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# The function $\omega_f$ on simple $n$-ods

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Abstract

*Given a discrete dynamical system $(X, f)$, we consider the function $\omega_f$-limit set from $X$ to $2^X$ as*

$$\omega_f(x) = \{y \in X : \text{ there exists a sequence of positive integers } n_1 < n_2 < \dots \text{ such that } \lim_{k\to\infty} f^{n_k}(x) = y\},$$

*for each $x \in X$. In the article [1], A. M. Bruckner and J. Ceder established several conditions which are equivalent to the continuity of the function $\omega_f$ where $f : [0,1] \to [0,1]$ is continuous surjection. It is natural to ask whether or not some results of [1] can be extended to finite graphs. In this direction, we study the function $\omega_f$ when the phase space is a $n$-od simple $T$. We prove that if $\omega_f$ is a continuous map, then $Fix(f^2)$ and $Fix(f^3)$ are connected sets. We will provide examples to show that the inverse implication fails when the phase space is a simple triod. However, we will prove that:*

**Theorem A 2.** *If $f : T \to T$ is a continuous function where $T$ is a simple triod, then $\omega_f$ is a continuous set valued function iff the family $\{f^0, f^1, f^2, \dots\}$ is equicontinuous.*

*As a consequence of our results concerning the $\omega_f$ function on the simple triod, we obtain the following characterization of the unit interval.*

**Theorem A 1.** *Let $G$ be a finite graph. Then $G$ is an arc iff for each continuous function $f : G \to G$ the following conditions are equivalent:*

*(1) The function $\omega_f$ is continuous.*

*(2) The set of all fixed points of $f^2$ is nonempty and connected.*

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## 1. Introduction

In this article, a *continuum* is a nonempty compact connected metric space. We shall consider discrete dynamical systems $(X, f)$ where the space $X$ is a continuum and $f : X \to X$ is a continuous map. Given a dynamical system $(X, f)$, we define: $f^0$ as the identity map of $X$, $f^n = f \circ f^{n-1}$ for every positive $n \in \mathbb{N}$, and the function $\omega_f$-limit set from $X$ to $2^X$ as

$$\omega_f(x) = \{y \in X : \text{ there is a sequence of positive integers}$$
$$n_1 < n_2 < \ldots \text{ such that } \lim_{k\to\infty} f^{n_k}(x) = y\},$$

for each $x \in X$. We remark that $\omega_f(x) = \bigcap_{m\geq 0} \overline{\{f^n(x) : n \geq m\}}$ for every $x \in X$.

Given $([0, 1], f)$ a discrete dynamical system, the authors of [1] proved that the following conditions are equivalence:

(1) $\omega_f$ is a continuous function.
(2) The set of fixed points of $f^2$, $Fix(f^2)$, is connected and nonempty.
(3) $f$ is equicontinuous.

We wonder whether or not this result can be extended to discrete dynamical systems $(G, f)$, where $G$ is finite graph. We answer this question in negative form, actually, we prove that the unique finite graph that satisfies the equivalence (1) $\Leftrightarrow$ (2) is the arc. To obtain this result, we start proving some properties that satisfies $\omega_f$, when the phase space is a dendroid, a fan and finally a simple triod $T$, the latter continuum is the union of 3 arcs emanating from a point $v$ such that the intersection of any two of the arcs is $v$. For a simple triod, we prove that if $\omega_f$ is a continuous function, then $Fix(f^2)$ and $Fix(f^3)$ are connected and nonempty. Moreover, we give examples to show that the connectivity of $Fix(f^2)$ and $Fix(f^3)$ does not imply the continuity of the map $\omega_f$. The proofs of these assertions will be given in the third Section. In the fourth Section, we will prove the equivalence (1) $\Leftrightarrow$ (3) when the phase space is a simple triod, $T$. This assertion requires some properties of $Fix(f)$ when $f$ is a surjective map and $\omega_f$ is a continuous function. More precisely, we prove that $f^{-1}(Fix(f)) = Fix(f)$ and $Fix(f)$ coincides with one of the following sets: $T$, the vertex of $T$ and some edge of $T$. We also show that each point of $T$ is a periodic point of any continuous surjection $f : T \to T$, with period at most 3. In general is interesting to find conditions equivalent to the equicontinuity of a map (the papers [1], [2], [7], and [9] contain results in this direction).

## 2. Preliminaries

Given a discrete dynamical system $(X, f)$. For a point $x \in X$, the *orbit* of $x$ under $f$ is the set $\mathcal{O}_f(x) = \{f^n(x) : n \in \mathbb{N}\}$; $x$ is said to be a *fixed point* of $f$ if $f(x) = x$, $x$ is said to be *n-periodic point*, if $f^n(x) = x$ and $f^i(x) \neq x$ for every $1 \leq i < n$ with $n \geq 1$; $x$ is said to be *periodic point* if there exists an $n \in \mathbb{N}$ such that $x$ is an $n$-periodic point. The sets of fixed points, $n$-periodic points and periodic points of $f$ are denoted by $Fix(f)$, $Per_n(f)$ and $Per(f)$, respectively. A function $f : X \to X$ is said to be *equicontinuous* (relative to the metric $d$) if for each $\varepsilon > 0$, there exists $\delta > 0$ such that $d(f^n(x), f^n(y)) < \varepsilon$ for each $x, y \in X$ with $d(x, y) < \delta$ and all $n \in \mathbb{N}$. A function $f : X \to X$ is said to be *topological transitivity* if for every pair of nonempty open sets $U$ and $V$ in $X$, there exists $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \varnothing$.

For a continuum $X$ we denote the collection of all nonempty compact subsets of $X$ by $2^X$, we consider $2^X$ equipped with the Hausdorff metric. An *arc* is a continuum homeomorphic to $[0, 1]$. Given an arc $A$ and a homeomorphism $h : [0, 1] \to A$, the points $h(0)$ and $h(1)$ are called the *end points* of $A$. A *simple closed curve* is a continuum homeomorphic to a circle. Given a point $x \in X$, $x$ is said to be an *end point* of $X$, if for each arc $A$ in $X$ such that $x \in A$, then $x$ is an end point of the arc $A$. Let $n \in \mathbb{N} \cup \{\omega\}$, $x$ is said to be a *point of order $n$ in the classical sense*, or here briefly a *point of order $n$*, $ord_X(x) = n$, if $x$ is a unique common end point of every two of exactly $n$ arcs contained in $X$. If $ord_X(x) \geq 3$, then $x$ is said to be a *ramification point of $X$*. The sets of end points and ramification points of $X$ are denoted by $\mathcal{E}(X)$ and $\mathcal{R}(X)$, respectively. A continuum $X$ is said to be *arc-wise connected* provided that for every two points $a, b \in X$ there exists an arc in $X$ with end points $a$ and $b$. $X$ is said to be *unicoherent* provided that for every two proper subcontinua $A$ and $B$ of $X$ such that $X = A \cup B$, we have that $A \cap B$ is connected. A continuum $X$ is *irreducible between $a$* and $b$ if no proper subcontinuum of $X$ contains $a$ and $b$. Given $A$, $B$ and $C$ subontinua of $X$, we said that $C$ is *irreducible from $A$* to $B$ if $A \cap C \neq \varnothing \neq B \cap C$ and no proper subcontinuum of $C$ intersects both $A$ and $B$. Given a property $P$, a continuum $X$ is said to be *hereditarily $P$*, provided that every non degenerate subcontinuum of $X$ has the property $P$. A *dendroid* is a hereditarily arc-wise connected and hereditarily unicoherent continuum. It is well know that a dendroid is unique arc-wise connected. If $X$ is a dendroid we denote by $[a, b]$ the arc in $X$ with end points $a$ and $b$. A *fan* is a dendroid with exactly one ramification point. A *finite graph* is a continuum which can be written as the union of finitely many arcs any two of which are either disjoint or intersect only in one or both of their end points. A *tree* is finite graph without simple closed curves. Given $n \in \mathbb{N}$, $n > 2$ a *simple $n$−od* with *vertex $v$* is the union of $n$ arcs emanating from the point $v$ and such that $v$ is the intersection of any two of the arcs, $v$ is called de vertex of the simple $n$-od. Throughout this paper, $T$ will be denote a simple $n$-od with vertex $v$ and set of end points $\mathcal{E}(T) = \{e_1, e_2, \ldots, e_n\}$, we consider $T$ with convex metric $d$. In order to define some specific functions on a simple triod we will consider the special case when $T = ([-1, 1] \times \{0\}) \cup (\{0\} \times [0, 1])$, $T$ is a subspace of the Euclidian plane $\mathbb{R}^2$, and its vertex will be $v = (0, 0)$ and its end points will be $e_1 = (-1, 0)$, $e_2 = (1, 0)$, $e_3 = (0, 1)$.

The following result is well known and will very useful throughout this work, we present a proof for the convenience of the reader.

**Proposition 2.1.** *Let $X$ be a dendroid. If $F$ be a closed non-connected subset of $X$, then there exist points $a, b \in X$ such that $[a, b] \cap F = \{a, b\}$.*

*Proof.* Let $A$ and $B$ components of $F$, and let $Y$ be the irreducible continuum from $A$ to $B$. By [8, 11.37(a)] there exist $a_0 \in A$ and $b_0 \in B$ such that $Y$ be the irreducible continuum between $a_0$ and $b_0$. Since $X$ is a dendroid, we have that $Y = [a_0, b_0]$. If $[a_0, b_0] \cap F = \{a_0, b_0\}$, then the proposition follows. Suppose that $[a_0, b_0] \cap F \neq \{a_0, b_0\}$. Let $p \in [a_0, b_0] - F$ and let $U \subset [a_0, b_0] - F$ be the component of $[a_0, b_0] - F$ that contains $p$. We observe that $U = (a, b)$. By [8, 5.6], we have that $F \cap Bd(U) \neq \varnothing$ and then $F \cap \{a, b\} = \{a, b\}$, hence $[a, b] \cap F = \{a, b\}$. □

The implicit properties of continua that we will use in this article may be found in the book [8].

## 3. Connectivity of the set of fixed points of an iterate

To start this section, we shall study those discrete dynamical systems $(X, f)$ where $X$ is a fan and the function $\omega_f$ is continuous. The next, four lemmas will provide the basic information that we need for our purposes.

**Lemma 3.1.** *Let $(X, f)$ be a discrete dynamical system where $X$ is a dendroid. If $n \in \mathbb{N}$ and $a, x \in X$ such that $[a, x] \cap Fix(f^n) = \{a\}$ and $f^n(z) \in [a, z)$ for each $z \in (a, x]$, then we have that $a \in \omega_f(z)$ for each $z \in [a, x]$.*

*Proof.* Let $z \in [a, x]$. If $z = a$ the result is immediately. Assume that $z \neq a$, it follows from the hypothesis that $f^{nk}(z) \in [a, f^{n(k-1)}(z))$ for all $k \in \mathbb{N}$. Then we have that $\{f^{nk}(z)\}_{k=1}^{\infty}$ is a monotone sequence and hence this sequence converges. Set $c := \lim_{k\to\infty} f^{nk}(z)$ and notice that $\lim_{k\to\infty} f^n(f^{nk}(z)) = c$. Then we have that $f^n(c) = c$. Since $f^{nk}(z) \in [a, x]$ for each $k \in \mathbb{N}$, we must have that $c \in [a, x]$ and since $[a, x] \cap Fix(f^n) = \{a\}$, $c = a$. Thus, $a \in \omega_f(z)$. □

**Lemma 3.2.** *Let $(X, f)$ be a discrete dynamical system where $X$ is a dendroid such that $\omega_f$ is a continuous function. If $n \in \mathbb{N}$ and $a, b, c \in X$ such that $b \in (a, c)$, $[a, c] \cap Fix(f^n) = \{a, c\}$, $f^n(b) = a$, then $\omega_f(y) = \omega_f(c)$ for each $y \in [a, c]$.*

*Proof.* Fix an arbitrary point $y_0 \in [a, c]$. Since $f^n(b) = a$ and $f^n(c) = c$, we have that $y_0 \in [a, c] \subset f^n([b, c])$. Hence, there exists $y_1 \in [b, c]$ such that $f^n(y_1) = y_0$. Notice that $y_1 \in (y_0, c]$; otherwise, $y_0 \in [y_1, c]$ and since $f^n(b) = a$, $[b, y_1] \cap Fix(f^n) \neq \varnothing$ which contradicts the hypothesis. As $y_1 \in [y_0, c] \subset f^n([y_1, c])$, then there exists $y_2 \in [y_1, c]$ such that $f(y_2) = y_1$, and since $y_2 \in [y_1, c] \subset f^n([y_2, c])$, we can find $y_3 \in [y_2, c]$ so that $f(y_3) = y_2$. By continuing with this process, we may construct a sequence $\{y_k\}_{k=1}^{\infty}$ such that $y_{k+1} \in [y_k, c]$ and $f^n(y_k) = y_{k-1}$ for each $k \in \mathbb{N}$. Since $\{y_k\}_{k=1}^{\infty}$ is a monotone sequence, it converges and $\omega_f(y_k) = \omega_f(y_0)$, for each $k \in \mathbb{N}$. Put $\lim_{k\to\infty} y_k = d$ and notice that $\lim_{k\to\infty} f^n(y_k) = d \in [b, c]$. Then $f^n(d) = d$ and since $[b, c] \cap Fix(f^n) = \{c\}$, we have that $d = c$. By the continuity of $\omega_f$, we obtain that $\lim_{k\to\infty}\omega_f(y_k)=\omega_f(c)$, and as $\omega_f(y_k)=\omega_f(y_0)$ for each $k\in\mathbb{N}$, then $\omega_f(y_0)=\omega_f(c)$. □

**Lemma 3.3.** *Let $(X,f)$ be a discrete dynamical system where $X$ is a dendroid such that $\omega_f$ is a continuous function. If $n\in\mathbb{N}$ and $a,b,c\in X$ such that $b\in(a,c)$ and $[a,b]\cap Fix(f^n)=\{a\}$, $f^n(b)=c$, then $\omega_f(y)=\omega_f(a)$ for each $y\in[a,c]$.*

*Proof.* By a procedure similar to the one used in the proof of Lemma 3.2, we may construct a sequence $\{y_k\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty}y_k=d$ exists, $y_{k+1}\in[a,y_k]$ and $f^n(y_k)=y_{k-1}$ for every $k\in\mathbb{N}$. Notice that $\lim_{k\to\infty}f^n(y_k)=d\in[a,b]$. Then $f^n(d)=d$ and since $[a,c]\cap Fix(f^n)=\{a\}$ we have that $d=a$. Since $\omega_f$ is a continuous function we have that $\lim_{k\to\infty}\omega_f(y_k)=\omega_f(a)$, and as $\omega_f(y_k)=\omega_f(y_0)$ for each $k\in\mathbb{N}$, then $\omega_f(y_0)=\omega_f(a)$. □

**Lemma 3.4.** *Let $(X,f)$ be a discrete dynamical system where $X$ is a dendroid such that $\omega_f$ is a continuous function. If $b,c,d\in X$ such that $b\neq d$, $c\in(b,d)$, $f(c)=d$, $f(d)=b$, then $\omega_f(c)\subset[c,d]$.*

*Proof.* As $f(c)=d$ and $f(d)=b$, then $[b,d]\subset f([c,d])$. Hence, there exist $r_1,s_1\in[c,b]$ and $r_1\neq s_1$ such that $f(r_1)=c$, $f(s_1)=d$ and $f((r_1,s_1))=(c,d)$. It is easy to proof that $r_1\neq c$ and $s_1\neq d$. Following with this procedure, we may construct two sequences $\{r_n\}_{n=1}^{\infty}$ and $\{s_n\}_{n=1}^{\infty}$ such that $[r_{n+1},s_{n+1}]\subset[r_n,s_n]$, $r_n\neq r_{n+1}\neq s_n$, $s_{n+1}\neq s_n$, $f^n(r_n)=c$, $f^n(s_n)=b$, $f^n((r_n,s_n))=(c,b)$ for each $n\in\mathbb{N}$. Then, we have that $\omega_f(r_n)=\omega_f(c)$ and $\omega_f(s_n)=\omega_f(b)$ for each $n\in\mathbb{N}$. Without loss of generality, suppose that $\lim_{n\to\infty}r_n=r$. Observe that $r\in\bigcap_{n=1}^{\infty}[r_n,s_n]$. Since $\omega_f$ is a continuous function $\lim_{n\to\infty}\omega_f(r_n)=\omega_f(r)$. On the other hand, since $\omega_f(r_n)=\omega_f(c)$ for each $n\in\mathbb{N}$, $\omega_f(r)=\omega_f(c)$. Since $f^n(r)\in f^n([r_n,s_n])=[c,d]$ for each $n\in\mathbb{N}$, then we obtain that $\omega_f(r)\subset[c,d]$. Therefore, $\omega_f(c)\subset[c,d]$. □

**Theorem 3.5.** *Let $(X,f)$ be a discrete dynamical system where $X$ is a fan such that $\omega_f$ is a continuous function. If $n\in\mathbb{N}$ and $a,b\in Fix(f^n)$ such that $a\neq b$ and $Fix(f^n)\cap[a,b]=\{a,b\}$ then $\omega_f(a)=\omega_f(b)$.*

*Proof.* Let $v$ be the ramification point of $X$. We shall consider the following cases:

**Case 1.** $f^n(x)\in[a,b]$ for every $x\in(a,b)$. Since $Fix(f^n)\cap[a,b]=\{a,b\}$, we have that either $f^n(x)\in[a,x)$ for every $x\in(a,b)$ or $f^n(x)\in(x,b]$ for every $x\in(a,b)$. Without loss of generality suppose that $f^n(x)\in[a,x)$ for every $x\in(a,b)$. It follows from Lemma 3.1 that $a\in\omega_f(x)$, for each $x\in(a,b)$. As $\omega_f$ is a continuous function, we obtain that $a\in\omega_f(b)$. Since $f^n(b)=b$ and $f^n(a)=a$, and $a\in\omega_f(b)=\{f(b),\dots,f^n(b)\}$, we must have that $\omega_f(b)=\omega_f(a)$.

**Case 2.** There exists $x\in(a,b)$ such that $f^n(x)\notin[a,b]$. Choose $e_1,e_2\in\mathcal{E}(X)$ such that $[a,b]\subset[e_1,e_2]$ and $a\in[e_1,v]$.

**Case 2.1.** $f^n(x)\in[e_1,e_2]$. Since $f^n(x)\in[e_1,e_2]-[a,b]$, without loss of generality, we may assume that $f^n(x)\in[e_1,a)$. Then, it follows from the continuity of $f^n$ that there exists $c\in[a,b]$ such that $f^n(c)=a$. Thus, by

Lemma 3.2, we obtain that $\omega_f(y) = \omega_f(b)$ for each $y \in [a, b]$. As a consequence, $\omega_f(a) = \omega_f(b)$.

**Case 2.2.** $f^n(x) \in X - [e_1, e_2]$. Without loss of generality, we suppose that $x \in [a, f^n(x)]$. It follows from Lemma 3.3 that $\omega_f(y) = \omega_f(a)$ for each $y \in [a, f^n(x)]$. If $[a, b] \subset [a, v]$, then $\omega_f(b) = \omega_f(a)$ since $[a, v] \subset [a, f^n(x)]$.

Next, assume that $[a, b] \not\subset [a, v]$. Then, $b \neq v$ and so either $f^n((v, b]) \not\subset [v, e_2]$ or $f^n((v, b]) \subset [v, e_2]$.

**Case 2.2.1.** $f^n((v, b]) \not\subset [v, e_2]$. Then there exists $y \in [v, b]$ so that $f^n(y) = v$. According to Lemma 3.3, we have that $\omega_f(y) = \omega_f(b)$ for each $y \in [v, b]$. Since $\omega_f(v) = \omega_f(a)$ and $\omega_f(v) = \omega_f(b)$, we obtain that $\omega_f(a) = \omega_f(b)$.

**Case 2.2.2.** $f^n((v, b]) \subset [v, e_2]$.

- If $f^n(y) \in (y, b]$, for each $y \in [v, b]$, by Lemma 3.1, then we have that $b \in \omega_f(y)$ for each $y \in [v, b]$. In particular, $b \in \omega_f(v) = \omega_f(a)$. Therefore, $\omega_f(a) = \omega_f(b)$.
- There exists $c \in [v, b] \subset [a, b]$ such that $f^n(c) = b$. By Lemma 3.2, we know that $\omega_f(y) = \omega_f(a)$ for each $y \in [a, b]$. Hence, $\omega_f(b) = \omega_f(a)$.

In both subcases, we conclude that $\omega_f(a) = \omega_f(b)$. □

In [1] the authors studied the relation between the continuity of $\omega_f$ and the connectivity of the sets $Fix(f)$ and $Fix(f^2)$, where $f$ is a continuous map from $[0, 1]$ to $[0, 1]$ . The Corollary 3.6 is a generalization of [1, Lemma 1.1] when $f$ is a continuous function from a fan to itself. Further the Theorem 3.8 generalizes [1, Theorem 1.2 (1) $\rightarrow$ (6)], in the case where $f$ is a continuous function from a simple triod to itself.

**Corollary 3.6.** *Let $(X, f)$ be a discrete dynamical system where $X$ is a fan such that $\omega_f$ is a continuous function. Then $Fix(f)$ is connected.*

*Proof.* Suppose to the contrary, $Fix(f)$ is not connected. Then, by Proposition 2.1, there are points $a, b \in Fix(f)$, $a \neq b$ such that $[a, b] \cap Fix(f) = \{a, b\}$. It follows from Theorem 3.5 that $\omega_f(a) = \omega_f(b)$, but this is impossible because of $\omega_f(a) = \{a\}$ and $\omega_f(b) = \{b\}$. □

*Remark* 3.7. Let $(X, f)$ be a discrete dynamical system where $X$ is a fan. If $a, b \in Fix(f^n)$ are distinct, for some $1 < n \in \mathbb{N}$, and $\omega_f(b) = \omega_f(a)$, then the following statements $a \neq f(a)$ and $b \neq f(b)$.

From now on we will consider discrete dynamical systems on a simple triod. Our next task is to analyze the consequences when $\omega_f$ is a continuous function.

**Theorem 3.8.** *Let $(T, f)$ be a discrete dynamical system where $T$ is a simple $n$-od and such that $\omega_f$ is a continuous function, then $Fix(f^2)$ is connected.*

*Proof.* Let $v \in T$ the vertex of $T$. Suppose to the contrary that $Fix(f^2)$ is non-connected. By Proposition 2.1, there exist two points $a, b \in Fix(f^2)$ $a \neq b$ such that $[a, b] \cap Fix(f^2) = \{a, b\}$. Theorem 3.5 asserts we have that $\omega_f(a) = \omega_f(b)$. So, $f(a) = b$, and $f(b) = a$.

**Case 1.** $v \notin (a,b)$. Since $[a,b] \subset f([a,b])$ and $v \notin (a,b)$, then $f$ would have a fixed point on $(a,b)$, and hence $(a,b)$ also would have a fixed point of $f^2$, this is a contradiction.

**Case 2.** $v \in (a,b)$. Without loss of generality, we may assume that $a \in [e_1, v]$ and $b \in [v, e_2]$. Notice that $[a,b] \subset [e_1, e_2]$.

**Case 2.1.** $f(v) \in [e_1, e_2]$. Suppose that $f(v) \in (v, e_2]$. Since $[v, f(v)] \subset [a, f(v)] \subset f([v,b])$, we have that $[v,b)$ has a fixed point of $f$. Therefore $[v,b)$ has a fixed point of $f^2$, this contradicts our supposition. Similarly, we analyze the case when $f(v) \in (v, e_1]$.

**Case 2.2.** $f(v) \notin [e_1, e_2]$. Since $v \in [a, f(v)] = [f(b), f(v)] \subset f([v,b])$ there exists $v_1 \in [v,b]$ such that $f(v_1) = v$. Notice that $v \neq v_1$, in other case $v \in Fix(f^2)$ which is impossible because of $Fix(f^2) \cap [a,b] = \{a,b\}$. As $v_1 \in [v,b] \subset [b, f(v)]$ and $[v_1, b] \cap Fix(f^2) = \{b\}$ it follows from Lemma 3.3 that $\omega_f(y) = \omega_f(b)$, for each $y \in [b, f(v)]$, in particular $\omega_f(v) = \omega_f(a) = \{a,b\}$. Let $\varepsilon > 0$ such that $B_\varepsilon(b) \subset (v_1, e_2]$ and $B_\varepsilon(a) \subset (v, e_1]$, by continuity there exists $0 < \delta < \varepsilon$ such that if $d(x,y) < \delta$ then $d(f(x), f(y)) < \varepsilon$. Since $\omega_f(v) = \omega_f(b) = \{a,b\}$ there exists $m \in \mathbb{N}$ such that $f^m(v) \in B_\delta(a)$. So, $f^{m+1}(v) \in B_\varepsilon(b)$.

Since $[v, v_1] \subset [f^m(v), f^{m+1}(v)] = [f^{m+1}(v_1), f^{m+1}(v)] \subset f^{m+1}([v, v_1])$ we have that $[v, v_1] \cap Fix(f^{m+1}) \neq \varnothing$. Given $x \in [v, v_1] \cap Fix(f^{m+1})$, we obtain that $\omega_f(x) = \mathcal{O}_f(x)$. on the other hand as $x \in [v, v_1] \subset [b, f(v)]$, we have that $\omega_f(x) = \omega_f(b) = \{a,b\}$, then $\mathcal{O}_f(x) = \{a,b\}$ which is impossible because of $a \neq x \neq b$ and $x \in \mathcal{O}_f(x)$. □

The following example show that the converse of Corollary 3.6 and Theorem 3.8 are not true. Therefore, we have that [1, Theorem 1.2 (6) ⇔ (1)] is not satisfied, when $f$ is a continuous function from a simple triod to itself.

**Example 3.9.** We will give a function $f$ from $T$ to itself such that the sets $Fix(f)$ and $Fix(f^2)$ are connected but the function $\omega_f$ is not continuous and $Fix(f^3)$ is not connected. In order to have this done, we will assume that $T = ([-1,1] \times \{0\}) \cup (\{0\} \times [0,1])$. Define $f: T \to T$ given by

$$f((x,y)) = \begin{cases} (2x + \frac{3}{2}, 0) & (x,y) \in [-1, -\frac{3}{4}] \times \{0\}, \\ (3 + 4x, 0) & (x,y) \in [-\frac{3}{4}, -\frac{1}{2}] \times \{0\}, \\ (-2x, 0) & (x,y) \in [-\frac{1}{2}, 0] \times \{0\}, \\ (0, 2x) & (x,y) \in [0, \frac{1}{2}] \times \{0\}, \\ (0, 3 - 4x) & (x,y) \in [\frac{1}{2}, -\frac{3}{4}] \times \{0\}, \\ (2x - \frac{3}{2}, 0) & (x,y) \in [-\frac{3}{4}, 1] \times \{0\}, \\ (-2y, 0) & (x,y) \in \{0\} \times [0, \frac{1}{2}], \\ (3 - 4y, 0) & (x,y) \in \{0\} \times [\frac{1}{2}, \frac{3}{4}], \\ (0, 2y - \frac{3}{2}) & (x,y) \in \{0\} \times [\frac{3}{4}, 1]. \end{cases}$$

In the Figure 1 we have the graph of $f$.

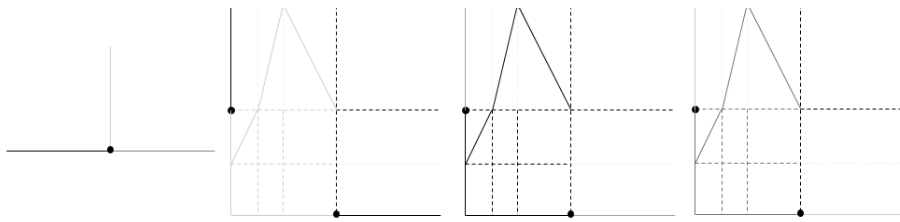

Figure 1.

In the Figure 2 we have the graph of $f^2$.

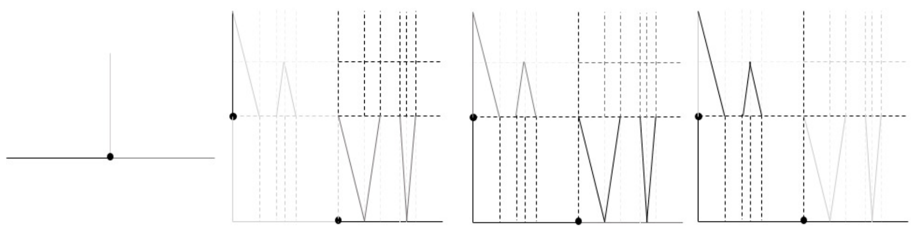

Figure 2.

We will see that $Fix(f)$ and $Fix(f^2)$ are connected sets however $\omega_f$ is not a continuous function. Indeed, it follows from the definition that $Fix(f) = \{v\} = Fix(f^2)$. Further, if $A$ is a subset of $T$, with $T \neq A \neq \{v\}$, then we have that $f(A) \neq A$. By [5, Theorem 2.2.2 (1) $\Leftrightarrow$ (9)], we know that $f$ is a transitive continuous function. Hence, according to [5, Theorem 2.2.2 (1) $\Leftrightarrow$ (16)], we have that there exists $x_0 \in T$ such that $\overline{\mathcal{O}_f(x_0)} = T$. So, for every $\varepsilon > 0$, we can find $y_0 \in \mathcal{O}_f(x_0)$ so that $d(y_0, v) < \varepsilon$, but $\omega_f(v) = \{v\}$ and $\omega_f(y_0) = \omega_f(x_0) = T$, which shows that $\omega_f$ is not a continuous function at $v$. Finally, its easy to see that $Fix(f^3)$ is not a connected set (compare with Theorem 3.11 below).

Next let see how Example 3.9 gives a new characterization of the arc in terms of the continuity of $\omega_f$ and the connectivity of $Fix(f^2)$.

*Theorem A 1. Let $G$ be a finite graph. Then $G$ is an arc if and only if for each continuous function $f : G \to G$, the following conditions are equivalent:*

(1) *$\omega_f$ is a continuous function,*
(2) *$Fix(f^2)$ is connected and nonempty.*

*Proof.* Necessity. Suppose that $G$ is an arc and that $f : G \to G$ is a continuous function, since an arc has the fixed point property, we have that $Fix(f^2)$ is nonempty. In accordance with [1, Theorem 1.2 (1) $\leftrightarrow$ (6)], we have that the conditions (1) and (2) are equivalents.

Sufficiency. Assume that conditions (1) and (2) are equivalents for each continuous function $f : G \to G$. Suppose that $G$ is not an arc, we need analyze the following two cases:

**Case 1.** $G$ has a simple closed curve. Let $S \subset G$ be a simple closed curve. Since $G$ is 1-dimensional, by [3, Theorem VI 4], there is retraction $r : G \to S$ and we consider an irrational rotation $g : S \to S$. Then, the function $f = g \circ r$ is a continuous and satisfies that $\omega_f(x) = S$ for each $x \in G$. That is, $\omega_f$ is a continuous function. On the other hand, since $g$ is irrational rotation we have that $Fix(f^2) = \varnothing$, this is a contradiction.

**Case 2.** $G$ has not a simple closed curve. In this case, $G$ contains a simple triod $T$ with vertex $v$. According to [6, Theorem 2.1], there is a retraction $r : G \to T$. If $f : T \to T$ is as in Example 3.9, then $h = f \circ r$ is a continuous function such that $Fix(h^2) = \{v\}$ and $\omega_h$ is not a continuous function, but this is impossible.

In any case we obtain a contradiction. Therefore, $G$ is an arc. □

The following theorem shows that the statement of Theorem 3.8 is valid for $f^3$.

**Lemma 3.10.** *Let $(X, f)$ be a discrete dynamical system. If $a, b \in Fix(f^3)$, $a \neq b$ and $\omega_f(a) = \omega_f(b)$. Then the following conditions hold:*

(1) *$a \neq f^i(a)$, with $i \in \{1, 2\}$,*
(2) *$b \neq f^i(b)$, with $i \in \{1, 2\}$.*

*Proof.* As $\omega_f(a) = \omega_f(b)$, we know that either $a = f(b)$ or $a = f^2(b)$.

(1). Suppose that $a = f(a)$. If $a = f(b)$, then $a = f^2(a) = f^3(b) = b$, but this is a contradiction. Now, if $a = f^2(b)$, then $a = f(a) = f^3(b) = b$ and so we obtain a contradiction. Assume that $f^2(a) = a$. If $a = f(b)$, then $a = f^2(a) = f^3(b) = b$, which is impossible. If $a = f^2(b)$, then $f(a) = f^3(b) = b$ and $a = f^2(a) = f(b)$, then $b = f^2(b)$, hence $a = b$ and we have a contradiction.

Clause (2) is established in a similar way. □

The function $f : [0, 1] \to [0, 1]$, define by $f(x) = 1 - x$ for each $x \in [0, 1]$, witnesses that Lemma 3.10 does not work for $n = 4$.

**Theorem 3.11.** *Let $(T, f)$ be a discrete dynamical system where $T$ is a simple $n$-od and such that $\omega_f$ is a continuous function, then $Fix(f^3)$ is connected.*

*Proof.* Let $v \in T$ the vertex of $T$. Suppose that $Fix(f^3)$ is not connected. By Proposition 2.1, there are points $a, b \in Fix(f^3)$ such that $[a, b] \cap Fix(f^3) = \{a, b\}$. As a consequence of Theorem 3.5, we obtain that $\omega_f(a) = \omega_f(b)$. By Lemma 3.10, we can suppose that $\omega_f(b) = \{b, f(b), a\}$.

**Case 1.** There exists an arc $A \subset T$ such that $\omega_f(b) \subset A$.

**Case 1.1.** $a \in (b, f(b))$. Being that $f(a) = b$ and $f(b) = f(b)$, by Lemma 3.4, we obtain that $\omega_f(a) \subset [b, a]$, which contradicts $f(b) \notin [b, a]$.

**Case 1.2.** $f(b) \in (a, b)$. Since $f(f(b)) = a$ and $f(a) = b$, we obtain of Lemma 3.4 that $\omega_f(f(b)) \subset [a, f(b)]$, this is impossible because of $b \notin [a, f(b)]$.

**Case 1.3.** $b \in (a, f(b))$. As $f(b) = f(b)$ and $f(f(b)) = a$, by Lemma 3.4 we have that $\omega_f(b) \subset [b, f(b)]$, this is a contradiction to $a \notin [b, f(b)]$.

**Case 2.** $\omega_f(b) \not\subset A$ for any arc $A \subset T$. Without loss of generality, suppose that $a \in [v, e_1]$, $b \in [v, e_2]$ and $f(b) \in [v, e_3]$.

**Case 2.1.** $f(v) \in [e_1, e_2]$. Assume that $f(v) \in (v, e_2]$. As $[f(v), v] \subset [f(v), f(b)] \subset f([v, b])$, we must have that $[v, b)$ has a fixed point of $f$; hence, $(a, b)$ has a fixed point of $f^3$ which is a contradiction. Similarly, we obtain a contradiction if $f(v) \in (v, e_1]$.

**Case 2.2.** $f(v) \notin [e_1, e_2]$.

**Case 2.2.1.** $f(v) \in (v, e_3]$. Since $[v, f(v)] \subset [a, f(v)] = [f(f(b)), f(v)] \subset f([v, f(b)])$, we have that $[v, f(b)]$ has a fixed point of $f$. Let $z \in [v, f(b)]$ be a fixed point of $f$. As $[v, f(b)] \subset [f(v), b] = [f(v), f(a)] \subset f([a, v])$, there exits $y \in [a, v]$ for which $f(y) = z$. Then, $\omega_f(y) = \{z\}$. On the other hand, as $v \in [a, f(v)] = [f(f(b)), f(v)] \subset f([f(b), v])$ there is $v_1 \in (v, f(b))$ such that $f(v_1) = v$. As $v_1 \in [b, f(b)] = [f(a), f(b)] \subset f([a, b])$ there is $v_2 \in (a, b)$ such that $f(v_2) = v_1$ and so $f^3(v_2) = f(v)$. As $[a, b] \cap Fix(f^3) = \{a, b\}$, if $v_2 \in [a, v]$, then $[a, v_2] \cap Fix(f^3) = \{a\}$ and $v_2 \in [a, f(v)]$ it follows from Lemma 3.3 that $\omega_f(x) = \omega_f(a)$ for each $x \in [a, f(v)]$ in particular $\omega_f(y) = \omega_f(a)$. Hence, $\omega_f(a) = \{z\}$ but this is impossible because of $\omega_f(a) = \{b, f(b), a\}$. Similarly in the case that $v_2 \in [v, b]$ we obtain a contradiction.

**Case 2.2.2.** $f(v) \notin (v, e_3]$.

Since $v \in [f(b), f(v)] \subset f([v, b])$ there exists $v_1 \in [v, b]$ such that $f(v_1) = v$. Since $[v, b] \cap Fix(f) = \varnothing$ we have that $v \neq v_1$. As $v_1 \in [v, b] \subset [f(v), b]$ and $[v_1, b] \cap Fix(f^2) = \{b\}$ it follows from Lemma 3.3 that $\omega_f(y) = \omega_f(b)$, for each $y \in [b, f(v)]$. Thus, $\omega_f(v) = \omega_f(b) = \{b, f(b), a\}$. Let $\varepsilon > 0$ such that $B_\varepsilon(b) \subset (v_1, e_2]$ and $B_\varepsilon(a) \subset (v, e_1]$, since $f$ is a continuous function there exists $0 < \delta < \varepsilon$ such that if $d(x, y) < \delta$ then $d(f(x), f(y)) < \varepsilon$. Since $\omega_f(v) = \omega_f(b) = \{b, f(b), a\}$ there exists $m \in \mathbb{N}$ such that $f^m(v) \in B_\delta(a)$. Hence, as $f(a) = b$ we have that $f^{m+1}(v) \in B_\varepsilon(b)$.

On the other hand $[v, v_1] \subset [f^m(v), f^{m+1}(v)] = [f^{m+1}(v_1), f^{m+1}(v)] \subset f^{m+1}([v, v_1])$ we have that $[v, v_1] \cap Fix(f^{m+1}) \neq \varnothing$. Let $x \in [v, v_1] \cap Fix(f^{m+1})$. We have that $\omega_f(x) = \mathcal{O}_f(x)$. As $x \in [v, v_1] \subset [b, f(v)]$, we obtain that $\omega_f(x) = \omega_f(b) = \{b, f(b), a\}$ which is not possible because of $a \neq x \neq b$, $f(b) \neq x$ and $x \in \mathcal{O}_f(x)$ □

The following example shows that the converse of Theorem 3.11 is not true.

**Example 3.12.** We will give a continuous function $f$ from $T$ to itself such that the sets $Fix(f)$ and $Fix(f^3)$ are connected but the function $\omega_f$ is not continuous and $Fix(f^2)$ is not connected. Define $f : T \to T$ given by

$$f((x,y)) = \begin{cases} (-\frac{3}{2}(x+\frac{1}{3}),0) & (x,y)\in[-1,-\frac{1}{3}]\times\{0\}, \\ (0,\frac{1}{3}+x) & (x,y)\in[-\frac{1}{3},0]\times\{0\}, \\ (0,\frac{1}{3}-x) & (x,y)\in[0,\frac{1}{3}]\times\{0\}, \\ (\frac{3}{2}(\frac{1}{3}-x),0) & (x,y)\in[\frac{1}{3},1]\times\{0\}, \\ (0,\frac{1}{3}) & (x,y)\in\{0\}\times[0,1]. \end{cases}$$

The graph of $f$, $f^2$ and $f^3$ appear in the Figure 3.

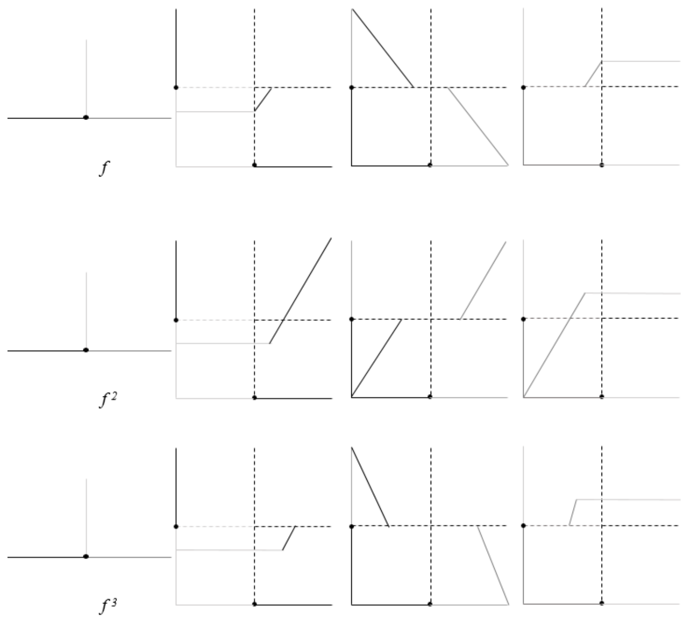


FIGURE 3.

It follows from the definition that $Fix(f) = \{(0,\frac{1}{3})\} = Fix(f^3)$ which are both connected. Since $Fix(f^2) = \{(0,\frac{1}{3}), e_1, e_2\}$, then $Fix(f^2)$ cannot be connected. Hence, by Theorem 3.8 we conclude that the function $\omega_f$ is not continuous.

By a procedure similar to the one used in the construction of Example 3.9, we can define a function $f$ from a simple 4-od to itself, for which the sets $Fix(f)$, $Fix(f^2)$ and $Fix(f^3)$ are connected, but $Fix(f^4)$ is not connected and $\omega_f$ is not continuous.

**Question 3.13.** *Let $T$ be a simple $n$-od and $f : T \to T$ be a continuous map. If there is $m \in \mathbb{N}$ such that $Fix(f^n)$ is connected, for all $n > m$, must $\omega_f$ be a continuous map?*

In relation with Theorem 3.8 and Theorem 3.11 we have the following questions.

**Question 3.14.** *Let $T$ be a simple $n$-od and let $f : T \to T$ be a continuous map. If $\omega_f$ is a continuous map, must $Fix(f^m)$ be connected for all $m \geq 4$?*

**Question 3.15.** *Let $X$ be a fan and let $f : X \to X$ be a continuous map. If $\omega_f$ is a continuous map, must $Fix(f^m)$ be connected for all $m \geq 2$?*

In connection with Question 3.13, the following example shows that, there is a fan $X$ and a continuous function $f : X \to X$ that satisfy: $Fix(f^n)$ is a connected set, for all $n \in \mathbb{N}$ and $\omega_f$ is not continuous.

**Example 3.16.** Let $L$ be the segment of the line that join $(0,0)$ with $(1,0)$ and for each $n \in \mathbb{N}$ let $L_n$ be the segment of the line that join $(0,0)$ with $(1, \frac{1}{n})$. Define $X = \bigcup_{n=1}^{\infty} L_n \cup L$. For each $n \in \mathbb{N}$ define $f_n : [0,1] \to L_n$ given by $f_n(t) = (t, \frac{t}{n})$, $f_n$ is an homeomorphism.

Define $f : X \to X$ given by

$$f((x,y)) = \begin{cases} f_n(\frac{n}{n+1}(f_n^{-1}(x,y))) & (x,y) \in L_n, \\ (x,y) & (x,y) \in L. \end{cases}$$

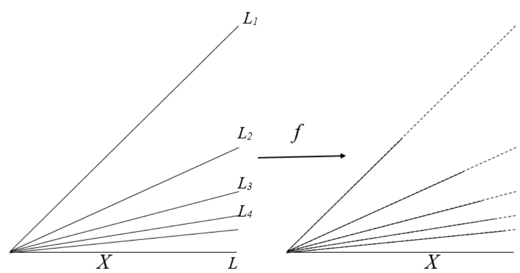


Figure 4.

Then, we have that

$$\omega_f((x,y)) = \begin{cases} (0,0) & (x,y) \in L_n, \\ (x,y) & (x,y) \in L. \end{cases}$$

Hence, $\omega_f$ is not continuous. On the other hand $Fix(f^n) = L$, so $Fix(f^n)$ is connected, for each $n \in \mathbb{N}$.

## 4. Equicontinuity of functions on the simple triod

Now, we address our attention to discrete dynamical systems on a simple triod. Through this section $T$ will be denote a simple triod with vertex $v$ and set of end points $\mathcal{E}(T) = \{e_1, e_2, e_3\}$, we consider $T$ with convex metric $d$ . Mainly we prove that $\omega_f$ is a continuous function if and only if $f$ is equicontinuous. To have this done we need the following preliminary results.

**Lemma 4.1.** *If $(T, f)$ is a discrete dynamical system such that $f$ is a surjective map and $\omega_f$ is a continuous function. If $x_0 \in T - Fix(f)$ satisfies that $f(x_0) \in Fix(f)$, then the following conditions hold:*

(1) *There exists a sequence* $\{x_n\}_{n=1}^{\infty}$ *with the following properties:*
 (a) $x_n \in T - Fix(f)$ *for each* $n \in \mathbb{N}$,
 (b) $x_i \neq x_j$ *for each* $i, j \in \mathbb{N}$ *such that* $i \neq j$,
 (c) $f(x_n) = x_{n-1}$ *for each* $n \in \mathbb{N}$, *and*
 (d) $\lim_{n\to\infty} x_n = f(x_0)$.
(2) $f(x_0) \in Fr(Fix(f))$,
(3) $[f(x_0), x_0] \cap Fix(f) = \{f(x_0)\}$.
 *Even more, when* $Fix(f) = \{f(x_0)\}$ *we have stronger properties:*
(4) *If* $f(x_0) \in (v, e_i)$ *for some* $i\{1, 2, 3\}$, *then there exists a strictly growing sequence* $\{k_n\}_{n=1}^{\infty}$, *when* $k_n \in \mathbb{N}$, *that satisfies the following conditions:*
 (i) *If* $n$ *is odd* $x_{k_n} \in (e_i, f(x_0))$ *and* $x_{k_j+1} \in (e_i, x_{k_j})$ *for each* $j \in \{k_n, \ldots, k_{n+1} - 2\}$,
 (ii) *if* $n$ *is even* $x_{k_n} \in (f(x_0), x_0)$ *and* $x_{k_j+1} \in (x_{k_j}, x_0)$ *for all* $j \in \{k_n, \ldots, k_{n+1} - 2\}$.
(5) *If* $f(x_0) = v$ *and the sequence* $\{x_n\}_{n=1}^{\infty}$ *satisfies that* $\{x_n : n \in \mathbb{N}\} \cap (v, e_i]$ *is infinite, for each* $i \in \{1, 2, 3\}$, *then there are strictly increasing sequences* $\{k_n\}_{n=1}^{\infty}$ *and* $\{l_n\}_{n=1}^{\infty}$ *that satisfy* $k_n \leq l_n < k_{n+1}$, *for every* $n \in \mathbb{N}$, *and the following conditions:*
 (i) *If* $n \in \mathbb{N} \cup \{0\}$ *and* $i \in \{1, 2, 3\}$, *then* $x_{k_{3n+i}} \in (e_i, v)$,
 (ii) *If* $n \in \mathbb{N}$, $i \in \{1, 2, 3\}$, $l_n > k_n$ *and* $x_{k_n} \in (e_i, v)$, *then* $x_{k_j+1} \in (x_{k_j}, e_i]$, *for each* $j \in \{k_n, \ldots, l_n - 1\}$,
 (iii) *If* $n \in \mathbb{N}$ *and* $i \in \{1, 2, 3\}$, *then* $x_{l_{3n+i}} \in (v, e_i)$ *and* $x_{l_{3n+i}+1} \notin (v, e_i)$.

*Proof.* (1). Since $f$ is surjective there exists $x_1 \in T$ such that $f(x_1) = x_0$, notice that $x_1 \neq x_0$ and $x_1 \in T - Fix(f)$. Assume that we constructed $x_1, x_2, \ldots, x_{n-1}$ that satisfy $(a) - (c)$. Being as $f$ is surjective there exists $x_n \in T$ such that $f(x_n) = x_{n-1}$, it is clear that $x_n$ satisfy $(a) - (c)$. Thus, we construct our sequence $\{x_n\}_{n=1}^{\infty}$. It follows from $(c)$ that $f^{n+1}(x_n) = f(x_0)$ for each $n \in \mathbb{N}$. Hence, $\omega_f(x_n) = \{f(x_0)\}$ for each $n \in \mathbb{N}$. Now, we procedure to prove $(d)$. Without loss of generality, suppose that $\lim_{n\to\infty} x_n = z$, then $f(z) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n-1} = z$. On the other hand, since $\omega_f$ is a continuous function, we have that $\lim_{n\to\infty} \omega_f(x_n) = \{f(x_0)\}$ for each $n \in \mathbb{N}$, then $\omega_f(z) = \{f(x_0)\}$. Then, $z = f(x_0)$ and $(d)$ is proved.

(2). By $(a)$ and $(d)$, we have that $x_n \in T - Fix(f)$ for each $n \in \mathbb{N}$, and $\lim_{n\to\infty} x_n = f(x_0)$, further, by hypothesis $f(x_0) \in Fix(f)$. Therefore, $f(x_0) \in Fr(Fix(f))$ and this shows (2).

To prove (3) suppose that $[f(x_0), x_0] \cap Fix(f) \neq \{f(x_0)\}$. Since $Fix(f)$ is connected, there is $z \in Fix(f) - \{f(x_0)\}$, for which $[f(x_0), x_0] \cap Fix(f) = [f(x_0), z]$. Let $y \in (f(x_0), z) \subset Fix(f)$. Since $[f(x_0), z] \subset f([z, x_0])$, there exists $x \in (z, x_0)$ such that $f(x) = y$ and $x \notin Fix(f)$. According to $(d)$ and (2), we have that there exists a sequence $\{y_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} y_n = u$, and $y \in Fr(fix(f))$, but this is a contradiction because of $u \in (f(x_0), z)$. So, (3) is proved.

(4) Without loss of generality, we assume that $f(x_0) \in (e_1, v)$. Notice that $[e_1, f(x_0)) \cap Fix(f) = \varnothing = (f(x_0), v] \cap Fix(f)$. Hence, we must have that

- $f(y) \in (y, v] \cup [v, e_2] \cup [v, e_3]$ for each $y \in [e_1, f(x_0))$ and
- $f(y) \in [e_1, y)$, for each $y \in (f(x_0), v]$.

Let $\varepsilon > 0$ such that $B_\varepsilon(f(x_0)) \subset (e_1, v)$. Since $\lim_{n\to\infty} x_n = f(x_0)$, there is $m \in \mathbb{N}$ that satisfies that $x_n \in B_\varepsilon(f(x_0))$ for each $n > m$. Fix $k_1 > m$ so that $x_{k_1} \in (e_1, f(x_0))$. Since $f(x_{k_1+1}) = x_{k_1}$ we have that $x_{k_1+1} \in [e_1, x_{k_1}) \cup (f(x_0), x_0)$. If $x_{k_1+1} \in (f(x_0), x_0)$, then put $k_2 = k_1 + 1$. If not, then $x_{k_1+1} \in [e_1, x_{k_1})$, and since $\lim_{n\to\infty} x_n = f(x_0)$, there is $k_2 > k_1$ such that $x_{k_2} \in (f(x_0), v)$ and $x_{k_j+1} \in (e_1, x_{k_j})$ for each $j \in \{k_1, \ldots, k_2 - 2\}$. As $f(x_{k_2+1}) = x_{k_2}$, we must have that $x_{k_2+1} \in [e_1, f(x_0)) \cup (x_{k_2}, x_0)$. If $x_{k_2+1} \in [e_1, f(x_0))$, then we define $k_3 = k_2 + 1$. Assume that $x_{k_2+1} \in [e_1, f(x_0))$. Since $\lim_{n\to\infty} x_n = f(x_0)$, there exists $k_3 > k_1$ such that $x_{k_3} \in (e_1, f(x_0))$ and $x_{k_j+1} \in (x_{k_j}, x_0)$ for all $j \in \{k_2, \ldots, k_3 - 2\}$. By following with this process we obtain our desired sequence.

(5) Without loss of generality, we suppose that $x_0 \in [e_1, v)$. Since $Fix(f) = \{v\}$, we have that $f(y) \in (y, v] \cup [v, e_2] \cup [v, e_3]$ for each $y \in (x_0, v)$. Choose $\varepsilon > 0$ so that $B_\varepsilon(v)$ is connected and $B_\varepsilon(v) \subset T - \{x_0\}$. Since $\lim_{n\to\infty} x_n = f(x_0)$, there exist $m \in \mathbb{N}$ such that $x_n \in B_\varepsilon(f(x_0))$ for each $n > m$.

Fix $k_1 > m$ such that $x_{k_1} \in (x_0, v)$. Since $f(x_{k_1+1}) = x_{k_1}$, we obtain that $x_{k_1+1} \in (x_0, x_{k_1}) \cup (v, e_2] \cup (v, e_3]$. If $x_{k_1+1} \in (v, e_2]$, then put $l_1 = k_1$ and $k_2 = k_1 + 1$. Then assume that $x_{k_1+1} \in (x_0, x_{k_1})$. Since $\lim_{n\to\infty} x_n = f(x_0)$, there is $l_1 > k_1$ so that $x_{k_j+1} \in (x_{k_j}, x_0) \subset (x_{k_j}, e_1]$ and $x_{l_1+1} \notin (v, e_1)$ for $j \in \{k_1, \ldots, l_1 - 1\}$. If $x_{l_1+1} \in (v, e_2]$, we set $k_2 = l_1 + 1$. Suppose that $x_{l_1+1} \in (v, e_3]$. Since $\{x_n : n \in \mathbb{N}\} \cap (v, e_2]$ is infinite and $\lim_{n\to\infty} x_n = v$, there is $k_2 > l_1 + 1$ such that $x_{k_2} \in (v, e_2)$ and $x_j \notin (v, e_2)$ for each $j \in \{l_1 + 1, \ldots, k_2 - 1\}$. Thus, $f(x_{k_2+1}) = x_{k_2}$ and then $x_{k_2+1} \in (x_{k_2}, e_2) \cup (v, e_1] \cup (v, e_3]$. We have arrived to the conditions from the beginning. By this way we define our required sequences $\{k_n\}_{n=1}^\infty$ and $\{l_n\}_{n=1}^\infty$. □

**Theorem 4.2.** *Let $(T, f)$ be a discrete dynamical system such that $f$ is a surjective map and $\omega_f$ is a continuous function. If $Fix(f) \neq T$, then $f(T - Fix(f)) \subset T - Fix(f)$.*

*Proof.* Suppose that there is $x_0 \in T - Fix(f)$ such that $f(x_0) \in Fix(f)$. Without of generality, suppose that $x_0 \in [e_1, v]$. By (1) of Lemma 4.1, we obtain that there exists a sequence $\{x_n\}_{n=1}^\infty$ with the following properties:

(a) $x_n \in T - Fix(f)$ for each $n \in \mathbb{N}$,
(b) $x_i \neq x_j$ for each $i, j \in \mathbb{N}$ such that $i \neq j$,
(c) $f(x_n) = x_{n-1}$ for each $n \in \mathbb{N}$,
(d) $\lim_{n\to\infty} x_n = f(x_0)$.

We consider the following cases:

**Case 1.** Either $Fix(f)$ is not degenerate or $Fix(f) \in \mathcal{E}(T)$. Since $\omega_f$ is a continuous function, by Corollary 3.6, we have that $Fix(f)$ is connected and, then we have that either $Fix(f) \subset [e_1, x_0)$ or $Fix(f) \subset (x_0, v] \cup [v, e_2] \cup [v, e_3]$.

**Case 1.1.** $Fix(f) \subset [e_1, x_0)$. Suppose that $a, b \in [e_1, x_0)$, such that $b \in [a, x_0)$ and $Fix(f) = [a, b]$. By (2) and (3) of Lemma 4.1 we have that $f(x_0) \in Fr(Fix(f))$ and $[f(x_0), x_0] \cap Fix(f) = \{f(x_0)\}$, then $b = f(x_0)$. Let $\varepsilon > 0$ such that $B_\varepsilon(f(x_0)) \subset (a, x_0)$. Since $\lim_{n\to\infty} x_n = f(x_0)$, then there exists $k \in \mathbb{N}$ such that $x_n \in B_\varepsilon(f(x_0)) \cap (f(x_0), x_0)$ for each $n > k$, and there exists $l > k$ such that $x_{l+1} \in (f(x_0), x_l)$. Since $[b, x_l] \subset f([x_{l+1}, x_0])$, $[x_{l+1}, x_0]$ has a fixed point of $f$, but this is a contradiction.

**Case 1.2.** $Fix(f) \subset (x_0, v] \cup [v, e_2] \cup [v, e_3]$. Choose $\varepsilon > 0$ such that $B_\varepsilon(f(x_0)) \cap [e_1, f(x_0)] \subset (x_0, f(x_0)]$. Since $\lim_{n\to\infty} x_n = f(x_0)$, we can find $k \in \mathbb{N}$ such that $x_n \in B_\varepsilon(f(x_0)) \cap [x_0, f(x_0)]$ for each $n > k$, and there is $l > k$ such that $x_{l+1} \in (x_l, f(x_0))$. Since $[x_l, f(x_0)] \subset f([x_0, x_{l+1}])$, we obtain that $[x_0, x_{l+1}]$ has a fixed point of $f$, but this is impossible.

**Case 2.** $Fix(f) = \{f(x_0)\}$, $f(x_0) \notin \mathcal{E}(T)$ and $f(x_0) \neq v$. Suppose that $f(x_0) \in (e_i, v)$ for some $i \in \{1, 2, 3\}$, we obtain from (4) of Lemma 4.1 that there exists a strictly increasing sequence $\{k_n\}_{n=1}^\infty$, when $k_n \in \mathbb{N}$, that satisfies the following conditions:

(i) If $n$ is odd $x_{k_n} \in (e_i, f(x_0))$ and $x_{k_j+1} \in (e_i, x_{k_j})$ for each $j \in \{k_n, \dots, k_{n+1} - 2\}$,
(ii) if $n$ is even $x_{k_n} \in (f(x_0), x_0)$ and $x_{k_j+1} \in (x_{k_j}, x_0)$ for all $j \in \{k_n, \dots, k_{n+1} - 2\}$.

Pick an odd integer $m$ so that $x_{k_m} \in [e_i, f(x_0))$ and $x_{k_{m+2}} \in (e_i, x_{k_m})$. For definition, we have that $f(x_{k_{m+1}}) = x_{k_{m+1}-1}$ and $x_{k_{m+1}-1} \in (e_i, x_{k_m}) \subset [e_1, f(x_0))$. Then $[x_{k_{m+1}-1}, f(x_0)] \subset f([f(x_0), x_{k_{m+1}}])$. On the other hand, since $x_{k_{m+2}} \in (e_i, x_{k_m})$ and $x_{k_{m+1}-1} \in (e_i, x_{k_m}) \subset [e_i, f(x_0))$, then $x_{k_{m+2}} \in [x_{k_{m+1}-1}, f(x_0)] \subset f([f(x_0), x_{k_{m+1}}])$. Hence, there is $z \in [f(x_0), x_{k_{m+1}}]$ such that $f(z) = x_{k_{m+2}}$. Observe that $f^2(z) = f(x_{k_{m+2}}) = x_{k_{m+2}-1}$. Since $[f(x_0), x_{k_{m+2}-1}] \subset f^2([z, x_0])$, we must have that $f^2([z, x_0])$ has a fixed point of $f^2$. That is, there is $y \in [z, x_0]$, such that $f^2(y) = y$. By Theorem 3.8, we have that $Fix(f^2)$ is connected and, so $[f(x_0), y] \subset Fix(f^2)$. Hence, $f^2(z) = z$ which is impossible because of $f^2(z) = x_{k_{m+2}-1}$.

**Case 3.** $Fix(f) = \{f(x_0)\}$ and $f(x_0) = v$. If there exist $i, j \in \{1, 2, 3\}$ and $k \in \mathbb{N}$ such that $x_n \in [e_i, e_j]$ for each $n > k$, then the proof follows as in the Case 2. Thus, we may suppose that $\{x_n : n \in \mathbb{N}, n > K\} \cap (v, e_i]$ is infinite for each $i \in \{1, 2, 3\}$. It follows from (5) of Lemma 4.1 that there are strictly increasing sequences $\{k_n\}_{n=1}^\infty$ and $\{l_n\}_{n=1}^\infty$, that satisfy $k_n \leq l_n < k_{n+1}$, for every $n \in \mathbb{N}$, and the following conditions:

(i) for each $n \in \mathbb{N} \cup \{0\}$ and $i \in \{1, 2, 3\}$, $x_{k_{3n+i}} \in (e_i, v)$,
(ii) for each $n \in \mathbb{N}$ and $i \in \{1, 2, 3\}$, if $l_n > k_n$ and $x_{k_n} \in (e_i, v)$, then $x_{k_j+1} \in (x_{k_j}, e_i]$, for each $j \in \{k_n, \dots, l_n - 1\}$,
(iii) for each $n \in \mathbb{N}$ and $i \in \{1, 2, 3\}$, $x_{l_{3n+i}} \in (v, e_i)$ and $x_{l_{3n+i}+1} \notin (v, e_i)$.

If there exists $r \in \mathbb{N}$, such that $l_r + 1 \neq k_{r+1}$ the proof follows as in the Case 2. Suppose that $l_n + 1 = k_{n+1}$ for each $n \in \mathbb{N}$. Let $m \in \mathbb{N}$ such that $x_{k_m} \in [x_0, v]$, $x_{k_{m+3}} \in (x_{k_m}, v)$. Then we have that $x_{k_{m+2}} \in [e_3, v)$ and $x_{k_{m+1}} \in [e_2, v)$. By definition we obtain that $f(x_{k_{m+3}}) = x_{k_{m+3}-1}$, then $[x_{k_{m+3}-1}, v]) \subset f([x_{k_{m+3}}, v]$, by (ii) $x_{k_{m+3}} \in (x_{k_{m+3}-1}, v) \subset f([x_{k_{m+3}}, v]$. Then there is $z \in [x_{k_{m+1}}, v]$ such that $f(z) = x_{k_{m+2}}$. Moreover $f(x_{k_{m+2}}) = x_{k_{m+2}-1}$ then $[x_{k_{m+2}-1}, v] \subset f([x_{k_{m+2}}, v]) \subset f^2([z, v])$. By (ii) we have that $x_{k_{m+1}} \in (x_{k_{m+2}-1}, v)$, then there exists $w \in [z, v]$ such that $f^2(w) = x_{k_{m+1}}$. So $[x_{k_{m+1}-1}, v] \subset f([x_{k_{m+1}}, v]) \subset f^3([w, v])$, since $f(x_{k_{m+1}}) = x_{k_{m+1}-1}$. By (ii), we obtain that $x_{k_1} \in (x_{k_{m+1}-1}, v)$ and $x_{k_{m+3}} \in (x_{k_1}, v)$. Then, there exists $y \in [w, v]$ such that $f^3(y) = x_{k_{m+3}}$. Hence, we obtain that $[v, x_{k_{m+3}}] \subset f^3([x_0, y])$. Thus, $[x_0, y]$ has a fixed point of $f^3$, says $y' \in [x_0, y]$. By Theorem 3.11, we know that $Fix(f^3)$ is connected and, then $[y', v] \subset Fix(f^3)$, which implies that $f^3(y) = y$, but this contradicts the equality $f^3(y) = x_{k_{m+3}}$. □

By the Corollary 3.6, we know that $Fix(f)$ is connected, when $\omega_f$ is a continuous map and $f$ is a surjective map of a simple triod to itself. So, it can be a point, an arc or a simple triod. The following result limits these possibilities and clarifies that set it is.

**Theorem 4.3.** *Let $(T, f)$ be a discrete dynamical system such that $f$ is a surjective map and $\omega_f$ is a continuous function. Then, one of the following conditions holds:*

(1) $Fix(f) = T$,
(2) $Fix(f) = \{v\}$, *or*
(3) *There exists $i \in \{1, 2, 3\}$, such that $Fix(T) = [v, e_i]$.*

*Proof.* Suppose that neither of the conditions, (1)-(3), is true. Since $\omega_f$ is a continuous function, we obtain from Corollary 3.6 that $Fix(f)$ is connected. We consider the following cases:

**Case 1.** $v \in Fix(f)$. By assumption, for each $i \in \{1, 2, 3\}$ there is $c_i \in [v, e_i]$ so that $Fix(f) = [c_1, v] \cup [v, c_2] \cup [v, c_3]$. As (2) and (3) fail, we have that $c_i \neq v$ and $c_j \neq e_j$ for some $i, j \in \{1, 2, 3\}$. Suppose, without loss of generality, that $e_1 \neq c_1$. Since $f$ is a surjective map, we can find $e \in T$ such that $f(e) = e_1$. We may assume that $e \notin (e_1, c_1)$; otherwise, $(e_1, c_1)$ would have a fixed point of $f$. Suppose, without loss of generality, that $e \in (c_2, e_2]$.

**Case 1.1.** $c_1 \neq v$. Since $[e_1, c_1] \subset [e_1, v] \subset f([v, e])$, there exists $c \in [v, e]$ such that $f(c) = c_1$. As $c \in (v, e] \subset (v, e_2]$, then $c \notin Fix(f)$. It follows from Theorem 4.2 that $f(c) \notin Fix(f)$. This contradicts the fact $f(c) = c_1 \in Fix(f)$.

**Case 1.2.** $c_2 \neq v$. As $v \in [e_1, c_2] \subset f([c_2, e])$, there is $c \in [c_2, e]$ such that $f(c) = v$. Since $c \in [c_2, e]$ and $c_2 \neq v$, we have that $c \neq v$. Hence, $c \notin Fix(f)$, but this contradicts Theorem 4.2.

**Case 1.3.** $c_1 = v = c_2$. Since neither (2) nor (3) hold, we obtain that $v \neq c_3 \neq e_3$. As $f$ is a surjective function there exits $d \in T$ such that $f(d) = e_3$. Then we proceed as in Case 1.1. Thus, this case is impossible.

**Case 2.** $v \notin Fix(f)$. Suppose, without of generality, that $Fix(f) \subset [e_1, v)$. Pick $a, b \in [e_1, v)$ such that $b \in [a, v)$ and $Fix(f) = [a, b]$. Notice that $f(v) \in (b, v)$; in other case, $f(v) \in [e_2, e_3]$ and then $[e_2, e_3]$ has a fixed point of $f$, but this impossible because of $Fix(f) \subset [e_1, v)$. Since $f(v) \in (b, v)$, $f(y) \in (b, y)$ for each $y \in (b, v]$, we obtain from Lemma 3.1 that $b \in \omega_f(v)$.

**Case 2.1.** $a = e_1$. As $f$ is a surjective map, there exists $d_i \in T$ such that $f(d_i) = e_i$ for each $i \in \{2, 3\}$. Note that $d_i \notin (v, e_i)$; otherwise, $(v, e_i)$ has a fixed point of $f$. Thus, $d_2 \in (v, e_3)$ and $d_3 \in (v, e_2)$. Since $[e_2, v] \subset f([v, d_2])$, we can choose $c_2 \in [v, d_2]$ for which $f(c_3) = d_3$. So $f^2(c_3) = e_3$ and, then $(v, e_3]$ has a fixed point of $f^2$, says $y \in (v, e_3]$. We know from Theorem 3.5 that $Fix(f^2)$ is connected, and so $[b, y] \subset Fix(f^2)$ which implies that $v \in Fix(f^2)$. Therefore, $\omega_f(v) = \{v, f(v)\}$. As $b \in \omega_f(v)$, we have that $f(v) = b$, but this contradicts Theorem 4.2 since $v \notin Fix(f)$.

**Case 2.2.** $a \neq e_1$. Since $f$ is a surjective function, there is $e \in T$ so that $f(e) = e_1$. Notice that $e \notin [e_1, a)$; if not, $[e_1, a)$ has a fixed point of $f$. Remember that $f(y) \in (b, y)$ for each $y \in (b, v]$. Hence, we have that $e \in [e_2, e_3]$. As $[e_1, f(v)] \subset f([v, e])$, there exists $c \in [v, e]$ such that $f(c) = a$. Since $c \in [v, e] \subset [e_2, e_3]$, $c \notin [a, b] \subset [e_1, v)$. But this is impossible by the Theorem 4.2.

In each case we obtain a contradiction. Thus, we obtain that one of the conditions either (1), (2) or (3) holds. □

**Proposition 4.4.** *Let $(T, f)$ be a discrete dynamical system such that $f$ is a surjective map and $\omega_f$ is a continuous function. Then for each $i \in \{1, 2, 3\}$ there exists $j \in \{1, 2, 3\}$ such that $f([v, e_i]) = [v, e_j]$.*

*Proof.* Fix $i \in \{1, 2, 3\}$ and assume that $f([v, e_i]) \neq [v, e_j]$ for each $j \in \{1, 2, 3\}$. It follows from Theorem 4.3 that $v \in Fix(f)$. We consider the following cases.

**Case 1.** There exist distinct $j, k \in \{1, 2, 3\}$ so that $f([v, e_i]) \cap (v, e_j] \neq \varnothing \neq f([v, e_i]) \cap (v, e_k]$. It follows from assumption that there exists $c, d \in (v, e_i]$ such that $f(c) \in (v, e_j]$ and $f(d) \in (v, e_k]$. Then, $v \in f([c, e]) \subset f((v, e_i])$ and, hence there is $v' \in [c, e]$ such that $f(v') = v$. By Theorem 4.2, we have that $v = f(v') \notin Fix(f)$, which is a contradiction.

**Case 2.** There exists $j \in \{1, 2, 3\}$ such that $f([v, e_i]) \subset [v, e_j)$. As $f$ is a surjective function there exists $e \in T$ such that $f(e) = e_j$. Since $f([v, e_i]) \subset [v, e_j)$, $e \notin [v, e_i]$. Suppose that $e \in [v, e_k]$ for $k \in \{1, 2, 3\} \setminus \{i\}$.

• If $f([v, e_k]) \neq [v, e_j]$. Then, there exits $c \in (v, e_k]$ such that $f(c) \notin (v, e_j]$. Assume that $f(c) \in [v, e_l]$, where $l \neq j$. Then, $f([v, e_k]) \cap (v, e_j] \neq \varnothing \neq f([v, e_k]) \cap (v, e_l]$.

• Suppose that $f([v, e_k]) = [v, e_j]$. Then, $f([v, e_i]) \cup f([v, e_k]) = [v, e_j]$. First, if $i \neq j \neq k$, then $[v, e_k] \cup [v, e_i] \subset f([v, e_j])$ and so $f([v, e_j]) \cap (v, e_k] \neq \varnothing \neq f([v, e_j]) \cap (v, e_i]$. Now, if $i = j$ and $l \in \{1, 2, 3\} \setminus \{i, k\}$, then $[v, e_k] \cup [v, e_l] \subset f([v, e_l])$ and hence $f([v, e_l]) \cap (v, e_k] \neq \varnothing \neq f([v, e_l]) \cap (v, e_l]$. Finally if $j = k$ and $l \in \{1, 2, 3\} \setminus \{i, k\}$, then $[v, e_i] \cup [v, e_l] \subset f([v, e_l])$ which implies that $f([v, e_l]) \cap (v, e_i] \neq \varnothing \neq f([v, e_l]) \cap (v, e_l]$.

In each one of the previous case, we arrived to the conditions of Case 1. So we can obtain a contradiction.

□

**Theorem 4.5.** *Let $(T, f)$ be a discrete dynamical system such that $f$ is a surjective map and $\omega_f$ is a continuous function, then $f|_{\mathcal{E}(T)}$ is a permutation.*

*Proof.* Suppose that $Fix(f) \neq T$. By Proposition 4.4, we have that for each $i \in \{1,2,3\}$ there exists $j_i \in \{1,2,3\}$ such that $f([v,e_i]) = [v,e_{j_i}]$. By Theorem 4.3 we need consider the following cases.

**Case 1.** There exists $i \in \{1,2,3\}$ satisfying $Fix(f) = [v,e_i]$. Assume, without of generality, that $i = 1$. Then, we have that $f([v,e_2]) = [v,e_3]$ and $f([v,e_3]) = [v,e_2]$. We will prove, that $f(e_2) = e_3$ and $f(e_3) = e_2$. Suppose that $f(e_2) \neq e_3$, since $f([v,e_2]) = [v,e_3]$, we have that there exists $b_2 \in (v,e_2)$ such that $f(b_2) = e_3$.

**Case 1.1.** $f(e_3) = e_2$. Since $f(b_2) = e_3$ we have that $[v,e_2] \subset f^2([v,b_2])$. Then, $(b_2,e_2]$ has a fixed point of $f^2$, says $c \in (b_2,e_2]$. By Theorem 3.8, we have that $Fix(f^2)$ is connected. Thus, $[v,c] \subset Fix(f^2)$ for which $f^2(b_2) = b_2$, this is a contradiction, because of $f^2(b_2) = e_2$.

**Case 1.2.** $f(e_3) \neq e_2$. As $f([v,e_2]) = [v,e_3]$ we have that, there exists $b_3 \in (v,e_3)$ such that $f(b_3) = e_2$. Since $[v,e_3] \subset f([v,b_2])$, there is $c_2 \in [v,b_2]$ such that $f(c_2) = b_3$. Then $[v,e_2] \subset f^2([v,b_3]) =$ and so $f((c_2,e_2])$ has a fixed point of $f^2$, says $c \in (c_2,e_2]$. We know from the Theorem 3.8 that $Fix(f^2)$ is connected. Hence, $[v,c] \subset Fix(f^2)$ and then $f^2(c_2) = c_2$. This contradicts the fact $f^2(c_2) = e_2$.

**Case 2.** $Fix(f) = \{v\}$. By Proposition 4.4, we can assume, without of generality, that $f([v,e_1]) = [v,e_2]$, $f([v,e_2]) = [v,e_3]$, $f([v,e_3]) = [v,e_1]$. We will show, that $f(e_1) = e_2$, $f(e_2) = e_3$ and $f(e_3) = e_1$. Suppose that $f(e_1) \neq e_2$. Since $f([v,e_1]) = [v,e_2]$, there exists $b_1 \in (v,e_1)$ that satisfies $f(b_1) = e_2$.

**Case 2.1.** $f(e_2) = e_3$ and $f(e_3) = e_1$. Since $f(b_1) = e_2$, we obtain that $[v,e_1] = f([v,e_3]) = f^2([v,e_2]) = f^3([v,b_1])$. Hence, $(b_1,e_1]$ has a fixed point of $f^3$, says $c \in (b_1,e_1]$. It follows from Theorem 3.11 that $Fix(f^3)$ is connected. Thus, $[v,c] \subset Fix(f^3)$ for which $f^3(b_1) = b_1$, this is a contradiction, because of $f^3(b_1) = e_1$.

**Case 2.2.** $f(e_2) \neq e_3$ and $f(e_3) = e_1$. By assumption $f([v,e_2]) = [v,e_3]$, for which, there is $b_2 \in [v,e_2]$ such that $f(b_2) = e_3$. By other hand as $[v,e_2] = f([v,b_1])$, we can find $c_1 \in [v,e_1]$ such that $f(c_1) = b_2$. Hence, $[v,e_1] = f([v,e_3]) = f^2([v,b_2]) \subset f^3([v,c_1])$ and then $(c_1,e_1]$ has a fixed point of $f^3$, says $c \in (c_1,e_1]$. We know from the Theorem 3.11 that $Fix(f^3)$ is connected. Hence, $[v,c] \subset Fix(f^3)$ and so $f^3(c_1) = c_1$. This contradicts the fact $f^3(c_1) = e_1$.

**Case 2.3.** $f(e_2) = e_3$ and $f(e_3) \neq e_1$. Since $f([v,e_3]) = [v,e_1]$. So, we can choose $b_3 \in [v,e_3]$ such that $f(b_3) = e_1$. By other hand as $f(b_1) = e_2$, then $[v,e_3] = f([v,e_2]) = f^2([v,b_1])$, and so, we can pick $c_1 \in [v,b_1]$ such that $f^2(c_1) = b_3$. Hence, $[v,e_1] = f([v,b_3]) \subset f^3([v,c_1])$ $(c_1,e_1]$ has a fixed point of $f^3$, says $c \in (c_1,e_1]$. By Theorem 3.11, $Fix(f^3)$ is connected. Thus, $[v,c] \subset Fix(f^3)$ and then $f^3(c_1) = c_1$, but this is impossible since $f^3(c_1) = e_1$.

**Case 2.4.** $f(e_2) \neq e_3$ and $f(e_3) \neq e_1$. By supposition $f([v,e_2]) = [v,e_3]$ and $f([v,e_3]) = [v,e_1]$, then we can find $b_2 \in (v,e_2)$ and $b_3 \in (v,e_3)$ such that $f(b_2) = e_3$ and $f(b_3) = e_1$. Since $[v,e_2] \subset f([v,b_1])$, there is $c_1 \in (v,b_1)$ such that $f(c_1) = b_2$. So, $[v,e_3] \subset f^2([v,c_1])$, then choose $d_1 \in [v,c_1]$ such that $f^2(d_1) = b_3$. Hence, $[v,e_1] \subset f^3([v,d_1])$ for which $(d_1,e_1]$ has a fixed point of $f^3$, says $c \in (d_1,e_1]$. We know from the Theorem 3.11 that $Fix(f^3)$ is connected.

Hence, $[v, c] \subset Fix(f^3)$ and then $f^3(d_1) = d_1$. This is a contradiction because of $f^3(d_1) = e_1$.

To proof $f(e_2) = e_3$ and $f(e_3) = e_1$, we proceed in the similar way.

In each case we have that $f|_{E(T)}$ is a permutation. □

**Corollary 4.6.** *Let $(T, f)$ be a discrete dynamical system such that $f$ is a surjective map and $\omega_f$ is a continuous function. Then $T = Per(f)$ and each point of $T$ is $n-$periodic, where $n \in \{1, 2, 3\}$.*

*Proof.* It follows from Theorem 4.3 that one of the following conditions holds.

(1) $Fix(f) = T$,
(2) $Fix(f) = \{v\}$,
(3) There exists $i \in \{1, 2, 3\}$, such that $Fix(T) = [v, e_i]$.

If $Fix(f) = T$, then each point of $T$ is a fixed point, so 1-periodic and $T = Per(f)$.

**Case 1.** There exists $i \in \{1, 2, 3\}$ such that $Fix(f) = [v, e_i]$. Suppose that $i = 1$. We obtain from Proposition 4.4 that $f([v, e_2]) = [v, e_3]$ and $f([v, e_3]) = [v, e_2]$. It follows from Theorem 4.5 that $f(e_2) = e_3$ and $f(e_3) = e_2$. Thus, $f^2(e_2) = e_2$, $f^2(e_3) = e_3$ and $f^2(e_1) = e_1$. By Theorem 3.8 we have that $Fix(f^2)$ is connected, hence $Fix(f^2) = T$. Therefore, each point of $T$ is at most 2-periodic and $T = Per(f)$.

**Case 2.** $Fix(f) = \{v\}$. By Proposition 4.4, we have that for each $i \in \{1, 2, 3\}$ there exists $j_i \in \{1, 2, 3\}$ such that $f([v, e_i]) = [v, e_{j_i}]$. Without of generality, suppose that $f([v, e_1]) = [v, e_2]$, $f([v, e_2]) = [v, e_3]$ and $f([v, e_3]) = [v, e_1]$. We obtain from the Theorem 4.5 that $f(e_1) = e_2$, $f(e_2) = e_3$ and $f(e_3) = e_1$. So, $f^3(e_1) = e_1$, $f^3(e_2) = e_2$ and $f^3(e_3) = e_3$. We know from the Theorem 3.11 that $Fix(f^3)$ is connected. Hence, $Fix(f^3) = T$ and so each point of $T$ is 3-periodic and $T = Per(f)$. □

The following result is the version of the Theorem 4.5, when the phase space is the arc. Following the same way, of the proof, of Theorem 4.5 it is easy to prove it.

**Theorem 4.7.** *Let $f : [0, 1] \to [0, 1]$ be a surjective continuous function such that $\omega_f$ is a continuous function. Then $f(0) = 0$ and $f(1) = 1$ or $f(0) = 1$ and $f(1) = 0$.*

**Corollary 4.8.** *Let $f : [0, 1] \to [0, 1]$ be a surjective continuous function such that $\omega_f$ is a continuous function. Then $f = f^0$ or $f^2 = f^0$*

**Corollary 4.9.** *Let $f : [0, 1] \to [0, 1]$ be a surjective continuous function such that $\omega_f$ is a continuous function. Then each point of $[0, 1]$ is $n-$periodic with $n \in \{1, 2\}$.*

The following result shows that it is sufficient to request the equicontinuity of a function $f$, so that, the function $\omega_f$ will be a continuous map, when the phase space is a compact metric space.

 

**Proposition 4.10.** *Let $(X, f)$ be a discrete dynamical system, where $X$ is a metric compact space with metric $d$. If $f$ is equicontinuous, then $\omega_f$ is a continuous function.*

*Proof.* Let $\varepsilon > 0$. Since $f$ is equicontinuous there is $0 < \delta \leq \frac{\varepsilon}{2}$ such that $d(f^n(x), f^n(y)) < \frac{\varepsilon}{2}$, for each $x, y \in X$ with $d(x, y) \leq \delta$, and every $n \in \mathbb{N}$.

Fix $x, y \in X$ such that $d(x, y) < \delta$. Choose $z \in \omega_f(x)$, by definition, there is a sequence of positive integers $n_1 < n_2 < \ldots$, such that $\lim_{k\to\infty} f^{n_k}(x) = z$. As $X$ is compact, we can assume that $\lim_{k\to\infty} f^{n_k}(y) = y'$. Hence, $y' \in \omega_f(y)$. Since $d(f^{n_k}(x), f^{n_k}(y)) < \frac{\varepsilon}{2}$, for each $k \in \mathbb{N}$, we obtain that $d(z, y') \leq \frac{\varepsilon}{2} < \varepsilon$. Thus, $\omega_f(x) \subset \mathcal{N}_\varepsilon(\omega_f(y))$. Similarly we obtain that $\omega_f(y) \subset \mathcal{N}_\varepsilon(\omega_f(x))$. It follows from [4, 2.9] that $\mathcal{H}(\omega_f(x), \omega_f(y)) < \varepsilon$. Therefore, $\omega_f$ is a continuous function. □

Now we will prove the main result of this work, in which we show that the equicontinuity of a function $f$ is equivalent to the continuity of $\omega_f$, when the phase space is the simple triod, this result was proved in [1], when the phases space is the arc.

*Theorem A* 2. *Let $(T, f)$ be a discrete dynamical system. Then, $\omega_f$ is a continuous function if and only if $f$ is equicontinuous.*

*Proof.* If $f$ is equicontinuous by Proposition 4.10 we have that $\omega_f$ is a continuous function.

Suppose that $\omega_f$ is a continuous function, we consider the following cases.

**Case 1.** $f$ is a surjective function. We know from Theorem 4.6 that $T = Per(f)$. Thus, each point of $T$ has periodic orbit and so $\omega_f$ is a periodic orbit for every point of $X$. Moreover $\omega_f$ is a continuous function, then follows of [9, Theorem 3.8] that $f$ is equicontinuous.

**Case 2.** $f$ is not a surjective function. Define $R = \bigcap_{n=1}^{\infty} f^n(T)$. Notice that $f(R) = R \neq \varnothing$, then $f|_R : R \to R$ is a surjective continuous function.

- If $R$ is a point, then $f|_R$ is equicontinuous.
- If $R$ is an arc, it follows from [1, Theorem 1.2] that $f|_R$ is equicontinuous.
- If $R$ is a simple triod, we know from Case 1 that $f|_R$ is equicontinuous.

So, $f|_R$ is equicontinuous.

**Case 2.1.** $R$ is degenerate. Pick $a \in T$ such that $R = \{a\}$. Let $\varepsilon > 0$, since $\{a\} = \bigcap_{n=1}^{\infty} f^n(T) = \lim_{n\to\infty} f^n(T)$, then there exists $m \in \mathbb{N}$ such that $\mathcal{H}(f^n(T), \{a\}) < \frac{\varepsilon}{2}$ for each $n > m$. For which $d(f^n(x), a) < \frac{\varepsilon}{2}$ for each $x \in T$ and $n \in \mathbb{N}$. By other hand as $f^n$ is a continuous function for each $n \in \{1, \ldots, m\}$, there exists $\delta > 0$ such that $d(f^n(x), f^n(y)) < \varepsilon$, for each $x, y \in T$ with $d(x, y) \leq \delta$, and all $n \in \{1, \ldots, m\}$. It follows from the above that $d(f^n(x), f^n(y)) < \varepsilon$, for each $x, y \in T$ with $d(x, y) \leq \delta$, and every $n \in \mathbb{N}$. Therefore $f$ is equicontiuous.

**Case 2.2.** $R$ is not degenerate. By assumption, either $R$ is an arc or $R$ a a simple triod. We suppose that $R$ is a simple triod, the proof when $R$ is an arc it follows similarly. Since $R$ is a simple triod, by Corollary 4.6, we have that $R = Per(f|_R)$ and each point of $R$ is $n-$ periodic for some $n \in \{1, 2, 3\}$.

Thus, $f^6(x) = x$ for each $x \in R$. For $i \in \{1,2,3\}$, choose $c_i \in (v, e_i]$ such that $R = [c_1, v] \cup [v, c_2] \cup [v, c_3]$. Set $s_i = diam([v, c_i])$ for each $i \in \{1,2,3\}$, and $s = \min\{s_i : i \in \{1,2,3\}\}$. Let $0 < \varepsilon < s$ such that $\mathcal{N}_\varepsilon(R) \subseteq T$. Now, we will find positive numbers $\delta_3 < \delta_2 < \delta_1 < \frac{\varepsilon}{2}$ as follows:

• Since $f, f^2, \ldots, f^6$ are continuous maps, there exists $0 < \delta_1 < \frac{\varepsilon}{2}$ such that, if $x, y \in T$ and $d(x,y) < \delta_1$, then $d(f^n(x), f^n(y)) < \frac{\varepsilon}{2}$ for each $n \in \{1, 2, \ldots, 6\}$.

• $f|_R$ is equicontinuous, we can find $0 < \delta_2 < \delta_1$ such that $d(f^n(x), f^n(y)) < \frac{\varepsilon}{2}$ for each $x, y \in T$ with $d(x,y) \leq \delta_2$, and every $n \in \mathbb{N}$.

• Since $R = \lim_{n\to\infty} f^n(T) = R$, there is $m \in \mathbb{N}$, $m = 6k$ for some $k \in \mathbb{N}$, such that $\mathcal{H}(f^n(T), R) < \delta_2$ for each $n > m$.

• Since $f, f^2, \ldots, f^{m+1}$ are continuous functions, then there exists $0 < \delta_3 < \delta_2$ such that, if $x, y \in T$, $d(x,y) < \delta_3$ and $n \in \{1, 2, \ldots, m+1\}$; then $d(f^n(x), f^n(y)) < \delta_2$.

Fix $x, y \in T$ such that $d(x,y) < \delta_3$. We have that $d(f^{m+1}(x), f^{m+1}(y)) < \delta_2$ and $f^{m+1}(x), f^{m+1}(y) \in \mathcal{N}_{\delta_2}(R)$.

**Case 2.2.1.** $f^{m+1}(x), f^{m+1}(y) \in R$. Since $d(f^{m+1}(x), f^{m+1}(y)) < \delta_2$, we have that $d(f^n(f^{m+1}(x)), f^n(f^{m+1}(y))) < \frac{\varepsilon}{2}$ for each $n \in \mathbb{N}$. Moreover, since $d(x,y) < \delta_3$, we know that $d(f^n(x), f^n(y)) < \delta_2 < \frac{\varepsilon}{2}$ for each $n \in \{1, 2, \ldots, m+1\}$. Therefore, $d(f^n(x), f^n(y)) < \varepsilon$ for each $n \in \mathbb{N}$.

**Case 2.2.2.** $f^{m+1}(x) \in R$ and $f^{m+1}(y) \notin R$. Suppose, without of generality, that $c_1 \in [f^{m+1}(y), f^{m+1}(x)]$. Then we have that $d(f^{m+1}(y), c_1) < \delta_2$ and $d(f^{m+1}(x), c_1) < \delta_2$. For which $d(f^n(f^{m+1}(x)), f^n(c_1)) < \frac{\varepsilon}{2}$, for each $n \in \mathbb{N}$. Since $d(f^{m+1}(y), c_1) < \delta_2 < \delta_1$, we have that $d(f^{m+1+i}(y), f^i(c_1)) < \frac{\varepsilon}{2}$ for each $i \in \{1, 2, \ldots, 6\}$, but we know that $\mathcal{H}(f^n(T), R) < \delta_2$ for each $n > m$, then $d(f^{m+1+i}(y), f^i(c_1)) < \delta_2$ for each $i \in \{1, 2, \ldots, 6\}$. Now, if there is $i \in \{1, 2, \ldots, 6\}$ such that $f^{m+1+i}(y) \in R$, then $d(f^n(f^{m+1+i}(y)), f^n(f^i(c_1))) < \frac{\varepsilon}{2}$, for each $n \in \mathbb{N}$, so $d(f^n(f^{m+1}(x)), f^n(f^{m+1}(y))) < \varepsilon$. Further, since $d(x,y) < \delta_3$, we have that $d(f^n(x), f^n(y)) < \frac{\varepsilon}{2}$ for each $n \in \{1, 2, \ldots, m+1\}$. Therefore, $d(f^n(x), f^n(y)) < \varepsilon$ for each $n \in \mathbb{N}$.

Assume that $f^{m+1+i}(y) \notin R$ for each $i \in \{1, 2, \ldots, 6\}$. We know that $d(f^{m+7}(y), f^6(c_1)) = d(f^{m+7}(y), c_1) < \delta_2$, then we can proceed in a way similar to what was done previously, and so $d(f^n(f^{m+1}(y), f^n(c_1))) < \frac{\varepsilon}{2}$ for each $n \in \mathbb{N}$. Further, since $d(f^n(f^{m+1}(x)), f^n(c_1))) < \frac{\varepsilon}{2}$ and $d(f^n(f^{m+1}(y)), f^n(c_1))) < \frac{\varepsilon}{2}$, we obtain $d(f^n(f^{m+1}(x)), f^n(f^{m+1}(y))) < \varepsilon$. Moreover, as $d(x,y) < \delta_3$, we have that $d(f^n(x), f^n(y)) < \delta_2 < \frac{\varepsilon}{2}$ for each $n \in \{1, 2, \ldots, m+1\}$. Therefore, $d(f^n(x), f^n(y)) < \varepsilon$ for each $n \in \mathbb{N}$.

**Case 2.2.3.** $f^{m+1}(x), f^{m+1}(y) \notin R$. Since $d(f^{m+1}(x), f^{m+1}(y)) < \delta_2 < s$, and $\mathcal{H}(f^{m+1}(T), R) < \delta_2$ we can suppose, without loss of generality that $d(f^{m+1}(x), c_1) < \delta_2$, and $d(f^{m+1}(x), c_1) < \delta_2$. Following similarly to Case 2.2.2, we obtain the following inequality $d(f^n(f^{m+1}(x), f^n(c_1))) < \frac{\varepsilon}{2}$ and $d(f^n(f^{m+1}(y), f^n(c_1))) < \frac{\varepsilon}{2}$ for each $n \in \mathbb{N}$, so $d(f^n(f^{m+1}(x)), f^n(f^{m+1}(y))) < \varepsilon$. As $d(x,y) < \delta_3$, $d(f^n(x), f^n(y)) < \delta_2 < \frac{\varepsilon}{2}$ for each $n \in \{1, 2, \ldots, m+1\}$. Therefore, $d(f^n(x), f^n(y)) < \varepsilon$ for each $n \in \mathbb{N}$. □

**Corollary 4.11.** *Let $(T, f)$ be a discrete dynamical system such that $\omega_f$ is a continuous function. Then $\omega_{f^n}$ is a continuous function and $Fix(f^n)$ is connected, for each $n \in \mathbb{N}$.*

*Proof.* By Theorem A 2 we have that $f$ is equicontinuos, notice that if $f$ es equicontinuos then $f^n$ is equicontinuos then by Theorem A 2 we have that $\omega_{f^n}$ is a continuous function, and so it follows from Corollary 3.6 that $Fix(f^n)$ is connected. □

**Question 4.12.** *Can Theorem A 2 be extended when the phase space is a $n$-od with $n \geq 4$?*

To finish this paper, we give an example of a function $f$ that is equicontinuous, however the function $\omega_f$ is not continuous. When the phase space is a harmonic fan.

**Example 4.13.** Consider $X$, $L$, $L_n$ and $f_n : [0,1] \to L_n$ as in Example 3.16. Define $g_n : [0, \frac{1}{2^n}] \to [0, \frac{n+1}{2^n}]$ given by $g_n(t) = (n+1)(t)$; $h_n : [\frac{1}{2^n}, \frac{1}{2^{n-1}}] \to [0, \frac{n+1}{2^n}]$ given by $h_n(t) = (n+1)(\frac{1}{2^{n-1}} - t)$ and

$$f((x,y)) = \begin{cases} f_{n+1}(g_n(f_n^{-1}(x,y))) & (x,y) \in L_n \text{ and } f_n^{-1}(x,y) \in [0, \frac{1}{2^n}], \\ f_{n+1}(h_n(f_n^{-1}(x,y))) & (x,y) \in L_n \text{ and } f_n^{-1}(x,y) \in [\frac{1}{2^n}, \frac{1}{2^{n-1}}], \\ (0,0) & (x,y) \in L_n \text{ and } f_n^{-1}(x,y) \in [\frac{1}{2^{n-1}}, 1], \\ (0,0) & (x,y) \in L. \end{cases}$$

We have that for each $x \in X$, $\omega_f(x) = \{(0,0)\}$, hence $\omega_f$ is a continuous function. We will show that $f$ is not equicontinuous.

We consider $X$ with the maximum metric, $d_M$. Let $\varepsilon > 0$, given $m \in \mathbb{N}$ such that $\frac{1}{2^m} < \varepsilon < \frac{1}{2^{m-1}}$. We consider the point $(x,y) = (\frac{1}{2^m m!}, \frac{1}{2^m m!}) \in L_1$, then we have that $f^{m-1}((x,y)) = f_m(\frac{1}{2^m}) \in L_m$, so $f^m((x,y)) = f_{m+1}(\frac{m+1}{2^m})$, then $f^{m+1}((x,y)) = (0,0)$ because of $\frac{m+1}{2^m} > \frac{1}{2^{m-1}}$. Now, we have that $d_M((x,y),(0,0)) = \frac{1}{2^m m!} < \frac{1}{2^m} < \varepsilon$, and we have that $d_M(f^m(x,y), f^m(0,0)) = \frac{m+1}{2^m} > \frac{1}{2^{m-1}} > \varepsilon$. Therefore, $f$ is not equicontinuous.

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