger a^\dagger ab = yab(ab)^\dagger ab = yab = b$. Thus

$$bb^\dagger R = bb^\dagger a^\dagger aR \text{ and } a^\dagger aR = a^\dagger abb^\dagger R. \tag{3.5}$$

The condition $(ab)^\dagger = b^\dagger a^\dagger$ gives $(a^\dagger abb^\dagger)^2 = a^\dagger abb^\dagger$. By Lemma 3.4 we get $a^\dagger abb^\dagger = bb^\dagger a^\dagger a$. Therefore we have $a^\dagger a = bb^\dagger$ by equation (3.5) and Corollary 2.2.

(2) $\Leftrightarrow$ (6). It is easy to see that by Lemma 3.8. The proofs of (3) $\Leftrightarrow$ (7), (4) $\Leftrightarrow$ (8) and (5) $\Leftrightarrow$ (9) are similar to the proof of (2) $\Leftrightarrow$ (6). $\square$

The element $a \in R^\dagger$ is said to be *bi-EP* when $[a^\dagger a, aa^\dagger] = 0$. Let a be an element of an associative ring R with 1. In [4, Proposition 8.22], we have a has a group inverse if and only if $a^2x = a$ and $ya^2 = a$ both have solutions. Thus, taking $a = b$ in Theorem 3.9, we get the following corollary.

Corollary 3.10. *Let R be a $*$-reducing ring with $a \in R^{\dagger}$. Then the following are equivalent:*

(1) $a \in R^{\mathrm{EP}}$;

(2) $a \in R^{\#}$ *and a is bi-EP;*

(3) $a \in R^{\#}$ *and* $(a^{\dagger})^2 = (a^2)^{\dagger}$;

(4) $a \in R^{\#}$ *and* $(a^{\dagger})^2 \in a^2\{1\}$;

(5) $a \in R^{\#}$ *and* $a^2 \in (a^{\dagger})^2\{1\}$.

ACKNOWLEDGMENTS

This research is supported by the National Natural Science Foundation of China (No. 11371089). The first author is grateful to China Scholarship Council for giving him a scholarship for his further study in Universitat Politècnica de València, Spain.

We are very grateful to the referees for their carefully reading of the preliminary version of the manuscript.

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