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Irreducible fractal structures for Moran's theorems

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Abstract

In this paper, we deal with a classical problem in Fractal Geometry consisting of the calculation of the similarity dimension of IFS-attractors. The open set condition allows to easily calculate their similarity dimension though it depends on an external open set. We contribute a necessary condition to reach the equality among some fractal dimensions for the natural fractal structure for IFS-attractors and the similarity dimension. That condition, weaker than the SOSC, *becomes more representative of the attractor's self-similar structure.*

Keywords: *fractal structure; Hausdorff dimension; similarity dimension; open set condition; IFS-attractor.*
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1. Introduction

A classical problem in Fractal Geometry deals with determining under what conditions on the pieces of a strict self-similar set $\mathcal{K}$, the equality between the similarity and the Hausdorff dimensions of $\mathcal{K}$ stands. In this way, a classical result contributed by P. A. P. Moran in the forties (c.f. [12, Theorem III]) states that under the open set condition (OSC in the sequel), a property required to the pieces of $\mathcal{K}$ to guarantee that their overlaps are thin enough, the desired equality holds. Afterwards, Lalley introduced the strong open set condition (SOSC) by further requiring that the (feasible) open set provided by the OSC intersects the attractor $\mathcal{K}$. The next chain of implications and equivalences stands in the case of Euclidean self-similar sets and is best possible (c.f. [14]):

$$\text{SOSC} \Leftrightarrow \text{OSC} \Leftrightarrow \mathcal{H}_{\mathrm{H}}^{\alpha}(\mathcal{K}) > 0 \Rightarrow \dim_{\mathrm{H}}(\mathcal{K}) = \alpha, \tag{1}$$

where $\mathcal{H}_{\mathrm{H}}^{\alpha}$ is the $\alpha-$dimensional Hausdorff measure, $\dim_{\mathrm{H}}$ denotes the Hausdorff dimension, and α is the similarity dimension of $\mathcal{K}$. A counterexample due to Mattila allows to guarantee that the last implication in Eq. (1) cannot be inverted, in general. Accordingly, the OSC becomes only sufficient to reach the equality between those dimensions. A further extension of the problem above takes place in the more general context of attractors on complete metric spaces. Schief also explored such a problem and justified the following chain of implications (c.f. [15]):

$$\mathcal{H}_{\mathrm{H}}^{\alpha}(\mathcal{K}) > 0 \Rightarrow \text{SOSC} \Rightarrow \dim_{\mathrm{H}}(\mathcal{K}) = \alpha, \tag{2}$$

i.e., the SOSC is necessary for $\mathcal{H}_{\mathrm{H}}^{\alpha}(\mathcal{K}) > 0$ and only sufficient for $\dim_{\mathrm{H}}(\mathcal{K}) = \alpha$. Once again, the above-mentioned result due to Mattila implies that Eq. (2) is best possible. From both Eqs. (1) and (2), it holds that the SOSC is a sufficient condition on the pre-fractals of $\mathcal{K}$ leading to $\dim_{\mathrm{H}}(\mathcal{K}) = \alpha$.

In this paper, we shall make use of the concept of a fractal structure (first sketched in [3]) to explore and characterize a novel separation property in both contexts: Euclidean attractors and self-similar sets in complete metric spaces. Such a separation property, weaker than the OSC, becomes necessary to reach the equality between the similarity dimension of the attractor and its Hausdorff dimension.

2. Preliminaries

2.1. **The open set condition.** We say that $\mathcal{F} = \{f_1, \ldots, f_k\}$ (or its attractor $\mathcal{K}$, as well) is under the Moran's OSC (c.f. [12]) if there exists a nonempty open subset $\mathcal{V} \subseteq \mathbb{R}^d$ such that the images $f_i(\mathcal{V})$ are pairwise disjoint with all of them contained in $\mathcal{V}$, called a feasible open set. The strong open set condition (SOSC) stands, if and only if, it holds, in addition to the OSC assumptions, that $\mathcal{V} \cap \mathcal{K} \neq \emptyset$ (c.f. [10]). Schief proved that both the OSC and the SOSC are equivalent on Euclidean spaces (c.f. [14, Theorem 2.2]).

2.2. **Fractal structures.** Fractal structures were first sketched by Bandt and Retta in [3] and formally introduced afterwards by Arenas and Sánchez-Granero to characterize non-Archimedean quasi-metrization (c.f. [1]).

By a covering of a nonempty set X, we shall understand a family Γ of subsets such that $X = \cup\{A : A \in \Gamma\}$. Let Γ_1 and Γ_2 be two coverings of X. The notation $\Gamma_2 \prec \Gamma_1$ means that Γ_2 is a *refinement* of Γ_1, i.e., for all $A \in \Gamma_2$, there exists $B \in \Gamma_1$ such that $A \subseteq B$. Moreover, $\Gamma_2 \prec\prec \Gamma_1$ denotes that $\Gamma_2 \prec \Gamma_1$, and additionally, for all $B \in \Gamma_1$, it holds that $B = \cup\{A \in \Gamma_2 : A \subseteq B\}$. Thus, a fractal structure on X is a countable family of coverings $\mathbf{\Gamma} = \{\Gamma_n\}_{n\in\mathbb{N}}$ such that $\Gamma_{n+1} \prec\prec \Gamma_n$, for all natural number n. The covering Γ_n is called *level* n of $\mathbf{\Gamma}$. A fractal structure is said to be finite if all its levels are finite coverings.

Definition 1 (c.f. [2], Definition 4.4)**.** Let $\mathcal{F}$ be an IFS whose attractor is $\mathcal{K}$. The natural fractal structure on $\mathcal{K}$ as a self-similar set is given by the countable family of coverings $\mathbf{\Gamma} = \{\Gamma_n\}_{n\in\mathbb{N}}$, where $\Gamma_1 = \{f_i(\mathcal{K}) : i \in \Sigma\}$, and $\Gamma_{n+1} = \{f_i(A) : A \in \Gamma_n, i \in \Sigma\}$.

3. Fractal dimensions for fractal structures

Let $\mathbf{\Gamma}$ be a fractal structure on a metric space (X, ρ). We shall define $\mathcal{A}_n(F)$ as the collection consisting of all the elements in level n of $\mathbf{\Gamma}$ that intersect a subset F of X. Mathematically, $\mathcal{A}_n(F) = \{A \in \Gamma_n : A \cap F \neq \emptyset\}$. Further, let $\operatorname{diam}(\Gamma_n) = \sup\{\operatorname{diam}(A) : A \in \Gamma_n\}$, and $\operatorname{diam}(F, \Gamma_n) = \sup\{\operatorname{diam}(A) : A \in \mathcal{A}_n(F)\}$, as well.

Definition 2 (c.f. [5, Definition 4.2] and [7, Definition 3.2])**.** Assume that $\operatorname{diam}(F, \Gamma_n) \to 0$ and consider the following expression for $k = 3, 4$:

$$\mathcal{H}^s_{n,k}(F) = \inf \left\{ \sum \operatorname{diam}(A_i)^s : \{A_i\}_{i \in I} \in \mathcal{A}_{n,k}(F) \right\}, \text{where}$$

(i) $\mathcal{A}_{n,3}(F) = \{\mathcal{A}_l(F) : l \geq n\}$.

(ii) $\mathcal{A}_{n,4}(F) = \{\{A_i\}_{i \in I} : A_i \in \cup_{l \geq n} \Gamma_l, F \subseteq \cup_{i \in I} A_i, \operatorname{Card}(I) < \infty\}$. Here, $\operatorname{Card}(I)$ denotes the cardinal of I.

In addition, let $\mathcal{H}^s_k(F) = \lim_{n \to \infty} \mathcal{H}^s_{n,k}(F)$. By the fractal dimension III (resp., IV) of F, we shall understand the (unique) critical point satisfying the identity

$$\dim^k_{\mathbf{\Gamma}}(F) = \sup\{s \geq 0 : \mathcal{H}^s_k(F) = \infty\} = \inf\{s \geq 0 : \mathcal{H}^s_k(F) = 0\}.$$

4. Moran's type theorems under the OSC

One of the main goals in this paper is to explore some separation conditions for IFS−attractors in the context of fractal structures. It is worth pointing out that the main ideas contributed hereafter first appeared in [13].

IFS conditions. Let $(X, \mathcal{F})$ be an IFS, where X is a complete metric space, $\mathcal{F} = \{f_1, \ldots, f_k\}$ is a finite collection of similitudes on X, and $\mathcal{K}$ is the IFS−attractor of $\mathcal{F}$. Moreover, let $\mathbf{\Gamma}$ be the natural fractal structure on $\mathcal{K}$ as a self-similar set (c.f. Definition 1), and c_i be the similarity ratio of $f_i \in \mathcal{F}$.

All the results contributed along this paper stand under the IFS conditions above.

Next, we recall the concept of similarity dimension for IFS−attractors.

Definition 3. Let $\mathcal{F}$ be an IFS and $\mathcal{K}$ its attractor. By the similarity dimension of $\mathcal{K}$, we shall understand the unique solution $\alpha > 0$ of the equation $\sum_{i=1}^k c_i^s = 1$. In other words, the similarity dimension of $\mathcal{K}$ is the unique value $\alpha > 0$ such that $p(\alpha) = 0$, where $p(s) = \sum_{i=1}^k c_i^s - 1$.

Along the sequel, α will denote the similarity dimension of an IFS−attractor. It is worth noting that (without any additional assumption) $\mathcal{H}^\alpha_{\mathrm{H}}(\mathcal{K}) < \infty$ for any IFS-attractor $\mathcal{K}$ (c.f. [8, Proposition 4 (i)]).

Theorem 4 (IFS)**.** *(c.f.* [5, Theorem 4.20]) $\dim_{\mathbf{\Gamma}}^3(\mathcal{K}) = \alpha$, *and* $\mathcal{H}_3^\alpha(\mathcal{K}) \in (0, \infty)$.

Moran's Theorem (1946) (EIFS)**.** OSC $\Rightarrow \dim_{\mathrm{H}}(\mathcal{K}) = \alpha$, and $\mathcal{H}_{\mathrm{H}}^\alpha(\mathcal{K}) \in (0, \infty)$.

By a Moran's type theorem, we shall understand a result that yields the equality between a fractal dimension dim of an IFS$-$attractor $\mathcal{K}$ and its similarity dimension, namely, $\dim(\mathcal{K}) = \alpha$.

Corollary 5 (EIFS)**.** *(c.f.* [5, Corollary 4.22]) OSC $\Rightarrow \dim_{\mathrm{H}}(\mathcal{K}) = \dim_{\mathbf{\Gamma}}^3(\mathcal{K}) = \alpha$.

Lemma 6. *(c.f.* [7, Proposition 3.5 (3)]*) Let* $\mathbf{\Gamma}$ *be a finite fractal structure on a metric space* (X, ρ)*,* F *be a subset of* X*, and assume that* $\mathrm{diam}\,(F, \Gamma_n) \to 0$*. Then*

$$\dim_{\mathrm{H}}(F) \leq \dim_{\mathbf{\Gamma}}^4(F) \leq \dim_{\mathbf{\Gamma}}^3(F).$$

Corollary 7 (IFS)**.** $\dim_{\mathrm{H}}(\mathcal{K}) \leq \dim_{\mathbf{\Gamma}}^4(\mathcal{K}) \leq \dim_{\mathbf{\Gamma}}^3(\mathcal{K}) = \alpha$.

Theorem 8 (EIFS)**.** OSC $\Rightarrow \dim_{\mathrm{H}}(\mathcal{K}) = \dim_{\mathbf{\Gamma}}^4(\mathcal{K}) = \dim_{\mathbf{\Gamma}}^3(\mathcal{K}) = \alpha$.

To conclude this section, we recall two key results explored by Schief (c.f. [14, 15]).

Theorem 9.

(EIFS) SOSC $\Leftrightarrow$ OSC $\Leftrightarrow \mathcal{H}_{\mathrm{H}}^\alpha(\mathcal{K}) > 0 \Rightarrow \dim_{\mathrm{H}}(\mathcal{K}) = \alpha$.
(IFS) $\mathcal{H}_{\mathrm{H}}^\alpha(\mathcal{K}) > 0 \Rightarrow$ SOSC $\Rightarrow \dim_{\mathrm{H}}(\mathcal{K}) = \alpha$.

Theorem 9 is best possible due to Mattila's counterexample.

5. Towards a necessary condition for Moran's type theorems

In this section, we introduce a novel separation condition for each level of the natural fractal structure $\mathbf{\Gamma}$ that any IFS$-$attractor can be endowed with (c.f. Definition 1). Such a separation property is equivalent to $\mathbf{\Gamma}$ being irreducible.

Definition 10. We shall understand that $\mathcal{F}$ satisfies the level separation property (LSP) if the two following conditions hold for each level of $\mathbf{\Gamma}$:

LSP1: $A^\circ \cap B^\circ = \emptyset$, for all $A, B \in \Gamma_n : A \neq B$.
LSP2: $A^\circ \neq \emptyset$, for each $A \in \Gamma_n$,

where the interiors have been considered in $\mathcal{K}$.

It is worth pointing out that the LSP does not depend on an external open set, unlike the OSC. Let Γ be a covering of X. Recall that Γ is a tiling provided that all the elements of Γ have disjoint interiors and are regularly closed, i.e., $\overline{A^\circ} = A$ for each $A \in \Gamma$. A fractal structure $\mathbf{\Gamma}$ is called a tiling if each level Γ_n of $\mathbf{\Gamma}$ is a tiling itself.

Theorem 11 (IFS)**.** *The following are equivalent:*

(i) $\mathbf{\Gamma}$ *irreducible.*
(ii) $\dim_{\mathbf{\Gamma}}^4(\mathcal{K}) = \dim_{\mathbf{\Gamma}}^3(\mathcal{K}) = \alpha$*.*
(iii) LSP*.*
(iv) LSP2 *and* $A_{\mathbf{i}} \subseteq A_{\mathbf{j}}$ *implies* $\mathbf{j} \sqsubseteq \mathbf{i}$*.*
(v) $\mathbf{\Gamma}$ *tiling.*
(vi) $\mathcal{H}_4^\alpha(\mathcal{K}) > 0$*.*

Definition 12. We shall understand that $\mathcal{F}$ is under the weak separation condition (WSC) if any of the equivalent statements provided in Theorem 11 stands.

Corollary 13 (IFS)**.** SOSC $\Rightarrow$ WSC*, and the reciprocal is not true, in general.*

The following Moran's type theorem holds for both fractal dimensions III and IV provided that $\mathcal{F}$ is under the WSC.

Theorem 14 (IFS)**.** WSC $\Leftrightarrow \dim_{\mathbf{\Gamma}}^4(\mathcal{K}) = \dim_{\mathbf{\Gamma}}^3(\mathcal{K}) = \alpha$*.*

6. Conclusion

In this section, we summarize all the results contributed along this paper.

Theorem 15. *Consider the following statements:*

(i) $\mathcal{H}_{\mathrm{H}}^\alpha(\mathcal{K}) > 0$*.*
(ii) SOSC*.*
(iii) OSC*.*
(iv) $\dim_{\mathrm{H}}(\mathcal{K}) = \dim_{\mathbf{\Gamma}}^4(\mathcal{K}) = \dim_{\mathbf{\Gamma}}^3(\mathcal{K}) = \alpha$*.*
(v) $\dim_{\mathbf{\Gamma}}^4(\mathcal{K}) = \dim_{\mathbf{\Gamma}}^3(\mathcal{K}) = \alpha$*.*

(vi) $\mathbf{\Gamma}$ *irreducible.*
(vii) $\mathbf{\Gamma}$ *tiling.*
(viii) $\mathcal{H}_4^\alpha(\mathcal{K}) > 0$.

The next chains of implications and equivalences stand:

(EIFS) $(i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Rightarrow (iv) \Rightarrow (v) \Leftrightarrow (vi) \Leftrightarrow (vii) \Leftrightarrow (viii)$.
(IFS) $(i) \Rightarrow (ii) \Rightarrow (iv) \Rightarrow (v) \Leftrightarrow (vi) \Leftrightarrow (vii) \Leftrightarrow (viii)$.

To conclude this paper, we provide two comparative theorems (one for each context, EIFS or IFS) involving our results vs. those obtained by Schief.

Theorem 16 (EIFS, comparative theorem)**.**

$$\mathcal{H}_{\mathrm{H}}^\alpha(\mathcal{K}) > 0 \Leftrightarrow \mathrm{OSC} \Leftrightarrow \mathrm{SOSC} \Rightarrow \dim_{\mathrm{H}}(\mathcal{K}) = \alpha.$$
$$\mathrm{WSC} \Leftrightarrow \mathcal{H}_4^\alpha(\mathcal{K}) > 0 \Leftrightarrow \dim_{\mathbf{\Gamma}}^4(\mathcal{K}) = \alpha.$$

Theorem 17 (IFS, comparative theorem)**.**

$$\mathcal{H}_{\mathrm{H}}^\alpha(\mathcal{K}) > 0 \Rightarrow \mathrm{SOSC} \Rightarrow \dim_{\mathrm{H}}(\mathcal{K}) = \alpha.$$
$$\mathrm{WSC} \Leftrightarrow \mathcal{H}_4^\alpha(\mathcal{K}) > 0 \Leftrightarrow \dim_{\mathbf{\Gamma}}^4(\mathcal{K}) = \alpha.$$

Both statements in Theorem 17 (Schief's and our's) can be combined into the following summary result standing in the general case:

Corollary 18 (IFS)**.** $\mathcal{H}_{\mathrm{H}}^\alpha(\mathcal{K}) > 0 \Rightarrow \mathrm{SOSC} \Rightarrow \dim_{\mathrm{H}}(\mathcal{K}) = \alpha \Rightarrow \mathrm{WSC}$, *where*

$$\mathrm{WSC} \Leftrightarrow \mathcal{H}_4^\alpha(\mathcal{K}) > 0 \Leftrightarrow \dim_{\mathbf{\Gamma}}^4(\mathcal{K}) = \alpha.$$

Interestingly, Corollary 18 highlights that the WSC becomes necessary to reach the equality between the Hausdorff and the similarity dimensions of IFS-attractors. In other words, if the natural fractal structure which any IFS-attractor can be endowed with is not irreducible, then a Moran's type theorem cannot hold.

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