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Simple dynamical systems

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Abstract

In this paper, we study the class of simple systems on $\mathbb{R}$ induced by homeomorphisms having finitely many non-ordinary points. We characterize the family of homeomorphisms on $\mathbb{R}$ having finitely many non-ordinary points upto (order) conjugacy. For $x, y \in \mathbb{R}$, we say $x \sim y$ on a dynamical system $(\mathbb{R}, f)$ if x and y have same dynamical properties, which is an equivalence relation. Said precisely, $x \sim y$ if there exists an increasing homeomorphism $h : \mathbb{R} \to \mathbb{R}$ such that $h \circ f = f \circ h$ and $h(x) = y$. An element $x \in \mathbb{R}$ is ordinary *in $(\mathbb{R}, f)$ if its equivalence class $[x]$ is a neighbourhood of it.*

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1. Introduction

A dynamical system is a pair (X, f) where X is a metric space and f is a continuous self map on X. Two dynamical systems (X, f), (Y, g) are said to be topological conjugate if there exists a homeomorphism $h : X \to Y$ (called topological conjugacy) such that $h \circ f = g \circ h$. The properties of dynamical systems which are preserved by topological conjugacies are called dynamical properties.

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The points which are unique upto some dynamical property are called *dynamically special points*. Said differently, a special point has a dynamical property which no other point has. The idea of special points is relatively new to the literature (see [7]). In this paper, we introduce the notion of non-ordinary point.

Throughout this paper we will be working with continuous self homeomorphisms of the real line. Since $\mathbb{R}$ has order structure, we would like to consider the topological conjugacies (simply we call conjugacies) preserving the order. Hence the conjugacies which we mainly consider in this paper are order preserving conjugacies (increasing conjugacies). The increasing conjugacies are usually called order conjugacies. When we are working with a single system, any self conjugacy can utmost shuffle points with same dynamical behavior. Therefore a point which is unique upto its behavior must be fixed by every self conjugacy. On the other hand, if a point is fixed by all self conjugacies then it must have a special property (some times it may not be known explicitly). These ideas motivated us to call the set of all points fixed by all self conjugacies as set of *special points*. For $x, y \in \mathbb{R}$, we write $x \sim y$ if x and y have the same dynamical properties in the dynamical system $(\mathbb{R}, f)$. Said precisely, $x \sim y$ if there exists an increasing homeomorphism $h : \mathbb{R} \to \mathbb{R}$ such that $h \circ f = f \circ h$ and $h(x) = y$. It is easy to see that $\sim$ is an equivalence relation. Since the equivalence relation is coming from self conjugacy it is important in the field of topological dynamics. Let $[x]$ denote the equivalence class of $x \in \mathbb{R}$.

In a dynamical system (X, f), we say that a point x is ordinary if its "like" points are near to it. That is, an element $x \in \mathbb{R}$ is *ordinary* in $(\mathbb{R}, f)$ if its equivalence class $[x]$ is a neighbourhood of it, i.e., the equivalence class of x contains an open interval around x. A point which is not ordinary is called *non-ordinary*. Let $N(f)$ be the set of all non-ordinary points of f. We call a point to be *special* if $[x] = \{x\}$. Let $S(f)$ be the set of all special points of f. A point x in a topological space X is said to be *rigid* if it is fixed by every self homeomorphism of X. For example, the point 1 is rigid in $(0, 1]$. According to the above definition all rigid points are special even though there is no role for the map f. We make this as a convention. By definition, the points of $[x]$ are dynamically same. We consider systems for which there are only finitely many equivalence classes. This means there are only finitely many kinds of orbits upto conjugacy. In particular, their sets of periods $Per(f)$ are contained in $\{1, 2, 2^2, ...\}$. If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $\text{Per}(f)$ properly contained in $\{1, 2, 2^2, ...\}$ then f is not Li-Yorke chaotic (see [1]). Also note that if $f : \mathbb{R} \to \mathbb{R}$ is Devaney chaotic then $6 \in \text{Per}(f)$ (see [2]). Therefore, if $f : \mathbb{R} \to \mathbb{R}$ is a continuous map having finitely many non-ordinary points then it is neither Li-Yorke chaotic nor Devaney chaotic due to Sharkovskii's theorem. For these reasons, we call such systems as *simple systems*. These are the system in which the phase portrait can be drawn. Phase portraits (see [5]) are frequently used to graphically represent the dynamics of a system. A phase portrait consists of a diagram representing possible beginning positions in the system and arrows that indicate the change in these positions under iteration of the function. The drawable systems are interesting to physicists and for this reason the study of the class of simple dynamical systems can be useful.

Our main results characterize the family of homeomorphisms on $\mathbb{R}$ having finitely many non-ordinary points upto (order) conjugacy. In particular, we prove that:

(i) The number of all increasing continuous bijections (upto order conjugacy) on $\mathbb{R}$ with exactly n non-ordinary points is equal to $a_n = C_1(1+\sqrt{3})^n + C_2(1-\sqrt{3})^n$, where $C_1 = \frac{(5+3\sqrt{3})}{2\sqrt{3}}$ and $C_2 = \frac{(3\sqrt{3}-5)}{2\sqrt{3}}$.

(ii) If t_n denotes the number of increasing homeomorphisms (upto conjugacy) on $\mathbb{R}$ with exactly n non-ordinary points, then $t_0 = 2$, $t_1 = 5$ and $t_2 = 12$, and we have

$$t_n = \begin{cases} \frac{a_n + 2a_{\frac{n-4}{2}}}{2} & \text{if } n \text{ is even} \\ \frac{a_n + 2a_{\frac{n-3}{2}}}{2} & \text{if } n \text{ is odd} \end{cases}$$

for $n \geq 3$.

(iii) If s_n denotes the number of decreasing homeomorphisms (upto order conjugacy) on $\mathbb{R}$ with exactly n non-ordinary points, then

$$s_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ a_{\frac{n-1}{2}} & \text{if } n \text{ is odd} \end{cases}$$

for all n.

(iv) If k_n denotes the number of decreasing homeomorphisms (upto conjugacy) on $\mathbb{R}$ with exactly n non-ordinary points, then

$$k_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ t_{\frac{n-1}{2}} & \text{if } n \text{ is odd} \end{cases}$$

for all n.

2. Basic Results

Let (X, f) be a dynamical system. We denote the *full orbit* of a point $x \in X$ by the set $\widetilde{O}(x) = \{y \in X : f^n(x) = f^m(y) \text{ for some } m, n \in \mathbb{N}\}$. For any subset $A \subset \mathbb{R}$, let

$$\tilde{O}(A) = \bigcup_{x \in A} \widetilde{O}(x) = \bigcup_{x \in A} \{y \in \mathbb{R} : f^n(y) = f^m(x) \ for\ some\ m,\ n \in \mathbb{N}\}.$$

A point x in a dynamical system (X, f) is said to be a *critical point* if f fails to be one-one in every neighbourhood of x. The set of all critical points of f is denoted by $C(f)$, and by $P(f)$ we denote the set of all periodic points of f. Recall that a point x in a dynamical system (X, f) is said to be periodic if $f^n(x) = x$ for some $n \in \mathbb{N}$. Let $f : \mathbb{R} \to \mathbb{R}$ and $D(f)$ be the set $\widetilde{O}(C(f) \cup P(f) \cup \{f(\infty), f(-\infty)\})$, where $f(\infty)$ and $f(-\infty)$ are the limits of f at ∞ and $-\infty$ respectively, provide they are finite.

We will prove below (see Proposition 2.13) that if the map has only finitely many non-ordinary points then $N(f) = S(f)$. Hence the following characterization theorem holds for the set $N(f)$ since a similar type of characterization holds for $S(f)$.

For a continuous map $f : \mathbb{R} \to \mathbb{R}$, we define $S_p(\mathbb{R}, f) = \bigcap_h \{x \in \mathbb{R} : \ h(x) = x, \, h : \mathbb{R} \to \mathbb{R}$ is a homeomorphism such that $h \circ f = f \circ h\}$.

Theorem 2.1. *For continuous self maps of the real line $\mathbb{R}$, the set of all special points is contained in the closure of the union of full orbits of critical points, periodic points and the limits at infinity (if they exist and finite). That is, $N(f)$ is a subset of the closure of $D(f)$.*

Proof. For a continuous map $f : \mathbb{R} \to \mathbb{R}$, first observe that $S(f) \subset S_p(\mathbb{R}, f)$. By Theorem 1 of [7], we have $S_p(\mathbb{R}, f) \subset \overline{D(f)}$. Hence the proof follows. □

Remark 2.2. Consider the map $f(x) = x + sin(x)$ for all $x \in \mathbb{R}$. Observe that all integral multiples of π are fixed points for f but the increasing bijection $x \mapsto x + 2\pi$ commutes with f and fixes none of them. Hence in this case $N(f)$ is properly contained in the closure of $D(f)$.

Now consider the following theorems.

Theorem 2.3. *For polynomials of even degree the equality $\overline{D(f)} = S(f)$ holds.*

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be a polynomial map of even degree. By Theorem 2 of [7], we have $S_p(\mathbb{R}, f) = S(f) = \overline{D(f)}$. □

Theorem 2.4. *For polynomial maps f of $\mathbb{R}$, $S(f)$ has to be either empty or a singleton or the closure of $D(f)$.*

Proof. The ideas involved in the proof of Theorem 3 as in [7] can be adapted to order conjugacies. Hence the proof follows. □

From the definition of special points, it is clear that the set of special points $S(f)$ is always closed. The following theorem is about the converse.

Theorem 2.5. *Given any closed subset F of $\mathbb{R}$, there exists a continuous map $f : \mathbb{R} \to \mathbb{R}$ such that $S(f) = F$.*

Proof. The ideas involved in the proof of Lemma 2, Lemma 3, and Theorem 4 can be adapted to order conjugacies. Hence the proof follows. □

The following total order on $\mathbb{N}$ is called the Sharkovskii's ordering:
$3 \succ 5 \succ 7 \succ 9 \succ ... \succ 2 \times 3 \succ 2 \times 5 \succ 2 \times 7 \succ ...$
$\succ 2^n \times 3 \succ 2^n \times 5 \succ 2^n \times 7 \succ ...$
$... \succ 2^n \succ .... \succ 2^2 \succ 2 \succ 1.$
We write $m \succ n$ if m precedes n (not necessarily immediately) in this order. An n-cycle means a cycle of length n.

Theorem 2.6 (Sharkovskii's Theorem, see[8])**.** *Let $m \succ n$ in the Sharkovskii's ordering. For every continuous self map of $\mathbb{R}$, if there is an m-cycle, then there is an n-cycle.*

For any continuous map $f : \mathbb{R} \to \mathbb{R}$, topological conjugacy (respectively order conjugacy) of f is a homeomorphism (respectively increasing homeomorphism) $h : \mathbb{R} \to \mathbb{R}$ such that $h \circ f = f \circ h$. For any continuous map $f : \mathbb{R} \to \mathbb{R}$, let G_f be the set of all topological conjugacies of f and let $G_{f\uparrow}$ be the set of all order conjugacies of f.

Proposition 2.7. *If x is an ordinary point of f and if h is a self conjugacy of f, then $h(x)$ is ordinary.*

Proof. Since x is ordinary there exists an open interval V contained in $[x]$. We prove that the open interval (since h is a homeomorphism) $h(V)$ is contained in $[h(x)]$. Take $s \in h(V)$. Then $s = h(t)$ for some $t \in V$. Since $V \subset [x]$, there exists $\varphi \in G_{f\uparrow}$ such that $\varphi(t) = x$. Then the increasing homeomorphism $\psi = h\varphi h^{-1}$ carries s to $h(x)$ and commutes with f. □

Proposition 2.8. *If x is a non-ordinary point of f and if h is a self conjugacy of f, then $h(x)$ is non-ordinary.*

Proof. Note that if h is a self conjugacy of f then h^{-1} is also a self conjugacy of f. Now, the proof follows from Proposition 2.7. □

For any subset A of $\mathbb{R}$, we write $\partial A = \overline{A} \cap (\overline{X - A})$ for the boundary of A, where $\overline{A}$ denotes the closure of A in $\mathbb{R}$. Recall that the properties which are preserved under topological conjugacies are called dynamical properties. Hence, if two points x, y in the dynamical system (X, f) differ by a dynamical property, then no conjugacy can map one to the other. Hence the following proposition follows.

Proposition 2.9. *The points of ∂S_P are non-ordinary for any dynamical property P, where S_P denotes the set of all points in (X, f) having the dynamical property P.*

Corollary 2.10. *Let $f : \mathbb{R} \to \mathbb{R}$ be constant in a neighbourhood of a point x_0. Then the end points of the maximal interval around x_0 on which f is constant are non-ordinary.*

Remark 2.11. Note that, being a point in a particular equivalence class $[x]$ is a dynamical property. Therefore when there are n non-ordinary points then there are $n + 1$ equivalence classes. But the converse is not true. Consider the map $x \mapsto x + sin(x)$ on $\mathbb{R}$. There are two equivalence classes but infinitely many non-ordinary points.

Now we ask: For a continuous map $f : \mathbb{R} \to \mathbb{R}$, how the equivalence classes look like?

The following lemma answers this question.

Lemma 2.12. *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Suppose $a < b$ and $(a, b) \cap N(f) = \varnothing$. Then $x \sim y$ for all $x, y \in (a, b)$.*

Proof. Assume without loss of generality that $x < y$. Suppose $x \nsim y$, so $z = \sup([x] \cap (-\infty, y])$ exists. Clearly $z \in \overline{[x]}$. If $z = y$ then $z \in \overline{[y]} \subset \overline{\mathbb{R} \setminus [x]}$.

Otherwise $z < y$ and $[z, y) \cap (\mathbb{R} \setminus [x]) \neq \varnothing$ for every $y - x > \epsilon > 0$ which again shows $z \in \overline{\mathbb{R} \setminus [x]}$. Then $z \in \partial([x])$ and hence $z \in N(f)$ by Proposition 2.9. But $a < x \leq z \leq y < b$. Hence $z \in (a, b) \cap N(f)$ contradicting our hypothesis. □

Proposition 2.13. *If $f : \mathbb{R} \to \mathbb{R}$ has only finitely many non-ordinary points then every non-ordinary point is special.*

Proof. Since the set of all non-ordinary points $N(f)$ is finite, it follows from Proposition 2.7 and Proposition 2.8 that $h(N(f)) = N(f)$ for all $h \in G_{f\uparrow}$. Then we must have $h(x) = x$ for all $x \in N(f)$ because of the order preserving nature of h. Hence all points of $N(f)$ are special. □

Thus by above proposition the idea of special points and the idea of non-ordinary points coincide in the class of maps with finitely many non-ordinary points.

For a set A, we denote $|A|$ for the cardinality of A. Now we consider the following theorem.

Theorem 2.14. *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. If $|N(f)| = n$ then $|\{[x] : x \in \mathbb{R}\}| = 2n + 1$.*

Proof. Let $N(f) = \{x_1, x_2, \cdots, x_n\}$ where $x_1 < x_2 < \cdots < x_n$. By Proposition 2.13, each $\{x_i\}$ is an equivalence class. Then each of these intervals $(-\infty, x_1), (x_1, x_2), \cdots, (x_{n-1}, x_n), (x_n, \infty)$ is invariant under every element of $G_{f\uparrow}$. Hence all the remaining equivalence classes are contained in one of these intervals. Lemma 2.12 above now shows that each of these interval is an equivalence class, giving $|\{[x] : x \in \mathbb{R}\}| = 2n + 1$. □

Remark 2.15. Note that, being a point in a particular equivalence class $[x]$ is a dynamical property.

Remark 2.16. If $f : \mathbb{R} \to \mathbb{R}$ has a unique fixed point then it is non-ordinary and vice-versa.

Proof. Since the topological conjugacies carry fixed points to fixed points, the unique fixed point must be fixed by every self conjugacy and hence special. Next suppose $x_0 \in \mathbb{R}$ is the unique non-ordinary point of f. Then $h(x_0) = x_0$ for all $h \in G_{f\uparrow}$. Now, for any $h \in G_{f\uparrow}$ we have $h(f(x_0)) = f(h(x_0)) = f(x_0)$. That is, the point $f(x_0)$ is special. Then we have $f(x_0) = x_0$ since x_0 is the only special point. □

Remark 2.17. If $f : \mathbb{R} \to \mathbb{R}$ has finitely many fixed points (critical points) then all fixed (critical) points are special and hence non-ordinary.

Proof. This remark follows from the fact that under a topological conjugacy fixed points will be mapped to fixed points and critical points will be mapped to critical points and the fact that it takes the finite set F (of fixed points) to F bijectively preserving the order. □

Remark 2.18. If there are only finitely many periodic cycles then all periodic points are special.

The following remark says that in general non-ordinary points need not be special.

Remark 2.19. It is immediate from the definition that every special point is non-ordinary. But every non-ordinary point may not be special. For example, consider the map $x \mapsto x + \sin x$ on $\mathbb{R}$ which has countably many fixed points. Note that all the fixed points are non-ordinary and they all together form a single equivalence class. Hence they are not special.

Proposition 2.20. *For maps with finitely many non-ordinary points, $f(x)$ is non-ordinary whenever x is non-ordinary.*

Proof. Since x is non-ordinary and since there are only finitely many non-ordinary points, we have $h(x) = x$ for all $h \in G_{f\uparrow}$. Now for any $h \in G_{f\uparrow}$, we have $h(f(x)) = f(h(x)) = f(x)$. Hence $f(x)$ is non-ordinary. $\square$

For a map $f : A \to \mathbb{R}$, $A \subset \mathbb{R}$, we denote $\sup f$ and $\inf f$ for the supremum of $f(A)$ and infimum of $f(A)$ respectively. Recall that, if $f : \mathbb{R} \to \mathbb{R}$ has a unique non-ordinary point then it must be a fixed point.

Proposition 2.21. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and let $x \in \mathbb{R}$. Then*

- (i) *If $x \in \mathbb{R}$ is both critical and ordinary then f is locally constant at x.*
- (ii) *If x is ordinary then so is $f(x)$ unless f is constant in a neighbourhood of x.*

Proof. (i) Let $x_0 \in \mathbb{R}$ be both critical and ordinary.

Claim: f is constant in some neighbourhood of x_0.

Since x_0 is ordinary, there exists $\eta > 0$ such that all points in $(x_0 - \eta, x_0 + \eta)$ will look alike. So it is enough to prove that f is somewhere constant in $(x_0 - \eta, x_0 + \eta)$.

Case 1:

Suppose some point in $(x_0 - \eta,\ x_0 + \eta)$ is point of local maximum of f. Then we can prove easily that every point of $(x_0 - \eta,\ x_0 + \eta)$ is a point of local maximum. That is there exists $\epsilon > 0$ such that $f(x_0) \geq f(t)\ \forall\ t \in (x_0 - \epsilon,\ x_0 + \epsilon)$. Next choose $\delta < \epsilon,\ \eta$. Then there exists $y \in [x_0 - \delta,\ x_0 + \delta]$ such that

$$f(y) \leq f(t) \forall\ t \in [x_0 - \delta, x_0 + \delta]. \tag{2.1}$$

But y is a point of local maximum (since $\delta < \eta$). That is there exists $\alpha > 0$ such that

$$f(y) \geq f(s)\ \forall\ s \in (y - \alpha,\ y + \alpha) \tag{2.2}$$

From (1) and (2), it follows that f is constant in some neighbourhood y and hence constant in some neighbourhood of x_0.

Case 2:

No point is a point of local maximum of f. That is, f attains its maximum at one of the endpoints in every subinterval. If f assumes supremum always on the right end (or always on the left end) then f is strictly monotone. Note that, it is enough if we prove monotone somewhere. Take a neighbourhood (α, β) of x_0 such that $(\alpha, \beta) \subset (x_0 - \eta,\ x_0 + \eta)$ and let $\sup f$ on (α, β) be attained at the right end point β. Suppose $\sup f$ is attained at the right end point in every subinterval of (α, β) containing x_0. Then f is increasing in $(x_0,\ \beta)$. We are done. Suppose there is a subinterval say $(x_0 - \epsilon_1,\ x_0 + \epsilon_2)$ of (α, β) on which f attains its supremum at the left end point. Then f attains its infimum on $(x_0 - \epsilon_1, \beta)$ at some interior point. We now argue as in Case 1. This ends the proof of Part (i).

(ii) We make use of (i).

Assume that f is not constant on any neighbourhood of x. Because x is ordinary, there exists an open interval J around x, in which all points are equivalent to x, such that f is not constant on J. It follows that f is not constant on any non-trivial subinterval of J, because the endpoints of intervals of constancy are non-ordinary. From (i), it follows that J has no critical point. Therefore $f(J)$ is an open interval. We claim that any two elements of $f(J)$ are equivalent. Let $f(y)$ be a general element of $f(J)$, where $y \in J$, $y \neq x$. By choice of J, there exists a self conjugacy h of f such that $h(y) = x$, which implies $h \circ f(y) = f \circ h(y) = f(x)$. Therefore $f(y)$ is equivalent to $f(x)$. This proves $f(x)$ is ordinary. □

Remark 2.22. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Then $\sup f(\mathbb{R})$, $\inf f(\mathbb{R})$, $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ are non-ordinary provided they are finite. Note that, for maps with finitely many non-ordinary points both $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ always exist in $\mathbb{R} \cup \{\infty, -\infty\}$.

Proof. For any $h \in G_{f\uparrow}$, $h(f(\mathbb{R})) = f(h(\mathbb{R})) = f(\mathbb{R})$. That is, h takes the range of f to itself. Since h is increasing, $h(\sup f) = \sup f$ and $h(\inf f) = \inf f$. Now we prove that for maps with finitely many non-ordinary points, $\lim_{x\to\infty} f(x)$ always exists in $\mathbb{R} \cup \{\infty, -\infty\}$. For this, let t_0 be the largest non-ordinary point and let A be the set of all critical points $> t_0$. Suppose A is empty. Then f is monotone on $[t_0, \infty)$ and hence $\lim_{x\to\infty} f(x)$ exists. Suppose A is nonempty. Then ∂A is nonempty. But every element of ∂A is non-ordinary. Hence $\partial A = \{t_0\}$. Therefore $A = (t_0, \infty)$. Therefore f is constant on A (we argue as in the proof of Case 2 of (i) in Proposition 2.21). Hence $\lim_{x\to\infty} f(x)$ exists. Next we will prove $\lim_{x\to\infty} f(x)$ is special. We denote $\lim_{x\to\infty} f(x)$ by l. Let $h \in G_{f\uparrow}$. Note that for any sequence $(x_n) \to \infty$, we have $f(x_n) \to l$ and $h(x_n) \to \infty$. Hence $h(f(x_n)) = f(h(x_n)) \to h(l)$. Being $h(x_n) \to \infty$, by the definition of l we find $f(h(x_n)) = h(f(x_n)) \to l$. Hence $h(l) = l$. This completes the proof. □

Proposition 2.23. *The maps $x+1$ and $x-1$ on $\mathbb{R}$ are topologically conjugate; but not order conjugate.*

Proof. The maps $x+1$ and $x-1$ are conjugate to each other through the order conjugacy $-x + \frac{1}{2}$. If possible, let h be an order conjugacy from $f(x) = x + 1$ to $g(x) = x - 1$. Then $h(x + 1) = h(f(x)) = g(h(x)) = h(x) - 1$. i.e.,

$h(x+1) - h(x) = -1 < 0$, which is a contradiction to the assumption that h is increasing. □

Remark 2.24. Note that for the map $x+1$ on $\mathbb{R}$, all points are ordinary. This is because, if $a, b \in \mathbb{R}$ then the map $x + b - a$ is the order conjugacy of $x+1$ which maps a to b.

The following proposition is proved in [6]. For the sake of completeness, we included its proof.

Proposition 2.25. *Let $f : \mathbb{R} \to \mathbb{R}$ be a homeomorphism without fixed points. Then*

(i) *If $f(0) > 0$ then f is order conjugate to $x+1$.*
(ii) *If $f(0) < 0$ then f is order conjugate to $x-1$.*

Proof. Define $h : \mathbb{R} \to \mathbb{R}$ as follows. Assume $f(0) > 0$. Define $h(t) = \frac{t}{f(0)}$, $0 \le t < f(0)$. We know that $(f^n(0))$ increases and diverges to ∞ and $(f^{-n}(0))$ decreases and diverges to $-\infty$ for all $n \in \mathbb{N}$. Moreover for $t \in \mathbb{R}$ there exists unique $n \in \mathbb{Z}$ such that, $f^n(0) \le t < f^{n+1}(0)$. Define $h(t) = h(f^{-n}(t)) + n$. Then $h \circ f(t) = h(t) + 1$ for all $t \in \mathbb{R}$. This h gives a conjugacy from f to $x+1$. If $f(0) < 0$ then we can give a similar proof. □

For a map $f : A \to \mathbb{R}$, $A \subset \mathbb{R}$, we define $graph(f) := \{(x, f(x)) : x \in A\}$. For continuous maps $f, g : A \to \mathbb{R}$, we say that $graph(f)$ and $graph(g)$ are *above the diagonal* if $f(x) > x$ and $g(x) > x$ for all $x \in A$. Similarly, $graph(f)$ and $graph(g)$ are said to be *below the diagonal* if $f(x) < x$ and $g(x) < x$ for all $x \in A$, and $graph(f)$ and $graph(g)$ are said to be *on the diagonal* if $f(x) = g(x) = x$ for all $x \in A$. We say that $graph(f)$ and $graph(g)$ are on the *same side of the diagonal* if it is either above the diagonal or below the diagonal or on the diagonal.

Corollary 2.26. *Let $f, g : (a, b) \to (a, b)$ be homeomorphisms without fixed points. Then f is order conjugate to g if and only if both $graph(f)$ and $graph(g)$ are on the same side of the diagonal.*

In particular:

(i) *If $f(x) > x$ for all $x \in (a, b)$ then f is order conjugate to $x+1$.*
(ii) *If $f(x) < x$ for all $x \in (a, b)$ then f is order conjugate to $x-1$.*

Remark 2.27. In fact, the interval (a, b) involved in Corollary 2.26 can be replaced by any open ray in $\mathbb{R}$.

Remark 2.28. If $f : \mathbb{R} \to \mathbb{R}$ is an increasing bijection with finitely many non-ordinary points then all non-ordinary points are fixed points.

Proof. We know that for maps with finitely many non-ordinary points all non-ordinary points are fixed by every order conjugacy. Here f itself is a self conjugacy. □

For a continuous map $f : \mathbb{R} \to \mathbb{R}$, let $Fix(f)$ denote the set of all fixed points of f. It follows from the continuity of f that $Fix(f)$ is closed. For any subset A of a metric space X, we denote A^c for the complement of A and $int(A)$ for the interior of A. Recall that

$$(\partial A)^c = int(A) \cup int(A^c). \tag{2.3}$$

The following proposition provides a characterization for the non-ordinary points of increasing homeomorphisms.

Proposition 2.29. *Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing bijection and let $x \in \mathbb{R}$. Then x is non-ordinary if and only if $x \in \partial(Fix(f))$.*

Proof. Let $x \in \partial Fix(f)$. Then x is non-ordinary since every open interval around x contains fixed and non-fixed points. Now suppose $x \notin \partial Fix(f)$. We shall prove that x is ordinary. Now, $x \notin \partial Fix(f)$ implies $x \in (\partial Fix(f))^c = int(Fix(f)) \cup int((Fix(f))^c)$ by equation (1). Hence $x \in int(Fix(f))$ or $x \in int(Fix(f)^c)$.

Case 1: $x \in int(Fix(f))$

Suppose $x \in int(Fix(f))$. Then choose $a, b \in \mathbb{R}$ such that $x \in (a,\ b) \subset Fix(f)$. Let $y \in (a, b)$ be such that $y \neq x$. Then define an increasing continuous bijection $\phi_y : \mathbb{R} \to \mathbb{R}$ such that

$$\phi_y(t) = \begin{cases} t & \text{if } \ t \notin (a,b) \\ y & \text{if } t = x \\ \textit{piecewise linear} & \textit{otherwise.} \end{cases}$$

This ϕ_y maps x to y. Both $[a, b]$ and its complement are invariant under both ϕ_y and f. Note that f is identity on $[a, b]$ and ϕ_y is identity on the complement of $[a, b]$. Hence ϕ_y commutes with f on $[a, b]$. This proves x is an ordinary point.

Case 2: $x \in int(Fix(f)^c)$

Suppose $x \in int(Fix(f)^c)$. Let (a, b) be the component interval (open) of $(Fix(f))^c$ containing x. Then $f(a) = a$ and $f(b) = b$, and the map $f|_{(a,b)}$ is a fixed point free self map of (a, b) since f is increasing. Hence by Corollary 2.26, the map $f|_{(a,b)}$ is order conjugate to either $x + 1$ or $x - 1$, for which all points are ordinary. This completes the proof. ☐

Remark 2.30. There are continuous maps $f : \mathbb{R} \to \mathbb{R}$ having finitely many equivalence classes (hence only finitely many special points) but infinitely many non-ordinary points. For example, consider the map $f(x) = x + \sin x$ on $\mathbb{R}$. There are two classes of fixed points. Since increasing orbits must map to increasing orbits under increasing conjugacies, points like $\frac{\pi}{2}$ (increasing orbit) and $\frac{3\pi}{2}$ (decreasing orbit) cannot be equivalent. Hence there must be at least four equivalence classes. To see that there are exactly four equivalence classes, let $I_k = (2k\pi, (2k+1)\pi), D_k = ((2k+1)\pi, 2(k+1)\pi)$ and observe that $I_k \cap N(f) = \varnothing = D_k \cap N(f)$ for each $k \in \mathbb{Z}$ by Proposition 2.29. Hence by Lemma 2.12, each I_k and D_k is contained in a single equivalence class. Conjugacies of the form $x \mapsto x + 2k\pi$ complete the argument.

3. Main Results: Class of homeomorphisms

Note that, under a topological conjugacy a point can be mapped to a point with similar dynamics. By definition, the points of $[x]$ are dynamically the same, i.e., all have the dynamics similar to that of x. We now consider the systems for which there are only finitely many equivalence classes. This means there are only finitely many kinds of orbits upto conjugacy. For this reason, we call such systems as *simple systems*. In this paper, we try to understand some simple systems on $\mathbb{R}$. Recall that, if S_P denote the set of all points having the dynamical property P then the points of ∂S_P (the boundary of S_P) are non-ordinary. In particular, being a point in a particular equivalence class is a dynamical property of the point. Hence, by the very nature of the order conjugacies, it follows that when there are finitely many non-ordinary points (therefore special points) there are only finitely many equivalence classes. These are the simple systems we study in this paper. We describe completely, the homeomorphisms on $\mathbb{R}$, having finitely many non-ordinary points and give a general formula for counting. By Remark 2.11, for systems with finitely many non-ordinary points there are only finitely many equivalence classes. We now study, in the next subsections, the class of simple systems induced by homeomorphisms having finitely many non-ordinary points.

3.1. **Class of increasing homeomorphisms.** Note that the complement of $Fix(f)$ is a countable union of open intervals (including rays) whose end points are fixed points. Since f is increasing and the end points are fixed, no point in a component interval can be mapped to a point in any other component interval by f. Hence, it is observed that, for an increasing bijection f on $\mathbb{R}$, if $Fix(f)^c = \sqcup I_n$ then $f|_{I_n}$ is a self map of I_n, where $\sqcup$ denotes the disjoint union.

Proposition 3.1. *Let f, g be two increasing bijections such that $Fix(f) = Fix(g)$ and let $Fix(f)^c = \sqcup I_n$. If $f|_{I_n}$ is order conjugate to $g|_{I_n}$ for every n, then f is order conjugate to g.*

Proof. For each $n \in \mathbb{N}$, let $h_n : I_n \to I_n$ be an order conjugacy from $f|_{I_n}$ to $g|_{I_n}$.

Define $h : \mathbb{R} \to \mathbb{R}$ by

$$h(x) = \begin{cases} h_n(x) & \text{if } x \in I_n \\ x & \text{otherwise.} \end{cases}$$

Then h is an increasing bijection such that $h \circ f = g \circ h$. □

An *alphabet* is a finite set of letters with at least two elements. A finite sequence of letters from an alphabet is often referred to as a *word*. For example, if $\Sigma = \{a, b\}$ be an alphabet then *abab*, *aaabbbab* are words over Σ. Number of letters (may not be distinct) in a word is called its *length*. Any word of consecutive characters in a word w is said to be a *subword* of w. Throughout this section we will be working with the alphabet $\{\mathbf{A}, \mathbf{B}, \mathbf{O}\}$. Let $\tilde{\mathbf{A}} = \mathbf{B}$,

$\tilde{\mathbf{B}} = \mathbf{A}$ and $\tilde{\mathbf{O}} = \mathbf{O}$. If $w = w_1 w_2 ... w_n$ then the *dual* of w is defined as

$$\tilde{w} = \tilde{w}_n \tilde{w}_{n-1} ... \tilde{w}_1.$$

If $\tilde{w} = w$ then the word w is said to be *self conjugate*. Here **A** stands for "**a**bove the diagonal" and **B** stands for "**b**elow the diagonal" and **O** stands for "**o**n the diagonal". Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing homeomorphism with finitely many non-ordinary (hence special) $x_1, ..., x_n$ for some $n \in \mathbb{N}$. Without loss of generality, assume that $x_1 < x_2 < ... < x_n$ for some $n \in \mathbb{N}$. This finite set of points gives rise to an ordered partition $\{(-\infty, x_1), (x_1, x_2), ..., (x_n, \infty)\}$ of $\mathbb{R} \backslash \{x_1, x_2, ..., x_n\}$. Note that, on each component interval exactly one of the following holds by Proposition 2.29 (since the only subsets of $\mathbb{R}$ with empty boundary are the empty set and $\mathbb{R}$):

(i) $f(t) > t \; \forall \; t$ (ii) $f(t) < t \; \forall \; t$ (iii) $f(t) = t \; \forall \; t$.

This gives rise to a word $w(f)$ over $\{$**A, B, O**$\}$ of length $n + 1$ by associating **A** to (i), **B** to (ii) and **O** to (iii). Note that the subword **OO** is forbidden. For this, suppose **O** is occurring at i^{th} and $(i+1)^{th}$ place then in both(x_i, x_{i+1}) and (x_{i+1}, x_{i+2}) all points are fixed. Then x_{i+1} becomes ordinary, a contradiction to the assumption that x_{i+1} is a non-ordinary point. Conversely, suppose a word w of length $n + 1$ in which **OO** is forbidden is given. Then we can construct an increasing bijection on $\mathbb{R}$ such that its associated word is w, as follows:

Take the points $0, 1, 2, ..., n$ and consider $\{(-\infty, 0), (0, 1), (1, 2), ..., (n, \infty)\}$, a partition of $\mathbb{R}$. If $w = w_1 w_2 ..... w_{n+1}$ then associate w_1 to $(-\infty, 0)$, w_2 to $(0, 1)$, ..., and w_{n+1} to (n, ∞). Now it is easy to construct an increasing bijection $f : \mathbb{R} \to \mathbb{R}$ such that $w(f) = w$.

Firstly, define $f(t)$ on $(-\infty, 0)$ according to the value of w_1 as follows:

$$f(t) = \begin{cases} \frac{1}{2}t & \text{if } w_1 = \mathbf{A} \\ \frac{3}{2}t & \text{if } w_1 = \mathbf{B} \\ t & \text{if } w_1 = \mathbf{O}. \end{cases}$$

Secondly, define $f(t)$ on the remaining subintervals $(i-2, i-1)$ for $i = 2, ..., n$ as follows:

To be precise, if $i - 2 < t < i - 1$, $i = 2, 3, ..., n$, then consider

$$f(t) = \begin{cases} i - 2 + (t - i + 2)^2 & \text{if } w_i = \mathbf{B} \\ i - 2 + \sqrt{t - i + 2} & \text{if } w_i = \mathbf{A} \\ t & \text{if } w_i = \mathbf{O}. \end{cases}$$

And, finally, define $f(t)$ on (n, ∞) according to the value of w_{n+1} as follows:

$$f(t) = \begin{cases} \frac{1}{2}(t - n) + n & \text{if } w_{n+1} = \mathbf{B} \\ \frac{3}{2}(t - n) + n & \text{if } w_{n+1} = \mathbf{A} \\ t & \text{if } w_{n+1} = \mathbf{O}. \end{cases}$$

Now by the following proposition the increasing bijection constructed above is unique upto order conjugacy.

Proposition 3.2. *Let f, g be two increasing bijection on $\mathbb{R}$ with finitely many (same number of) non-ordinary points. Then f and g are order conjugate if and only if $w(f) = w(g)$.*

Proof. Suppose $w(f) = w(g) = w_1 w_2 ... w_n$. Let $x_1 < x_2 < ... < x_n$ and $y_1 < y_2 < ... < y_n$ be the non-ordinary points f and g respectively. The former gives the ordered partition $\{(-\infty, x_1), (x_1, x_2), ...(x_n, \infty)\}$ of $\mathbb{R}\backslash\{x_1, x_2, ..., x_n\}$ and the later gives the ordered partition $\{(-\infty, y_1), (y_1, y_2), ..., (y_n, \infty)\}$ of $\mathbb{R}\backslash\{y_1, y_2, ..., y_n\}$. Now by Proposition 2.29, it follows that both $f|_{(x_i, x_{i+1})}$ and $g|_{(y_i, y_{i+1})}$ are fixed point free self maps (homeomorphisms) for each i and hence by Corollary 2.26, both are order conjugate to $x + 1$ if $w_{i+1} = \mathbf{A}$, and order conjugate to $x - 1$ if $w_{i+1} = \mathbf{B}$. Hence, by Proposition 3.1, f is order conjugate to g. Converse follows from Corollary 2.26. □

Thus we have proved:

Proposition 3.3. *There is a one to one correspondence between the set of all increasing continuous bijections (upto order conjugacy) on $\mathbb{R}$ with exactly n non-ordinary points and the set of all words of length $n + 1$ on three symbols* ***A, B, O*** *such that* ***OO*** *is forbidden.*

Now we consider the following proposition.

Proposition 3.4. *Let a_n be the number of words of length $n+1$ over $\{\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{O}\}$, where* ***OO*** *is forbidden. Then $a_n = C_1(1 + \sqrt{3})^n + C_2(1 - \sqrt{3})^n$, where $C_1 = \frac{(5+3\sqrt{3})}{2\sqrt{3}}$ and $C_2 = \frac{(3\sqrt{3}-5)}{2\sqrt{3}}$.*

Proof. Let A_n be the set of all words of length $n + 1$ over $\{\mathbf{A}, \mathbf{B}, \mathbf{O}\}$ in which **OO** is forbidden. A general element in A_{n+2} is of the form

(i) $\mathbf{A}w$ or $\mathbf{B}w$ for some $w \in A_{n+1}$ or (ii) $\mathbf{OA}v$ or $\mathbf{OB}v$ for some $v \in A_n$.

Therefore $a_{n+2} = a_{n+1} + a_{n+1} + a_n + a_n$ since A_{n+2} is the disjoint union of four types of the elements described above. Hence $a_{n+2} = 2(a_n + a_{n+1})$. This is a linear homogeneous recurrence relation with constant coefficients. The corresponding characteristic equation is $\alpha^2 - 2\alpha - 2 = 0$ which has the two distinct roots $\alpha_1 = 1 + \sqrt{3}$ and $\alpha_2 = 1 - \sqrt{3}$. It follows that $a_n = C_1(1+\sqrt{3})^n + C_2(1-\sqrt{3})^n$, where the constants C_1 and C_2 can be determined by using the boundary conditions $a_0 = 3$ and $a_1 = 8$. Here $C_1 = \frac{(5+3\sqrt{3})}{2\sqrt{3}}$ and $C_2 = \frac{(3\sqrt{3}-5)}{2\sqrt{3}}$. □

The following result is one of our principal theorems. It follows from Propositions 3.3 and 3.4.

Theorem 3.5. *The number of all increasing continuous bijections (upto order conjugacy) on $\mathbb{R}$ with exactly n non-ordinary points is $a_n = C_1(1 + \sqrt{3})^n + C_2(1 - \sqrt{3})^n$, where $C_1 = \frac{(5+3\sqrt{3})}{2\sqrt{3}}$ and $C_2 = \frac{(3\sqrt{3}-5)}{2\sqrt{3}}$.*

For any two continuous map $f,\ g : \mathbb{R} \to \mathbb{R}$, we say that f is decreasingly conjugate to g if there is a decreasing homeomorphism $h : \mathbb{R} \to \mathbb{R}$ such that

$h \circ f = f \circ h$. The following proposition is an analogue of Proposition 3.2, and its proof will be omitted.

Proposition 3.6. *Let f, g be two increasing bijections on $\mathbb{R}$ with finitely many (same number of) non-ordinary points. Then f and g are decreasingly conjugate if and only if $w(g) = \overline{w(f)}$.*

Let t_n denote the number of increasing homeomorphisms on $\mathbb{R}$ with exactly n non-ordinary points upto topological conjugacy, a_n denote the number of increasing homeomorphisms on $\mathbb{R}$ with exactly n non-ordinary points upto order conjugacy, and ν_{n+1} denote the number of self conjugate words of length $n+1$ over $\{\mathbf{A},\ \mathbf{B},\ \mathbf{O}\}$. If $f,\ g : \mathbb{R} \to \mathbb{R}$ are two increasing homeomorphisms such that they are topologically conjugate then either $w(g) = w(f)$ or $w(g) = \overline{w(f)}$ by Proposition 3.2 and Proposition 3.6. First observe that $t_0 = 2$, $t_1 = 5$, and $t_2 = 12$. In general, we have $t_n = \frac{1}{2}(a_n - \nu_{n+1}) + \nu_{n+1}$ for all $n \in \mathbb{N}$.

Case 1: When n is even. Say $n = 2m$. A self conjugate word w of length $2m+1$ (**OO** is forbidden) is of the form $w_1 w_2 ... w_m w_{m+1} w_{m+2} ... w_{2m+1}$ such that $w_{m+1} = \mathbf{O}$ and $w_m, w_{m+2} \in \{\mathbf{A}, \mathbf{B}\}$ such that $w_m \neq w_{m+2}$. Therefore the number of self conjugate words is $2a_{m-2}$. Hence $t_{2m} = \frac{a_{2m} + 2a_{m-2}}{2}$ for all $m \geq 2$.

Case 2: When n is odd. Say $n = 2m+1$. In this case any self conjugate word of length $2m+2$ (**OO** is forbidden) is of the form $w_1 w_2 ... w_m w_{m+1} w_{m+2} ... w_{2m+2}$ such that $w_{m+1}, w_{m+2} \in \{\mathbf{A}, \mathbf{B}\}$ and $w_{m+1} \neq w_{m+2}$. Hence the number of self conjugate words of length $2m+2$ is $2a_{m-1}$. Therefore $t_{2m+1} = \frac{a_{2m+1} + 2a_{m-1}}{2}$ for all $m \geq 1$.

Thus we have proved:

Theorem 3.7. *If t_n denotes the number of increasing homeomorphisms upto topological conjugacy. Then $t_0 = 2$, $t_1 = 5$ and $t_2 = 12$ by direct computation and for all $n \geq 3$ we have:*

$$t_n = \begin{cases} \frac{a_n + 2a_{\frac{n-4}{2}}}{2} & \text{if } n \text{ is even} \\ \frac{a_n + 2a_{\frac{n-3}{2}}}{2} & \text{if } n \text{ is odd} \end{cases}$$

3.2. **Class of decreasing homeomorphisms.** We now ask: Given a whole number n, how many decreasing bijections are there on $\mathbb{R}$ upto order conjugacy having exactly n non-ordinary points?

For a map $f : \mathbb{R} \to \mathbb{R}$, we denote the composition of f with itself by f^2.

Proposition 3.8. *Two decreasing bijections f and g are order conjugate (respectively topologically conjugate) if and only if $f^2|_{[a,\infty)}$ and $g^2|_{[b,\infty)}$ are order conjugate (respectively topologically conjugate), where a and b are the fixed points of f and g respectively. Note that every decreasing homeomorphism has a unique fixed point.*

Proof. Suppose f and g are order conjugate (respectively topologically conjugate). Then the same conjugacy between f and g when we restrict forms an order conjugacy (respectively topological conjugacy) between $f^2|_{[a,\infty)}$ and

$g^2|_{[b,\infty)}$. Conversely, suppose $f^2|_{[a,\infty)}$ and $g^2|_{[b,\infty)}$ are order conjugate through the increasing homeomorphism h_1. Then $h_1([a,\infty)) = [b,\infty)$ and $h_1(a) = b$. Also note that $f((-\infty,a]) = [a,\infty)$ and $g((-\infty,b]) = [b,\infty)$. That is, $f^{-1}([a,\infty)) = (-\infty,a]$ and $g^{-1}([b,\infty)) = (-\infty,b]$. Define $h : \mathbb{R} \to \mathbb{R}$ such that

$$h(x) = \begin{cases} h_1(x) & \text{if } x \in [a,\infty) \\ g^{-1}hf(x) & \text{if } x < a. \end{cases}$$

If $t < a$ then by definition $h \circ f(t) = g \circ h(t)$. If $t > a$ then $f(t) < a$. Therefore $h(f(t)) = g^{-1}(h_1(f(f(t)))) = g^{-1}(h_1(f^2(t))) = g^{-1}(g^2(h_1(t))) = g(h_1(t)) = g(h(t))$. Hence h forms an order conjugacy from f to g. □

The following proposition is analogous to Proposition 3.8, and its proof will be omitted.

Proposition 3.9. *Two decreasing bijections f and g are order conjugate (respectively topologically conjugate) if and only if $f^2|_{(-\infty,a]}$ and $g^2|_{(-\infty,b]}$ are order conjugate (respectively topologically conjugate), where a and b are the fixed points of f and g respectively.*

A map $f : \mathbb{R} \to \mathbb{R}$ is said to be *odd* if $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.

Proposition 3.10. *Let $f : \mathbb{R} \to \mathbb{R}$ be an odd increasing bijection. Then, there exist a decreasing homeomorphism f_r such that $f_r^2 = f$. Such an f_r is called a decreasing square root of f.*

Proof. Note that $f(0) = 0$. Define f_r such that

$$f_r(x) = \begin{cases} -f(x) & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

Clearly, f_r is a decreasing bijection. Then $f_r(x) \leq 0$ for all $x \geq 0$. Therefore $f_r(f_r(x)) = -f_r(x) = f(x)$. Also we have $f_r(f_r(x)) = f_r(-x) = -f(-x) = f(x)$ for all $x < 0$. □

Remark 3.11. The conclusion of the above proposition is not true in general. For this, let

$$h(x) = \begin{cases} \frac{x}{2} & \text{if } x \geq 0 \\ x & \text{if } x < 0. \end{cases}$$

Clearly, h is an increasing bijection from $\mathbb{R}$ to $\mathbb{R}$. There is no decreasing bijection $f : \mathbb{R} \to \mathbb{R}$ such that $f \circ f = h$. Let if possible f be one such function. Then we have $f(f(x)) = h(x) = x$ for all $x < 0$. Choose $y > 0$ such that $f(y) < 0$. Therefore $f^2(f(y)) = f(y) = f(f^2(y))$. Then $f^2(y) = y$ since f is one-one. Therefore $h(y) = y$, a contradiction since $h(y) = \frac{y}{2}$.

Proposition 3.12. *Let $f : (0,\infty) \to (0,\infty)$ be an increasing bijection. Then there exists a unique decreasing bijection $f_r : \mathbb{R} \to \mathbb{R}$ upto order conjugacy such that $f_r^2|_{(0,\infty)} = f$.*

Proof. Let $f : (0, \infty) \to (0, \infty)$ be an increasing bijection. This forces that $f(0) := \lim_{x \to 0^+} f(x) = 0$. Any map $f : (0, \infty) \to (0, \infty)$ can be extended uniquely to an odd function $\tilde{f} : \mathbb{R} \to \mathbb{R}$. Then there exists $f_r : \mathbb{R} \to \mathbb{R}$ such that $f_r^2|_{(0,\infty)} = f$, by Proposition 3.10. This f is unique upto order conjugacy by Proposition 3.9. □

Proposition 3.13. *Let $f : \mathbb{R} \to \mathbb{R}$ be a decreasing bijection. Then the non-ordinary points of f^2 are precisely the non-ordinary points of f.*

Proof. Suppose x is an ordinary point for f. Then the resut follows from the fact that if h commutes with f then it commutes with f^2 also. Conversely, suppose x is an ordinary point of f^2. Let the unique fixed point of f be zero, i.e., $f(x) = 0$ if and only if $x = 0$. Now let $x > 0$. Then there exists a neighbourhood $(x-\delta, x+\delta)$ such that for y in $(x-\delta, x+\delta)$ there is an $h \in G_{f \circ f}$ for which $h(x) = y$. Then $h|_{(0,\infty)}$ is a topological conjugacy between $f \circ f|_{(0,\infty)}$ and $f \circ f|_{(0,\infty)}$. Then h induces $\tilde{h}$ a conjugacy between f and f by Proposition 3.9. By the way $\tilde{h}$ is defined, we have $\tilde{h}(x) = h(x) = y$. Therefore x is an ordinary point of f. □

Proposition 3.14. *Let f be a decreasing bijection from $\mathbb{R}$ to $\mathbb{R}$ with fixed point a. Then, f has $2n+1$ non-ordinary points if and only if $f^2|_{(a,\infty)} : (a, \infty) \to (a, \infty)$ has n non-ordinary points.*

Proof. Suppose that f has $2n + 1$ non-ordinary points. Let them be $x_1 < x_2 < ... < x_n < x_{n+1} < x_{n+2} < ... < x_{2n+1}$. Let $N = \{x_1, x_2, ..., x_{2n+1}\}$. Then $f(N) \subset N$ by Proposition 2.20. Since f is a decreasing bijection, we have $f(N) = N$ and $a = x_n$. Hence $f^2|_{(a,\infty)} : (a, \infty) \to (a, \infty)$ has n non-ordinary points. Conversely, suppose $f^2|_{(a,\infty)} : (a, \infty) \to (a, \infty)$ has n non-ordinary points. Then by Proposition 3.14, we have $N(f) = N(f^2|_{(a,\infty)}) \cup f(N(f^2|_{(a,\infty)})) \cup \{a\}$. Thus f has $2n + 1$ non-ordinary points. □

Remark 3.15. *From the above proposition it follows that there does not exist a decreasing homeomorphism with even number of non-ordinary points.*

Theorem 3.16. *If s_n denotes the number of decreasing homeomorphisms upto order conjugacy, then*

$$s_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ a_{\frac{n-1}{2}} & \text{if } n \text{ is odd} \end{cases}$$

for all n.

Proof. By Proposition 3.14, we have $s_{2n} = 0$ for each $n \in \mathbb{N}$. Now we will prove that $s_{2n+1} = a_n$ for all $n \in \mathbb{N}$. Let $f : \mathbb{R} \to \mathbb{R}$ be a decreasing bijection with $2n + 1$ non-ordinary points. Without loss of generality we can assume that 0 is the unique fixed point. Then $g = f^2|_{(0,\infty)} : (0, \infty) \to (0, \infty)$ is an increasing bijection with n non-ordinary points. Since $(0, \infty)$ is homeomorphic to $\mathbb{R}$, we get an increasing homeomorphism $g' : \mathbb{R} \to \mathbb{R}$ (unique upto order conjugacy) with n non-ordinary points and which is order conjugate to g. On the other hand, suppose $h : \mathbb{R} \to \mathbb{R}$ is an increasing bijection with n non-ordinary points. Since $(0, \infty)$ is homeomorphic to $\mathbb{R}$, corresponding to each h we have a unique (upto order conjugacy) increasing bijection $h' : (0, \infty) \to (0, \infty)$ with n non-ordinary points. Then by Proposition 3.10, there exists unique (upto order conjugacy) decreasing square root $f : \mathbb{R} \to \mathbb{R}$ for h' such that $f \circ f|_{(0,\infty)} = h'$. By proposition 3.14, f has $2n+1$ non-ordinary points. Thus, there is a one-one correspondence between the set of all increasing bijections with n non-ordinary points (upto order conjugacy) and the set of all decreasing bijections with $2n+1$ non-ordinary points (upto order conjugacy). Hence $s_{2n+1} = a_n$. □

Theorem 3.17. *If k_n denotes the number of decreasing homeomorphisms having n non-ordinary points upto topological conjugacy, then*

$$k_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ t_{\frac{n-1}{2}} & \text{if } n \text{ is odd} \end{cases}$$

for all n.

Proof. If n is even then we have $k_n = 0$ since $s_n = 0$. If n is odd then we will argue as in Theorem 3.16 to prove that there is a one-one correspondence between the set of all increasing bijections (upto topological conjugacy) on $\mathbb{R}$ having n non-ordinary points and the set of all decreasing bijections (upto topological conjugacy) on $\mathbb{R}$ with $2n + 1$ non-ordinary points. Thus $k_{2n+1} = t_n$. □

4. Summary

We conclude this paper with the following table:

n	a_n	s_n	t_n	k_n
0	3	0	2	0
1	8	0	5	2
2	22	0	12	0
3	60	8	33	5
4	164	0	85	0
5	448	22	232	12

where a_n be the number of increasing bijections on $\mathbb{R}$ with exactly n non-ordinary points upto order conjugacy, t_n be the number of increasing bijections on $\mathbb{R}$ with exactly n non-ordinary points upto topological conjugacy, s_n be the number of decreasing bijections on $\mathbb{R}$ with exactly n non-ordinary points upto order conjugacy, and k_n be the number of decreasing bijections on $\mathbb{R}$ with exactly n non-ordinary points upto topological conjugacy.

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